Chapter 2
A Variational Characterization of the Best Lyapunov Constants

Abstract This chapter is devoted to the definition and main properties of the L_p Lyapunov constant, $1 \leq p \leq \infty$, for scalar ordinary differential equations with different boundary conditions, in a given interval $(0, L)$. It includes resonant problems at the first eigenvalue and nonresonant problems. A main point is the characterization of such a constant as a minimum of some especial minimization problem, defined in appropriate subsets X_p of the Sobolev space $H^1(0, L)$. This variational characterization is a fundamental fact for several reasons: first, it allows to obtain an explicit expression for the L_p Lyapunov constant as a function of p and L; second, it allows the extension of the results to systems of equations (Chap. 5) and to PDEs (Chap. 4). For resonant problems (Neumann or periodic boundary conditions), it is necessary to impose an additional restriction to the definition of the spaces $X_p,\ 1 \leq p \leq \infty$, so that we will have constrained minimization problems. This is not necessary in the case of nonresonant problems (Dirichlet or antiperiodic boundary conditions) where we will find unconstrained minimization problems. For nonlinear equations, we combine the Schauder fixed point theorem with the obtained results for linear equations.

2.1 Neumann Boundary Conditions, As a Paradigm of Linear Resonant Problems

This section will be concerned with the existence of nontrivial solutions of the homogeneous linear problem with Neumann boundary conditions

$$u''(x) + a(x)u(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0. \tag{2.1}$$

If $a(\cdot)$ is a constant function $\lambda \in \mathbf{R}$, (2.1) has nontrivial solutions if and only if λ belongs to the set $\{\lambda_n = n^2\pi^2/L^2,\ n \in \mathbf{N} \cup \{0\}\}$, i.e., the set of eigenvalues of the eigenvalue problem:

$$u''(x) + \lambda u(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0. \tag{2.2}$$

Obviously, the problem is much more complicated if $a(\cdot)$ is not a constant function.

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To proceed to the definition of the Lyapunov constants, we will assume throughout this chapter that $a \in \Lambda$, where Λ is defined by

$$\Lambda = \{a \in L^1(0,L) \setminus \{0\} : \int_0^L a(x)\,dx \geq 0 \text{ and (2.1) has nontrivial solutions }\}. \tag{2.3}$$

Here, for each $p, 1 \leq p < \infty$, $L^p(0,L)$ denotes the usual Lebesgue space of measurable functions $a(\cdot)$ such that $|a(\cdot)|^p$ is integrable in $(0,L)$, while $L^\infty(0,L)$ denotes the set of measurable functions such that there exists a constant c satisfying $|a(x)| \leq c$, a.e. in $(0,L)$. On the other hand, $u \in H^1(0,L)$, the usual Sobolev space. The interested reader can consult the reference [2] for these concepts.

For each p with $1 \leq p \leq \infty$, we can define the functional $I_p : \Lambda \bigcap L^p(0,L) \to \mathbf{R}$ given by the expression

$$\begin{aligned} I_p(a) &= \|a^+\|_p = \left(\int_0^L |a^+(x)|^p\,dx \right)^{1/p}, \forall\, a \in \Lambda \bigcap L^p(0,L),\ 1 \leq p < \infty \\ I_\infty(a) &= \|a^+\|_\infty = \sup\ \text{ess}\ a^+,\ \forall a \in \Lambda \bigcap L^\infty(0,L), \end{aligned} \tag{2.4}$$

where a^+ is the positive part of the function a (i.e., $a^+(x) = \max\{0, a(x)\}$) and sup ess a^+ is the essential supremum of the function a^+.

Since the positive eigenvalues of the eigenvalue problem (2.2), belong to the set $\Lambda \bigcap L^p(0,L)$, the nonnegative constant

$$\beta_p \equiv \inf_{a \in \Lambda \bigcap L^p(0,L)} I_p(a),\ 1 \leq p \leq \infty \tag{2.5}$$

is well defined. Due to the pioneering work of Lyapunov for Dirichlet boundary conditions and $p = 1$ [16, 22, 23], we will call to the constant β_p, defined in (2.5), the *best (optimal)* L_p *Lyapunov constant*.

Remark 2.1. We need the positivity of $\int_0^L a(x)\,dx$ in order to prove that the constant β_p is strictly positive. In fact, if the set Λ in (2.3) is replaced by

$$\Upsilon = \{a \in L^1(0,L) \setminus \{0\} :\ \text{(2.1) has nontrivial solutions }\}$$

then the constant

$$\gamma_p \equiv \inf_{a \in \Upsilon \bigcap L^p(0,L)} I_p(a),\ 1 \leq p \leq \infty$$

is zero, for each $p, 1 \leq p \leq \infty$ (see Remark 4 in [3]). The real number 0 is the first eigenvalue of the eigenvalue problem (2.2). As it will be seen in Sect. 2.3, this *extra condition* on the sign of $\int_0^L a(x)\,dx$ is not necessary in nonresonant problems.

Remark 2.2. The study of the constant β_p can be seen as an optimal control problem: the admissible control set is $\Lambda \bigcap L^p(0,L)$ and the functional that we want to minimize is I_p. However, we caution that the condition

$$\text{(2.1) has nontrivial solutions} \tag{2.6}$$

is difficult to handle from a mathematical point of view and this is the main difficulty of the problem. Because of this, one of the main purposes of this chapter is to get a variational characterization of the best Lyapunov constant β_p. This will be very important for the possible extension of the results to PDEs and to systems of equations.

We begin with the easiest situation: $p = \infty$. In this case, the constant β_∞ is nothing but the first positive eigenvalue of (2.2). The proof is known and it uses two basic ideas: Hölder's inequality and the variational characterization of the eigenvalues of (2.2) [8].

Theorem 2.1.

$$\beta_\infty = \min_{v \in X_\infty \setminus \{0\}} \frac{\int_0^L (v')^2}{\int_0^L (v)^2} = \frac{\pi^2}{L^2}, \tag{2.7}$$

where $X_\infty = \{v \in H^1(0,L) : \int_0^L v = 0\}$.

Proof. If $a \in \Lambda$ and $u \in H^1(0,L)$ is a nontrivial solution of (2.1), then

$$\int_0^L u'v' = \int_0^L auv, \ \forall \ v \in H^1(0,L).$$

In particular, we have

$$\int_0^L u'^2 = \int_0^L au^2, \ \int_0^L au = 0. \tag{2.8}$$

Therefore, for each $k \in \mathbf{R}$, we have

$$\int_0^L (u+k)'^2 = \int_0^L u'^2 = \int_0^L au^2 \le \int_0^L au^2 + k^2 \int_0^L a$$

$$= \int_0^L au^2 + \int_0^L k^2 a + 2k \int_0^L au = \int_0^L a(u+k)^2 \le \int_0^L a^+(u+k)^2.$$

Hölder's inequality implies

$$\int_0^L (u+k)'^2 \leq \|a^+\|_\infty \int_0^L (u+k)^2.$$

Also, since the function a belongs to Λ, u is a nonconstant solution of (2.1), so that $u+k$ is a nontrivial function. Consequently

$$\|a^+\|_\infty \geq \frac{\displaystyle\int_0^L (u+k)'^2}{\displaystyle\int_0^L (u+k)^2}.$$

Now, choose $k_0 \in \mathbf{R}$ satisfying

$$\int_0^L (u+k_0) = 0. \tag{2.9}$$

Then,

$$\|a^+\|_\infty \geq \frac{\displaystyle\int_0^L (u+k_0)'^2}{\displaystyle\int_0^L (u+k_0)^2} \geq \inf_{v \in X_\infty \setminus \{0\}} \frac{\displaystyle\int_0^L (v')^2}{\displaystyle\int_0^L (v)^2} = \frac{\pi^2}{L^2}, \ \forall\, a \in \Lambda. \tag{2.10}$$

Moreover, it is very well known that the previous infimum is, in fact, a minimum and that the value of this minimum is $\frac{\pi^2}{L^2}$ [8]. The previous inequalities imply $\beta_\infty \geq \frac{\pi^2}{L^2}$. Since the constant function $\frac{\pi^2}{L^2}$ is an element of Λ, we deduce $\beta_\infty = \dfrac{\pi^2}{L^2}$. This completes the proof of the theorem.

Remark 2.3. The constant β_∞ was defined in (2.5) as an infimum, but it can be seen that this infimum is attained in a unique element $a_\infty \in \Lambda$, given by $a_\infty(x) \equiv \dfrac{\pi^2}{L^2}$ [3].

Now we deal with the case $p = 1$. It is the only case where the infimum β_p, defined in (2.5), is not attained. The proof is inspired by Borg [1], but next theorem additionally provides a variational characterization of β_1 [3].

Theorem 2.2.

$$\beta_1 = \min_{u \in X_1 \setminus \{0\}} \frac{\displaystyle\int_0^L u'^2}{\|u\|_\infty^2} = \frac{4}{L}, \tag{2.11}$$

where $X_1 = \{u \in H^1(0,L) : \max_{x \in [0,L]} u(x) + \min_{x \in [0,L]} u(x) = 0\}$.

Proof. First, we prove

$$\min_{u\in X_1\setminus\{0\}} \frac{\int_0^L u'^2}{\|u\|_\infty^2} = \frac{4}{L}. \tag{2.12}$$

To do this, if $u \in X_1 \setminus \{0\}$, and $x_1, x_2 \in [0, L]$ are such that $u(x_1) = \max_{[0,L]} u,\ u(x_2) = \min_{[0,L]} u$, then $\|u\|_\infty = \max_{[0,L]} u = -\min_{[0,L]} u$. Clearly, it is not restrictive to assume that $x_1 < x_2$. Let us denote $I = [x_1, x_2]$. Then, it follows from the Cauchy–Schwarz inequality

$$\begin{aligned}\int_0^L u'^2 \geq \int_I u'^2 \geq \frac{\left(\int_I |u'|\right)^2}{x_2 - x_1} \geq \frac{\left(\int_I u'\right)^2}{x_2 - x_1}\\ = \frac{(u(x_2) - u(x_1))^2}{x_2 - x_1} = \frac{4\|u\|_\infty^2}{x_2 - x_1} \geq \frac{4}{L}\|u\|_\infty^2.\end{aligned} \tag{2.13}$$

Therefore,

$$\inf_{u\in X_1\setminus\{0\}} \frac{\int_0^L u'^2}{\|u\|_\infty^2} \geq \frac{4}{L}.$$

On the other hand, if $v(x) = x - \frac{L}{2},\ \forall\, x \in [0, L]$, then $v \in X_1 \setminus \{0\}$ and $\dfrac{\int_0^L v'^2}{\|v\|_\infty^2} = \dfrac{4}{L}$. This proves (2.12).

Now, we prove that $\beta_1 = \dfrac{4}{L}$. To see this, if $a \in \Lambda$ and $u \in H^1(0, L)$ is a nontrivial solution of (2.1), then by using Hölder's inequality, we obtain for each $k \in \mathbf{R}$,

$$\int_0^L (u + k)'^2 \leq \int_0^L a(u + k)^2 \leq \|a^+\|_1 \|(u + k)\|_\infty^2$$

and consequently

$$\|a^+\|_1 \geq \frac{\int_0^L (u + k)'^2}{\|(u + k)\|_\infty^2}.$$

If we choose $k_0 \in \mathbf{R}$ satisfying $u + k_0 \in X_1$, we deduce

$$\|a^+\|_1 \geq \frac{\int_0^L (u + k_0)'^2}{\|u + k_0\|_\infty^2} \geq \frac{4}{L},\ \forall\, a \in \Lambda. \tag{2.14}$$

Therefore, $\beta_1 \geq \dfrac{4}{L}$. Also, we can define a minimizing sequence in the following way. Let $\{u_n\} \subset C^2[0,L]$ be a sequence such that $u_n(x) = \left(x - \frac{L}{2}\right), \ \forall\, x \in \left(\frac{1}{n}, L - \frac{1}{n}\right);\ u_n'(0) = u_n'(L) = 0;\ u_n''(x) > 0,\ \forall\, x \in \left[0, \frac{1}{n}\right);\ u_n''(x) < 0,\ \forall\, x \in \left(L - \frac{1}{n}, L\right]$. Then, if we define the sequence of continuous functions $a_n : [0,L] \to \mathbf{R}$, as $a_n(x) = 0,\ \forall\, x \in \left[\frac{1}{n}, L - \frac{1}{n}\right]; a_n(x) = \frac{-u_n''(x)}{u_n(x)},\ \forall\, x \in \left[0, \frac{1}{n}\right] \cup \left[L - \frac{1}{n}, L\right]$, we have that $a_n \in L^\infty(0,L),\ a_n \geq 0,$ a.e. in $(0,L),\ a_n$ is nontrivial and

$$u_n''(x) + a_n(x)u_n(x) = 0, \text{ in } (0,L),\ u_n'(0) = u_n'(L) = 0.$$

Therefore, $a_n \in \Lambda,\ \forall\, n \in \mathbf{N}$. Moreover,

$$\begin{aligned}
\int_0^L a_n^+ &= \int_0^{\frac{1}{n}} \frac{-u_n''(x)}{u_n(x)} + \int_{L-\frac{1}{n}}^L \frac{-u_n''(x)}{u_n(x)} \\
&\leq \int_0^{\frac{1}{n}} \frac{u_n''(x)}{\min\limits_{[0,\frac{1}{n}]} (-u_n)} + \int_{L-\frac{1}{n}}^L \frac{-u_n''(x)}{\min\limits_{[L-\frac{1}{n},L]} (u_n)} \\
&= \frac{u_n'\left(\frac{1}{n}\right)}{\frac{L}{2} - \frac{1}{n}} + \frac{u_n'\left(L - \frac{1}{n}\right)}{\frac{L}{2} - \frac{1}{n}} = \frac{1}{\frac{L}{2} - \frac{1}{n}} + \frac{1}{\frac{L}{2} - \frac{1}{n}}.
\end{aligned}$$

Taking limits as $n \to \infty$, we deduce $\beta_1 = \dfrac{4}{L}$.

Remark 2.4. The infimum β_1, defined in (2.5), is not attained, i.e.,

$$\|a^+\|_1 > \frac{4}{L},\ \forall\, a \in \Lambda. \tag{2.15}$$

To prove this, let $a \in \Lambda$ be such that $\|a^+\|_1 = \dfrac{4}{L}$. By choosing u a nontrivial solution of (2.1) and $k_0 \in \mathbf{R}$ such that $u + k_0 \in X_1$, we obtain

$$\int_0^L (u + k_0)'^2 \leq \frac{4}{L} \|(u + k_0)\|_\infty^2.$$

On the other hand, since $u + k_0 \in X_1$, we deduce from (2.12)

$$\int_0^L (u + k_0)'^2 \geq \frac{4}{L} \|(u + k_0)\|_\infty^2.$$

Therefore,

$$\int_0^L (u + k_0)'^2 = \frac{4}{L} \|(u + k_0)\|_\infty^2.$$

Then, for the function $u + k_0$, all the inequalities of (2.13) transform into equalities. In particular, $x_2 = L,\ x_1 = 0$ and $\left(\int_0^L (u + k_0)'\right)^2 = L \int_0^L (u + k_0)'^2$. Again, the Cauchy–Schwarz inequality (equality in this case) implies that the function $(u+k_0)'$ is constant in $[0, L]$. Taking into account that $u + k_0 \in X_1 \setminus \{0\}$, we have $u(x) + k_0 = k(x - \frac{L}{2})$, $\forall x \in [0, L]$ and for some nontrivial constant k. Then from (2.1) we deduce $a \equiv 0$, which is a contradiction.

Remark 2.5. The formula $\beta_1 = \dfrac{4}{L}$ was proved in [18] by using methods from Optimal Control Theory. More precisely, the authors used the Pontryagin's maximum principle. The variational proof that we have presented here motivates some of the main ideas that we will use in the case $1 < p < \infty$.

Remark 2.6. In [21] the authors study the problem with linear damping

$$u''(x) + b(x)u'(x) + a(x)u(x) = 0,\ u'(0) = u'(L) = 0 \tag{2.16}$$

obtaining the best L_1 Lyapunov constant.

As a first application of Theorems 2.1 and 2.2 to the linear problem

$$u''(x) + a(x)u(x) = f(x),\ x \in (0, L),\ u'(0) = u'(L) = 0 \tag{2.17}$$

we have the following corollary.

Corollary 2.1. *Let $a \in L^\infty \setminus \{0\}$, $0 \le \int_0^L a(x)$, satisfying one of the following conditions:*

1. $\|a^+\|_1 \le \beta_1 = \dfrac{4}{L}$
2. $\|a^+\|_\infty \le \beta_\infty = \dfrac{\pi^2}{L^2}$ *and a^+ is not identically to the constant β_∞*

Then for each $f \in L^\infty(0, L)$, the boundary value problem (2.17) has a unique solution.

Proof. The corollary is proved if the homogeneous problem

$$u''(x) + a(x)u(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0 \tag{2.18}$$

has only the trivial solution [16]. But this is clear from Theorem 2.1, Remark 2.3 and Theorem 2.2, Remark 2.4.

Remark 2.7. In the previous corollary, the conditions on the function $a(\cdot)$:

$$\begin{gathered} \|a^+\|_1 \le \beta_1 \\ \|a^+\|_\infty \le \beta_\infty = \text{ and } a^+ \text{ is not identically to the constant } \beta_\infty \end{gathered} \tag{2.19}$$

are given, respectively, in terms of the L^1 norm $\|a^+\|_1$ and L^∞ norm $\|a^+\|_\infty$. Clearly, they are not related in general, in the sense that none of them imply the other. In the next theorem, we consider the case $1 < p < \infty$, and we establish other different conditions given in terms of the L^p norm $\|a^+\|_p,\ 1 < p < \infty$. They will show a natural relation between the cases $p = 1$ and $p = \infty$ in (2.19) when one studies what happens for $p \to 1^+$ and $p \to \infty$.

In order to motivate the variational characterization of the constant $\beta_p,\ 1 < p < \infty$, which is discussed in the next theorem, take into account that if $a \in \Lambda \bigcap L^p(0,L)$ and $u \in H^1(0,L)$ is a nontrivial solution of (2.1) then

$$\int_0^L u'v' = \int_0^L auv,\ \forall\ v \in H^1(0,L).$$

In particular, choosing $v \equiv u$ and $v \equiv 1$, we have respectively

$$\int_0^L u'^2 = \int_0^L au^2,\ \int_0^L au = 0. \tag{2.20}$$

Therefore, for each $k \in \mathbf{R}$, we have (remember that $\int_0^L a \geq 0$)

$$\begin{aligned}\int_0^L (u+k)'^2 &= \int_0^L u'^2 = \int_0^L au^2 \leq \int_0^L au^2 + k^2 \int_0^L a \\ &= \int_0^L au^2 + \int_0^L k^2 a + 2k \int_0^L au = \int_0^L a(u+k)^2 \leq \int_0^L a^+(u+k)^2.\end{aligned}$$

From Hölder's inequality it follows

$$\int_0^L (u+k)'^2 \leq \|a^+\|_p \|(u+k)^2\|_{\frac{p}{p-1}}.$$

Moreover, since u is a nonconstant solution of (2.1), $u+k$ is not identically the zero function. Consequently

$$\|a^+\|_p \geq \frac{\displaystyle\int_0^L (u+k)'^2}{\|(u+k)^2\|_{\frac{p}{p-1}}},\ \forall\ a \in \Lambda. \tag{2.21}$$

This reasoning suggests the minimization of a functional like the previous one on some *appropriate subset* of $H^1(0,L)$. Motivated by the case $p = \infty$ (Theorem 2.1), this *appropriate subset* could be of the type

$$\left\{ u \in H^1(0,L) : \int_0^L |u|^{\lambda(p)} u = 0 \right\},$$

where $\lambda(\infty) = 0$. Here we choose $\lambda(p) = \dfrac{2}{p-1}$. To understand why this election is suitable, we must see in detail the proof of the next theorem, especially the part where the Lagrange multiplier Theorem is applied (see [3] for more details).

Theorem 2.3. *If* $1 < p < \infty$,

$$\beta_p = \min_{X_p\setminus\{0\}} J_p(u) = \min_{X_p\setminus\{0\}} \frac{\displaystyle\int_0^L u'^2}{\left(\displaystyle\int_0^L |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}$$

$$= \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}} p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\, dx\right)^2, \tag{2.22}$$

where

$$X_p = \left\{ u \in H^1(0,L) : \int_0^L |u|^{\frac{2}{p-1}} u = 0 \right\}.$$

Proof. The proof will be carried out into three steps:

1. **The minimization problem stated in (2.22) has solution.**

   The proof of this fact is standard: first we will demonstrate that any minimizing sequence is bounded in the Hilbert space $H^1(0,L)$. Then we will use that the considered functional is weak lower semi-continuous in order to conclude that the infimum is attained.

   Let us denote

$$m_p \equiv \inf_{X_p\setminus\{0\}} J_p. \tag{2.23}$$

   If $\{u_n\} \subset X_p \setminus \{0\}$ is a minimizing sequence, then $\{k_n u_n\}$ where $\{k_n\}$ is an arbitrary sequence of nonzero real numbers, is also a minimizing sequence, since $J_p(u_n) = J_p(k_n u_n)$. Therefore, we can assume without loss of generality that $\displaystyle\int_0^L |u_n|^{\frac{2p}{p-1}} = 1$. As $J_p(u_n)$ is bounded, $\left\{\displaystyle\int_0^L |u_n'^2|\right\}$ is also bounded. Moreover, since $\displaystyle\int_0^L |u_n|^{\frac{2}{p-1}} u_n = 0$, for each u_n there is $x_n \in (0,L)$ such that $u_n(x_n) = 0$. Now, $u_n(x) = \displaystyle\int_{x_n}^x u'(s)\, ds$, $\forall\, x \in (0,L)$ and Hölder's inequality implies that $\{u_n\}$ is bounded in $H^1(0,L)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(0,L)$ (weak convergence) and $u_n \to u_0$ in $C[0,L]$, with the uniform norm [2]. The strong convergence in $C[0,L]$ gives us $\displaystyle\int_0^L |u_0|^{\frac{2p}{p-1}} = 1$, $\displaystyle\int_0^L |u_0|^{\frac{2}{p-1}} u_0 = 0$, and consequently $u_0 \in X_p \setminus \{0\}$. As the functional J_p is weak lower semi-continuous [2], the weak convergence in $H^1(0,L)$ implies $J_p(u_0) \leq \liminf J_p(u_n) = m_p$. Then u_0 is a minimizer.

Since $X_p = \{ u \in H^1(0,L) : \varphi(u) = 0\}$, $\varphi(u) = \int_0^L |u|^{\frac{2}{p-1}} u$, if $u_0 \in X_p \setminus \{0\}$ is any minimizer of J_p, Lagrange multiplier Theorem [10] implies that there is $\lambda \in \mathbf{R}$ such that

$$H'(u_0) + \lambda \varphi'(u_0) = 0,$$

where $H : H^1(0,L) \to \mathbf{R}$ is defined by

$$H(u) = \int_0^L u'^2 - m_p \left(\int_0^L |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}.$$

Also, as $u_0 \in X_p$ we have $H'(u_0)(1) = 0$. Moreover $H'(u_0)(v) = 0, \ \forall\, v \in H^1(0,L) : \ \varphi'(u_0)(v) = 0$. Finally, as any $v \in H^1(0,L)$ may be written in the form $v = \alpha + w$, $\alpha \in \mathbf{R}$, and w satisfying $\varphi'(u_0)(w) = 0$, we conclude $H'(u_0)(v) = 0, \ \forall\, v \in H^1(0,L)$, i.e., $H'(u_0) = 0$ which implies that u_0 satisfies the problem

$$v''(x) + m_p \left(\int_0^L |v|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}} |v(x)|^{\frac{2}{p-1}} v(x) = 0, \ \ v'(0) = v'(L) = 0. \tag{2.24}$$

2. **The constant β_p is equal to the constant m_p** (this fact implies the characterization of β_p as the minimum value of J_p on $X_p \setminus \{0\}$ and will be of special interest in the extension of the results to systems of equations in Chap. 5).

   In fact, previously to the theorem, we have proved that if $a \in \Lambda \bigcap L^p(0,L)$ and $u \in H^1(0,L)$ is a nontrivial solution of (2.1), then (2.21) is satisfied for each $k \in \mathbf{R}$. Then, if for each $a \in \Lambda \bigcap L^p(0,L)$ and each u, nontrivial solution of (2.1), we choose $k_0 \in \mathbf{R}$ satisfying $u + k_0 \in X_p$, we deduce $\beta_p \geq m_p$. Reciprocally, if $u_p \in X_p \setminus \{0\}$ is any minimizer of J_p, then u_p satisfies (2.24). Therefore, if we denote

$$A_p(v) = m_p \left(\int_0^L |v|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}} \tag{2.25}$$

   we have that $A_p(u_p)|u_p|^{\frac{2}{p-1}} \in \Lambda \bigcap L^p(0,L)$ and

$$\| A_p(u_p)|u_p|^{\frac{2}{p-1}} \|_p = m_p.$$

   Then $\beta_p \leq m_p$. The conclusion is that $\beta_p = m_p$.

3. **Integrating the Euler's equation (2.24) to obtain** m_p.

The explicit calculus of m_p is a very delicate and technical matter, but we emphasize that the same ideas can be used in many other situations, as it will be seen in Sect. 2.3 (see [3] for further details). In fact, this method can be used whenever we have a detailed knowledge about the number and distribution of zeros of nontrivial solutions v of Eq. (2.26) below and their first derivatives v'.

Start with the method: if $u_p \in X_p \setminus \{0\}$ is a minimizer of J_p, then we have proved that u_p satisfies a problem of the type

$$v''(x) + B|v(x)|^{\frac{2}{p-1}} v(x) = 0, \ x \in, (0, L), \ \ v'(0) = v'(L) = 0, \tag{2.26}$$

where B is some positive real constant. Also, let us observe that if v is a nontrivial solution of (2.26), then $\int_0^L |v(x)|^{\frac{2}{p-1}} v(x) = 0$. Therefore, v belongs to $X_p \setminus \{0\}$ and consequently,

$$\inf_{B \in \mathbf{R}^+} \inf_{v \in S_B} J_p(v) = m_p,$$

where, for a given $B \in \mathbf{R}^+$, S_B denotes the set of all nontrivial solutions of (2.26).

Now, let $B \in \mathbf{R}^+$ be a fixed number and v a nontrivial solution of (2.26). First, our main purpose is *to calculate* v in the interval $[0, L]$ and then, *to calculate* $J_p(v)$. It is clear that we may assume without loss of generality that $v(0) > 0$. Moreover, since $v \in X_p \setminus \{0\}$, v must change its sign in $(0, L)$. Let x_0 be the first zero point of v in $(0, L)$.

a. The function v in $[0, x_0]$.

The function v satisfies the initial value problem

$$w''(x) + B|w(x)|^{\frac{2}{p-1}} w(x) = 0, \ \ w(0) = v(0), \ w'(0) = 0 \tag{2.27}$$

and this problem has a unique solution defined in $\mathbf{R}$ (see Proposition 2.1. in [14]).

If $x \in (0, x_0)$ is fixed, multiplying both terms of (2.26) by v' and integrating in the interval $[0, x]$ we obtain

$$-\frac{(v'(x))^2}{2} = \frac{B(p-1)}{2p}\left(|v(x)|^{\frac{2p}{p-1}} - |v(0)|^{\frac{2p}{p-1}}\right). \tag{2.28}$$

On the interval $(0, x_0)$ the function v satisfies $v(x) > 0$ and $v'(x) \leq 0$ (see (2.26)) and thus

$$v'(x) = -\left[\frac{B(p-1)}{p}\right]^{1/2}\left[|v(0)|^{\frac{2p}{p-1}} - |v(x)|^{\frac{2p}{p-1}}\right]^{1/2}. \tag{2.29}$$

Therefore,

$$\int_0^x \frac{v'(t)}{\left[|v(0)|^{\frac{2p}{p-1}} - |v(t)|^{\frac{2p}{p-1}}\right]^{1/2}}\, dt = -\left[\frac{B(p-1)}{p}\right]^{1/2} x$$

for any $x \in (0, x_0)$. Doing the change of variables $s = \dfrac{v(t)}{v(0)}$, previous relation can be written as

$$-\varphi(1) + \varphi\left(\frac{v(x)}{v(0)}\right) = -v(0)^{\frac{1}{p-1}} \left[\frac{B(p-1)}{p}\right]^{1/2} x, \ \forall\, x \in (0, x_0).$$

Here $\varphi : [0, 1] \to \mathbf{R}$ is the strictly increasing function defined by

$$\varphi(t) = \int_0^t \frac{ds}{\left(1 - s^{\frac{2p}{p-1}}\right)^{1/2}}.$$

If $\varphi\ [0, 1] = [0, I]$, then we find

$$\frac{v(x)}{v(0)} = \varphi^{-1}\left[I - v(0)^{\frac{1}{p-1}} \left(\frac{B(p-1)}{p}\right)^{1/2} x\right] \ \forall\, x \in (0, x_0). \tag{2.30}$$

Moreover, since $v(x_0) = 0$, we obtain

$$I - v(0)^{\frac{1}{p-1}} \left(\frac{B(p-1)}{p}\right)^{1/2} x_0 = 0.$$

Hence,

$$v(0) = \left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{p-1}. \tag{2.31}$$

Finally,

$$v(x) = \left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{p-1} \varphi^{-1}\left(I - \frac{I}{x_0}x\right), \ \forall\, x \in [0, x_0] \tag{2.32}$$

b. Now, we can calculate v in $[x_0, 2x_0], [2x_0, 3x_0], \ldots$

To do this, the initial value problem

$$w''(x) + B|w(x)|^{\frac{2}{p-1}} w(x) = 0, \ \ w(x_0) = v(x_0) = 0, \ w'(x_0) = v'(x_0) \tag{2.33}$$

has a unique solution defined in $\mathbf{R}$ [14]. Since the function $-v(2x_0 - x)$, $x \in (x_0, 2x_0)$, is a solution of (2.33), this provides $v(x) = -v(2x_0 - x)$, $\forall\, x \in (x_0, 2x_0)$.

In an analogous way, the initial value problem

$$w''(x) + B|w(x)|^{\frac{2}{p-1}} w(x) = 0, \;\; w(2x_0) = v(2x_0), \;\; w'(2x_0) = v'(2x_0) = 0 \tag{2.34}$$

has a unique solution defined in $\mathbf{R}$. Since the function $v(4x_0 - x)$, $x \in (2x_0, 3x_0)$, is a solution of (2.34), this provides $v(x) = v(4x_0 - x)$, $\forall\, x \in (2x_0, 3x_0)$.

Now, we can repeat this procedure in the intervals $[nx_0, (n+1)x_0]$, $\forall\, n \in \mathbf{N}$, obtaining:

$$\begin{aligned} v(x) &= -v(2x_0 - x), \;\; \forall\, x \in [x_0, 2x_0], \\ v(x) &= v(4x_0 - x), \;\; \forall\, x \in [2x_0, 3x_0], \\ v(x) &= -v(6x_0 - x), \;\; \forall\, x \in [3x_0, 4x_0], \\ &\dots \end{aligned} \tag{2.35}$$

The conclusion is that if v is a nontrivial solution of (2.26) for some $B \in \mathbf{R}^+$, and x_0 is the first zero point of v in $(0, L)$, then $L = 2nx_0$ for some $n \in \mathbf{N}$. Next we calculate $J_p(v)$.

It follows from previous reasonings that

$$J_p(v) = \frac{\displaystyle\int_0^L v'^2}{\left(\displaystyle\int_0^L |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}} = \frac{2n \displaystyle\int_0^{x_0} v'^2}{\left(2n \displaystyle\int_0^{x_0} |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}. \tag{2.36}$$

From (2.28) we obtain

$$\int_0^{x_0} (v'(x))^2 \, dx = \frac{B(p-1)}{p} \left[-\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} \, dx + x_0 |v(0)|^{\frac{2p}{p-1}} \right] \tag{2.37}$$

and from (2.32) we obtain

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \int_0^{x_0} \left(\frac{I}{x_0} \left(\frac{p}{B(p-1)} \right)^{1/2} \right)^{2p} \left[\varphi^{-1} \left(I - \frac{I}{x_0} x \right) \right]^{\frac{2p}{p-1}} dx. \tag{2.38}$$

Doing the change of variables $s = \varphi^{-1}(I(1 - \frac{x}{x_0}))$, we have

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \left(\frac{I}{x_0} \left(\frac{p}{B(p-1)} \right)^{1/2} \right)^{2p} \frac{x_0}{I} \int_0^1 s^{\frac{2p}{p-1}} \left(1 - s^{\frac{2p}{p-1}} \right)^{-1/2} ds. \tag{2.39}$$

Integrating by parts the previous expression with $f(s) = s$, $g'(s) = s^{\frac{p+1}{p-1}}\left(1 - s^{\frac{2p}{p-1}}\right)^{-1/2}$, we deduce

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p} \frac{x_0}{I}\frac{p-1}{2p-1}I. \tag{2.40}$$

If we substitute this expression in (2.37) and, moreover, we take into account (2.31), we obtain (think that $L = 2nx_0$)

$$\int_0^{x_0} |v'(x)|^2\, dx = \frac{B(p-1)}{p}x_0\left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p}\frac{p}{2p-1}. \tag{2.41}$$

Now we can substitute (2.40) and (2.41) in (2.36). After some elementary calculations we deduce

$$J_p(v) = \frac{4n^2I^2p}{L^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}. \tag{2.42}$$

At this point, one may observe two things. First, $J_p(v)$ does not depend on B. Second, all values of $n \in \mathbf{N}$ are possible in (2.42). In fact if $x_0 = \dfrac{L}{2n}$, formula (2.32) defines a nontrivial solution of (2.26). Therefore, the infimum m_p is attained if $n = 1$. Finally, doing the change of variables $s^{\frac{p}{p-1}} = \sin t$, we obtain $I = \displaystyle\int_0^1 \frac{ds}{\left(1 - s^{\frac{2p}{p-1}}\right)^{1/2}} = \frac{p-1}{p}K$, where $K = \displaystyle\int_0^{\pi/2} (\sin t)^{-1/p}\, dt$. This gives

$$m_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}}p(2p-1)^{1/p}}\left(\int_0^{\pi/2}(\sin x)^{-1/p}\, dx\right)^2. \tag{2.43}$$

Remark 2.8. In order to study other boundary conditions (Sect. 2.3), it seems essential to highlight the basic facts of the previous procedure.

We emphasize that if v is a nontrivial solution of (2.26) and x_0 is the first zero point of v in $(0, L)$, then $L = 2nx_0$ for some natural number $n \geq 1$ and, in addition,

$$\begin{aligned} v'(0) &= v'(2x_0) = \ldots = v'(2nx_0) = 0,\\ v(x_0) &= \ldots = v((2n-1)x_0) = 0, \end{aligned} \tag{2.44}$$

and $v(x) \neq 0,\ v'(x) \neq 0,\ \forall\, x \in (jx_0, (j+1)x_0),\ 0 \leq j \leq 2n-1$. These properties allow to calculate, in a explicit way, the functions v' and v in $[0, L]$ and consequently, to find the value of $J_p(v)$ given in (2.42).

Remark 2.9. It is proved in [3] that β_p, as a function of $p \in [1, +\infty]$, is continuous.

Remark 2.10. If $L = 1$ and $1 \le p < q < \infty$, then $\beta_p < \beta_q$ (see [3]). As a trivial consequence, if L is an arbitrary positive number, the mapping $(1, \infty) \to \mathbf{R},\ p \to L^{-1/p}\beta_p$ is strictly increasing.

Now, we return to the linear boundary value problem (2.17), corollary 2.1, and remark 2.7. The following result establishes a *natural link* between the cases $p = 1$ and $p = \infty$. Previously, remember that from Theorems 2.1, 2.2 and 2.3 the constant β_p, defined as an infimum in 2.5, is attained, if and only if $1 < p \le \infty$.

Corollary 2.2. *Let* $a \in L^\infty \setminus \{0\},\ 0 \le \displaystyle\int_0^L a(x)$, *satisfying one of the following conditions:*

1. $\|a^+\|_1 \le \beta_1$,
2. *There is some* $p \in (1, \infty)$ *such that* $\|a^+\|_p < \beta_p$,
3. $\|a^+\|_\infty < \beta_\infty$ *or* $\|a^+\|_\infty = \beta_\infty$ *and* $a^+ \not\equiv a_\infty$.

Then for each $f \in L^\infty(0, L)$, *the boundary value problem (2.17) has a unique solution.*

Remark 2.11. We have shown that the best Sobolev constant β_p, defined in (2.5), can be computed by using a certain minimization problem given in Theorems 2.1, 2.2, and 2.3. Motivated by a completely different problem (an isoperimetric inequality known as Wulff theorem, of interest in crystallography), the authors studied in [9] a similar variational problem for the case of periodic or Dirichlet boundary conditions (see also [11] for more general minimization problems). Our treatment of the Euler equation associated with the mentioned minimization problem is different from that of Croce and Dacorogna [9].

2.2 Nonlinear Neumann Problems

Lyapunov inequalities can be used in the study of nonlinear resonant problems. To accomplish this, the linear results are combined with Schauder fixed point theorem.

We focus on a resonant nonlinear problem with Neumann boundary conditions, but the same ideas and methods can be used for other situations (see Sect. 2.3).

More precisely, let us consider the problem

$$u''(x) + f(x, u(x)) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0, \tag{2.45}$$

where $f : [0, L] \times \mathbf{R} \to \mathbf{R},\ (x, u) \to f(x, u)$ is continuous.

The *associated linear problem*

$$u''(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0 \tag{2.46}$$

has nontrivial solutions (any constant function) and this is the reason why we call (2.45) a resonant problem.

If (2.45) is linear, i.e., it is of the type

$$u''(x) + a(x)u(x) = 0,\ x \in (0, L),\ u'(0) = u'(L) = 0 \tag{2.47}$$

and for some integer $n \geq 0$ there is a positive number δ such that

$$\lambda_n + \delta \leq a(x) \leq \lambda_{n+1} - \delta,\ \text{in } [0, L], \tag{2.48}$$

where λ_n is an eigenvalue of the eigenvalue problem (2.2), then (2.47) has only the trivial solution $u \equiv 0$ (see, for instance, [20]). In particular, for the first eigenvalue $\lambda_0 = 0$, (2.48) becomes

$$\delta \leq a(x) \leq \frac{\pi^2}{L^2} - \delta,\ \text{in } [0, L]. \tag{2.49}$$

We must remark that (2.48) does not allow to the function $a(\cdot)$ to cross any eigenvalue of (2.2). Using Lyapunov inequalities, it is possible that $f_u(x, u)$ in (2.45) crosses the eigenvalues λ_n (f_u means the partial derivative of the function $f(x, u)$ with respect to the variable u.) and it is possible to provide some extensions of Corollary 2.2 to nonlinear situations.

To this respect, we will assume throughout this section that the following hypothesis is satisfied

(H) f, f_u are continuous on $[0, L] \times \mathbf{R}$ and $0 \leq f_u(x, u)$ on $[0, L] \times \mathbf{R}$.

Then, the existence of a solution u of (2.45) implies

$$\int_0^L f(x, u(x))\, dx = 0. \tag{2.50}$$

Now, the previous hypothesis **(H)** implies that $f(x, u)$ is increasing with respect to u. Therefore,

$$\int_0^L f(x, m)\, dx \leq \int_0^L f(x, u(x))\, dx = 0 \leq \int_0^L f(x, M)\, dx,$$

where $m = \min\limits_{[0,L]} u$ and $M = \max\limits_{[0,L]} u$ and consequently

$$\int_0^L f(x, z)\, dx = 0 \tag{2.51}$$

for some $z \in \mathbf{R}$. However, conditions (H) and (2.51) are not sufficient for the existence of solutions of (2.45). Indeed, if $n \in \mathbf{N}$ is any natural number, consider the problem

$$u''(x) + n^2\pi^2 u(x) + \cos(n\pi x) = 0,\ x \in (0, 1),\ u'(0) = u'(1) = 0. \tag{2.52}$$

The function $f(x, u) = n^2\pi^2 u + \cos(n\pi x)$ satisfies (H) and (2.51), but the Fredholm alternative theorem [16] shows that there is no solution of (2.52).

If (H) and (2.51) are assumed, and for instance, $L = 1$ for simplicity, different supplementary assumptions can be given which imply the existence of a solution of (2.45). For example

(h1) $f_u(x,u) \le \beta(x)$ on $[0,1] \times \mathbf{R}$ with $\beta \in L^\infty(0,1)$, $\beta(x) \le \pi^2$ on $[0,1]$ and $\beta(x) < \pi^2$ on a subset of $(0,1)$ of positive measure.

Conditions of this type are referred to as nonuniform nonresonance conditions with respect to the first positive eigenvalue of the associated linear homogeneous problem. By using variational methods, it is proved in [26] that (H), (2.51), and (h1) imply the existence of solutions of (2.45). Restriction (h1) is related to Lyapunov-type inequalities: the number π^2 is the best L_∞ Lyapunov constant, β_∞, for $L = 1$ (Theorem 2.1).

On the other hand, in [18] it is supposed

(h2) $f_u(x,u) \le \beta(x)$ on $[0,1] \times \mathbf{R}$ with $\beta \in L^1(0,1)$ and $\int_0^1 \beta(x)\, dx \le 4$

The authors use Optimal Control theory methods to prove that (H), (2.51), and (h2) imply the existence and uniqueness of solutions of (2.45). Restriction (h2) is also related to Lyapunov-type inequalities: the number 4 is the best L_1 Lyapunov constant, β_1, for $L = 1$ (Theorem 2.2).

Let us observe that supplementary conditions (h1) and (h2) are given respectively in terms of $\|\beta\|_\infty$ and $\|\beta\|_1$, the usual norms in the spaces $L^\infty(0,1)$ and $L^1(0,1)$. Also, it is trivial that under the hypotheses (H) and (2.51), (h1) and (h2) are not related (i.e., none of these hypotheses implies the other).

In the next theorem we provide supplementary conditions in terms of $\|\beta\|_p$, $1 \le p \le \infty$. As a consequence, a natural relation between (h1) and (h2) arises if one takes into account Remark (2.9) and studies the limits of $\|\beta\|_p$ for $p \to 1^+$ and $p \to \infty$ (see [3] for further details).

Theorem 2.4. *Let us consider (2.45) where the following requirements are fulfilled:*

1. *f and f_u are continuous on* $[0,L] \times \mathbf{R}$.
2. $0 \le f_u(x,u)$ *in* $[0,L] \times \mathbf{R}$. *Moreover, for each $u \in C[0,L]$ one has $f_u(x,u(x)) \ne 0$, a.e. on* $[0,L]$ *and* $\int_0^L f(x,0)\, dx = 0$.
3. *For some function $\beta \in L^\infty(0,L)$, we have $f_u(x,u) \le \beta(x)$ on* $[0,L] \times \mathbf{R}$ *and β satisfies some of the conditions given in Corollary 2.2.*

Then, problem (2.45) has a unique solution.

Proof. The proof consists of two parts: existence and uniqueness of solutions of (2.45). We begin with the second one.

Uniqueness of Solutions We assume that (2.45) has two solutions. Then, the mean value theorem [8] and Corollary (2.2) are used to prove that they are the same.

Let u_1 and u_2 be two solutions of (2.45). Then,

$$\begin{aligned} -(u_1-u_2)''(x) &= f(x,u_1(x)) - f(x,u_2(x)) \\ &= \int_0^1 \frac{d}{d\theta}\left[f(x,u_2(x)+\theta(u_1(x)-u_2(x))\right] d\theta \\ &= \left[\int_0^1 f_u(x,u_2(x)+\theta(u_1(x)-u_2(x))\, d\theta\right](u_1(x)-u_2(x)),\ x\in[0,L]. \end{aligned} \tag{2.53}$$

Hence, the function $u = u_1 - u_2$ is a solution of a homogeneous problem of the type (2.17) with $a(x) = \int_0^1 f_u(x,u_2(x)+\theta u(x))\, d\theta$. From the hypotheses of the theorem and applying Corollary 2.2, we obtain $u \equiv 0$.

Existence of Solutions The main idea is to rewrite (2.45) in an equivalent form, such that the solutions of (2.45) be the fixed points of a certain completely continuous operator, and then, to apply the Schauder fixed point theorem [12]. To see this, by using the same idea that in (2.53), we rewrite (2.45) as

$$\begin{aligned} 0 = u''(x) + f(x,u(x)) &= u''(x) + f(x,u(x)) - f(x,0) + f(x,0) \\ &= u''(x) + \int_0^1 \frac{d}{d\theta}\left[f(x,\theta u(x))\right] d\theta + f(x,0) \\ &= u''(x) + \left[\int_0^1 f_u(x,\theta u(x))\, d\theta\right] u(x) + f(x,0). \end{aligned} \tag{2.54}$$

Therefore, u is a solution of (2.45) if and only if u satisfies

$$u''(x) + b(x,u(x))u(x) = -f(x,0),\ x\in\ [0,L],\ u'(0) = u'(L) = 0, \tag{2.55}$$

where the continuous function $b : [0,L]\times\mathbf{R}\to\mathbf{R}$ is defined by

$$b(x,z) = \int_0^1 f_u(x,\theta z)\, d\theta.$$

From the hypotheses of the theorem, it is deduced that for each function $y \in C^1([0,L],\mathbf{R})$, the linear equation

$$u''(x) + b(x,y(x))u(x) = -f(x,0),\ x\in\ [0,L],\ u'(0) = u'(L) = 0 \tag{2.56}$$

satisfies all the hypotheses of Corollary 2.2 and consequently, (2.56) has a unique solution u_y. Then, if $X = C^1([0,L],\mathbf{R})$ with the usual norm, i.e.,

$$\|y\|_X = \max_{x\in[0,L]} |y(x)| + \max_{x\in[0,L]} |y'(x)|,\ \forall\ y\in X$$

we can define the operator $T : X\to X$, by $Ty = u_y$. Clearly, u is a solution of (2.45) if and only if y is a fixed point of T.

We claim that T is completely continuous (T is continuous and if $B \subset X$ is bounded, then $T(B)$ is relatively compact in X) and that $T(X)$ is bounded. Then, the Schauder fixed point theorem ensures that T has a fixed point which provides a solution of (2.45).

To prove the claim, if $T(X)$ is not bounded, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. Moreover, from the hypotheses of the theorem, the sequence of functions $\{b(\cdot, y_n(\cdot))\}$ is bounded in $L^2(0, L)$ and, passing to a subsequence if necessary, we may assume that $\{b(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^2(0, L)$ to a function β_0 satisfying $0 \le \beta_0(x) \le \beta(x)$, a.e. in $[0, L]$ (see [2] for the properties of the convergence in $L^2(0, L)$).

In addition, each u_{y_n} satisfies

$$u''_{y_n}(x) + b(x, y_n(x))u_{y_n}(x) = -f(x, 0),\ x \in\ [0, L],\ u'(0) = u'(L) = 0. \tag{2.57}$$

Since the embedding $H^1(0, L) \subset\ C[0, L]$ is compact (in $C[0, L]$ we take the uniform norm), if $z_n \equiv \dfrac{u_{y_n}}{\|u_{y_n}\|_X}$, then passing to a subsequence if necessary, we may assume that $z_n \to z_0$, uniformly in $[0, L]$, where z_0 satisfies $\|z_0\|_X = 1$ and

$$z''_0(x) + \beta_0(x)z_0(x) = 0,\ x \in\ [0, L],\ z'_0(0) = z'_0(L) = 0. \tag{2.58}$$

Moreover, from the hypotheses of the theorem, we have for each $n \in \mathbf{N}$,

$$\int_0^L b(x, y_n(x))u_{y_n}(x)\, dx = -\int_0^L f(x, 0)\, dx = 0.$$

Also, the function $b(\cdot, y_n(\cdot))$ is nonnegative and not identically zero. Therefore, for each $n \in \mathbf{N}$, the function u_{y_n} has a zero in $[0, L]$. This implies that for each $n \in \mathbf{N}$, the function z_n has a zero in $[0, L]$ and hence so does z_0. Taking into account (2.58), $\beta_0 \in L^\infty(0, L) \setminus \{0\}$. This is a contradiction with Corollary 2.2.

Now, let us prove that the operator T is continuous. To see this, if $\{y_n\} \to y_0$ in the space X and u_{y_n} does not converge to u_{y_0}, passing to a subsequence if necessary, there exists a constant $\delta > 0$ such that $u_{y_n} \notin B_X(u_{y_0}; \delta),\ \forall\ n \in \mathbf{N}$, where $B_X(u_{y_0}; \delta)$ denotes the open ball in X of center u_{y_0} and radius δ. Also, taking into account (2.56) and the boundness of the operator T, we obtain that the sequence $\{u''_{y_n}\}$ is uniformly bounded. Thus, by Arzela–Ascoli Theorem [7], again passing to a subsequence if necessary, we deduce that u_{y_n} converges to some function u_0. But, by the uniqueness of solution for problem (2.56), we must have $u_0 = u_{y_0}$, which is a contradiction.

Finally, by using again the Arzela–Ascoli theorem, it is trivial from (2.56) that if $B \subset X$ is bounded, then $T(B)$ is relatively compact in X.

Remark 2.12. If $f(x, u) = a(x)u$, the second hypothesis in the previous theorem becomes $0 \le a(x)$ and $a(x) \ne 0$, a.e. on $[0, L]$.

Remark 2.13. Since the change of variables $u(x) = v(x) + z,\ z \in \mathbf{R}$, transforms (2.45) into the problem

$$v''(x) + f(x, v(x) + z) = 0,\ x \in (0, L),\ v'(0) = v'(L) = 0,$$

the condition $\int_0^L f(x,0)\ dx = 0$ in the previous theorem may be substituted by $\int_0^L f(x,z)\ dx = 0$, for some $z \in \mathbf{R}$.

Remark 2.14. Taking into account Remark 2.9, previous result establishes a clear relationship between Theorem B in [18] and Theorem 2 in [26] for the case of ordinary differential equations.

Remark 2.15. Let us remark that the hypothesis of the previous theorem allows the function $f_u(x,u)$ to cross an arbitrary number of different eigenvalues λ_n of the eigenvalue problem (2.2) (see [3, 18]).

2.3 The Variational Method for Other Boundary Conditions

The variational method that we have used in Sect. 2.1 (Theorem 2.3), to obtain the explicit value of the constant $\beta_p,\ 1 < p < \infty$, is valid for many other boundary conditions. Remember the two key points for Neumann problem (2.1).

1. The set of boundary value problems

$$v''(x) + B|v(x)|^{\frac{2}{p-1}} v(x) = 0,\ x \in (0,L),\ \ v'(0) = v'(L) = 0,\ \ B \in \mathbf{R}^+ \tag{2.59}$$

provides

$$\beta_p = \inf_{B\in\mathbf{R}^+}\ \inf_{v\in S_B}\ J_p(v), \tag{2.60}$$

where

$$J_p(v) = \frac{\displaystyle\int_0^L v'^2}{\left(\displaystyle\int_0^L |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}$$

and for a given $B \in \mathbf{R}^+,\ S_B$ denotes the set of all nontrivial solutions of (2.59).

2. If v is a nontrivial solution of (2.59) for some $B \in \mathbf{R}^+$, then

$$J_p(v) = \frac{4n^2I^2p}{L^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}, \tag{2.61}$$

where

$$I = \frac{p-1}{p}\int_0^{\pi/2}(\sin x)^{-1/p}\ dx \tag{2.62}$$

and n is the unique natural number (depending on v), satisfying the properties:

$$\begin{gathered} x_0 \text{ is the first zero point of } v \text{ in } (0,L),\ L = 2nx_0, \\ v'(0) = v'(2x_0) = \ldots = v'(2nx_0) = 0, \\ v(x_0) = \ldots = v((2n-1)x_0) = 0, \\ v(x) \neq 0,\ v'(x) \neq 0,\ \forall\, x \in (jx_0, (j+1)x_0),\ 0 \leq j \leq 2n-1. \end{gathered} \tag{2.63}$$

Let us emphasize that the value of $J_p(v)$ in (2.61) does not depend, explicitly, on the positive constant B and that to obtain β_p we must find the minimum value of n in the expression (2.61). For instance, for Neumann boundary conditions this minimum value is $n = 1$ (see the last part of Theorem 2.3).

Below we describe the main ideas for other boundary conditions.

In the remainder of the chapter we will denote as β_p^N the constant β_p obtained above for Neumann boundary conditions.

Dirichlet Boundary Conditions This case is very similar to the Neumann one. If we consider the linear problem

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u(0) = u(L) = 0, \tag{2.64}$$

where $a \in \Lambda^D$ and Λ^D is defined by

$$\Lambda^D = \{a \in L^1(0,L) \text{ such that (2.64) has nontrivial solutions }\} \tag{2.65}$$

then, for each p with $1 \leq p \leq \infty$, we can define the functional $I_p : \Lambda^D \bigcap L^p(0,L) \rightarrow \mathbf{R}$ given by $I_p(a) = \|a^+\|_p$ (the same expression as in (2.4)), and in a similar form, we can define the constant

$$\beta_p^D \equiv \inf_{a \in \Lambda^D \bigcap L^p(0,L)} I_p(a),\ 1 \leq p \leq \infty. \tag{2.66}$$

Taking into account the same ideas that for the Neumann problem, it can be easily proved that

$$\beta_p^D = \beta_p^N,\ 1 \leq p \leq \infty. \tag{2.67}$$

In the proof, we must simply replace the spaces X_p of Theorems 2.1–2.3 by the Sobolev space $H_0^1(0,L)$ and (2.63) by

$$\begin{gathered} v(0) = v(2x_0) = \ldots = v(2nx_0) = 0, \\ v'(x_0) = \ldots = v'((2n-1)x_0) = 0, \\ v(x) \neq 0,\ v'(x) \neq 0,\ \forall\, x \in (jx_0, (j+1)x_0),\ 0 \leq j \leq 2n-1. \end{gathered} \tag{2.68}$$

Remark 2.16. Let us note that, contrary to what happens for Neumann problems, in the minimization problems associated with Dirichlet boundary conditions, we do not need to impose any additional restriction to the space $H_0^1(0, L)$ (see [28]). This is due to the fact that the homogeneous linear part of (2.64)

$$u''(x) = 0, \ x \in (0, L), \ u(0) = u(L) = 0 \tag{2.69}$$

has only the trivial solution $u \equiv 0$. In this work, we will call to this type of problems *nonresonant problems.*

Periodic Boundary Conditions In the case of the periodic boundary value problem

$$u''(t) + a(t)u(t) = 0, \ t \in (0, T), \ u(0) - u(T) = u'(0) - u'(T) = 0 \tag{2.70}$$

we assume that $a \in L_T(\mathbf{R}, \mathbf{R})$, the set of T-periodic functions $a : \mathbf{R} \to \mathbf{R}$ such that $a|_{[0,T]} \in L^1(0, T)$ (due to the applications to stability, it is convenient to use t as the independent variable, instead of x).

If we define the set

$$\Lambda^{\text{per}} = \{a \in L_T(\mathbf{R}, \mathbf{R}) \setminus \{0\} : \int_0^T a(t)\, dt \geq 0 \text{ and (2.70) has nontrivial solutions } \} \tag{2.71}$$

the positive eigenvalues of the eigenvalue problem

$$u''(t) + \lambda u(t) = 0, \ t \in (0, T), \ u(0) - u(T) = u'(0) - u'(T) = 0 \tag{2.72}$$

belong to Λ^{per}. Therefore, for each p with $1 \leq p \leq \infty$, we can define the L^p Lyapunov constant for the periodic problem, β_p^{per}, as the real number

$$\beta_p^{\text{per}} \equiv \inf_{a \in \Lambda^{\text{per}} \bigcap L^p(0,T)} \|a^+\|_p. \tag{2.73}$$

An explicit expression for the constant β_p^{per}, as a function of p and T, has been obtained in [30]. As in the Neumann case, we can obtain a characterization of β_p^{per} as a minimum of a convenient minimization problem, where only some appropriate subsets of the space $H^1(0, T)$ are used (see [6] for further details).

Since (2.72) is, as (2.1), a resonant problem, just to get a variational characterization of β_p^{per} we need an additional restriction to the space $H^1(0, T)$. This is shown in the next theorem.

Theorem 2.5. *If $1 \leq p \leq \infty$ is a given number, let us define the sets X_p^{per} and the functionals $I_p^{\text{per}} : X_p^{per} \setminus \{0\} \to \mathbf{R}$ as*

$$X_1^{\text{per}} = \{v \in H^1(0, T) : v(0) - v(T) = 0, \max_{t \in [0,T]} v(t) + \min_{t \in [0,T]} v(t) = 0\},$$

$$X_p^{\text{per}} = \left\{ v \in H^1(0, T) : v(0) - v(T) = 0, \int_0^T |v|^{\frac{2}{p-1}} v = 0 \right\}, \ \textit{if} \ 1 < p < \infty,$$

$$X_\infty^{\text{per}} = \{v \in H^1(0,T) : \ v(0) - v(T) = 0, \ \int_0^T v = 0\},$$

$$I_1^{\text{per}}(v) = \frac{\int_0^T v'^2}{\|v\|_\infty^2}, \ \ I_p^{\text{per}}(v) = \frac{\int_0^T v'^2}{\left(\int_0^T |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}, \ \textit{if} \ 1 < p < \infty, \ \ I_\infty^{\text{per}}(v) = \frac{\int_0^T v'^2}{\int_0^T v^2}. \tag{2.74}$$

Then, the L_p Lyapunov constant β_p^{per} defined in (2.73), satisfies

$$\beta_p^{\text{per}} = \min_{X_p^{\text{per}} \setminus \{0\}} I_p^{\text{per}}, \ \ 1 \le p \le \infty. \tag{2.75}$$

Proof. Only those innovative details with respect to the Neumann case are shown [6].

The case $p = 1$. It is very well known that $\beta_1^{\text{per}} = \frac{16}{T}$ [17, 30]. Now, if $u \in X_1^{\text{per}} \setminus \{0\}$, then there exists $x_0 \in [0,T]$ such that $u(x_0) = 0$. Taking into account that u can be extended as a T-periodic function to $\mathbf{R}$, if we define the function $v(x) = u(x + x_0), \ \forall \ x \in \mathbf{R}$, then $v|_{[0,T]} \in X_1^{\text{per}} \setminus \{0\}, \ v(0) = v(T) = 0$ and $I_1^{\text{per}}(u) = I_1^{\text{per}}(v)$. In addition (if it is necessary, we can choose $-v$ instead of v), there exist $0 < x_1 < x_2 < x_3 < T$ such that

$$v(x_1) = \max_{[0,T]} v, \ \ v(x_2) = 0, \ \ v(x_3) = \min_{[0,T]} v.$$

If $x_0 = 0, x_4 = T$, it follows from the Cauchy–Schwarz inequality

$$\begin{aligned}
\int_0^T v'^2 &= \sum_{i=0}^{3} \int_{x_i}^{x_{i+1}} v'^2 \ge \sum_{i=0}^{3} \frac{\left(\int_{x_i}^{x_{i+1}} |v'|\right)^2}{x_{i+1} - x_i} \\
&\ge \sum_{i=0}^{3} \frac{\left(\int_{x_i}^{x_{i+1}} v'\right)^2}{x_{i+1} - x_i} = \sum_{i=0}^{3} \frac{(v(x_{i+1}) - v(x_i))^2}{x_{i+1} - x_i} \\
&= \|v\|_\infty^2 \sum_{i=0}^{3} \frac{1}{x_{i+1} - x_i} \ge \frac{16}{T} \|v\|_\infty^2.
\end{aligned} \tag{2.76}$$

Consequently

$$I_1^{\text{per}}(u) = \frac{\int_0^T u'^2}{\|u\|_\infty^2} = I_1^{\text{per}}(v) \ge \frac{16}{T}, \ \forall \ u \in X_1^{\text{per}} \setminus \{0\}. \tag{2.77}$$

On the other hand, the function $w \in X_1^{\text{per}} \setminus \{0\}$ defined as

$$w(x) = \begin{cases} x, \text{ if } 0 \le x \le T/4, \\ -(x - \frac{T}{2}), \text{ if } T/4 \le x \le 3T/4, \\ (x - T), \text{ if } 3T/4 \le x \le T, \end{cases} \tag{2.78}$$

satisfies

$$\frac{\int_0^T w'^2}{\|w\|_\infty^2} = \frac{16}{T}.$$

Consequently, the case $p = 1$ is proved.

The case $p = \infty$. It is very well known that $\beta_\infty^{\text{per}} = \frac{4\pi^2}{T^2}$, the first positive eigenvalue of the eigenvalue problem (2.72) (see [30]). From its variational characterization, we obtain

$$\beta_\infty^{\text{per}} = \min_{X_\infty^{\text{per}} \setminus \{0\}} I_\infty^{\text{per}}.$$

The case $1 < p < \infty$. The ideas are similar to those used in the case of Neumann boundary conditions. If we denote

$$m_p^{\text{per}} = \inf_{X_p^{\text{per}} \setminus \{0\}} I_p^{\text{per}}$$

then this infimum is attained in some function u_0 which satisfies

$$\begin{gathered} u_0''(x) + A_p(u_0)|u_0(x)|^{\frac{2}{p-1}} u_0(x) = 0, \; x \in (0, T), \\ u_0(0) - u_0(T) = 0, \; u_0'(0) - u_0'(T) = 0, \end{gathered} \tag{2.79}$$

where

$$A_p(u_0) = m_p^{\text{per}} \left(\int_0^T |u_0|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}}. \tag{2.80}$$

Let us observe that the previous equation is of the type (2.59), but with periodic boundary conditions instead of Neumann ones. As it was commented at the beginning of this section, this is not a problem. If one has an exact knowledge about the number and distribution of the zeros of the functions u_0 and u_0', the Euler equation (2.79) can be integrated (see [3], Lemma 2.7). In our case, it is not restrictive to assume $u_0(0) = u_0(T) = 0$ (see the previous case $p = 1$). Then, if we denote the zeros of u_0 in $[0, T]$ by $0 = x_0 < x_2 < \ldots < x_{2n} = T$ and the zeros of u_0' in $(0, T)$ by $x_1 < x_3 < \ldots < x_{2n-1}$, we obtain

$$m_p^{\text{per}} = \frac{4n^2 I^2 p}{T^{2-\frac{1}{p}} (p-1)^{1-\frac{1}{p}} (2p-1)^{1/p}}, \tag{2.81}$$

where I is defined in (2.62).

The novelty here is that, for the periodic boundary value problem (2.79), $n \geq 2$ (see the relations (2.35)), while for the Neumann and Dirichlet problem $n \geq 1$.

The conclusion is that

$$m_p^{\mathrm{per}} = \frac{16I^2 p}{T^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}} \tag{2.82}$$

that is, four times the corresponding L^p Lyapunov constant for the Dirichlet and the Neumann problem. Finally, in [30] it is shown that this is, exactly, the L^p Lyapunov constant for the periodic problem. Consequently, $m_p^{\mathrm{per}} = \beta_p^{\mathrm{per}},\ 1 < p < \infty$.

Finally, we treat in this section with antiperiodic boundary conditions, another important case due to its applications to stability theory (Chap. 3). As we will show, in some aspects this case is similar to the case of periodic boundary conditions, but in others it is similar to Neumann or Dirichlet boundary conditions.

Antiperiodic Boundary Conditions Let us consider the antiperiodic boundary value problem

$$u''(t) + a(t)u(t) = 0,\ t \in (0,T),\ u(0) + u(T) = u'(0) + u'(T) = 0 \tag{2.83}$$

where $a \in L_T(\mathbf{R}, \mathbf{R})$.

If we define the set

$$\Lambda^{\mathrm{ant}} = \{a \in L_T(\mathbf{R}, \mathbf{R}) :\ (2.83) \text{ has nontrivial solutions } \} \tag{2.84}$$

the positive eigenvalues of the eigenvalue problem

$$u''(t) + \lambda u(t) = 0,\ t \in (0,T),\ u(0) + u(T) = u'(0) + u'(T) = 0 \tag{2.85}$$

belong to Λ^{ant}. Therefore, for each p with $1 \leq p \leq \infty$, we can define the L^p Lyapunov constant for the antiperiodic problem, β_p^{ant}, as the real number

$$\beta_p^{\mathrm{ant}} \equiv \inf_{a \in \Lambda^{\mathrm{ant}} \bigcap L^p(0,T)} \|a^+\|_p \tag{2.86}$$

An explicit expression for the constant β_p^{ant}, as a function of p and T, has been obtained in [30]. As in the cases of Neumann, Dirichlet, or periodic boundary conditions, it is possible to prove a characterization of β_p^{ant} as a minimum of a convenient minimization problem, where only some appropriate subsets of the space $H^1(0,T)$ are used (see [6] for further details). Since (2.83) is, as (2.64), a no resonant problem, i.e., the linear part

$$u''(t) = 0,\ t \in (0,T),\ u(0) + u(T) = u'(0) + u'(T) = 0 \tag{2.87}$$

has only the trivial solution, just to get a variational characterization of β_p^{ant} we do not need any additional restriction to the space $H^1(0,T)$, except $u(0) + u(T) = 0$. This is shown in the next theorem, where the proof is omitted (see [6]).

Theorem 2.6. *If $1 \le p \le \infty$ is a given number, let us define the sets X_p^{ant} and the functional $I_p^{ant} : X_p^{ant} \setminus \{0\} \to \mathbf{R}$, as*

$$X_p^{ant} = \left\{v \in H^1(0,T) : v(0) + v(T) = 0\right\},\ 1 \le p \le \infty,$$

$$I_1^{ant}(v) = \frac{\int_0^T v'^2}{\|v\|_\infty^2}, I_p^{ant}(v) = \frac{\int_0^T v'^2}{\left(\int_0^T |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}},\ \text{if } 1 < p < \infty,\ I_\infty^{ant}(v) = \frac{\int_0^T v'^2}{\int_0^T v^2}. \tag{2.88}$$

Then, the L_p Lyapunov constant β_p^{ant} defined in (2.86) satisfies

$$\beta_p^{ant} = \min_{X_p^{ant}\setminus\{0\}} I_p^{ant},\ \ 1 \le p \le \infty. \tag{2.89}$$

Remark 2.17. Using the procedure described in Sect. 2.1 of this chapter, many other boundary conditions can be studied. We bring out the case of problems of mixed type

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u(L) = 0,$$

where the number n of the relation (2.61) must be chosen as $n = 1/2$. However, due to the important relationship of this case with the notion of disfocality and its applications to resonant nonlinear problems and the theory of stability, such problems will be treated in the next section.

2.4 Disfocality

Under the natural restrictions $a \in L^1(0,L) \setminus \{0\}$ and $\int_0^L a(x)\,dx \ge 0$, the relation between Neumann boundary conditions and disfocality arises in a natural way, since if $u \in H^1(0,L)$ is any nontrivial solution of

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u'(L) = 0 \tag{2.90}$$

then u must have a zero c in the interval $(0,L)$. In fact, $u(0) \ne 0$ and $u(L) \ne 0$. Then, if u has not zeros in the interval $(0,L)$, we can assume that u is, for example, a positive (nonconstant) solution of (2.90). Considering $v = \dfrac{1}{u}$ as test function in the weak formulation of (2.90), we obtain

$$\int_0^L a = \int_0^L au\frac{1}{u} = \int_0^L u'(\frac{1}{u})' = -\int_0^L \frac{u'^2}{u^2} < 0$$

which is a contradiction with the hypothesis $\int_0^L a(x)\,dx \geq 0$.

In consequence both problems

$$v''(x) + a(x)v(x) = 0,\ x \in (0,c),\ v'(0) = v(c) = 0 \qquad \textbf{PM(0,c)}$$

and

$$v''(x) + a(x)v(x) = 0,\ x \in (c,L),\ v(c) = v'(L) = 0 \qquad \textbf{PM(c,L)}$$

have nontrivial solutions.

This simple observation (which has been previously employed in the case of Dirichlet boundary conditions, [15, 19]) can be used to deduce the following conclusion: if $a \in L^1(0,L) \setminus \{0\}$ with $\int_0^L a \geq 0$ is any function such that for any $c \in (0,L)$, either problem **PM(0,c)** or problem **PM(c,L)** has only the trivial solution, then problem (2.90) has only the trivial solution.

Below we study the relation between the best L_p Lyapunov constants for the problems (2.90) and

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u(L) = 0, \tag{2.91}$$

where for Neumann problem (2.90), function $a \in \Lambda$ and Λ is defined by

$$\Lambda = \{a \in L^1(0,L) \setminus \{0\} : \int_0^L a(x)\,dx \geq 0 \text{ and (2.90) has nontrivial solutions}\} \tag{2.92}$$

whereas for mixed problem (2.91), function $a \in \Lambda^*$ and Λ^* is defined by

$$\Lambda^* = \{a \in L^1(0,L) :\ \text{(2.91) has nontrivial solutions}\} \tag{2.93}$$

Here $u \in H = H^1(0,L)$ (the usual Sobolev space) in the case of Neumann conditions and $u \in H^* = \{u \in H : u(L) = 0\}$ in the case of mixed boundary conditions. Obviously, the positive eigenvalues of the problems

$$u''(x) + \lambda u(x) = 0,\ x \in (0,L),\ u'(0) = u'(L) = 0 \tag{2.94}$$

and

$$u''(x) + \lambda u(x) = 0,\ x \in (0,L),\ u'(0) = u(L) = 0 \tag{2.95}$$

belong, respectively, to Λ and Λ^*. Therefore Λ and Λ^* are both not empty and the quantities

$$\beta_p \equiv \inf_{a \in \Lambda \bigcap L^p(0,L)} \|a^+\|_{L^p(0,L)}, \ 1 \le p \le \infty \tag{2.96}$$

and

$$\beta_p^* \equiv \inf_{a \in \Lambda^* \bigcap L^p(0,L)} \|a^+\|_{L^p(0,L)}, \ 1 \le p \le \infty \tag{2.97}$$

are well defined.

The next theorem establishes a clear relation between β_p and β_p^*. When it is necessary, we will write $\Lambda(0,L), \beta_p(0,L), \ldots$ to show up the explicit dependence of these quantities with respect to the interval $(0,L)$. Also, it is possible to define, in an analogous manner, $\Lambda(c,d), \beta_p(c,d), \ldots$ for arbitrary real numbers $c < d$.

Theorem 2.7. *If* $1 \le p \le \infty$, *we have* $\beta_p^* = \beta_p/4$.

Proof. By using the definition of β_p and β_p^*, and doing a trivial change of variables, it is easy to prove the equalities

$$\beta_p(0,c) = \left(\frac{c}{d}\right)^{\frac{1}{p}-2} \beta_p(0,d), \ \forall c,d \in \mathbf{R}^+, \ \ \forall p, \ 1 \le p \le \infty \tag{2.98}$$

and

$$\beta_p^*(0,c) = \left(\frac{c}{d}\right)^{\frac{1}{p}-2} \beta_p^*(0,d), \ \forall c,d \in \mathbf{R}^+, \ \forall p, \ 1 \le p \le \infty. \tag{2.99}$$

Moreover, problem (2.91) becomes

$$v''(x) + a(L-x)v(x) = 0, \ x \in (0,L), \ v(0) = v'(L) = 0 \tag{2.100}$$

through the variable change $y = L - x$ and it is clear that

$$\|a(\cdot)\|_{L^p(0,L)} = \|a(L-\cdot)\|_{L^p(0,L)}$$

Lemma 2.1. *If* $1 \le p \le \infty$, *we have* $\beta_p^* \le \beta_p/4$.

Proof. If $a \in \Lambda(0,L) \bigcap L^p(0,L)$ and u is a nontrivial solution of (2.90), there exists $c \in (0,L)$ such that $u(c) = 0$. Therefore both problems **PM(0,c)** and **PM(c,L)** have nontrivial solutions. In consequence, function a, restricted to the interval $[0,c]$ belongs to $\Lambda^*(0,c)$ and function $a(L+c-\cdot)$, restricted to the interval $[c,L]$ belongs to $\Lambda^*(c,L)$. Let us assume $1 \le p < \infty$. Then taking into account the definition of β_p^*, (2.99) and (2.100), we have

$$
\begin{aligned}
\|a^+\|^p_{L^p(0,L)} &= \|a^+\|^p_{L^p(0,c)} + \|a^+\|^p_{L^p(c,L)} \\
&\geq (\beta^*_p(0,c))^p + (\beta^*_p(c,L))^p \\
&= \left(\frac{L}{c}\right)^{2p-1} (\beta^*_p(0,L))^p + \left(\frac{L}{L-c}\right)^{2p-1} (\beta^*_p(0,L))^p \\
&= \left[\left(\frac{L}{c}\right)^{2p-1} + \left(\frac{L}{L-c}\right)^{2p-1}\right] (\beta^*_p(0,L))^p \\
&\geq (\inf_{c\in(0,L)} g(c))(\beta^*_p(0,L))^p,
\end{aligned}
\tag{2.101}
$$

where $g : (0,L) \to \mathbf{R}$ is defined by

$$
g(c) = \left[\left(\frac{L}{c}\right)^{2p-1} + \left(\frac{L}{L-c}\right)^{2p-1}\right], \ \forall\, c \in (0,L).
$$

It is easily checked that

$$
g'(c) < 0, \ \forall\, c \in (0,L/2) \text{ and } g'(c) > 0, \ \forall\, c \in (L/2,L).
$$

Thus

$$
\inf_{c\in(0,L)} g(c) = g(L/2) = 4^p. \tag{2.102}
$$

Therefore, from (2.101) and (2.102) we deduce

$$
\|a^+\|^p_{L^p(0,L)} \geq 4^p (\beta^*_p(0,L))^p, \ \forall\, a \in \Lambda \cap L^p(0,L).
$$

Similar ideas may be used in the case $p = \infty$. This proves the lemma.

Lemma 2.2. *If $1 \leq p \leq \infty$, we have $\beta^*_p \geq \beta_p/4$.*

Proof. If $a \in \Lambda^*(0,L) \bigcap L^p(0,L)$ and u is a nontrivial solution of (2.91), let us define the functions

$$
\begin{aligned}
&\tilde{a}, \tilde{u} : [0,2L] \to \mathbf{R} \\
&\tilde{a}(x) = \begin{cases} a(x), \ x \in [0,L], \\ a(2L-x), \ x \in (L,2L] \end{cases} \\
&\tilde{u}(x) = \begin{cases} u(x), \ x \in [0,L], \\ -u(2L-x), \ x \in (L,2L]. \end{cases}
\end{aligned}
\tag{2.103}
$$

Then $\tilde{u} \in H^1(0,2L)$ and we claim that $\tilde{u}$ is a (nontrivial) solution of

$$
w''(x) + \tilde{a}(x)w(x) = 0, \ x \in (0,2L), \ w'(0) = w'(2L) = 0. \tag{2.104}
$$

To see this, we need to demonstrate

$$\int_0^{2L} \tilde{u}'(x)z'(x)\,dx = \int_0^{2L} \tilde{a}(x)\tilde{u}(x)z(x)\,dx, \ \forall\, z \in H^1(0,2L). \tag{2.105}$$

If $z \in H^1(0,2L)$ satisfies

$$z(L) = 0 \tag{2.106}$$

then z, restricted to the interval $[0,L]$ is a test function for mixed problem (2.91) and therefore

$$\begin{aligned}\int_0^{L} \tilde{u}'(x)z'(x)\,dx &= \int_0^{L} u'(x)z'(x)\,dx \\ &= \int_0^{L} a(x)u(x)z(x)\,dx = \int_0^{L} \tilde{a}(x)\tilde{u}(x)z(x)\,dx.\end{aligned} \tag{2.107}$$

Moreover, since function $z(2L-y), y \in [0,L]$, is also a test function for mixed problem (2.91), we have

$$\begin{aligned}\int_L^{2L} \tilde{u}'(x)z'(x)\,dx &= \int_0^{L} u'(y)z'(2L-y)\,dy \\ &= -\int_0^{L} a(y)u(y)z(2L-y)\,dy = \int_L^{2L} \tilde{a}(x)\tilde{u}(x)z(x)\,dx.\end{aligned} \tag{2.108}$$

From (2.107) and (2.108) we deduce (2.105) when z satisfies (2.106). But in the interval $[0,2L]$, function $\tilde{a}(x)\tilde{u}(x)$ is an odd function with respect to L. This implies (2.105) when $z \equiv 1$. Finally, as any $z \in H^1(0,2L)$ may be written in the form $z(x) = (z(x) - z(L)) + z(L)$, we conclude (2.105).

Once we have proved that $\tilde{u}$ is a (nontrivial) solution of (2.104) associated with function $\tilde{a}$, we would need to have the sign condition

$$\int_0^{2L} \tilde{a}(x)\,dx \geq 0 \tag{2.109}$$

since this property is included into the definition of the set $\Lambda(0,2L)$. But

$$\int_0^{2L} \tilde{a}(x)\,dx = 2\int_0^{L} a(x)\,dx$$

and $a \in \Lambda^*(0,L)$, a set where no sign conditions is assumed. This difficulty may be overcome by using some eigenvalue ideas. In fact, we will prove

$$\forall\, a \in \Lambda^*(0,L),\ \exists\, k \in (0,1] : ka^+ \in \Lambda^*(0,L). \tag{2.110}$$

To establish this, note that if $a \in \Lambda^*(0, L)$ then $\int_0^L u'^2(x)\,dx = \int_0^L a(x)u^2(x)\,dx$ for some nontrivial function $u \in H^*$. Therefore the set $\{x \in (0, L) : a(x) > 0\}$ has positive measure. As a consequence, the eigenvalue problem

$$w''(x) + \lambda a(x)w(x) = 0,\ x \in (0, L),\ \ w'(0) = w(L) = 0 \tag{2.111}$$

has a sequence of positive eigenvalues $\lambda_1(a) < \lambda_2(a) < \ldots$

Moreover, $a \in \Lambda^*(0, L)$ implies $\lambda_1(a) \le 1$. Since $a^+ \ge a$, we have $\lambda_1(a^+) \le \lambda_1(a)$. As $\lambda_1(a^+)a^+ \in \Lambda^*(0, L)$, this proves (2.110).

Now, from (2.110) we have $\|ka^+\|_{L^p(0,L)} \le \|a^+\|_{L^p(0,L)}$. Therefore, it is clearly not restrictive to assume from the beginning of the lemma we are proving that $a(x) \ge 0$. This fact implies (2.109) and as a consequence, function $\tilde{a}$ defined in (2.103) belongs to the set $\Lambda(0, 2L)$. Moreover, if $1 \le p < \infty$,

$$2\|a\|^p_{L^p(0,L)} = \|\tilde{a}\|^p_{L^p(0,2L)} \ge \beta_p^p(0, 2L) = 2^{1-2p}\beta_p^p(0, L)$$

which imply

$$\|a\|_{L^p(0,L)} \ge \frac{1}{4}\beta_p(0, L),$$

for each function $a \in \Lambda^*(0, L) \bigcap L^p(0, L)$ such that $a(x) \ge 0$. From this and (2.110) we obtain the conclusion of the lemma if $1 \le p < \infty$. Similar ideas may be used in the case $p = \infty$. This finishes also the proof of Theorem 2.7.

Remark 2.18. The proof of Theorem 2.7 that we have given here is based on an appropriate change of variables, but it is possible to carry out a different approach by using similar ideas to those contained in Sect. 2.1. In this way, some additional results for $\beta_p^*(0, L)$ may be proved. For instance, $\beta_p^*(0, L)$ is attained if and only if $1 < p \le \infty$

Next, we present some results on the existence and uniqueness of solutions of linear b.v.p.

$$u''(x) + a(x)u(x) = f(x),\ x \in (0, L),\ \ u'(0) = u'(L) = 0. \tag{2.112}$$

Previously, if $a \in L^1(c, d) \setminus \{0\}$, $\int_c^d a(x)\,dx \ge 0$ and $1 \le p \le \infty$, it may be convenient to introduce hypothesis **(Hp)***.

Hypothesis (Hp)* It is established as:

1. $\|a^+\|_{L^1(c,d)} \le \beta_1^*(c, d)$ if $p = 1$.
2. $a \in L^p(c, d),\ \|a^+\|_{L^p(c,d)} < \beta_p^*(c, d)$.

Remark 2.19. Let us observe that if a function a satisfies hypothesis **(Hp)*** for some $p,\ 1 \le p \le \infty$, then the unique solution of the boundary problems

$$u''(x) + a(x)u(x) = 0,\ x \in (c, d),\ u'(c) = u(d) = 0 \tag{2.113}$$

and

$$u''(x) + a(x)u(x) = 0,\ x \in (c,d),\ u(c) = u'(d) = 0 \tag{2.114}$$

is the trivial one.

Theorem 2.8. *Let* $a \in L^1(0,L) \setminus \{0\}$ *with* $\int_0^L a(x)\,dx \geq 0$, *satisfying:*

For each $c \in (0,L)$ *either hypothesis* **(Hp)*** *in the interval* $(0,c)$ *or hypothesis* **(Hq)*** *in the interval* (c,L) *(here,* $p,q \in [1,\infty]$ *may depend on* c*).*

Then for each $f \in L^1(0,L)$, *the boundary value problem (2.112) has a unique solution.*

Proof. Since (2.112) is a linear problem, it is sufficient to see that the unique solution of the homogeneous problem

$$u''(x) + a(x)u(x) = 0,\ x \in (0,L),\ u'(0) = u'(L) = 0 \tag{2.115}$$

is the trivial one. Now, if (2.115) has some nontrivial solution u, it was shown at the beginning of this chapter that u must have a zero d in the interval $(0,L)$. In this case, both problems **PM(0,d)** and **PM(d,L)** have nontrivial solutions. But by using either hypothesis **(Hp)*** in $(0,d)$ or hypothesis **(Hq)*** in (d,L), we have a contradiction.

In concrete examples, it may be convenient to choose $p = q$, independent from $c \in (0,L)$. To this respect, the following proposition may be of interest.

Proposition 2.1. *Let* $1 < p < \infty$ *and* $a \in L^p(0,L)$. *Then the following statements are equivalent:*

$$\forall\ c \in (0,L),\ \textit{either}\ \|a^+\|_{L^p(0,c)} < \beta_p^*(0,c)\ \textit{or}\ \|a^+\|_{L^p(c,L)} < \beta_p^*(c,L) \tag{2.116}$$

$$\exists\ x_0 \in (0,L):\ \|a^+\|_{L^p(0,x_0)} < \beta_p^*(0,x_0)\ \textit{and}\ \|a^+\|_{L^p(x_0,L)} < \beta_p^*(x_0,L). \tag{2.117}$$

Proof. Let (2.116) be satisfied. Function $c^{2-\frac{1}{p}}\ \|a^+\|_{L^p(0,c)}$ is continuous and increasing in the interval $c \in (0,L)$ whereas function $(L-c)^{2-\frac{1}{p}}\ \|a^+\|_{L^p(c,L)}$ is continuous and decreasing in $c \in (0,L)$. Then, choose x_0 as a point in $(0,L)$ such that

$$x_0^{2-\frac{1}{p}}\ \|a^+\|_{L^p(0,x_0)} = (L-x_0)^{2-\frac{1}{p}}\ \|a^+\|_{L^p(x_0,L)}. \tag{2.118}$$

Since (2.116) is fulfilled $\forall\ c \in (0,L)$, it is true in particular for $c = x_0$. But taking into account (2.118) and the relation (2.99)

$$\beta_p^*(x_0,L) = \beta_p^*(0,L-x_0) = \left(\frac{L-x_0}{x_0}\right)^{\frac{1}{p}-2} \beta_p^*(0,x_0)$$

if x_0 is as in (2.118), both inequalities in (2.116) are really the same inequality and, moreover, they are identical to (2.117).

Reciprocally, if (2.117) is satisfied and $c \in (0, L)$, we can distinguish two cases: $c \in (0, x_0]$ and $c \in (x_0, L)$. In the first case, we have

$$\begin{aligned}\|a^+\|_{L^p(0,c)} &\leq \|a^+\|_{L^p(0,x_0)} < \beta_p^*(0,x_0)\\ &= \left(\frac{x_0}{L}\right)^{\frac{1}{p}-2} \beta_p^*(0,L) \leq \left(\frac{c}{L}\right)^{\frac{1}{p}-2} \beta_p^*(0,L) = \beta_p^*(0,c).\end{aligned}$$

A similar reasoning is valid if $c \in (x_0, L)$.

Remark 2.20. An analogous result may be demonstrated for $p = 1$ by replacing strict inequalities in (2.116) and (2.117) with non-strict ones. If $p = \infty$, (2.117) implies (2.116). As a consequence, if (2.117) is satisfied for $p = \infty$, the unique solution of (2.115) is the trivial one. However, in this last case, a more precise condition may be obtained. This is shown in the next proposition.

Proposition 2.2. *If function a fulfills*

$$\begin{aligned}&a \in L^\infty(0,L) \setminus \{0\}, \ \int_0^L a \geq 0 \ \text{ and } \ \exists\, x_0 \in (0,L): \\ &\max\{x_0^2 \|a^+\|_{L^\infty(0,x_0)},\ (L-x_0)^2\|a^+\|_{L^\infty(x_0,L)}\} \leq \tfrac{\pi^2}{4}\end{aligned} \tag{H}$$

and, in addition, either a^+ is not the constant $\pi^2/4x_0^2$ in the interval $[0, x_0]$ or a^+ is not the constant $\pi^2/4(L-x_0)^2$ in the interval $[x_0, L]$, then for each $f \in L^1(0, L)$, the boundary value problem (2.112) has a unique solution.

Proof. To prove this proposition, take into account that $\beta_\infty^*(0, x_0) = \pi^2/4x_0^2$ and that $\beta_\infty^*(x_0, L) = \pi^2/4(L-x_0)^2$. Then, if $d \in (0, x_0)$ we have

$$\|a^+\|_{L^\infty(0,d)} \leq \|a^+\|_{L^\infty(0,x_0)} \leq \beta_\infty^*(0,x_0) < \beta_\infty^*(0,d)$$

Therefore, problem **PM(0,d)** has only the trivial solution. If $d \in (x_0, L)$ a similar reasoning is valid and we obtain that problem **PM(d,L)** has only the trivial solution. Finally, if $d = x_0$, we would have

$$\|a^+\|_{L^\infty(0,x_0)} \leq \beta_\infty^*(0,x_0), \ \ \|a^+\|_{L^\infty(x_0,L)} \leq \beta_\infty^*(x_0,L).$$

But since, in addition, we have that either a^+ is not the constant $\pi^2/4x_0^2$ in the interval $[0, x_0]$ or a^+ is not the constant $\pi^2/4(L-x_0)^2$ in the interval $[x_0, L]$, we deduce that either problem **PM(0,**x_0**)** or problem **PM(**x_0**,L)** has only the trivial solution. This proves that (2.115) has only the trivial solution and therefore we have the desired conclusion.

In particular, if $x_0 = L/2$ in Proposition 2.2, we obtain the classical result related to the so-called nonuniform nonresonance conditions with respect to the first positive eigenvalue $\frac{\pi^2}{L^2}$ [24–26]. However, if for instance, $x_0 \in (0, L/2)$, it is allowed the equality $\|a^+\|_{L^\infty(0,x_0)} = \pi^2/4x_0^2$ (which is a quantity greater than $\frac{\pi^2}{L^2}$) as long as $\|a^+\|_{L^\infty(x_0,L)} < \pi^2/4(L-x_0)^2$.

Remark 2.21. Hypothesis (**H**) is optimal in the sense that if a^+ is the constant $\pi^2/4x_0^2$ in the interval $[0,x_0]$ and a^+ is the constant $\pi^2/4(L-x_0)^2$ in the interval $[x_0,L]$, then (2.115) has the $C^2[0,L]$ nontrivial solution:

$$u(x) = \begin{cases} \dfrac{-x_0}{L-x_0} \cos \dfrac{\pi x}{2x_0}, & \text{if } x \in (0,x_0), \\ \cos \dfrac{\pi(L-x)}{2(L-x_0)}, & \text{if } x \in (x_0,L), \end{cases}$$

Remark 2.22. By using the definition of β_p, it is clear that if for some p, with $1 \le p < \infty$, function a satisfies

$$\|a^+\|_{L^p(0,L)} < \beta_p(0,L) \tag{2.119}$$

then the unique solution of (2.115) is the trivial one. It is easy to prove that (2.119) implies (2.116). In fact, if (2.116) is not true for some $c \in (0,L)$, taking into account (2.102) and Theorem 2.7 we obtain

$$\begin{aligned} \|a^+\|^p_{L^p(0,L)} &= \|a^+\|^p_{L^p(0,c)} + \|a^+\|^p_{L^p(c,L)} \\ &\ge (\beta^*_p(0,c))^p + (\beta^*_p(c,L))^p \\ &= \left[\left(\frac{c}{L}\right)^{1-2p} + \left(\frac{L-c}{L}\right)^{1-2p} \right] \frac{(\beta_p(0,L))^p}{4^p} \ge (\beta_p(0,L))^p \end{aligned}$$

which is a contradiction with (2.119).

Previous remark shows that if we want to have a criterion implying that (2.115) has only the trivial solution, then (2.116) is better than (2.119). In order to prove that (2.116) is a strict generalization of (2.119), we show the following example (see [5] for more details).

Example. Let c_1, c_2 be two positive numbers and let us consider the two step potential

$$a(x) = \begin{cases} c_1^2, & \text{if } 0 \le x < \dfrac{Lc_2}{c_1+c_2}, \\ c_2^2, & \text{if } \dfrac{Lc_2}{c_1+c_2} \le x \le L. \end{cases} \tag{2.120}$$

Then, for each p, $1 \le p \le \infty$, there exist c_1, c_2 such that

$$\text{function } a \text{ satisfies (2.117) (and therefore (2.116)) for } x_0 = \frac{Lc_2}{c_1+c_2} \tag{2.121}$$

and

$$\|a^+\|_{L^q(0,L)} > \beta_q(0,L), \ \forall q \in [1,\infty]. \tag{2.122}$$

Let us remark that from (2.122) we cannot deduce that (2.115) has only the trivial solution. However, we can affirm that this fact is true from (2.121).

We finish this section with some results on the existence and uniqueness of solutions of nonlinear b.v.p.

$$u''(x) + f(x,u(x)) = 0, \ x \in (0,L), \ \ u'(0) = u'(L) = 0. \tag{2.123}$$

Taking into account the previous discussion, next theorem is a strict generalization of Theorem 3.1 in [3], Theorem B in [18] and (for ordinary differential equations) Theorem 7.1 in [4] and Theorem 2 in [26]. The proof, which is similar to the one given in [3, 4], combines the linear results of this section with Schauder's fixed point theorem. We omit the details.

Theorem 2.9. *Let us consider (2.123) where the following requirements are supposed:*

1. *f and f_u are Caratheodory functions on $[0,L] \times \mathbf{R}$ and $f(\cdot,0) \in L^1(0,L)$.*
2. *There exist functions $\alpha, \beta \in L^\infty(0,L)$, satisfying*

$$\alpha(x) \le f_u(x,u) \le \beta(x)$$

on $[0,L] \times \mathbf{R}$ and β satisfies for each $c \in (0,L)$ either hypothesis **(Hp)*** *in the interval $(0,c)$ (for some $p \in [1,\infty]$), or hypothesis* **(Hq)*** *in the interval (c,L) (for some $q \in [1,\infty]$).*

3. *Moreover, we assume one of the following conditions:*

a.

$$\int_\Omega \alpha \ge 0, \ \ \alpha \not\equiv 0$$

b.

$$\alpha \equiv 0, \ \exists s_0 \in \mathbf{R} \text{ s.t. } \int_\Omega f(x,s_0)\, dx = 0, \ \text{and } f_u(x,u(x)) \not\equiv 0, \ \forall u \in C(\overline{\Omega}).$$

Then, problem (2.123) has a unique solution.

Remark 2.23. The idea of using qualitative properties of the mixed problem **PM(0,c)** in the study of resonant nonlinear problems like (2.123) has been previously employed by different authors. The interested reader may consult [13, 25, 27, 29] for the case where the nonlinearity f is restricted in one direction.

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