Chapter 2
Discrete Delta Fractional Calculus and Laplace Transforms

2.1 Introduction

At the outset of this chapter we will be concerned with the (delta) Laplace transform, which is a special case of the Laplace transform studied in the book by Bohner and Peterson [62]. We will not assume the reader has any knowledge of the material in that book. The delta Laplace transform is equivalent under a transformation to the Z-transform, but we prefer the definition of the Laplace transform given here, which has the property that many of the Laplace transform formulas will be analogous to the Laplace transform formulas in the continuous setting. We will show how we can use the (delta) Laplace transform to solve initial value problems for difference equations and to solve summation equations. We then develop the discrete delta fractional calculus. Finally, we apply the Laplace transform method to solve fractional initial value problems and fractional summation equations.

The continuous fractional calculus has been well developed (see the books by Miller and Ross [147], Oldham and Spanier [152], and Podlubny [153]). But only recently has there been a great deal of interest in the discrete fractional calculus (see the papers by Atici and Eloe [32–36], Goodrich [88–96], Miller and Ross [146], and M. Holm [123–125]). More specifically, the discrete delta fractional calculus has been recently studied by a variety of authors such as Atici and Eloe [31, 32, 34, 35], Goodrich [88, 89, 91, 92, 94, 95], Miller and Ross [147], and M. Holm [123–125]. As we shall see in this chapter, one of the peculiarities of the delta fractional difference is its domain shifting properties. This property makes, in certain ways, the study of the delta fractional difference more complicated than its nabla counterpart, as a comparison of the present chapter to Chap. 3 will demonstrate.

C. Goodrich, A.C. Peterson, *Discrete Fractional Calculus*,
DOI 10.1007/978-3-319-25562-0_2

2.2 The Delta Laplace Transform

In this section we develop properties of the (delta) Laplace transform. First we give an abstract definition of this transform.

Definition 2.1 (Bohner–Peterson [62]). Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then we define the (delta) **Laplace transform** of f based at a by

$$\mathcal{L}_a\{f\}(s) = \int_a^\infty e_{\ominus s}(\sigma(t), a) f(t) \Delta t$$

for all complex numbers $s \neq -1$ such that this improper integral converges.

The following theorem gives two useful expressions for the Laplace transform of f.

Theorem 2.2. *Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then*

$$\mathcal{L}_a\{f\}(s) = F_a(s) := \int_0^\infty \frac{f(a+k)}{(s+1)^{k+1}} \Delta k \tag{2.1}$$

$$= \sum_{k=0}^\infty \frac{f(a+k)}{(s+1)^{k+1}}, \tag{2.2}$$

for all complex numbers $s \neq -1$ such that this improper integral (infinite series) converges.

Proof. To see that (2.1) holds note that

$$\begin{aligned}
\mathcal{L}_a\{f\}(s) &= \int_a^\infty e_{\ominus s}(\sigma(t), a) f(t) \Delta t \\
&= \sum_{t=a}^\infty e_{\ominus s}(\sigma(t), a) f(t) \\
&= \sum_{t=a}^\infty [1 + \ominus s]^{\sigma(t)-a} f(t) \\
&= \sum_{t=a}^\infty \frac{f(t)}{(1+s)^{t-a+1}} \\
&= \sum_{k=0}^\infty \frac{f(a+k)}{(1+s)^{k+1}}.
\end{aligned}$$

This also gives us that

$$\mathcal{L}_a\{f\}(s) = \int_0^\infty \frac{f(a+k)}{(1+s)^{k+1}}\Delta k.$$

□

To find functions such that the Laplace transform exists on a nonempty set we make the following definition.

Definition 2.3. We say that a function $f : \mathbb{N}_a \to \mathbb{R}$ is of **exponential order** $r > 0$ (at ∞) if there exists a constant $A > 0$ such that

$$|f(t)| \leq Ar^t, \quad \text{for } t \in \mathbb{N}_a, \quad \text{sufficiently large.}$$

Now we can prove the following existence theorem.

Theorem 2.4 (Existence Theorem). *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then $\mathcal{L}_a\{f\}(s)$ converges absolutely for $|s+1| > r$.*

Proof. Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then there is a constant $A > 0$ and an $m \in \mathbb{N}_0$ such that for each $t \in \mathbb{N}_{a+m}$, $|f(t)| \leq Ar^t$. Hence for $|s+1| > r$,

$$\begin{aligned}
\sum_{k=m}^{\infty}\left|\frac{f(k+a)}{(s+1)^{k+1}}\right| &= \sum_{k=m}^{\infty}\frac{|f(k+a)|}{|s+1|^{k+1}} \\
&\leq \sum_{k=m}^{\infty}\frac{Ar^{k+a}}{|s+1|^{k+1}} \\
&= \frac{Ar^a}{|s+1|}\sum_{k=m}^{\infty}\left(\frac{r}{|s+1|}\right)^k \\
&= \frac{Ar^a}{|s+1|}\frac{\left(\frac{r}{|s+1|}\right)^m}{1-\left(\frac{r}{|s+1|}\right)} \\
&= \frac{A}{|s+1|^m}\frac{r^{a+m}}{|s+1|-r} \\
&< \infty.
\end{aligned}$$

Hence, the Laplace transform of f converges absolutely for $|s+1| > r$. □

We will see later (see Remark 2.57) that the converse of Theorem 2.4 does not hold in general.

In this chapter, we will usually consider functions f of some exponential order $r > 0$, ensuring that the Laplace transform of f does in fact converge somewhere in the complex plane—specifically, it converges for all complex numbers outside the closed ball of radius r centered at negative one, that is, for $|s+1| > r$. We will abuse the notation by sometimes writing $\mathcal{L}_a\{f(t)\}(s)$ instead of the preferred notation $\mathcal{L}_a\{f\}(s)$.

Example 2.5. Clearly, $e_p(t,a)$, $p \neq -1$, a constant, is of exponential order $r = |1+p| > 0$. Therefore, we have for $|s+1| > r = |1+p|$,

$$\begin{aligned}
\mathcal{L}_a\left\{e_p(t,a)\right\}(s) &= \mathcal{L}_a\left\{(1+p)^{t-a}\right\}(s) \\
&= \sum_{k=0}^{\infty} \frac{(1+p)^k}{(s+1)^{k+1}} \\
&= \frac{1}{s+1} \sum_{k=0}^{\infty} \left(\frac{p+1}{s+1}\right)^k \\
&= \frac{1}{s+1} \left(\frac{1}{1-\frac{p+1}{s+1}}\right) \\
&= \frac{1}{s-p}.
\end{aligned}$$

Hence

$$\mathcal{L}_a\{e_p(t,a)\}(s) = \frac{1}{s-p}, \quad |s+1| > |1+p|.$$

An important special case ($p = 0$) of the above formula is

$$\mathcal{L}_a\{1\}(s) = \frac{1}{s}, \quad \text{for} \quad |s+1| > 1.$$

In the next theorem we see that the Laplace transform operator $\mathcal{L}_a$ is a linear operator.

Theorem 2.6 (Linearity). *Suppose $f, g : \mathbb{N}_a \to \mathbb{R}$ and the Laplace transforms of f and g converge for $|s+1| > r$, where $r > 0$, and let $c_1, c_2 \in \mathbb{C}$. Then the Laplace transform of $c_1 f + c_2 g$ converges for $|s+1| > r$ and*

$$\mathcal{L}_a\{c_1 f + c_2 g\}(s) = c_1 \mathcal{L}_a\{f\}(s) + c_2 \mathcal{L}_a\{g\}(s), \tag{2.3}$$

for $|s+1| > r$.

Proof. Since $f, g : \mathbb{N}_a \to \mathbb{R}$ and the Laplace transforms of f and g converge for $|s+1| > r$, where $r > 0$, we have that for $|s+1| > r$

$$\begin{aligned}
c_1\mathcal{L}_a\{f\}(s) + c_2\mathcal{L}_a\{g\}(s) \\
&= c_1\sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+1}} + c_2\sum_{k=0}^{\infty}\frac{g(a+k)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{(c_1f+c_2g)(a+k)}{(s+1)^{k+1}} \\
&= \mathcal{L}_a\{c_1f+c_2g\}(s).
\end{aligned}$$

This completes the proof. □

The following uniqueness theorem is very useful.

Theorem 2.7 (Uniqueness). *Assume $f, g : \mathbb{N}_a \to \mathbb{R}$ and there is an $r > 0$ such that*

$$\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s)$$

for $|s+1| > r$. Then

$$f(t) = g(t), \quad \textit{for all} \quad t \in \mathbb{N}_a.$$

Proof. By hypothesis we have that

$$\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s)$$

for $|s+1| > r$. This implies that

$$\sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+1}} = \sum_{k=0}^{\infty}\frac{g(a+k)}{(s+1)^{k+1}}$$

for $|s+1| > r$. It follows from this that

$$f(a+k) = g(a+k), \quad k \in \mathbb{N}_0,$$

and this completes the proof. □

Next we give the Laplace transforms of the (delta) hyperbolic sine and cosine functions.

Theorem 2.8. *Assume $p \neq \pm 1$ is a constant. Then*

(i) $\mathcal{L}_a\{\cosh_p(t,a)\}(s) = \frac{s}{s^2-p^2}$;
(ii) $\mathcal{L}_a\{\sinh_p(t,a)\}(s) = \frac{p}{s^2-p^2}$,

for $|s+1| > \max\{|1+p|, |1-p|\}$.

Proof. To see that (ii) holds, consider

$$\begin{aligned}\mathcal{L}_a\{\sinh_p(t,a)\}(s) &= \frac{1}{2}\left[\mathcal{L}_a\{e_p(t,a)\}(s) - \mathcal{L}\{e_{-p}(t,a)\}(s)\right]\\ &= \frac{1}{2}\frac{1}{s-p} - \frac{1}{2}\frac{1}{s+p}\\ &= \frac{p}{s^2-p^2}\end{aligned}$$

for $|s+1| > \max\{|1+p|, |1-p|\}$. The proof of (i) is similar (see Exercise 2.5). □

Next, we give the Laplace transforms of the (discrete) sine and cosine functions.

Theorem 2.9. *Assume $p \neq \pm i$. Then*

(i) $\mathcal{L}_a\{\cos_p(t,a)\}(s) = \frac{s}{s^2+p^2}$;
(ii) $\mathcal{L}_a\{\sin_p(t,a)\}(s) = \frac{p}{s^2+p^2}$,

for $|s+1| > \max\{|1+ip|, |1-ip|\}$.

Proof. To see that (i) holds, note that

$$\begin{aligned}\mathcal{L}_a\{\cos_p(t,a)\}(s) &= \mathcal{L}_a\{\cosh_{ip}(t,a)\}(s)\\ &= \frac{1}{2}\left[\mathcal{L}_a\{e_{ip}(t,a)\}(s) + \mathcal{L}\{e_{-ip}(t,a)\}(s)\right]\\ &= \frac{1}{2}\frac{1}{s-ip} + \frac{1}{2}\frac{1}{s+ip}\\ &= \frac{s}{s^2+p^2},\end{aligned}$$

for $|s+1| > \max\{|1+ip|, |1-ip|\}$. For the proof of part (ii) see Exercise 2.6. □

Theorem 2.10. *Assume $\alpha \neq -1$ and $\frac{\beta}{1+\alpha} \neq \pm 1$. Then*

(i) $\mathcal{L}_a\{e_\alpha(t,a)\cosh_{\frac{\beta}{1+\alpha}}(t,a)\}(s) = \frac{s-\alpha}{(s-\alpha)^2-\beta^2}$;
(ii) $\mathcal{L}_a\{e_\alpha(t,a)\sinh_{\frac{\beta}{1+\alpha}}(t,a)\}(s) = \frac{\beta}{(s-\alpha)^2-\beta^2}$,

for $|s+1| > \max\{|1+\alpha+\beta|, |1+\alpha-\beta|\}$.

Proof. To see that (i) holds, for $|s+1| > \max\{|1+\alpha+\beta|, |1+\alpha-\beta|\}$, consider

$$\begin{aligned}&\mathcal{L}_a\{e_\alpha(t,a)\cosh_{\frac{\beta}{1+\alpha}}(t,a)\}(s)\\ &\quad= \frac{1}{2}\mathcal{L}_a\{e_\alpha(t,a)e_{\frac{\beta}{1+\alpha}}(t,a)\}(s) + \frac{1}{2}\mathcal{L}_a\{e_\alpha(t,a)e_{\frac{-\beta}{1+\alpha}}(t,a)\}(s)\\ &\quad= \frac{1}{2}\mathcal{L}_a\{e_{\alpha\oplus\frac{\beta}{1+\alpha}}(t,a)\}(s) + \frac{1}{2}\mathcal{L}_a\{e_{\alpha\oplus\frac{-\beta}{1+\alpha}}(t,a)\}(s)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\mathcal{L}_a\{e_{\alpha+\beta}(t,a)\}(s) + \frac{1}{2}\mathcal{L}_a\{e_{\alpha-\beta}(t,a)\}(s) \\
&= \frac{1}{2}\frac{1}{s-\alpha-\beta} + \frac{1}{2}\frac{1}{s-\alpha+\beta} \\
&= \frac{s-\alpha}{(s-\alpha)^2-\beta^2}.
\end{aligned}
$$

The proof of (ii) is Exercise 2.7. □

Similar to the proof of Theorem 2.10 one can prove the following theorem.

Theorem 2.11. *Assume* $\alpha \neq -1$ *and* $\frac{\beta}{1+\alpha} \neq \pm i$. *Then*

(i) $\mathcal{L}_a\{e_\alpha(t,a)\cos_{\frac{\beta}{1+\alpha}}(t,a)\}(s) = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$;

(ii) $\mathcal{L}_a\{e_\alpha(t,a)\sin_{\frac{\beta}{1+\alpha}}(t,a)\}(s) = \frac{\beta}{(s-\alpha)^2+\beta^2}$,

for $|s+1| > \max\{|1+\alpha+i\beta|, |1+\alpha-i\beta|\}$.

When solving certain difference equations one frequently uses the following theorem.

Theorem 2.12. *Assume that* f *is of exponential order* $r > 0$. *Then for any positive integer* N

$$
\mathcal{L}_a\left\{\Delta^N f\right\}(s) = s^N F_a(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a), \tag{2.4}
$$

for $|s+1| > r$.

Proof. By Exercise 2.2 we have for each positive integer N, the function $\Delta^N f$ is of exponential order r. Hence, by Theorem 2.4 the Laplace transform of $\Delta^N f$ for each $N \geq 1$ exists for $|s+1| > r$. Now integrating by parts we get

$$
\begin{aligned}
\mathcal{L}_a\{\Delta f\}(s) &= \int_a^\infty e_{\ominus s}(\sigma(t),a)\Delta f(t)\Delta t \\
&= e_{\ominus s}(t,a)f(t)|_a^{b\to\infty} - \int_a^\infty \ominus s e_{\ominus s}(t,a)f(t)\Delta t \\
&= -f(a) + s\int_a^\infty e_{\ominus s}(\sigma(t),a)f(t)\Delta t \\
&= sF_a(s) - f(a)
\end{aligned}
$$

for $|s+1| > r$. Hence (2.4) holds for $N=1$. Now assume $N \geq 1$ and (2.4) holds. Then

$$
\begin{aligned}
\mathcal{L}_a\{\Delta^{N+1}f\}(s) &= \mathcal{L}_a\{\Delta\left(\Delta^N f\right)\}(s) \\
&= s\mathcal{L}_a\{\Delta^N f\}(s) - \Delta^N f(a)
\end{aligned}
$$

$$= s\left[s^N F_a(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a)\right] - \Delta^N f(a)$$

$$= s^{N+1} F_a(s) - \sum_{j=0}^{(N+1)-1} s^j \Delta^{(N+1)-1-j} f(a).$$

Hence (2.4) holds for each positive integer by mathematical induction. □

The following example is an application of formula (2.4).

Example 2.13. Use Laplace transforms to solve the IVP

$$\Delta^2 y(t) - 3\Delta y(t) + 2y(t) = 2 \cdot 4^t, \quad t \in \mathbb{N}_0$$

$$y(0) = 2, \quad \Delta y(0) = 4.$$

Assume $y(t)$ is the solution of the above IVP. We have, by taking the Laplace transform of both sides of the difference equation in this example,

$$[s^2 Y_0(s) - sy(0) - \Delta y(0)] - 3[sY_0(s) - y(0)] + 2Y_0(s) = \frac{2}{s-3}.$$

Applying the initial conditions and simplifying we get

$$(s^2 - 3s + 2)Y_0(s) = 2s - 2 + \frac{2}{s-3}.$$

Further simplification leads to

$$(s-1)(s-2)Y_0(s) = \frac{2(s-2)^2}{s-3}.$$

Hence

$$Y_0(s) = \frac{2(s-2)}{(s-1)(s-3)}$$

$$= \frac{1}{s-1} + \frac{1}{s-3}.$$

It follows that the solution of our IVP is given by

$$y(t) = e_1(t,0) + e_3(t,0)$$

$$= 2^t + 4^t, \quad t \in \mathbb{N}_0.$$

Now that we see that our solution is of exponential order we see that the steps we did above are valid.

The following corollary gives us a useful formula for solving certain summation (delta integral) equations.

Corollary 2.14. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$. Then*

$$\mathcal{L}_a\left\{\int_a^t f(\tau)\Delta\tau\right\}(s) = \frac{1}{s}\mathcal{L}_a\{f\}(s) = \frac{F_a(s)}{s}$$

for $|s+1| > r$.

Proof. Since $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$, we have by Exercise 2.3 that the function h defined by

$$h(t) := \int_a^t f(\tau)\Delta\tau, \quad t \in \mathbb{N}_a$$

is also of exponential order $r > 1$. Hence the Laplace transform of h exists for $|s+1| > r$. Then

$$\begin{aligned}\mathcal{L}_a\{f\}(s) &= \mathcal{L}_a\{\Delta h\}(s)\\ &= s\mathcal{L}_a\{h\}(s) - h(a)\\ &= s\mathcal{L}_a\left\{\int_a^t f(\tau)\Delta\tau\right\}(s).\end{aligned}$$

It follows that

$$\mathcal{L}_a\left\{\int_a^t f(\tau)\Delta\tau\right\}(s) = \frac{1}{s}\mathcal{L}_a\{f\}(s) = \frac{F_a(s)}{s}$$

for $|s+1| > r$. □

Example 2.15. Solve the summation equation

$$y(t) = 2\cdot 4^t + 2\sum_{k=0}^{t-1} y(k), \quad t \in \mathbb{N}_0. \tag{2.5}$$

Equation (2.5) can be written in the equivalent form

$$y(t) = 2\cdot e_3(t,0) + 2\int_0^t y(k)\Delta k, \quad t \in \mathbb{N}_0. \tag{2.6}$$

Taking the Laplace transform of both sides of (2.6) we get, using Corollary 2.14,

$$Y_0(s) = \frac{2}{s-3} + \frac{2}{s}\,Y_0(s).$$

Solving for $Y_0(s)$ we get

$$\begin{aligned} Y_0(s) &= \frac{2s}{(s-2)(s-3)} \\ &= \frac{6}{s-3} - \frac{4}{s-2}. \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= 6e_3(t,0) - 4e_2(t,0) \\ &= 6 \cdot 4^t - 4 \cdot 3^t, \quad t \in \mathbb{N}_0. \end{aligned}$$

is the solution of (2.5).

Next we introduce the Dirac delta function and find its Laplace transform.

Definition 2.16. Let $c \in \mathbb{N}_a$. We define the Dirac delta function at c on $\mathbb{N}_a$ by

$$\delta_c(t) = \begin{cases} 1, & t = c \\ 0, & t \neq c. \end{cases}$$

Theorem 2.17. *Assume $c \in \mathbb{N}_a$. Then*

$$\mathcal{L}_a\{\delta_c\}(s) = \frac{1}{(s+1)^{c-a+1}} \quad \textit{for} \quad |s+1| > 0.$$

Proof. For $|s+1| > 0$,

$$\begin{aligned} \mathcal{L}_a\{\delta_c\}(s) &= \sum_{k=0}^{\infty} \frac{\delta_c(a+k)}{(s+1)^{k+1}} \\ &= \frac{1}{(s+1)^{c-a+1}}. \end{aligned}$$

This completes the proof. □

Next we define the unit step function and later find its Laplace transform.

Definition 2.18. Let $c \in \mathbb{N}_a$. We define the **unit step function** on $\mathbb{N}_a$ by

$$u_c(t) = \begin{cases} 0, & t \in \mathbb{N}_a^{c-1} \\ 1, & t \in \mathbb{N}_c. \end{cases}$$

We now prove the following shifting theorem.

Theorem 2.19 (Shifting Theorem). *Let $c \in \mathbb{N}_a$ and assume the Laplace transform of $f : \mathbb{N}_a \to \mathbb{R}$ exists for $|s+1| > r$. Then the following hold:*

(i) $\mathcal{L}_a\{f(t-(c-a))u_c(t)\}(s) = \frac{1}{(s+1)^{c-a}}\mathcal{L}_a\{f\}(s);$

(ii) $\mathcal{L}_a\{f(t+(c-a))\}(s) = (s+1)^{c-a}\left[\mathcal{L}_a\{f\}(s) - \sum_{k=0}^{c-a-1}\frac{f(a+k)}{(s+1)^{k+1}}\right],$

for $|s+1| > r$. (In (i) we have the convention that $f(t-(c-a))u_c(t) = 0$ for $t \in \mathbb{N}_a^{c-1}$ if $c \geq a+1$.)

Proof. To see that (i) holds, consider

$$\begin{aligned}
\mathcal{L}_a\{f(t+a-c)u_c(t)\}(s) &= \sum_{k=0}^{\infty}\frac{f(2a+k-c)u_c(a+k)}{(s+1)^{k+1}} \\
&= \sum_{k=c-a}^{\infty}\frac{f(2a+k-c)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{f(2a+k+c-a-c)}{(s+1)^{k+c-a+1}} \\
&= \sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+c-a+1}} \\
&= \frac{1}{(s+1)^{c-a}}\sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+1}} \\
&= \frac{1}{(s+1)^{c-a}}\mathcal{L}_a\{f\}(s)
\end{aligned}$$

for $|s+1| > r$.

Part (ii) holds since

$$\begin{aligned}
\mathcal{L}_a\{f(t+(c-a))\}(s) &= \sum_{k=0}^{\infty}\frac{f(a+k+c-a)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{f(k+c)}{(s+1)^{k+1}} \\
&= \sum_{k=c-a}^{\infty}\frac{f(a+k)}{(s+1)^{k-c+a+1}} \\
&= (s+1)^{c-a}\sum_{k=c-a}^{\infty}\frac{f(a+k)}{(s+1)^{k+1}}
\end{aligned}$$

$$= (s+1)^{c-a}\left[\sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+1}} - \sum_{k=0}^{c-a-1}\frac{f(a+k)}{(s+1)^{k+1}}\right]$$
$$= (s+1)^{c-a}\left[\mathcal{L}_a\{f\}(s) - \sum_{k=0}^{c-a-1}\frac{f(a+k)}{(s+1)^{k+1}}\right]$$

for $|s+1| > r$. □

In the following example we will use part (i) of Theorem 2.19 to solve an IVP.

Example 2.20. Solve the IVP

$$\Delta y(t) - 3y(t) = 2\delta_{50}(t), \quad t \in \mathbb{N}_0$$
$$y(0) = 5.$$

Taking the Laplace transform of both sides, we get

$$sY_0(s) - y(0) - 3Y_0(s) = \frac{2}{(s+1)^{51}}.$$

Using the initial condition and solving for $Y_0(s)$ we have that

$$Y_0(s) = \frac{5}{s-3} + \frac{2}{s-3}\frac{1}{(s+1)^{51}}.$$

Taking the inverse transform of both sides we get the desired solution

$$y(t) = 5e_3(t,0) + 2e_3(t-51,0)u_{51}(t)$$
$$= 5(4^t) + 2(4)^{t-51}u_{51}(t), \quad t \in \mathbb{N}_0.$$

In the following example we will use part (ii) of Theorem 2.19 to solve an IVP.

Example 2.21. Use Laplace transforms to solve the IVP

$$y(t+2) + y(t+1) - 6y(t) = 0, \quad t \in \mathbb{N}_0$$
$$y(0) = 5, \quad y(1) = 2.$$

Assume $y(t)$ is the solution of this IVP and take the Laplace transform of both sides of the given difference equation to get (using part (ii) of Theorem 2.19) that

$$(s+1)^2\left[Y_0(s) - \frac{5}{s+1} - \frac{2}{(s+1)^2}\right] + (s+1)\left[Y_0(s) - \frac{5}{s+1}\right] - 6Y_0(s) = 0.$$

Solving for $Y_0(s)$ we get

$$Y_0(s) = \frac{5s+12}{(s-1)(s+4)}$$
$$= \frac{17}{5}\frac{1}{s-1} + \frac{8}{5}\frac{1}{s+4}.$$

Taking the inverse transform of both sides we get

$$y(t) = \frac{17}{5}e_1(t,0) + \frac{8}{5}e_{-4}(t,0)$$
$$= \frac{17}{5}2^t + \frac{8}{5}(-3)^t, \quad t \in \mathbb{N}_0.$$

Theorem 2.22. *The following hold for $n \geq 0$:*

(i) $\mathcal{L}_a\{h_n(t,a)\}(s) = \frac{1}{s^{n+1}}$ *for* $|s+1| > 1$;
(ii) $\mathcal{L}_a\{(t-a)^{\underline{n}}\}(s) = \frac{n!}{s^{n+1}}$ *for* $|s+1| > 1$.

Proof. The proof of this theorem follows from Corollary 2.14 and the fact that $\mathcal{L}\{1\}(s) = \frac{1}{s}$ for $|s+1| > 1$. □

2.3 Fractional Sums and Differences

The following theorem will motivate the definition of the n-th integer sum, which will in turn motivate the definition of the ν-th fractional sum. We will then define the ν-th fractional difference in terms of the ν-th fractional sum.

Theorem 2.23 (Repeated Summation Rule). *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, then*

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1 = \int_a^t h_{n-1}(t,\sigma(s))f(s)\Delta s. \tag{2.7}$$

Proof. We will prove this by induction on n for $n \geq 1$. The case $n = 1$ is trivially true. Assume (2.7) holds for some $n \geq 1$. It remains to show that (2.7) then holds when n is replaced by $n+1$. To this end, let

$$y(t) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} \int_a^{\tau_n} f(\tau_{n+1})\Delta\tau_{n+1}\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1.$$

Let $g(\tau_n) = \int_a^{\tau_n} f(\tau_{n+1})\Delta\tau_{n+1}$, then it follows from the induction assumption that

$$\begin{aligned} y(t) &= \int_a^t h_{n-1}(t,\sigma(s))g(s)\Delta s \\ &= \int_a^t u(s)\Delta v(s)\Delta s, \end{aligned}$$

where

$$u(s) := g(s), \quad \Delta v(s) = h_{n-1}(t,\sigma(s)).$$

It follows (using Theorem 1.61, (v)) that

$$\Delta u(s) = f(s) \quad v(s) = -h_n(t,s), \quad v(\sigma(s)) = -h_n(t,\sigma(s)).$$

Hence, integrating by parts, it follows that

$$\begin{aligned} y(t) &= -h_n(t,s)\int_a^s f(\tau_{n+1})\Delta\tau_{n+1}\Big|_a^t \\ &\quad + \int_a^t h_n(t,\sigma(s))f(s)\Delta s \\ &= \int_a^t h_n(t,\sigma(s))f(s)\Delta s. \end{aligned}$$

This completes the proof. □

Motivated by (2.7), we define the n-th integer sum $\Delta_a^{-n}f(t)$ for positive integers n, by

$$\Delta_a^{-n}f(t) = \int_a^t h_{n-1}(t,\sigma(s))f(s)\Delta s.$$

But, since

$$h_{n-1}(t,\sigma(s)) = 0, \quad s = t-1, t-2, \cdots, t-n+1,$$

we obtain

$$\Delta_a^{-n}f(t) = \int_a^{t-n+1} h_{n-1}(t,\sigma(s))f(s)\Delta s, \tag{2.8}$$

which we consider the correct form of the n-th integer sum of $f(t)$. Before we use the definition (2.8) of the n-th integer sum to motivate the definition of the ν-th fractional sum, we define the ν-th fractional Taylor monomial as follows.

Definition 2.24. The ν-th fractional Taylor monomial based at s is defined by

$$h_\nu(t,s) = \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)},$$

whenever the right-hand side is well defined.

We can now define the ν-th fractional sum.

Definition 2.25. Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then the ν-th fractional sum of f (based at a) is defined by

$$\begin{aligned}\Delta_a^{-\nu} f(t) &:= \int_a^{t-\nu+1} h_{\nu-1}(t,\sigma(\tau)) f(\tau)\Delta\tau \\ &= \sum_{\tau=a}^{t-\nu} h_{\nu-1}(t,\sigma(\tau)) f(\tau),\end{aligned}$$

for $t \in \mathbb{N}_{a+\nu}$. Note that by our convention on delta integrals (sums) we can extend the domain of $\Delta_a^{-\nu} f$ to $\mathbb{N}_{a+\nu-N}$, where N is the unique positive integer satisfying $N-1 < \nu \le N$, by noting that

$$\Delta_a^{-\nu} f(t) = 0, \quad t \in \mathbb{N}_{a+\nu-N}^{a+\nu-1}.$$

The expression "fractional sum" is actually is misnomer as we define the ν-th fractional sum of a function for any $\nu > 0$. Expressions like $\Delta_a^{\sqrt{3}} y(t)$ and $\Delta_a^{\pi} y(t)$ are well defined.

Remark 2.26. Note that the value of the ν-th fractional sum of f based at a is a linear combination of $f(a), f(a+1), \cdots, f(t-\nu)$, where the coefficient of $f(t-\nu)$ is one. In particular one can check that $\Delta_a^{-\nu} f(t)$ has the form

$$\Delta_a^{-\nu} f(t) = h_{\nu-1}(t,\sigma(a)) f(a) + \cdots + \nu f(t-\nu-1) + f(t-\nu). \tag{2.9}$$

The following formulas concerning the fractional Taylor monomials generalize the integer version of this theorem (Theorem 1.61).

Theorem 2.27. *Let $t, s \in \mathbb{N}_a$. Then*

(i) $h_\nu(t,t) = 0$
(ii) $\Delta h_\nu(t,a) = h_{\nu-1}(t,a)$;
(iii) $\Delta_s h_\nu(t,s) = -h_{\nu-1}(t,\sigma(s))$;
(iv) $\int h_\nu(t,a)\Delta t = h_{\nu+1}(t,a) + C$;
(v) $\int h_\nu(t,\sigma(s))\Delta s = -h_{\nu+1}(t,s) + C$,

whenever these expressions make sense.

Proof. To see that (iii) holds, note that

$$\begin{aligned}
\Delta_s h_\nu(t,s) &= h_\nu(t,s+1) - h_\nu(t,s) \\
&= \frac{(t-s-1)^{\underline{\nu}}}{\Gamma(\nu+1)} - \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)} \\
&= \frac{\Gamma(t-s)}{\Gamma(t-s-\nu)\Gamma(\nu+1)} - \frac{\Gamma(t-s+1)}{\Gamma(t-s+1-\nu)\Gamma(\nu+1)} \\
&= \Big[(t-s-\nu) - (t-s)\Big]\frac{\Gamma(t-s)}{\Gamma(\nu+1)\Gamma(t-s-\nu+1)} \\
&= -\frac{(\nu+1)\Gamma(t-s)}{\Gamma(\nu)\Gamma(t-s-\nu+1)} \\
&= -\frac{\Gamma(t-s)}{\Gamma(\nu)\Gamma(t-s-\nu+1)} \\
&= -\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \\
&= -h_{\nu-1}(t,\sigma(s)).
\end{aligned}$$

The rest of the proof of this theorem is Exercise 2.16. □

Example 2.28. Using the definition of the fractional sum (Definition 2.25), find $\Delta_0^{-\frac{1}{2}}1$.

Using Theorem 2.27, part (v), we get

$$\begin{aligned}
\Delta_0^{-\frac{1}{2}}1 &= \int_0^{t+\frac{1}{2}} h_{-\frac{1}{2}}(t,\sigma(s))\cdot 1\ \Delta s \\
&= -h_{\frac{1}{2}}(t,s)\Big|_{s=0}^{s=t+\frac{1}{2}} \\
&= -h_{\frac{1}{2}}(t,t+\frac{1}{2}) + h_{\frac{1}{2}}(t,0) \\
&= -\frac{(-\frac{1}{2})^{\underline{\frac{1}{2}}}}{\Gamma(\frac{3}{2})} + \frac{t^{\underline{\frac{1}{2}}}}{\Gamma(\frac{3}{2})} \\
&= \frac{2}{\sqrt{\pi}}\, t^{\underline{\frac{1}{2}}}.
\end{aligned}$$

Later we will give a formula (2.16) that also gives us this result.

Next we define the fractional difference in terms of the fractional sum.

Definition 2.29. Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Choose a positive integer N such that $N - 1 < \nu \leq N$. Then we define the ν-th fractional difference by

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N-\nu}.$$

Note that our fractional difference agrees with our prior understanding of whole-order differences—that is, for any $\nu = N \in \mathbb{N}_0$

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t), \tag{2.10}$$

for $t \in \mathbb{N}_a$. This is called the **Riemann–Liouville** definition of the ν-th delta fractional difference.

Remark 2.30. We will see in the proof of Theorem 2.35 below that the value of the fractional difference $\Delta_a^\nu f(t)$ depends on the values of f on $\mathbb{N}_{a+\nu-N}^{t+\nu}$. This full history nature of the value of the ν-th fractional difference of f is one of the important features of this fractional difference. In contrast if one is studying an n-th order difference equation, the term $\Delta^n f(t)$ only depends on the values of f at the $n+1$ points $t, t+1, t+2, \cdots, t+n$.

Example 2.31. Use Definition 2.29 to find $\Delta_0^{\frac{1}{2}} 1$. Using Example 2.28, we have that

$$\begin{aligned}\Delta_0^{\frac{1}{2}} 1 &= \Delta \Delta_0^{-\frac{1}{2}} 1 \\ &= \Delta \frac{2}{\sqrt{\pi}} t^{\underline{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{\pi}} t^{\underline{-\frac{1}{2}}}.\end{aligned}$$

Later we will give a formula (see (2.22)) that also gives us this result.

The following **Leibniz formulas** will be very useful.

Lemma 2.32 (Leibniz Formulas). *Assume $f : \mathbb{N}_{a+\mu} \times \mathbb{N}_a \to \mathbb{R}$. Then*

$$\Delta\left[\int_a^{t-\mu+1} f(t,\tau)\Delta\tau\right] = \int_a^{t-\mu+1} \Delta_t f(t,\tau)\Delta\tau + f(t+1, t-\mu+1) \tag{2.11}$$

and

$$\Delta\left[\int_a^{t-\mu+1} f(t,\tau)\Delta\tau\right] = \int_a^{t-\mu+2} \Delta_t f(t,\tau)\Delta\tau + f(t, t-\mu+1) \tag{2.12}$$

for $t \in \mathbb{N}_{a+\mu}$, where the $\Delta_t f(t,s)$ inside the integral means the difference of $f(t,\tau)$ with respect to t.

Proof. To see that (2.11) holds, note that, for $t \in \mathbb{N}_{a+\mu}$,

$$\Delta\left[\int_a^{t-\mu+1} f(t,\tau)\Delta\tau\right] = \int_a^{t-\mu+2} f(t+1,\tau)\Delta\tau - \int_a^{t-\mu+1} f(t,\tau)\Delta\tau$$
$$= \int_a^{t-\mu+1} \Delta_t f(t,\tau)\Delta\tau + f(t+1, t+1-\mu).$$

The proof of (2.12) is Exercise 2.19. □

In the next theorem we give a very useful formula for $\Delta_a^\nu f(t)$. We call this formula the alternate definition of $\Delta_a^\nu f(t)$ (see Holm [123, 124]).

Theorem 2.33. *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, with $N-1 < \nu \leq N$. Then*

$$\Delta_a^\nu f(t) := \begin{cases} \int_a^{t+\nu+1} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau, & N-1 < \nu < N \\ \Delta^N f(t), & \nu = N \end{cases} \tag{2.13}$$

for $t \in \mathbb{N}_{a+N-\nu}$.

Proof. First note that if $\nu = N \in \mathbb{N}_0$, then using (2.10), we have that

$$\Delta_a^\nu f(t) = \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t).$$

Now assume $N-1 < \nu < N$. Our proof of (2.13) will follow from N applications of the Leibniz formula (2.12). To see this we have for $t \in \mathbb{N}_{a+N-\nu}$,

$$\Delta_a^\nu f(t) = \Delta^N \Delta_a^{-(N-\nu)} f(t)$$
$$= \Delta^N \left[\int_a^{t-(N-\nu)+1} h_{N-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau\right]$$
$$= \Delta^{N-1} \cdot \Delta\left[\int_a^{t-(N-\nu)+1} h_{N-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau\right].$$

Using the Leibniz rule (2.12), we get

$$\Delta_a^\nu f(t) = \Delta^{N-1}\left[\int_a^{t-(N-\nu-1)+1} h_{N-\nu-2}(t,\sigma(\tau))f(\tau)\Delta\tau \right.$$
$$\left. + h_{N-\nu-1}(t, t-(N-\nu-2))f(t-(N-\nu-1))\right]$$
$$= \Delta^{N-1}\left[\int_a^{t-(N-\nu-1)+1} h_{N-\nu-2}(t,\sigma(\tau))f(\tau)\Delta\tau\right].$$

Applying the Leibniz formula (2.12) again we get

$$\begin{aligned}\Delta_a^\nu f(t) &= \Delta^{N-2}\Bigg[\int_a^{t-(N-\nu-2)+1} h_{N-\nu-3}(t,\sigma(\tau))f(\tau)\Delta\tau \\ &\qquad + h_{N-\nu-2}(t,t-(N-\nu-3))f(t-(N-\nu-2))\Bigg] \\ &= \Delta^{N-2}\left[\int_a^{t-(N-\nu-2)+1} h_{N-\nu-3}(t,\sigma(\tau))f(\tau)\Delta\tau\right].\end{aligned}$$

Repeating these steps $N-2$ more times, we find that

$$\begin{aligned}\Delta_a^\nu f(t) &= \Delta^{N-N}\Bigg[\int_a^{t-(N-\nu-N)+1} h_{N-\nu-N-1}(t,\sigma(\tau))f(\tau)\Delta\tau \\ &\qquad + h_{N-\nu-N}(t,t-(N-\nu-(N+1))f(t-(N-\nu-N))\Bigg] \\ &= \int_a^{t+\nu+1} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau + h_{-\nu}(t,t+\nu+1)f(t+\nu) \\ &= \int_a^{t+\nu+1} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau.\end{aligned}$$

This completes the proof. □

Remark 2.34. By Theorem 2.33 we get for all $\nu > 0$, $\nu \notin \mathbb{N}_1$ that the formula for $\Delta_a^\nu f(t)$ can be obtained from the formula for $\Delta_a^{-\nu} f(t)$ in Definition 2.25 by replacing ν by $-\nu$ and vice-versa, but the domains are different.

Theorem 2.35 (Existence-Uniqueness Theorem). *Assume $q, f : \mathbb{N}_0 \to \mathbb{R}$, $\nu > 0$ and N is a positive integer such that $N-1 < \nu \le N$. Then the initial value problem*

$$\Delta_{\nu-N}^\nu y(t) + q(t)y(t+\nu-N) = f(t), \quad t \in \mathbb{N}_0 \tag{2.14}$$

$$y(\nu-N+i) = A_i, \quad 0 \le i \le N-1, \tag{2.15}$$

where A_i, $0 \le i \le N-1$, are given constants, has a unique solution on $\mathbb{N}_{\nu-N}$.

Proof. Note that by Remark 2.26, for each fixed t, $\Delta_{\nu-N}^{-(N-\nu)} y(t)$ is a linear combination of $y(\nu-N), y(\nu-N+1), \cdots, y(t-N+\nu)$ with the coefficient of $y(t-N+\nu)$ being one. Since

$$\Delta^{\nu}_{\nu-N}y(t) = \Delta^{N}\Delta^{-(N-\nu)}_{\nu-N}y(t),$$

we have for each fixed t, $\Delta^{\nu}_{\nu-N}y(t)$ is a linear combination of $y(\nu-N), y(\nu-N+1)$, $\cdots, y(t+\nu)$, where the coefficient of $y(t+\nu)$ is one. Now define $y(t)$ on $\mathbb{N}^{\nu-1}_{\nu-N}$ by the initial conditions (2.15). Then note that $y(t)$ satisfies the fractional difference equation (2.14) at $t = 0$ iff

$$\Delta^{\nu}_{\nu-N}y(0) + q(0)y(\nu - N) = f(0).$$

But this holds iff

$$(\cdots)y(\nu - N) + (\cdots)y(\nu - N + 1) + \cdots + y(\nu) + q(0)y(\nu - N) = f(0),$$

which is equivalent to the equation

$$(\cdots)A_0 + (\cdots)A_1 + \cdots + (\cdots)A_{n-1} + y(\nu) + q(0)A_0 = f(0).$$

Hence if we define $y(\nu)$ to be the solution of this last equation, then $y(t)$ satisfies the fractional difference equation at $t = 0$. Summarizing, we have shown that knowing $y(t)$ at the points $\nu - N + i$, $0 \le i \le N - 1$ uniquely determines what the value of the solution is at the next point ν. Next one uses the fact that the values of $y(t)$ on $\mathbb{N}^{\nu}_{\nu-N}$ uniquely determine the value of the solution at $\nu + 1$. An induction argument shows that the solution is uniquely determined on $\mathbb{N}_{\nu-N}$. □

Remark 2.36. We could easily extend Theorem 2.35 to the case when $f, q : \mathbb{N}_a \to \mathbb{R}$ instead of the special case $a = 0$ that we considered in Theorem 2.35. Also, the term $q(t)y(t+\nu-N)$ in equation (2.14) could be replaced by $q(t)y(t+\nu-N+i)$ for any $0 \le i \le N-1$. Note that we picked the nice set $\mathbb{N}_0$ so that the fractional difference equation needs to be satisfied for all $t \in \mathbb{N}_0$, but then solutions are defined on the shifted set $\mathbb{N}_{\nu-N}$. By shifting the set on which the fractional difference equation is defined, we can evidently obtain solutions that are defined on the nicer set $\mathbb{N}_0$. In this book our convention when considering fractional difference equations is to assume the fractional difference equation is satisfied for $t \in \mathbb{N}_a$ and the solutions are defined on $\mathbb{N}_{a+\nu-N}$.

In a standard manner one gets the following result that follows from Theorem 2.35.

Theorem 2.37. *Assume $q : \mathbb{N}_0 \to \mathbb{R}$. Then the homogeneous fractional difference equation*

$$\Delta^{\nu}_{\nu-N}u(t) + q(t)u(t + \nu - N) = 0, \quad t \in \mathbb{N}_{\nu-N}$$

has N linearly independent solutions $u_i(t)$, $1 \le i \le N$, on $\mathbb{N}_0$ and

$$u(t) = c_1u_1(t) + c_2u_2(t) + \cdots + c_Nu_N(t),$$

where $c_1, c_2, \cdots, c_N$ are arbitrary constants, is a general solution of this homogeneous fractional difference equation on $\mathbb{N}_0$. Furthermore, if in addition, $y_p(t)$ is a particular solution of the nonhomogeneous fractional difference equation (2.14) *on $\mathbb{N}_0$, then*

$$y(t) = c_1u_1(t) + c_2u_2(t) + \cdots + c_Nu_N(t) + y_p(t),$$

where $c_1, c_2, \cdots, c_N$ are arbitrary constants, is a general solution of the nonhomogeneous fractional difference equation (2.14).

2.4 Fractional Power Rules

Using the Leibniz formula we will prove the following fractional sum power rule. Later in this chapter (see Theorem 2.71) we will use discrete Laplace transforms to give an easier proof of this theorem. Later we will see that the fractional difference power rule (Theorem 2.40) will follow from this fractional sum power rule.

Theorem 2.38 (Fractional Sum Power Rule). *Assume $\mu \geq 0$ and $\nu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}} \tag{2.16}$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Proof. Let

$$g_1(t) := \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}},$$

and

$$g_2(t) := \Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \sum_{s=a+\mu}^{t-\nu} h_{\nu-1}(t,\sigma(s))(s-a)^{\underline{\mu}}, \tag{2.17}$$

for $t \in \mathbb{N}_{a+\mu+\nu}$. To complete the proof we will show that both of these functions satisfy the initial value problem

$$(t-a-(\mu+\nu)+1)\Delta g(t) = (\mu+\nu)g(t) \tag{2.18}$$

$$g(a+\mu+\nu) = \Gamma(\mu+1). \tag{2.19}$$

Since

$$\begin{aligned} g_1(a+\mu+\nu) &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(\mu+\nu)^{\underline{\mu+\nu}} \\ &= \Gamma(\mu+1) \end{aligned}$$

and

$$\begin{aligned} g_2(a+\mu+\nu) &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{a+\mu} (a+\mu+\nu-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}} \\ &= \frac{1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}\mu^{\underline{\mu}} \\ &= \Gamma(\mu+1) \end{aligned}$$

we have that $g_i(t)$, $i=1,2$ both satisfy the initial condition (2.19).

We next show that $g_1(t)$ satisfies the difference equation (2.18). Note that

$$\Delta g_1(t) = (\mu+\nu)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu-1}}.$$

Multiplying both sides by $t-a-(\mu+\nu)+1$ we obtain

$$\begin{aligned} &(t-a-(\mu+\nu)+1)\Delta g_1(t) \\ &= (\mu+\nu)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}[t-a-(\mu+\nu-1)](t-a)^{\underline{\mu+\nu-1}} \\ &= (\mu+\nu)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}} \qquad \text{by Exercise (1.9)} \\ &= (\mu+\nu)g_1(t) \end{aligned}$$

for $t \in \mathbb{N}_{a+\mu+\nu}$. That is, $g_1(t)$ is a solution of (2.18).

It remains to show that $g_2(t)$ satisfies (2.18). Using (2.17) we have that

$$\begin{aligned} &g_2(t) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} \big[(t-\sigma(s))-(\nu-2)\big](t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} \big[(t-a-(\mu+\nu)+1)-(s-a-\mu)\big](t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\ &= \frac{t-a-(\mu+\nu)+1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\ &- \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a-\mu)(s-a)^{\underline{\mu}} \\ &= h(t)-k(t), \end{aligned}$$

where

$$h(t) := \frac{t-a-(\mu+\nu)+1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}}$$

and

$$\begin{aligned} k(t) &:= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a-\mu)(s-a)^{\underline{\mu}} \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu+1}}. \end{aligned}$$

Using (2.17) and (2.11) we get

$$\begin{aligned} &\Delta g_2(t) \\ &= \frac{\nu-1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} + \frac{1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}(t+1-\nu-a)^{\underline{\mu}} \\ &= \frac{\nu-1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} + (t+1-\nu-a)^{\underline{\mu}}. \end{aligned}$$

It follows that

$$(t-a+(\mu+\nu)+1)\Delta g_2(t) = (\nu-1)h(t) + (t+1-\nu-a)^{\underline{\mu+1}}. \tag{2.20}$$

Also, integrating by parts we get (here we also use Lemma 2.32)

$$\begin{aligned} k(t) &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu+1}} \\ &= \frac{1}{\Gamma(\nu)} \left[-\frac{(s-a)^{\underline{\mu+1}}(t-s)^{\underline{\nu-1}}}{\nu-1} \right]_{s=a+\mu}^{s=t+1-\nu} \\ &\quad + \frac{\mu+1}{(\nu-1)\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}} \\ &= -\frac{(t+1-\nu-a)^{\underline{\mu+1}}}{\nu-1} + \frac{\mu+1}{(\nu-1)\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}}. \end{aligned}$$

It follows that

$$(t+1-\nu-a)^{\underline{\mu+1}} = -(\nu-1)k(t) + (\mu+1)g_2(t). \tag{2.21}$$

Finally, from (2.21) and (2.20), we get

$$\begin{aligned}(t-a+(\mu+\nu)+1)\Delta g_2(t) &= (\nu-1)h(t) + (t+1-\nu-a)^{\underline{\mu+1}} \\ &= (\nu-1)h(t) - (\nu-1)k(t) + (\mu+1)g_2(t) \\ &= (\mu+\nu)g_2(t).\end{aligned}$$

This completes the proof. □

Example 2.39. Find

$$\Delta_{\frac{5}{2}}^{-\frac{3}{2}}(t-2)^{\underline{\frac{1}{2}}}, \quad t \in \mathbb{N}_2.$$

Consider

$$\begin{aligned}\Delta_{\frac{5}{2}}^{-\frac{3}{2}}(t-2)^{\underline{\frac{1}{2}}} &= \Delta_{2+\frac{1}{2}}^{-\frac{3}{2}}(t-2)^{\underline{\frac{1}{2}}} \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(3)}(t-2)^{\underline{2}} \\ &= \frac{\sqrt{\pi}}{4}(t-2)^{\underline{2}} \\ &= \frac{\sqrt{\pi}}{4}(t^2-5t+6),\end{aligned}$$

for $t \in \mathbb{N}_2$.

Theorem 2.40 (Fractional Difference Power Rule). *Assume $\mu > 0$ and $\nu \geq 0$, $N-1 < \nu < N$. Then*

$$\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\underline{\mu-\nu}} \tag{2.22}$$

for $t \in \mathbb{N}_{a+\mu+N-\nu}$.

Proof. To see that (2.22) holds, note that

$$\begin{aligned}\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} &= \Delta^N \Delta_{a+\mu}^{-(N-\nu)}(t-a)^{\underline{\mu}} \\ &= \Delta^N\left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}\ (t-a)^{\underline{\mu+N-\nu}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}\ \Delta^N (t-a)^{\underline{\mu+N-\nu}} \\
&= \frac{\Gamma(\mu+1)(\mu+N-\nu)^{\underline{N}}}{\Gamma(\mu+1+N-\nu)}(t-a)^{\underline{\mu-\nu}} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}(t-a)^{\underline{\mu-\nu}}.
\end{aligned}
$$

This completes the proof. □

Example 2.41. Find

$$\Delta^{\frac{1}{2}}_{\frac{5}{2}}(t-1)^{\underline{\frac{3}{2}}}, \quad t \in \mathbb{N}_1.$$

Consider

$$
\begin{aligned}
\Delta^{\frac{1}{2}}_{\frac{5}{2}}(t-1)^{\underline{\frac{3}{2}}} &= \Delta^{\frac{1}{2}}_{1+\frac{3}{2}}(t-1)^{\underline{\frac{3}{2}}} \\
&= \frac{\Gamma(\frac{5}{2})}{\Gamma(2)}(t-1)^{\underline{1}} \\
&= \frac{3\sqrt{\pi}}{4}(t-1),
\end{aligned}
$$

for $t \in \mathbb{N}_1$.

The fractional power rules in terms of Taylor monomials take a nice form as we see in the following theorem.

Theorem 2.42. *Assume* $\mu > 0$, $\nu > 0$*, then the following hold:*

(i) $\Delta^{-\nu}_{a+\mu} h_\mu(t,a) = h_{\mu+\nu}(t,a), \quad t \in \mathbb{N}_{a+\mu+\nu};$
(ii) $\Delta^{\nu}_{a+\mu} h_\mu(t,a) = h_{\mu-\nu}(t,a), \quad t \in \mathbb{N}_{a+\mu-\nu}.$

Proof. To see that (i) follows from Theorem 2.38 note that for $t \in \mathbb{N}_{a+\mu+\nu}$

$$
\begin{aligned}
\Delta^{-\nu}_{a+\mu} h_\mu(t,a) &= \Delta^{-\nu}_{a+\mu} \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} \\
&= \frac{1}{\Gamma(\mu+1)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}} \\
&= \frac{(t-a)^{\underline{\mu+\nu}}}{\Gamma(\mu+\nu+1)} \\
&= h_{\mu+\nu}(t,a).
\end{aligned}
$$

Similarly, part (ii) follows from Theorem 2.40 (see Exercise 2.22). □

Theorem 2.43. *Assume $\mu > 0$ and N is a positive integer such that $N-1 < \mu \le N$. Then for any constant a*

$$x(t) = c_1(t-a)^{\underline{\mu-1}} + c_2(t-a)^{\underline{\mu-2}} + \cdots + c_N(t-a)^{\underline{\mu-N}}$$

for all constants $c_1, c_2, \cdots, c_N$, is a solution of the fractional difference equation $\Delta^{\mu}_{a+\mu-N} y(t) = 0$ on $\mathbb{N}_{a+\mu-N}$.

Proof. Let μ and N be as in the statement of this theorem. If $\mu = N$, then for $1 \le k \le N$, we have that

$$\Delta^{\mu}_{a+\mu-N}(t-a)^{\underline{\mu-k}} = \Delta^{N}(t-a)^{\underline{N-k}} = 0.$$

Now assume that $N-1 < \mu < N$. Then we want to consider the expression

$$\Delta^{\mu}_{a+\mu-N}(t-a)^{\underline{\mu-k}}.$$

Note that since the subscript and the exponent do not match up in the correct way we cannot immediately apply formula (2.22) to the above expression. To compensate for this we do the following.

$$\begin{aligned}\Delta^{\mu}_{a+\mu-N}\,(t-a)^{\underline{\mu-k}} &= \sum_{s=a+\mu-N}^{t+\mu} h_{-\mu-1}(t,\sigma(s))(s-a)^{\underline{\mu-k}} \\ &= \sum_{s=a+\mu-k}^{t+\mu} h_{-\mu-1}(t,\sigma(s))(s-a)^{\underline{\mu-k}},\end{aligned}$$

since

$$(s-a)^{\underline{\mu-k}} = 0, \quad \text{for} \;\; s = a+\mu-N, a+\mu-N+1, \cdots, a+\mu-k-1.$$

Therefore, we have that

$$\begin{aligned}\Delta^{\mu}_{a+\mu-N}\,(t-a)^{\underline{\mu-k}} &= \Delta^{\mu}_{a+\mu-k}(t-a)^{\underline{\mu-k}} \\ &= \frac{\Gamma(\mu-k+1)}{\Gamma(1-k)}(t-a)^{\underline{-k}} \\ &= 0.\end{aligned}$$

The conclusion of the theorem then follows from the fact that Δ^{μ}_{a} is a linear operator. □

It follows from Theorem 2.43 that

$$x(t) = a_1 h_{\mu-1}(t,a) + a_2 h_{\mu-2}(t,a) + \cdots + a_N h_{\mu-N}(t,a)$$

is a general solution of $\Delta_{a+\mu-N} y(t) = 0$.

Theorem 2.44 (Continuity of Fractional Differences). *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. Then the fractional difference $\Delta_a^\nu f$ is continuous with respect to ν for $\nu > 0$. By this we mean for each fixed $m \in \mathbb{N}_0$,*

$$\Delta_a^\nu f(a + \lceil \nu \rceil - \nu + m),$$

where $\lceil \nu \rceil$ denotes the ceiling of ν, is continuous for $\nu > 0$.

Proof. To prove this theorem it suffices to prove the following:

(i) $\Delta_a^\nu f(a + N - \nu + m)$ is continuous with respect to ν on $(N-1, N)$;
(ii) $\lim_{\nu \to N^-} \Delta_a^\nu f(a + N - \nu + m) = \Delta^N f(a + m)$;
(iii) $\lim_{\nu \to (N-1)^+} \Delta_a^\nu f(a + N - \nu + m) = \Delta^{N-1} f(a + m + 1)$.

First we show that (i) holds. For any fixed $\nu > 0$ with $N - 1 < \nu < N$, we have

$$\begin{aligned}
\Delta_a^\nu f(a + N - \nu + m) &= \sum_{s=a}^{t+\nu} h_{-\nu-1}(t, \sigma(s)) f(s) \Big|_{t=a+N-\nu+m} \\
&= \sum_{s=a}^{a+N+m} h_{-\nu-1}(a + N - \nu + m, \sigma(s)) f(s) \\
&= \sum_{s=a}^{a+N+m-1} h_{-\nu-1}(a + N - \nu + m, \sigma(s)) f(s) + f(a + N + m) \\
&= \sum_{s=a}^{a+N+m-1} \frac{(a + N - \nu + m - \sigma(s))^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(s) + f(a + N + m) \\
&= \sum_{s=a}^{a+N+m-1} \frac{\Gamma(a + N - \nu + m - s)}{\Gamma(a + N + m - s + 1)\Gamma(-\nu)} f(s) + f(a + N + m) \\
&= \sum_{s=a}^{a+N+m-1} \left(\frac{(a + N - \nu + m - s - 1) \cdots (-\nu)}{(a + N + m - s)!} f(s) \right) \\
&\quad + f(a + N + m) \\
&= \sum_{i=1}^{N+m} \left(\frac{(i - 1 - \nu) \cdots (-\nu + 1)\,(-\nu)}{i!} f(a + N + m - i) \right) \\
&\quad + f(a + N + m).
\end{aligned}$$

It follows from this last expression that $\Delta_a^\nu f(a+N-\nu+m)$ is a continuous function of ν, for $N-1<\nu<N$.

$$
\begin{aligned}
&\lim_{\nu\to N^-} \Delta_a^\nu f(a+N-\nu+m)\\
&= \lim_{\nu\to N^-}\Big[\sum_{i=1}^{N+m}\left(\frac{(i-1-\nu)\cdots(-\nu)}{i!}f(a+N+m-i)\right)\\
&\qquad\qquad +f(a+N+m)\Big]\\
&= \sum_{i=1}^{N+m}\left(\frac{(i-1-N)\cdots(-N)}{i!}f(a+N+m-i)\right)+f(a+N+m)\\
&= \sum_{i=1}^{N}\left(\frac{(i-1-N)\cdots(-N)}{i!}f(a+N+m-i)\right)+f(a+N+m),\\
&= \sum_{i=1}^{N}\left((-1)^i\frac{(N)\cdots(N-i+1)}{i!}f(a+N+m-i)\right)+f(a+N+m)\\
&= \sum_{i=1}^{N}\left((-1)^i\binom{N}{i}f(a+N+m-i)\right)\\
&\qquad +f(a+N+m)\\
&= \sum_{i=0}^{N}(-1)^i\binom{N}{i}f(a+N+m-i)\\
&= \sum_{i=0}^{N}(-1)^i\binom{N}{i}f((a+m)+N-i)\\
&= \Delta^N f(a+m).
\end{aligned}
$$

Hence, (ii) holds.

Finally, we show (iii) holds. To see this consider

$$
\begin{aligned}
&\lim_{\nu\to (N-1)^+} \Delta_a^\nu f(a+N-\nu+m)\\
&= \lim_{\nu\to (N-1)^+}\Big[\sum_{i=1}^{N+m}\left(\frac{(i-1-\nu)\cdots(-\nu)}{i!}f(a+N+m-i)\right)\\
&\qquad\qquad\qquad\qquad +f(a+N+m)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N+m} \left(\frac{(i-N)\cdots(-N+1)}{i!} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=1}^{N-1} \left(\frac{(i-N)\cdots(-N+1)}{i!} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=1}^{N-1} \left((-1)^i \frac{(N-1)\cdots(N-i)}{i!} f(a+N+m-i) \right) \\
&\qquad\qquad + f(a+N+m) \\
&= \sum_{i=1}^{N-1} \left((-1)^i \binom{N-1}{i} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=0}^{N-1} \left((-1)^i \binom{N-1}{i} f(a+m+1+(N-1)-i) \right) \\
&= \Delta^{N-1} f(a+m+1).
\end{aligned}
$$

Hence, (iii) holds. □

The binomial expression for $\Delta^N f(t)$ is given by

$$\Delta^N f(t) = \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(t+N-i).$$

In the following theorem we give the binomial expressions for fractional differences and fractional sums.

Theorem 2.45 (Fractional Binomial Formulas). *Assume* $N-1 < \nu \leq N$ *and* $f : \mathbb{N}_a \to \mathbb{R}$. *Then*

$$\Delta_a^\nu f(t) = \sum_{k=0}^{t+\nu-a} (-1)^k \binom{\nu}{k} f(t+\nu-k), \qquad t \in \mathbb{N}_{a+N-\nu} \tag{2.23}$$

and

$$\Delta_a^{-\nu} f(t) = \sum_{k=0}^{t-a-\nu} (-1)^k \binom{-\nu}{k} f(t-\nu-k) \tag{2.24}$$

$$= \sum_{k=0}^{t-a-\nu} \binom{\nu+k-1}{k} f(t-\nu-k), \qquad t \in \mathbb{N}_{a+\nu}. \tag{2.25}$$

Proof. Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $0 \leq \nu \leq N$. Fix $t \in \mathbb{N}_{a+N-\nu}$. Then $t = a + N - \nu + m$, for some $m \in \mathbb{N}_0$. Then

$$\begin{aligned}
\Delta_a^\nu f(t) &= \int_a^{t+\nu+1} h_{-\nu-1}(t, \sigma(\tau)) f(\tau) \Delta\tau \\
&= \sum_{\tau=a}^{t+\nu} \frac{(t-\sigma(\tau))^{\underline{-\nu-1}}}{\Gamma(-\nu)} f(\tau) \\
&= \sum_{\tau=a}^{t+\nu} \frac{\Gamma(t-\tau)}{\Gamma(t-\tau+\nu+1)\Gamma(-\nu)} f(\tau) \\
&= \sum_{\tau=a}^{a+N+m} \frac{\Gamma(a+N-\nu+m-\tau)}{\Gamma(a+N+m-\tau+1)\Gamma(-\nu)} f(\tau) \\
&= \sum_{\tau=0}^{N+m} \frac{\Gamma(N+m-\tau-\nu)}{\Gamma(N+m-\tau+1)\Gamma(-\nu)} f(a+\tau) \\
&= f(a+N+m) + \sum_{\tau=0}^{N+m-1} \frac{(N+m-1-\tau-\nu)\cdots(-\nu)}{\Gamma(N+m-\tau+1)} f(a+\tau) \\
&= f(a+N+m) \\
&\quad + \sum_{\tau=0}^{N+m-1} (-1)^{N+m-\tau} \frac{(\nu)\cdots(\nu-(N+m-\tau)+1)}{\Gamma(N+m-\tau+1)} f(a+\tau) \\
&= \sum_{\tau=0}^{N+m} (-1)^{N+m-\tau} \binom{\nu}{N+m-\tau} f(a+\tau) \\
&= \sum_{k=0}^{N+m} (-1)^k \binom{\nu}{k} f(a+N+m-k) \\
&= \sum_{k=0}^{N+m} (-1)^k \binom{\nu}{k} f((a+N-\nu+m)+\nu-k) \\
&= \sum_{k=0}^{t-a+\nu} (-1)^k \binom{\nu}{k} f(t+\nu-k).
\end{aligned}$$

Hence (2.23) holds. Since we can obtain the formula for $\Delta_a^{-\nu} f(t)$ from the formula for $\Delta_a^\nu f(t)$ by replacing ν by $-\nu$ we get that (2.24) holds with the appropriate change in domains. Finally, since

$$\binom{-\nu}{k} = (-1)^k \binom{\nu+k-1}{k},$$

(2.25) follows immediately from (2.24). □

Note that if we let $\nu = N$ in (2.23), we get the following integer binomial expression for $\Delta^N f(t)$, that is

$$\Delta^N f(t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(t+N-k), \qquad t \in \mathbb{N}_a.$$

2.5 Composition Rules

Theorem 2.46 (Composition of Fractional Sums). *Assume f is defined on $\mathbb{N}_a$ and μ, ν are positive numbers. Then*

$$\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) = \left(\Delta_a^{-(\mu+\nu)} f\right)(t) = \left[\Delta_{a+\mu}^{-\nu}\left(\Delta_a^{-\mu} f\right)\right](t)$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Proof. For $t \in \mathbb{N}_{a+\mu+\nu}$, consider

$$\begin{aligned}
\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) &= \sum_{s=a+\nu}^{t-\mu} h_{\mu-1}(t,\sigma(s)) \left(\Delta_a^{-\nu} f\right)(s) \\
&= \sum_{s=a+\nu}^{t-\mu} h_{\mu-1}(t,\sigma(s)) \sum_{r=a}^{s-\nu} h_{\nu-1}(s,\sigma(r)) f(r) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{s=a+\nu}^{t-\mu} \sum_{r=a}^{s-\nu} (t-\sigma(s))^{\underline{\mu-1}} (s-\sigma(r))^{\underline{\nu-1}} f(r) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \sum_{s=r+\nu}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} (s-\sigma(r))^{\underline{\nu-1}} f(r),
\end{aligned}$$

where in the last step we interchanged the order of summation. Letting $x = s - \sigma(r)$ we obtain

$$\begin{aligned}
&\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\sum_{x=\nu-1}^{t-\mu-r-1} (t-x-r-2)^{\underline{\mu-1}} x^{\underline{\nu-1}} \right] f(r)
\end{aligned}$$

$$= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\frac{1}{\Gamma(\mu)} \sum_{x=\nu-1}^{(t-r-1)-\mu} (t-r-1-\sigma(x))^{\underline{\mu-1}} x^{\underline{\nu-1}} \right] f(r)$$

$$= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \left[\Delta_{\nu-1}^{-\mu} t^{\underline{\nu-1}}\right]_{t\to t-r-1} f(r).$$

But by Theorem 2.38

$$\Delta_{\nu-1}^{-\mu} t^{\underline{\nu-1}} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} t^{\underline{\mu+\nu-1}}$$

and therefore

$$\begin{aligned}
\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) &= \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} (t-r-1)^{\underline{\mu+\nu-1}} f(r) \\
&= \frac{1}{\Gamma(\mu+\nu)} \sum_{r=a}^{t-(\mu+\nu)} (t-\sigma(r))^{\underline{\mu+\nu-1}} f(r) \\
&= \left(\Delta_a^{-(\mu+\nu)} f\right)(t),
\end{aligned}$$

$t \in \mathbb{N}_{a+\nu+\mu}$, which is one of the desired conclusions. Interchanging μ and ν in the above formula we also get the result

$$\left[\Delta_{a+\mu}^{-\nu}\left(\Delta_a^{-\mu} f\right)\right](t) = \left(\Delta_a^{-(\mu+\nu)} f\right)(t)$$

for $t \in \mathbb{N}_{a+\mu+\nu}$. □

In the next lemma we give composition rules for an integer difference with a fractional sum and with a fractional difference. Atici and Eloe proved (2.26) with the additional assumption that $0 < k < \nu$ and Holm [123, 125] proved (2.26) in this more general setting.

Lemma 2.47. *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, $N - 1 < \nu \le N$. Then*

$$\left[\Delta^k\left(\Delta_a^{-\nu} f\right)\right](t) = \left(\Delta_a^{k-\nu} f\right)(t), \qquad t \in \mathbb{N}_{a+\nu}. \tag{2.26}$$

and

$$\left[\Delta^k\left(\Delta_a^{\nu} f\right)\right](t) = \left(\Delta_a^{k+\nu} f\right)(t), \qquad t \in \mathbb{N}_{a+N-\nu}. \tag{2.27}$$

Proof. First we prove that

$$\left[\Delta^k\left(\Delta_a^{-k} f\right)\right](t) = f(t), \qquad t \in \mathbb{N}_{a+k} \tag{2.28}$$

by induction for $k \in \mathbb{N}_1$. For the base case we have

$$\Delta\Delta_a^{-1}f(t) = \Delta\left[\int_a^t f(\tau)\Delta\tau\right] = f(t)$$

for $t \in \mathbb{N}_{a+1}$. Now assume $k \geq 1$ and (2.28) holds. Then

$$\begin{aligned}
\Delta^{k+1}\Delta_a^{k+1}f(t) &= \Delta^{k+1}\Delta_{a+k}^{-1}\Delta_a^{-k}f(t) \qquad \text{using Theorem 2.46} \\
&= \Delta^k[\Delta\Delta_{a+k}^{-1}]\Delta_a^{-k}f(t) \\
&= \Delta^k\Delta_a^{-k}f(t) \qquad \text{by the base case with base } a+k \\
&= f(t) \qquad \text{by the induction assumption (2.28)}
\end{aligned}$$

for $t \in \mathbb{N}_{a+k+1}$. Therefore, for $k \geq N$

$$\Delta^k\Delta_a^{-N}f(t) = \Delta^{k-N}[\Delta^N\Delta_a^{-N}]f(t) = \Delta^{k-N}f(t)$$

and for $k < N$

$$\Delta^k\Delta_a^{-N}f(t) = \Delta^k\Delta_{a+N-k}^{-k}[\Delta_a^{-(N-k)}]f(t) = \Delta_a^{-(N-k)}f(t) = \Delta_a^{k-N}f(t)$$

for $t \in \mathbb{N}_{a+N}$. Hence for all $k \in \mathbb{N}_1$ we have that (2.26) holds for the case $\nu = N$. It is also true that (2.27) holds when $\nu = N$. Assume for the rest of this proof that $N-1 < \nu < N$. We will now show by induction that (2.27) holds for $k \in \mathbb{N}_1$. For the base case $k = 1$ we have using the Leibniz rule (2.11)

$$\begin{aligned}
&\Delta\Delta_a^{\nu}f(t) \\
&= \Delta\left[\int_a^{t+\nu+1} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau\right] \\
&= \int_a^{t+\nu+1} h_{-\nu-2}(t,\sigma(\tau))f(\tau)\Delta\tau + h_{\nu-1}(\sigma(t), t+\nu+1)f(t+\nu+1) \\
&= \int_a^{t+\nu+1} h_{-\nu-2}(t,\sigma(\tau))f(\tau)\Delta\tau + f(t+\nu+1) \\
&= \int_a^{t+\nu+2} h_{-\nu-2}(t,\sigma(\tau))f(\tau)\Delta\tau \\
&= \Delta_a^{-(-\nu-1)}f(t) \\
&= \Delta_a^{1+\nu}f(t).
\end{aligned}$$

Hence the base case

$$\Delta\Delta_a^{\nu}f(t) = \Delta_a^{1+\nu}f(t)$$

holds. Now assume $k \geq 1$ and

$$\Delta^k \Delta_a^\nu f(t) = \Delta_a^{k+\nu} f(t) \tag{2.29}$$

holds. It follows from the induction hypothesis (2.29) and the base case that

$$\begin{aligned}\Delta^{k+1}\Delta_a^\nu f(t) &= \Delta\Delta^k\Delta_a^{1+\nu} f(t)\\ &= \Delta\Delta_a^{k+\nu} f(t)\\ &= \Delta_a^{k+1+\nu} f(t).\end{aligned}$$

Hence (2.27) holds for all $k \in \mathbb{N}_1$. The proof of (2.26) is very similar and is left as an exercise (Exercise 2.23). □

We now prove a composition rule that appears in Holm [125] for a fractional difference with a fractional sum.

Theorem 2.48. *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\nu, \mu > 0$ and $N - 1 < \nu \leq N$, $N \in \mathbb{N}_1$. Then*

$$\Delta_{a+\mu}^\nu \Delta_a^{-\mu} f(t) = \Delta_a^{\nu-\mu} f(t), \qquad t \in \mathbb{N}_{a+\mu+N-\nu}. \tag{2.30}$$

Proof. Note that for $t \in \mathbb{N}_{a+\mu+N-\nu}$,

$$\begin{aligned}\Delta_{a+\mu}^\nu \Delta_a^{-\mu} f(t) &= \Delta^N \Delta_{a+\mu}^{-(N-\nu)} \Delta_a^{-\mu} f(t)\\ &= \Delta^N \Delta_a^{-(N-\nu+\mu)} f(t) \qquad \text{by Theorem 2.46}\\ &= \Delta_a^{N-(N-\nu+\mu)} f(t) \qquad \text{by (2.26)}\\ &= \Delta_a^{\nu-\mu} f(t).\end{aligned}$$

Hence (2.30) holds. □

Remark 2.49. From Theorem 2.46 we saw that we can take fractional sums of fractional sums by adding exponents and by Theorem 2.48 we can take fractional differences of fractional sums by adding exponents. The fundamental theorem of calculus gives us that

$$\Delta_a^{-1}\Delta f(\tau) = \int_a^t \Delta f(\tau) = f(t) - f(a) = \Delta_a^0 f(t) - f(a).$$

Hence we should not expect the fractional sum of a fractional difference can be obtained by adding exponents.

In the next theorem we give a formula for a fractional sum of an integer difference. The first formula in the following Theorem 2.50 is given in Atici et al. [34] and the second formula appears in Holm [125].

Theorem 2.50. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$, $k \in \mathbb{N}_0$ *and* $\nu, \mu > 0$ *with* $N - 1 < \mu \leq N$. *Then*

$$\Delta_a^{-\nu} \Delta^k f(t) = \Delta_a^{k-\nu} f(t) - \sum_{j=0}^{k-1} h_{\nu-k+j}(t, a) \Delta^j f(a), \tag{2.31}$$

for $t \in \mathbb{N}_{a+\nu}$, *and*

$$\Delta_{a+N-\mu}^{-\nu} \Delta_a^{\mu} f(t) = \Delta_a^{\mu-\nu} f(t) - \sum_{j=0}^{N-1} h_{\nu-N+j}(t - N + \nu, a) \Delta_a^{j-(N-\mu)} f(a + N - \mu), \tag{2.32}$$

for $t \in \mathbb{N}_{a+N-\mu+\nu}$.

Proof. We first prove that (2.31) holds by induction for $k \in \mathbb{N}_1$. For the base case $k = 1$ we have using integration by parts and

$$h_{\nu-1}(t, t - \nu + 1) = 1 = h_{\nu-2}(t, t - \nu + 2)$$

that for $t \in \mathbb{N}_{a+\nu}$

$$\begin{aligned}
\Delta_a^{-\nu} \Delta f(t) &= \int_a^{t-\nu+1} h_{\nu-1}(t, \sigma(\tau)) \Delta f(\tau) \Delta\tau \\
&= h_{\nu-1}(t, \tau) f(t) \Big|_{\tau=a}^{t-\nu+1} + \int_a^{t-\nu+1} h_{\nu-2}(t, \sigma(\tau)) f(\tau) \Delta\tau \\
&= h_{\nu-1}(t, t - \nu + 1) f(t - \nu + 1) - h_{\nu-1}(t, a) f(a) \\
&\qquad + \int_a^{t-\nu+1} h_{\nu-2}(t, \sigma(\tau)) \Delta\tau \\
&= f(t - \nu + 1) - h_{\nu-1}(t, a) f(a) + \int_a^{t-\nu+1} h_{\nu-2}(t, \sigma(\tau)) f(\tau) \Delta\tau \\
&= \int_a^{t-\nu+2} h_{\nu-2}(t, \sigma(\tau)) f(\tau) \Delta\tau - h_{\nu-1}(t, a) f(a) \\
&= \Delta_a^{1-\nu} f(t) - h_{\nu-1}(t, a) f(a)
\end{aligned}$$

which proves (2.31) for the base case $k = 1$. Now assume $k \geq 1$ and (2.31) holds for that k. Then we have that

$$\begin{aligned}
\Delta_a^{-\nu}\Delta^{k+1}f(t) &= \Delta_a^{-\nu}\Delta^k\Delta f(t)\\
&= \Delta_a^{k-\nu}\Delta f(t) - \sum_{j=0}^{k-1} h_{\nu-k-j}(t,a)\Delta^{j+1}f(a) \quad \text{(by (2.31))}\\
&= \Delta_a^{k-\nu}f(t) - \sum_{j=0}^{k-1} h_{\nu-k+j}(t,a)\Delta^{j+1}f(a) - h_{\nu-k-1}(t,a)f(a)\\
&= \Delta_a^{-\nu}\Delta_a^{k+1-\nu}f(t) - \sum_{j=0}^{k} h_{\nu-k-1+j}(t,a)\Delta^j f(a),
\end{aligned}$$

for $t \in \mathbb{N}_{a+N-\nu}$. Hence (2.31) holds. Next we show that (2.32) holds. To see this suppose now that $\nu > 0$ and $\mu > 0$ with $N-1 < \mu \le N$. Letting $g(t) = \Delta_a^{-(N-\mu)}f(t)$ and $b = a + N - \mu$ (the first point in the domain of g), we have for $t \in \mathbb{N}_{a+N-\mu+\nu}$,

$$\begin{aligned}
&\Delta_{a+N-\mu}^{-\nu}\Delta_a^{\mu}f(t)\\
&= \Delta_{a+N-\mu}^{-\nu}\Delta^N\left(\Delta_a^{-(N-\mu)}f(t)\right)\\
&= \Delta_{a+N-\mu}^{-\nu}\Delta^N g(t)\\
&= \Delta_{a+N-\mu}^{N-\nu}g(t) - \sum_{j=0}^{N-1} h_{\nu-N+j}(t,b)\Delta^j g(b) \qquad \text{by (2.32)}\\
&= \Delta_{a+N-\mu}^{N-\nu}\Delta_a^{-(N-\mu)}f(t) - \sum_{j=0}^{N-1} h_{\nu-N+j}(t,b)\Delta^j\Delta_a^{-(N-\mu)}f(b)\\
&= \Delta_a^{\mu-\nu}f(t) - \sum_{j=0}^{N-1} h_{\nu-N+j}(t-N+\nu,a)\Delta_a^{j-N+\mu}f(a+N-\mu),
\end{aligned}$$

where in this last step, we applied 2.31. □

Finally, we give a composition formula for composing two fractional differences. Note that the rule for this composition is nearly identical to the rule (2.32) for the composition $\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu}$. Theorem 2.51 is given for the specific case $\mu \in \mathbb{N}_0$ by Atici and Eloe in [34].

Theorem 2.51. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$, with $N-1 < \nu \le N$ and $M-1 < \mu \le M$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$,*

$$\Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu}f(t) = \Delta_a^{\nu+\mu}f(t) - \sum_{j=0}^{M-1} h_{-\nu-M+j}(t-M+\mu,a)\Delta_a^{j-M+\mu}f(a+M-\mu) \qquad (2.33)$$

for $N-1<\nu<N$. *If* $\nu=N$, *then* (2.33) *simplifies to*

$$\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t)=\Delta^{\nu+\mu}_{a}f(t),\quad t\in\mathbb{N}_{a+M-\mu}.$$

Proof. Let f, ν, and μ be given as in the statement of the theorem. Lemma 2.47 has already proven the case when $\nu=N$.

If $N-1<\nu<N$, then for $t\in\mathbb{N}_{a+M-\mu+N-\nu}$, we have

$$\begin{aligned}
&\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t)\\
&=\Delta^{N}\left[\Delta^{-(N-\nu)}_{a+M-\mu}\Delta^{\mu}_{a}f(t)\right],\text{ and now using (2.50),}\\
&=\Delta^{N}\Bigg[\Delta^{-N+\nu+\mu}_{a}f(t)\\
&\quad-\sum_{j=0}^{M-1}\Delta^{j-M+\mu}_{a}f(a+M-\mu)h_{N-\nu-M+j}(t-M+\mu,a)\Bigg]\\
&=\Delta^{\nu+\mu}_{a}\Delta^{N}h_{N-\nu-M+j}(t-M+\mu)f(t)-\\
&\quad\sum_{j=0}^{M-1}\Delta^{j-M+\mu}_{a}f(a+M-\mu)\Delta^{N}h_{N-\nu-M+j}(t-M+\mu,a)\text{ (Lemma 2.47)}\\
&=\Delta^{\nu+\mu}_{a}f(t)-\\
&\quad\sum_{j=0}^{M-1}\Delta^{j-M+\mu}_{a}f(a+M-\mu)h_{-\nu-M+j}(t-M+\mu,a)\\
&=\Delta^{\nu+\mu}_{a}f(t)\\
&\quad-\sum_{j=0}^{M-1}\Delta^{j-M+\mu}_{a}h_{-\nu-M+\mu}(t-M+\mu)f(a+M-\mu).
\end{aligned}$$

□

Theorem 2.52 (Variation of Constants Formula). *Assume* $N\geq 1$ *is an integer and* $N-1<\nu\leq N$. *If* $f:\mathbb{N}_0\to\mathbb{R}$, *then the solution of the IVP*

$$\Delta^{\nu}_{\nu-N}y(t)=f(t),\quad t\in\mathbb{N}_0 \tag{2.34}$$

$$y(\nu-N+i)=0,\quad 0\leq i\leq N-1 \tag{2.35}$$

is given by

$$y(t)=\Delta^{-\nu}_{0}f(t)=\sum_{s=0}^{t-\nu}h_{\nu-1}(t,\sigma(s))f(s),\quad t\in\mathbb{N}_{\nu-N}.$$

Proof. Let

$$y(t) = \Delta_0^{-\nu} f(t) = \sum_{s=0}^{t-\nu} h_{\nu-1}(t, \sigma(s)) f(s).$$

Then by our convention on sums

$$y(\nu - N + i) = \sum_{s=0}^{-N+i} h_{\nu-1}(\nu - N + i, \sigma(s)) f(s) = 0$$

for $0 \leq i \leq N-1$, and hence the initial conditions (2.35) are satisfied.

Also, for $t \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta_{\nu-N}^{\nu} y(t) &= \Delta^N \Delta_{\nu-N}^{-(N-\nu)} y(t) \\
&= \Delta^N \sum_{s=\nu-N}^{t-(N-\nu)} h_{N-\nu-1}(t, \sigma(s))\, y(s) \\
&= \Delta^N \sum_{s=\nu}^{t-(N-\nu)} h_{N-\nu-1}(t, \sigma(s))\, y(s),
\end{aligned}$$

where in the last step we used the initial conditions (2.35). Hence,

$$\begin{aligned}
\Delta_{\nu-N}^{\nu} y(t) &= \Delta^N \Delta_{\nu}^{-(N-\nu)} y(t) \\
&= \Delta^N \Delta_{0+\nu}^{-(N-\nu)} \Delta_0^{-\nu} f(t) \\
&= \Delta^N \Delta_0^{-N} f(t) \\
&= f(t).
\end{aligned}$$

Therefore y is a solution of the fractional difference equation (2.34) on $\mathbb{N}_0$. □

Next we use the fractional variation of constants formula to solve a simple fractional IVP.

Example 2.53. Use the variation of constants formula in Theorem 2.52 to solve the fractional IVP

$$\begin{aligned}
&\Delta_{-\frac{1}{2}}^{\frac{1}{2}} y(t) = 5, \quad t \in \mathbb{N}_0 \\
&y\left(-\frac{1}{2}\right) = 3\sqrt{\pi}.
\end{aligned}$$

The solution of this IVP is defined on $\mathbb{N}_{-\frac{1}{2}}$. Note that the corresponding homogeneous fractional difference equation

$$\Delta_{-\frac{1}{2}}^{\frac{1}{2}} y(t) = 0, \quad t \in \mathbb{N}_0$$

has the general fractional equation form

$$\Delta_{a+\nu-N}^{\nu} y(t) = 0, \quad t \in \mathbb{N}_a$$

in Theorem 2.43, where

$$a = 0, \quad \nu = \frac{1}{2} \quad N = 1, \quad a + \nu - N = -\frac{1}{2}.$$

Hence $t^{\underline{-\frac{1}{2}}}$ is a solution of the homogeneous equation $\Delta_{-\frac{1}{2}}^{\frac{1}{2}} y(t) = 0$ and hence (using Theorem 2.52) a general solution of $\Delta_{-\frac{1}{2}}^{\frac{1}{2}} y(t) = 5$ is given by

$$\begin{aligned} y(t) &= ct^{\underline{-\frac{1}{2}}} + \Delta_0^{-\frac{1}{2}} 5 \\ &= ct^{\underline{-\frac{1}{2}}} + 5\Delta_0^{-\frac{1}{2}} 1 \end{aligned} \tag{2.36}$$

By formula (2.16) we have that

$$\Delta_0^{-\frac{1}{2}} 1 = \Delta_0^{-\frac{1}{2}} t^{\underline{0}} = \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} t^{\underline{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}} t^{\underline{\frac{1}{2}}},$$

which is the expression that we got for $\Delta_0^{-\frac{1}{2}} 1$ in Example 2.28. It follows from (2.36) that

$$y(t) = ct^{\underline{-\frac{1}{2}}} + \frac{10}{\sqrt{\pi}} t^{\underline{\frac{1}{2}}}.$$

Using the initial condition $y\left(-\frac{1}{2}\right) = 3\sqrt{\pi}$ we get that $c = 3$. Therefore, the solution of the given IVP is

$$y(t) = 3t^{\underline{-\frac{1}{2}}} + \frac{10}{\sqrt{\pi}} t^{\underline{\frac{1}{2}}},$$

for $t \in \mathbb{N}_{-\frac{1}{2}}$.

Also, it is often necessary to know how a shifted Laplace transform with respect to its base relates to the original Laplace transform with base a, as is described in the following theorem.

Theorem 2.54. *Let $m \in \mathbb{N}_0$ be given and suppose $f : \mathbb{N}_{a-m} \to \mathbb{R}$ and $g : \mathbb{N}_a \to \mathbb{R}$ are of exponential order $r > 0$. Then for $|s+1| > r$,*

$$\mathcal{L}_{a-m}\{f\}(s) = \frac{1}{(s+1)^m}\mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1}\frac{f(a+k-m)}{(s+1)^{k+1}} \tag{2.37}$$

and

$$\mathcal{L}_{a+m}\{g\}(s) = (s+1)^m \mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1}(s+1)^{m-1-k} g(a+k). \tag{2.38}$$

Proof. Let f, g, r, and m be given as in the statement of this theorem. Then for $|s+1| > r$,

$$\begin{aligned}
\mathcal{L}_{a-m}\{f\}(s) &= \sum_{k=0}^{\infty}\frac{f(a-m+k)}{(s+1)^{k+1}} \\
&= \sum_{k=m}^{\infty}\frac{f(a-m+k)}{(s+1)^{k+1}} + \sum_{k=0}^{m-1}\frac{f(a-m+k)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty}\frac{f(a+k)}{(s+1)^{k+m+1}} + \sum_{k=0}^{m-1}\frac{f(a+k-m)}{(s+1)^{k+1}} \\
&= \frac{1}{(s+1)^m}\mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1}\frac{f(a+k-m)}{(s+1)^{k+1}},
\end{aligned}$$

and hence (2.37) holds.

Next, consider

$$\begin{aligned}
\mathcal{L}_{a+m}\{g\}(s) &= \sum_{k=0}^{\infty}\frac{g(a+m+k)}{(s+1)^{k+1}} \\
&= \sum_{k=m}^{\infty}\frac{g(a+k)}{(s+1)^{k-m+1}} \\
&= \sum_{k=0}^{\infty}\frac{g(a+k)}{(s+1)^{k-m+1}} - \sum_{k=0}^{m-1}\frac{g(a+k)}{(s+1)^{k-m+1}} \\
&= (s+1)^m \mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1}(s+1)^{m-1-k} g(a+k),
\end{aligned}$$

and thus (2.38) holds. □

We leave it as an exercise to verify that applying formulas (2.37) and (2.38) yields

$$\mathcal{L}_{(a+m)-m}\{f\}(s) = \mathcal{L}_{(a-m)+m}\{f\}(s) = \mathcal{L}_a\{f\}(s),$$

for $|s+1| > r$.

Recall the definition of the fractional Taylor monomials (Definition 2.24).

Definition 2.55. For each $\mu \in \mathbb{R}\setminus(-\mathbb{N}_1)$, define the μ-th order **Taylor monomial**, $h_\mu(t,a)$, by

$$h_\mu(t,a) := \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)}, \qquad \text{for} \qquad t \in \mathbb{N}_a.$$

Theorem 2.56. *If $\mu \le 0$ and $\mu \notin (-\mathbb{N}_1)$, then $h_\mu(t,a)$ is bounded (and hence is of exponential order $r = 1$). If $\mu > 0$, then for every $r > 1$, $h_\mu(t,a)$ is of exponential order r.*

Proof. First consider the case that $\mu \le 0$ with $\mu \notin (-\mathbb{N}_0)$. Then for all large $t \in \mathbb{N}_a$,

$$h_\mu(t,a) = \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \le \frac{1}{\Gamma(\mu+1)},$$

implying that h_μ is of exponential order one (i.e., bounded).

Next assume that $\mu > 0$, with $N \in \mathbb{N}_0$ chosen so that $N-1 < \mu \le N$. Then for any fixed $r > 1$,

$$\begin{aligned}
h_\mu(t,a) = \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} &= \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \\
&\le \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-N)} \\
&= \frac{(t-a)\cdots(t-a-N+1)}{\Gamma(\mu+1)} \\
&\le \frac{(t-a)^N}{\Gamma(\mu+1)} \\
&\le \frac{r^t}{\Gamma(\mu+1)},
\end{aligned}$$

for sufficiently large $t \in \mathbb{N}_a$.

Therefore, $h_\mu(t,a)$ is of exponential order r for each $\mu \in \mathbb{R}\setminus(-\mathbb{N}_1)$ and $r > 1$. It follows from Theorem 2.4 that $\mathcal{L}_a\left\{h_\mu(t,a)\right\}(s)$ exists for $|s+1| > 1$. □

Remark 2.57. Note that the fractional Taylor monomials, $h_\mu(t,a)$ for $\mu > 0$ are examples of functions that are of order r for all $r > 1$, but are not of order 1 (see Exercise 2.4).

Theorem 2.58. *Let* $\mu \in \mathbb{R}\setminus(-\mathbb{N}_1)$ *. Then*

$$\mathcal{L}_{a+\mu}\left\{h_\mu(t,a)\right\}(s) = \frac{(s+1)^\mu}{s^{\mu+1}} \tag{2.39}$$

for $|s+1| > 1$.

Proof. For $|s+1| > 1$, consider

$$\frac{(s+1)^\mu}{s^{\mu+1}} = \frac{1}{s+1}\left(\frac{s+1}{s}\right)^{\mu+1} = \frac{1}{s+1}\left(1-\frac{1}{s+1}\right)^{-\mu-1}.$$

Since $|\frac{1}{s+1}| < 1$, we have by the binomial theorem that

$$\begin{aligned}\frac{(s+1)^\mu}{s^{\mu+1}} &= \frac{1}{s+1}\sum_{k=0}^{\infty}(-1)^k\binom{-\mu-1}{k}\left(\frac{1}{s+1}\right)^k \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{-\mu-1}{k}\frac{1}{(s+1)^{k+1}}.\end{aligned} \tag{2.40}$$

But

$$\begin{aligned}(-1)^k\binom{-\mu-1}{k} &= (-1)^k\frac{(-\mu-1)^{\underline{k}}}{k!} \\ &= (-1)^k\frac{(-\mu-1)(-\mu-2)\cdots(-\mu-k)}{k!} \\ &= \frac{(\mu+k)(\mu+k-1)\cdots(\mu+1)}{k!} \\ &= \frac{(\mu+k)^{\underline{k}}}{k!} \\ &= \binom{\mu+k}{k} = \binom{\mu+k}{\mu} \quad \text{by Exercise 1.12, (v)} \\ &= \frac{(\mu+k)^{\underline{\mu}}}{\Gamma(\mu+1)} \\ &= \frac{[(a+\mu+k)-a]^{\underline{\mu}}}{\Gamma(\mu+1)} \\ &= h_\mu(a+\mu+k,a).\end{aligned} \tag{2.41}$$

Using (2.40) and (2.41), we have that

$$\begin{aligned}\frac{(s+1)^{\mu}}{s^{\mu+1}} &= \sum_{k=0}^{\infty} \frac{h_{\mu}(a+\mu+k,a)}{(s+1)^{k+1}} \\ &= \mathcal{L}_{a+\mu}\left\{h_{\mu}\left(t,a\right)\right\}(s),\end{aligned}$$

for $|s+1| > 1$. □

2.6 The Convolution Product

The following definition of the convolution product agrees with the convolution product defined for general time scales in [62], but it differs from the convolution product defined by Atici and Eloe in [32] (in the upper limit). We demonstrate several advantages of using Definition 2.59 in the following results.

Definition 2.59. Let $f, g : \mathbb{N}_a \to \mathbb{R}$ be given. Define the **convolution product** of f and g to be

$$(f * g)(t) := \sum_{r=a}^{t-1} f(r)g(t-\sigma(r)+a), \quad \text{for } t \in \mathbb{N}_a \tag{2.42}$$

(note that $(f * g)(a) = 0$ by our convention on sums).

Example 2.60. For $p \neq 0, -1$, find the convolution product $e_p(t,a) * 1$, and use your answer to find $\mathcal{L}\{e_p(t,a) * 1\}(s)$. By the definition of the convolution product

$$\begin{aligned}(e_p(t,a) * 1)(t) &= \sum_{r=a}^{t-1} e_p(r,a) \\ &= \int_a^t e_p(r,a)\Delta r \\ &= \frac{1}{p} e_p(r,a)\Big|_a^t \\ &= \frac{1}{p} e_p(t,a) - \frac{1}{p}.\end{aligned}$$

It follows that

$$\mathcal{L}_a\{e_p(t,a) * 1\}(s) = \frac{1}{p}\frac{1}{s-p} - \frac{1}{p}\frac{1}{s} = \frac{1}{(s-p)s}.$$

Note from Example 2.60 we get that

$$\mathcal{L}_a\{e_p(t,a) * 1\}(s) = \frac{1}{(s-p)s} = \frac{1}{s-p}\frac{1}{s} = \mathcal{L}_a\{e_p(t,a)\}(s)\mathcal{L}_a\{1\}(s),$$

which is a special case of the following theorem which gives a formula for the Laplace transform of the convolution product of two functions. Later we will show that this formula is useful in solving fractional initial value problems. In this theorem we use the notation $F_a(s) := \mathcal{L}_a\{f\}(s)$, which was introduced earlier.

Theorem 2.61 (Convolution Theorem). *Let $f, g : \mathbb{N}_a \to \mathbb{R}$ be of exponential order $r_0 > 0$. Then*

$$\mathcal{L}_a\{f * g\}(s) = F_a(s)G_a(s), \quad \textit{for } |s+1| > r_0\,. \tag{2.43}$$

Proof. We have

$$\begin{aligned}
\mathcal{L}_a\{f * g\}(s) &= \sum_{k=0}^{\infty} \frac{(f * g)(a+k)}{(s+1)^{k+1}} = \sum_{k=1}^{\infty} \frac{(f * g)(a+k)}{(s+1)^{k+1}} \\
&= \sum_{k=1}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{a+k-1} f(r)g(a+k-\sigma(r)+a) \\
&= \sum_{k=1}^{\infty}\sum_{r=0}^{k-1} \frac{f(a+r)g(a+k-r-1)}{(s+1)^{k+1}} \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{f(a+r)g(a+k-r-1)}{(s+1)^{k+1}}.
\end{aligned}$$

Making the change of variables $\tau = k - r - 1$ gives us that

$$\begin{aligned}
\mathcal{L}_a\{f * g\}(s) &= \sum_{\tau=0}^{\infty}\sum_{r=0}^{\infty} \frac{f(a+r)g(a+\tau)}{(s+1)^{\tau+r+2}} \\
&= \sum_{r=0}^{\infty} \frac{f(a+r)}{(s+1)^{r+1}} \sum_{\tau=0}^{\infty} \frac{g(a+\tau)}{(s+1)^{\tau+1}} \\
&= F_a(s)G_a(s),
\end{aligned}$$

for $|s+1| > r_0$. □

Example 2.62. Solve the (Volterra) summation equation

$$y(t) = 3 + 12\sum_{r=0}^{t-1} \left[2^{t-r-1} - 1\right] y(r), \quad t \in \mathbb{N}_0 \tag{2.44}$$

using Laplace transforms. We can write equation (2.44) in the equivalent form

$$\begin{aligned} y(t) &= 3 + 12 \sum_{r=0}^{t-1} [e_1(t-r-1,0) - 1] y(r) \\ &= 3 + 12\,[(e_1(t,0) - 1) * y(t)], \quad t \in \mathbb{N}_0. \end{aligned} \tag{2.45}$$

Taking the Laplace transform (based at 0) of both sides of (2.45), we obtain

$$\begin{aligned} Y_0(s) &= \frac{3}{s} + 12 \left[\frac{1}{s-1} - \frac{1}{s}\right] Y_0(s) \\ &= \frac{3}{s} + \frac{12}{s(s-1)} Y_0(s). \end{aligned}$$

Solving for $Y_0(s)$, we get

$$\begin{aligned} Y_0(s) &= \frac{3(s-1)}{(s+3)(s-4)} \\ &= \frac{12/7}{s+3} + \frac{9/7}{s-4}. \end{aligned}$$

Taking the inverse Laplace transform of both sides, we get

$$\begin{aligned} y(t) &= \frac{12}{7} e_{-3}(t,0) + \frac{9}{7} e_4(t,0) \\ &= \frac{12}{7}(-2)^t + \frac{9}{7}\, 5^t. \end{aligned}$$

2.7 Using Laplace Transforms to Solve Fractional Equations

When solving certain summation equations one uses the formula

$$\mathcal{L}_a \left\{\Delta_a^{-N} f\right\}(s) = \frac{F_a(s)}{s^N}, \tag{2.46}$$

where N is a positive integer. Since the summation equation (2.5) can be written in the form

$$y(t) = 2 \cdot 4^t + 2 \int_0^t y(s)\, \Delta s, \quad t \in \mathbb{N}_0,$$

this is an example of a summation equation for which we want to use the formula (2.46) with $N = 1$.

We will now set out to generalize formulas (2.4) and (2.46) to the fractional case so that we can solve fractional difference and summation equations using Laplace transforms.

We will show (see Theorem 2.65) that if $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order, then $\Delta_a^{-\nu} f$ and $\Delta_a^{\nu} f$ are of a certain exponential order and hence their Laplace transforms will exist. We will use the following lemma, which gives an estimate for $t^{\underline{\nu}}$ in the proof of Theorem 2.65.

Lemma 2.63. *Assume* $\nu > -1$ *and* $N - 1 < \nu \leq N$. *Then*

$$t^{\underline{\nu}} \leq t^N, \quad \textit{for } t \textit{ sufficiently large}. \tag{2.47}$$

Proof. In this proof we use the fact that $\Gamma(x) > 0$ for $x > 0$ and $\Gamma(x)$ is strictly increasing for $x \geq 2$. First consider the case $-1 < \nu \leq 0$. Then, since $t + 1 - \nu \geq t + 1$, we have for large t

$$\begin{aligned} t^{\underline{\nu}} &= \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \\ &\leq 1 = t^0 = t^N. \end{aligned}$$

Next, consider the case $\nu > 0$. Then for large t we have

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \leq \frac{\Gamma(t+1)}{\Gamma(t+1-N)} = t\,(t-1)\cdots(t-(N-1)) \leq t^N.$$

This completes the proof. □

Remark 2.64. Thus far whenever we have considered a function $f : \mathbb{N}_a \to \mathbb{R}$, we have always taken the domain of $\Delta_a^{-\nu} f$ to be the set $\mathbb{N}_{a+\nu}$. However, it is sometimes convenient to take the domain of $\Delta_a^{-\nu} f$ to be the set $\mathbb{N}_{a+\nu-N}$, where $\nu > 0$, and $N - 1 < \nu \leq N$. By our convention on sums we see that

$$\Delta_a^{-\nu} f(a + \nu - N + k) = 0, \quad \text{for} \quad 0 \leq k \leq N - 1.$$

Later (see, for example, Theorem 2.67) we will consider both of the

$$\mathcal{L}_{a+\nu}\{\Delta_a^{-\nu} f\}(s) \quad \text{and} \quad \mathcal{L}_{a+\nu-N}\{\Delta_a^{-\nu} f\}(s).$$

Note that $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$ and $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$ are of the same exponential order. Theorem 2.67 will give a relationship between these two Laplace transforms.

Theorem 2.65. *Suppose that* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$, *and let* $\nu > 0$, $N - 1 < \nu \leq N$, *be given. Then for each fixed* $\epsilon > 0$, $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$, $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$, *and* $\Delta_a^{\nu} f : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$ *are of exponential order* $r + \epsilon$.

Proof. First we show if $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r = 1$, then $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$ is of exponential order $r = 1 + \epsilon$, for each $\epsilon > 0$. By Exercise 2.1 it suffices to show that f is bounded on $\mathbb{N}_a$ implies $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$ is of exponential order $r = 1 + \epsilon$, for each $\epsilon > 0$. To this end assume

$$|f(t)| \leq N, \quad t \in \mathbb{N}_a.$$

Then, for $t \in \mathbb{N}_{a+\nu}$,

$$\begin{aligned}
|\Delta_a^{-\nu} f(t)| &= \left| \int_a^{t-\nu+1} h_{\nu-1}(t, \sigma(s)) f(s) \Delta s \right| \\
&\leq \int_a^{t-\nu+1} h_{\nu-1}(t, \sigma(s)) |f(s)| \Delta s \\
&\leq N \int_a^{t-\nu+1} h_{\nu-1}(t, \sigma(s)) \Delta s \\
&= -N h_\nu(t, s)|_{s=a}^{s=t-\nu+1}, \qquad \text{by Theorem 2.27, part (v)} \\
&= -N h_\nu(t, t-\nu+1) + N h_\nu(t, a) \\
&= N h_\nu(t, a).
\end{aligned}$$

Since, by Theorem 2.56, $h_\nu(t, a)$ is of exponential order $1 + \epsilon$ for each $\epsilon > 0$, it follows that $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \to \mathbb{R}$ is of exponential order $1 + \epsilon$, for each $\epsilon > 0$.

Next assume f is of exponential order $r > 1$, there exist an $A > 0$ and a $T \in \mathbb{N}_a$ such that

$$|f(t)| \leq A r^t, \quad \text{for all} \quad t \in \mathbb{N}_T. \tag{2.48}$$

For $t \in \mathbb{N}_{T+\nu}$, sufficiently large, consider

$$\begin{aligned}
&\left|\Delta_a^{-\nu} f(t)\right| = \left| \sum_{s=a}^{t-\nu} h_{\nu-1}(t, \sigma(s)) f(s) \right| \\
&\leq \sum_{s=a}^{t-\nu} h_{\nu-1}(t, \sigma(s)) |f(s)| \\
&= \sum_{s=a}^{T-1} h_{\nu-1}(t, \sigma(s)) |f(s)| + \sum_{s=T}^{t-\nu} h_{\nu-1}(t, \sigma(s)) |f(s)| \\
&\leq \left(\sum_{s=a}^{T-1} \frac{|f(s)|}{\Gamma(\nu)} \right) (t-a)^{N-1} + \frac{A(t-a)^{N-1}}{\Gamma(\nu)} \int_T^{t-\nu+1} r^s \Delta s \\
&= \left(\sum_{s=a}^{T-1} \frac{|f(s)|}{\Gamma(\nu)} \right) (t-a)^{N-1} + \frac{A(t-a)^{N-1}}{\Gamma(\nu)} \left[\frac{r^s}{r-1} \right]_{s=T}^{s=t-\nu+1}
\end{aligned}$$

$$= \left(\sum_{s=a}^{T-1} \frac{|f(s)|}{\Gamma(\nu)}\right)(t-a)^{N-1} + \frac{A(t-a)^{N-1}}{(r-1)\Gamma(\nu)}[r^{t-\nu+1} - r^T]$$
$$\leq \left(\sum_{s=a}^{T-1} \frac{|f(s)|}{\Gamma(\nu)}\right)(t-a)^{N-1} + \frac{A(t-a)^{N-1}r^{1-\nu}}{(r-1)\Gamma(\nu)}r^t$$
$$= B(t-a)^{N-1} + C(t-a)^{N-1}r^t,$$

where B and C are constants. But for any fixed $\epsilon > 0$ we get by applying L'Hôpital's rule, that

$$\lim_{t\to\infty} \frac{B(t-a)^{N-1} + C(t-a)^{N-1}r^t}{(r+\epsilon)^t} = 0.$$

Therefore, $\Delta_a^{-\nu}f : \mathbb{N}_{a+\nu} \to \mathbb{R}$ is of exponential order $r+\epsilon$ for each fixed $\epsilon > 0$. By Remark 2.64, we also have $\Delta_a^{-\nu}f : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$ is of exponential order $r+\epsilon$ for each fixed $\epsilon > 0$.

Finally, we show $\Delta_a^{\nu}f : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$, where $N-1 < \nu \leq N$, is of exponential order $r+\epsilon$ for each fixed $\epsilon > 0$. Since

$$\Delta_a^{\nu}f(t) = \Delta^N \Delta_a^{-(N-\nu)}f(t)$$

and by the first part of the proof, $\Delta_a^{-(N-\nu)}f(t)$ is of exponential order $r+\epsilon$, we have by Exercise 2.2 that $\Delta_a^{\nu}f$ is of exponential order $r+\epsilon$. □

Corollary 2.66. *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ be given with $N-1 < \nu \leq N$. Then*

$$\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu}f\right\}(s), \quad \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s), \quad \textit{and} \quad \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu}f\right\}(s)$$

converge for all $|s+1| > r$.

Proof. Suppose f, r, and ν are as in the statement of this corollary and fix s_0 so that $|s_0+1| > r$. Then there is an $\epsilon_0 > 0$ so that $|s_0+1| > r+\epsilon_0$. Since we know by Theorem 2.65 that $\Delta_a^{-\nu}f : \mathbb{N}_{a+\nu} \to \mathbb{R}$, $\Delta_a^{-\nu}f : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$, and $\Delta_a^{\nu}f : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$ are of exponential order $r+\epsilon_0$, it follows from Theorem 2.4 that $\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu}f\right\}(s_0)$, $\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s_0)$, and $\mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu}f\right\}(s_0)$ converge. Since $|s_0+1| > r$ is arbitrary, we have that

$$\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu}f\right\}(s), \quad \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s), \quad \text{and} \quad \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu}f\right\}(s)$$

all converge for all $|s+1| > r$. □

2.8 The Laplace Transform of Fractional Operators

With Corollary 2.66 in hand to insure the correct domain of convergence for the Laplace transform of any fractional operator, we may now safely develop formulas for applying the Laplace transform to fractional operators. This is the content of the next theorem.

Theorem 2.67. *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$, and let $\nu > 0$ be given with $N-1 < \nu \leq N$. Then for $|s+1| > r$,*

$$\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) = \frac{(s+1)^\nu}{s^\nu} F_a(s), \tag{2.49}$$

and

$$\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) = \frac{(s+1)^{\nu-N}}{s^\nu} F_a(s). \tag{2.50}$$

Proof. Since $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$, $F_a(s)$ exists for $|s+1| > r$ and by Corollary 2.66 both $\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s)$ and $\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s)$ exist for $|s+1| > r$. First, we find a relationship between the left-hand sides of equations (2.49) and (2.50). Using (2.37), we get

$$\begin{aligned}
&\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) \\
&\qquad = \frac{1}{(s+1)^N}\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) + \sum_{k=0}^{N-1} \frac{\Delta_a^{-\nu} f(a+\nu-N+k)}{(s+1)^{k+1}} \\
&\qquad = \frac{1}{(s+1)^N}\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s),
\end{aligned} \tag{2.51}$$

using the fact that $\Delta_a^{-\nu} f(a+\nu-N+k) = 0$ for $0 \leq k \leq N-1$, by our convention on sums.

To see that (2.49) holds, note that

$$\begin{aligned}
&\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) \\
&= \sum_{k=0}^{\infty} \frac{\Delta_a^{-\nu} f(a+k+\nu)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} h_{\nu-1}(a+k+\nu, \sigma(r)) f(r) \\
&= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} f(r) h_{\nu-1}\left((a+k+1) - \sigma(r) + a, a-(\nu-1)\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(f * h_{\nu-1}(t, a-(\nu-1)))\,(a+1+k)}{(s+1)^{k+1}}, \qquad \text{by (2.42)}$$

$$= \mathcal{L}_{a+1}\,\{f * h_{\nu-1}(t, a-(\nu-1))\}\,(s)$$

$$= (s+1)\,\mathcal{L}_a\,\{f * h_{\nu-1}(t, a-(\nu-1))\}\,(s)\,, \qquad \text{using (2.38) and (2.42)}$$

$$= (s+1)\,F_a\,(s)\,\mathcal{L}_a\,\{h_{\nu-1}(t, a-(\nu-1))\}\,(s)\,, \qquad \text{by (2.43)}$$

$$= \frac{(s+1)^{\nu}}{s^{\nu}} F_a\,(s)\,, \text{ applying (2.38), since } r \geq 1$$

proving (2.49). Finally, using (2.51) and (2.49), we get

$$\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) = \frac{1}{(s+1)^N}\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s)$$
$$= \frac{(s+1)^{\nu-N}}{s^{\nu}} F_a\,(s)\,,$$

for $|s+1| > r$, proving (2.50). □

Example 2.68. Find $\mathcal{L}_{2+\pi+e}\{\Delta_{5+\pi}^{-e} f\}(s)$ given that

$$f(t) = (t-5)^{\underline{\pi}}, \quad t \in \mathbb{N}_{5+\pi}.$$

First note that

$$f(t) = \Gamma(\pi+1) h_{\pi}(t, 5), \quad t \in \mathbb{N}_{5+\pi},$$

and hence using (2.39) we have that

$$F_{5+\pi}(s) = \Gamma(\pi+1)\mathcal{L}_{5+\pi}\{h_{\pi}(t,5)\}(s) = \Gamma(\pi+1)\frac{(s+1)^{\pi}}{s^{\pi+1}}$$

for $|s+1| > 1$.

Then using (2.50) gives us

$$\mathcal{L}_{2+\pi+e}\left\{\Delta_{5+\pi}^{-e} f\right\}(s) = \mathcal{L}_{(5+\pi)+e-3}\{\Delta_{5+\pi}^{-e} f\}(s)$$
$$= \frac{(s+1)^{e-3}}{s^e}\left(\Gamma(\pi+1)\frac{(s+1)^{\pi}}{s^{\pi+1}}\right)$$
$$= \Gamma(\pi+1)\frac{(s+1)^{\pi+e-3}}{s^{\pi+e+1}}$$

for $|s+1| > 1$.

Remark 2.69. Note that when $\nu = N$ in (2.50), the correct well-known formula (2.46) for $N = 1$, is obtained. This holds true for the Laplace transform of a fractional difference as well, as the following theorem shows (Holm [123]).

Theorem 2.70. *Suppose* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$*, and let* $\nu > 0$ *be given with* $N - 1 < \nu \leq N$*. Then for* $|s + 1| > r$

$$\mathcal{L}_{a+N-\nu}\left\{\Delta_a^\nu f\right\}(s) = s^\nu (s+1)^{N-\nu} F_a(s) - \sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} f(a+N-\nu). \tag{2.52}$$

Proof. Let f, r, ν, and N be given as in the statement of the theorem. By Exercise 2.28 we have that (2.52) holds when $\nu = N$. Hence we assume $N-1 < \nu < N$. To see this, consider

$$\begin{aligned}
&\mathcal{L}_{a+N-\nu}\left\{\Delta_a^\nu f\right\}(s) \\
&= \mathcal{L}_{a+N-\nu}\left\{\Delta^N \Delta_a^{-(N-\nu)} f\right\}(s) \\
&= s^N \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{-(N-\nu)} f\right\}(s) \\
&\quad - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} \Delta_a^{-(N-\nu)} f\,(a+N-\nu) \\
&= s^N \frac{(s+1)^{N-\nu}}{s^{N-\nu}} F_a(s) \\
&\quad - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} \Delta_a^{-(N-\nu)} f\,(a+N-\nu) \\
&= s^\nu (s+1)^{N-\nu} F_a(s) - \sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} f\,(a+N-\nu).
\end{aligned}$$

This completes the proof. □

2.9 Power Rule and Composition Rule

In this section (see Atici and Eloe [34], Holm [123, 125]), we present a number of properties and formulas concerning fractional sum and difference operators are developed. These include composition rules and fractional power rules, whose proofs employ a variety of tools, none of which involves the Laplace transform. However, some of these results may also be proved using the Laplace transform. The following are two previously known results for which the Laplace transform provides a significantly shorter and cleaner proof than the original ones found in [34, 123].

Theorem 2.71 (Power Rule). *Let $\nu, \mu > 0$ be given. Then for $t \in \mathbb{N}_{a+\mu+\nu}$,*

$$\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)} (t-a)^{\underline{\mu+\nu}}$$

or equivalently

$$\Delta_{a+\mu}^{-\nu} h_\mu(t,a) = h_{\mu+\nu}(t,a).$$

Proof. Applying Remark 2.57 together with Lemma 2.63, we conclude that for each $\epsilon > 0$, $(t-a)^{\underline{\mu}}$ is of exponential order $1+\epsilon$ and therefore we have that $\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}}$ is of exponential order $1+2\epsilon$. Thus, after employing an argument similar to that given in Corollary 2.66, we conclude that both $\mathcal{L}_{a+\mu} \{(t-a)^{\underline{\mu}}\}$ and $\mathcal{L}_{a+\mu+\nu} \left\{\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}}\right\}$ converge for $|s+1| > 1$. Hence, for $|s+1| > 1$, we have

$$\begin{aligned}
&\mathcal{L}_{a+\mu+\nu} \left\{\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}}\right\} (s) \\
&\quad = \frac{(s+1)^\nu}{s^\nu} \mathcal{L}_{a+\mu} \{(t-a)^{\underline{\mu}}\} (s), \qquad \text{using (2.49)} \\
&\quad = \frac{(s+1)^\nu}{s^\nu} \Gamma(\mu+1) \mathcal{L}_{a+\mu} \left\{h_\mu (t,a)\right\} (s) \\
&\quad = \frac{(s+1)^\nu}{s^\nu} \Gamma(\mu+1) \frac{(s+1)^\mu}{s^{\mu+1}}, \qquad \text{applying (2.39)} \\
&\quad = \Gamma(\mu+1) \frac{(s+1)^{\mu+\nu}}{s^{\mu+\nu+1}} \\
&\quad = \Gamma(\mu+1) \mathcal{L}_{a+\mu+\nu} \left\{h_{\mu+\nu} (t,a)\right\} (s) \\
&\quad = \mathcal{L}_{a+\mu+\nu} \left\{ \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\underline{\mu+\nu}} \right\} (s).
\end{aligned}$$

Since the Laplace transform is injective, it follows that

$$\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)} (t-a)^{\underline{\mu+\nu}}, \text{ for } t \in \mathbb{N}_{a+\mu+\nu}.$$

This completes the proof. □

Theorem 2.72. *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$, and let $\nu, \mu > 0$ be given. Then*

$$\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu}f(t) = \Delta_a^{-\nu-\mu}f(t) = \Delta_{a+\nu}^{-\mu}\Delta_a^{-\nu}f(t), \textit{ for all } t \in \mathbb{N}_{a+\mu+\nu}.$$

Proof. Let f, r, ν, and μ be given as in the statement of the theorem. It follows from Corollary 2.66 that each of

$$\mathcal{L}_{a+\mu+\nu}\left\{\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu}f\right\}, \; \mathcal{L}_{a+\mu}\left\{\Delta_a^{-\mu}f\right\} \text{ and } \mathcal{L}_{a+(\nu+\mu)}\left\{\Delta_a^{-(\nu+\mu)}f\right\}$$

exists for $|s+1| > r$. Therefore, we may apply (2.49) multiple times to write for $|s+1| > r$,

$$\begin{aligned}
\mathcal{L}_{a+\mu+\nu}\left\{\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu}f\right\}(s) &= \frac{(s+1)^\nu}{s^\nu}\mathcal{L}_{a+\mu}\left\{\Delta_a^{-\mu}f\right\}(s)\\
&= \frac{(s+1)^\nu}{s^\nu}\frac{(s+1)^\mu}{s^\mu}\mathcal{L}_a\{f\}(s)\\
&= \frac{(s+1)^{\nu+\mu}}{s^{\nu+\mu}}\mathcal{L}_a\{f\}(s)\\
&= \mathcal{L}_{a+(\nu+\mu)}\left\{\Delta_a^{-(\nu+\mu)}f\right\}(s)\\
&= \mathcal{L}_{a+\mu+\nu}\left\{\Delta_a^{-\nu-\mu}f\right\}(s).
\end{aligned}$$

The result follows from symmetry and the fact that the operator $\mathcal{L}_{a+\mu+\nu}$ is injective (see Theorem 2.7). □

2.10 The Laplace Transform Method

The tools developed in the previous sections of this chapter enable us to solve a general fractional initial value problem using the Laplace transform. The initial value problem (2.53) below is identical to that studied and solved using the composition rules in Holm [123, 125]. In Theorem 2.76 below, we present only that part of the proof involving the Laplace transform method.

Theorem 2.73. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and $\nu > 0$ with $N-1 < \nu \leq N$. Then the unique solution of the IVP*

$$\begin{aligned}
\Delta_{a+\nu-N}^{\nu}y(t) &= f(t), \quad t \in \mathbb{N}_a\\
\Delta^i y(a+\nu-N) &= 0, \quad 0 \leq i \leq N-1,
\end{aligned}$$

is given by

$$y(t) = \Delta_a^{-\nu} f(t) = \int_a^t h_{\nu-1}(t, \sigma(k)) f(k) \Delta k,$$

for $t \in \mathbb{N}_{a+\nu-N}$.

Proof. Since

$$\Delta_{a+\nu-N}^{\nu} y(t) = f(t), \quad t \in \mathbb{N}_a,$$

we have that

$$\mathcal{L}_a\{\Delta_{a+\nu-N}^{\nu} y\}(s) = F_a(s)$$

for $|s+1| > r$. Assume for the moment that the Laplace transform (based at $a+\nu-N$) of the solution of the given IVP converges for $|s+1| > r$. It follows from (2.52) that

$$\begin{aligned}\mathcal{L}_a\{\Delta_{a+\nu-N}^{\nu} y\}(s) &= s^{\nu}(s+1)^{N-\nu} Y_{a+\nu-N}(s) - \sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} y(a) \\ &= s^{\nu}(s+1)^{N-\nu} Y_{a+\nu-N}(s),\end{aligned}$$

where we have used the initial conditions. It follows that

$$\begin{aligned}\mathcal{L}_{a+\nu-N}\{y\}(s) &= Y_{a+\nu-N}(s) \\ &= \frac{(s+1)^{\nu-N}}{s^{\nu}} F_a(s) \\ &= \mathcal{L}_{a+\nu-N}\{\Delta_a^{-\nu} f\}(s), \quad \text{by (2.50)}.\end{aligned}$$

It then follows from the uniqueness theorem for Laplace transforms, Theorem 2.7, that

$$y(t) = \Delta_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a+\nu-N}.$$

From this we now know that y is of exponential order r and hence the above arguments hold and the proof is complete. □

Using Theorem 2.73 and Theorem 2.43 it is easy to prove the following result.

Theorem 2.74. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$ *and* $\nu > 0$ *with* $N-1 < \nu \leq N$. *Then a general solution of the nonhomogeneous equation*

$$\Delta_{a+\nu-N}^{\nu} y(t) = f(t), \quad t \in \mathbb{N}_a$$

is given by

$$y(t) = \sum_{k=1}^{N} c_k (t-a)^{\underline{\nu-k}} + \Delta_a^{-\nu} f(t)$$

for $t \in \mathbb{N}_{a+\nu-N}$.

Example 2.75. Solve the IVP

$$\Delta_{a-\frac{1}{2}}^{\frac{1}{2}} y(t) = h_{\frac{1}{2}}(t,a), \quad t \in \mathbb{N}_a$$
$$y\left(a - \frac{1}{2}\right) = \frac{1}{2}.$$

Note this IVP is of the form of the IVP in Theorem 2.74, where

$$\nu = \frac{1}{2}, \quad N = 1, \quad a + N - \nu = a - \frac{1}{2}, \quad f(t) = h_{\frac{1}{2}}(t,a).$$

From Theorem 2.74 a general solution of the fractional equation $\Delta_{a-\frac{1}{2}}^{\frac{1}{2}} y(t) = h_{\frac{1}{2}}(t,a)$ is given by

$$\begin{aligned} y(t) &= c_1 (t-a)^{\underline{\nu-1}} + \Delta_a^{-\frac{1}{2}} h_{\frac{1}{2}}(t,a) \\ &= c_1 (t-a)^{\underline{-\frac{1}{2}}} + (t-a). \end{aligned}$$

Applying the initial condition we get $c_1 = \frac{1}{\sqrt{\pi}}$. Hence the solution of the given IVP in this example is given by

$$y(t) = \frac{1}{\sqrt{\pi}} (t-a)^{\underline{-\frac{1}{2}}} + (t-a)$$

for $t \in \mathbb{N}_{a-\frac{1}{2}}$.

The following theorem appears in Ahrendt et al. [3].

Theorem 2.76. *Suppose that* $f : \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$, *and let* $\nu > 0$ *be given with* $N - 1 < \nu \leq N$. *The unique solution to the fractional initial value problem*

$$\begin{cases} \Delta_{a+\nu-N}^{\nu} y(t) = f(t), & t \in \mathbb{N}_a \\ \Delta^i y(a+\nu-N) = A_i, & i \in \{0, 1, \cdots, N-1\}; A_i \in \mathbb{R} \end{cases} \tag{2.53}$$

is given by

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \ \textit{for } t \in \mathbb{N}_{a+\nu-N},$$

where

$$\alpha_i := \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{N-\nu}} \binom{i}{p} \binom{i-p}{k} A_p,$$

for $i \in \{0, 1, \cdots, N-1\}$.

Proof. Since f is of exponential order r, we know that $F_a(s) = \mathcal{L}_a\{f\}(s)$ exists for $|s+1| > r$. So, applying the Laplace transform to both sides of the difference equation in (2.53), we have for $|s+1| > r$

$$\mathcal{L}_a \left\{ \Delta_{a+\nu-N}^{\nu} y \right\}(s) = F_a(s).$$

Using (2.52), we get

$$s^{\nu} (s+1)^{N-\nu} Y_{a+\nu-N}(s) - \sum_{j=0}^{N-1} s^j \Delta_{a+\nu-N}^{\nu-j-1} y(a) = F_a(s).$$

This implies that

$$Y_{a+\nu-N}(s) = \frac{F_a(s)}{s^{\nu}(s+1)^{N-\nu}} + \sum_{j=0}^{N-1} \frac{\Delta_{a+\nu-N}^{\nu-j-1} y(a)}{s^{\nu-j}(s+1)^{N-\nu}}.$$

From (2.50), we have immediately that

$$\frac{F_a(s)}{s^{\nu}(s+1)^{N-\nu}} = \mathcal{L}_{a+\nu-N} \left\{ \Delta_a^{-\nu} f \right\}(s).$$

Considering next the terms in the summation, we have for each fixed $j \in \{0, \cdots, N-1\}$,

$$\begin{aligned}
\frac{1}{s^{\nu-j}(s+1)^{N-\nu}} &= \frac{1}{(s+1)^{N-j-1}} \frac{(s+1)^{\nu-j-1}}{s^{\nu-j}} \\
&= \frac{1}{(s+1)^{N-j-1}} \mathcal{L}_{a+\nu-j-1} \left\{ h_{\nu-j-1}(t,a) \right\}(s), \qquad \text{by (2.39)} \\
&= \mathcal{L}_{a+\nu-N} \left\{ h_{\nu-j-1}(t,a) \right\}(s)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=0}^{N-j-2}\frac{h_{\nu-j-1}\left(k+a+\nu-N,a\right)}{(s+1)^{k+1}}, \text{by (2.37)}\\
&=\mathcal{L}_{a+\nu-N}\left\{h_{\nu-j-1}\left(t,a\right)\right\}(s),
\end{aligned}
$$

since

$$
\begin{aligned}
h_{\nu-j-1}\left(k+a+\nu-N,a\right) &= \frac{(k+\nu-N)^{\underline{\nu-j-1}}}{\Gamma(\nu-j)}\\
&= \frac{\Gamma(k+\nu-N+1)}{\Gamma(k-(N-j-2))\,\Gamma(\nu-j)}\\
&= 0,
\end{aligned}
$$

for $k\in\{0,\cdots,N-j-2\}$. It follows that for $|s+1|>r$,

$$
\begin{aligned}
&\mathcal{L}_{a+\nu-N}\left\{y\right\}(s)\\
&=\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s)+\sum_{j=0}^{N-1}\Delta_{a+\nu-N}^{\nu-j-1}y(a)\mathcal{L}_{a+\nu-N}\left\{h_{\nu-j-1}\left(t,a\right)\right\}(s)\\
&=\mathcal{L}_{a+\nu-N}\left\{\sum_{j=0}^{N-1}\Delta_{a+\nu-N}^{\nu-j-1}y(a)h_{\nu-j-1}\left(t,a\right)+\Delta_a^{-\nu}f\right\}(s).
\end{aligned}
$$

Since the Laplace transform is injective, we conclude that for $t\in\mathbb{N}_{a+\nu-N}$,

$$
\begin{aligned}
y\left(t\right) &= \sum_{j=0}^{N-1}\Delta_{a+\nu-N}^{\nu-j-1}y(a)h_{\nu-j-1}\left(t,a\right)+\Delta_a^{-\nu}f\left(t\right)\\
&=\sum_{j=0}^{N-1}\frac{\Delta_{a+\nu-N}^{\nu-j-1}y(a)}{\Gamma(\nu-j)}\left(t-a\right)^{\underline{\nu-j-1}}+\Delta_a^{-\nu}f\left(t\right)\\
&=\sum_{i=0}^{N-1}\left(\frac{\Delta_{a+\nu-N}^{i+\nu-N}y(a)}{\Gamma(i+\nu-N+1)}\right)\left(t-a\right)^{\underline{i+\nu-N}}+\Delta_a^{-\nu}f(t).
\end{aligned}
$$

Moreover, Holm [125] showed that

$$
\frac{\Delta_{a+\nu-N}^{i+\nu-N}y(a)}{\Gamma(i+\nu-N+1)}=\sum_{p=0}^{i}\sum_{k=0}^{i-p}\frac{(-1)^k}{i!}(i-k)^{\underline{N-\nu}}\binom{i}{p}\binom{i-p}{k}\Delta^i y(a+\nu-N),
$$

for $i\in\{0,1,\cdots,N-1\}$, concluding the proof. □

Theorem 2.76 shows how we can solve the general IVP (2.53) using the discrete Laplace transform method. We offer a brief example.

Example 2.77. Consider the IVP given by

$$\begin{cases} \Delta^{\pi}_{\pi-4}y(t) = \pi^4 t^{\underline{2}}, \ t \in \mathbb{N}_0 \\ y(\pi-4) = 2, \ \Delta y(\pi-4) = 3, \ \Delta^2 y(\pi-4) = 5, \ \Delta^3 y(\pi-4) = 7. \end{cases} \tag{2.54}$$

Note that (2.54) is a specific case of (2.53) from Theorem 2.76, with

$$\begin{aligned} &a = 0, \quad \nu = \pi, \ \ N = 4, f(t) = \pi^4 t^{\underline{2}} \\ &A_0 = 2, A_1 = 3, A_2 = 5 \ A_3 = 7. \end{aligned}$$

After applying the discrete Laplace transform method as described in Theorem 2.76, we have

$$\begin{aligned} y(t) &= \sum_{i=0}^{3} \alpha_i t^{\underline{i+\pi-4}} + \Delta_0^{-\pi}\left(\pi^4 t^{\underline{2}}\right) \\ &= \sum_{i=0}^{3} \alpha_i t^{\underline{i+\pi-4}} + \Delta_2^{-\pi}\left(\pi^4 t^{\underline{2}}\right), \text{ since } t^{\underline{2}} = t(t-1), \\ &\approx 0.303 t^{\underline{\pi-4}} + 5.040 t^{\underline{\pi-3}} + 6.977 t^{\underline{\pi-2}} + 4.876 t^{\underline{\pi-1}} + 3.272 t^{\underline{\pi+2}}, \end{aligned}$$

where in this last step, we calculated

$$\alpha_i = \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{4-\pi}} \binom{i}{p} \binom{i-p}{k} A_p, \text{ for } i = 0, 1, 2, 3,$$

for the first four terms and applied the power rule (Theorem 2.71) on the last term.

2.11 Exercises

2.1. Show that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r = 1$ iff f is bounded on $\mathbb{N}_a$.

2.2. Prove that if $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$, then $\Delta^n f : \mathbb{N}_a \to \mathbb{R}$ is also of exponential order r for $n \in \mathbb{N}_0$.

2.3. Show that if $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 1$, then $h(t) := \int_a^t f(\tau)\Delta\tau$, $t \in \mathbb{N}_a$ is also of exponential order r.

2.4. Show that $h_0(t,a)$ is of exponential order 1 and for each $n \geq 0$, $h_n(t,a)$ is of exponential order $1+\epsilon$ for all $\epsilon > 0$.

2.5. Prove formula (i) in Theorem 2.8, that is

$$\mathcal{L}_a\{\cosh_p(t,a)\}(s) = \frac{s}{s^2 - p^2}$$

for $|s+1| > \max\{|1+p|, |1-p|\}$.

2.6. Prove formula (ii) in Theorem 2.9, that is

$$\mathcal{L}_a\{\sin_p(t,a)\}(s) = \frac{p}{s^2 + p^2}$$

for $|s+1| > \max\{|1+ip|, |1-ip|\}$.

2.7. Prove formula (ii) in Theorem 2.10, that is

$$\mathcal{L}_a\{e_\alpha(t,a)\sinh_{\frac{\beta}{1+\alpha}}(t,a)\}(s) = \frac{\beta}{(s-\alpha)^2 - \beta^2},$$

for $|s+1| > \max\{|1+\alpha+\beta|, |1+\alpha-\beta|\}$.

2.8. Prove Theorem 2.11.

2.9. For each of the following find $y(t)$ given that

(i) $Y_a(s) = \frac{14-s}{s^2+2s-8}$;
(ii) $Y_0(s) = \frac{2s^2}{s^2-\sqrt{2}s+1}$.

2.10. Use Laplace transforms to solve the following IVPs

(i)

$$y(t+2) - 7y(t+1) + 12y(t) = 0, \quad t \in \mathbb{N}_0;$$
$$y(0) = 2, \quad y(1) = 4.$$

(ii)

$$y(t+1) - 2y(t) = 3^t, \quad t \in \mathbb{N}_0;$$
$$y(0) = 5.$$

(iii)

$$y(t+2) - 6y(t+1) + 8y(t) = 20(4)^t, \quad t \in \mathbb{N}_0$$
$$y(0) = 0, \quad y(1) = 4.$$

2.11. Use Laplace transforms to solve the IVP

$$\begin{aligned} u(t+1) + v(t) &= 0 \\ -u(t) + v(t+1) &= 0 \\ u(0) = 1, \quad v(0) &= 0. \end{aligned}$$

2.12. Solve each of the following IVPs:

(i)

$$\begin{aligned} &\Delta y(t) - 2y(t) = \delta_4(t), \quad t \in \mathbb{N}_0; \\ &y(0) = 2, \end{aligned}$$

(ii)

$$\begin{aligned} &\Delta y(t) - 5y(t) = 3u_{60}(t), \quad t \in \mathbb{N}_0 \\ &y(0) = 4, \quad t \in \mathbb{N}_0. \end{aligned}$$

2.13. Solve the following summation equations using Laplace transforms:

(i) $y(t) = 2 + 4\sum_{r=0}^{t-1} 3^{t-r-1} y(r), \quad t \in \mathbb{N}_0;$
(ii) $y(t) = 3 \cdot 5^t - 4\sum_{r=0}^{t-1} 5^{t-r-1} y(r), \quad t \in \mathbb{N}_0;$
(iii) $y(t) = t + \sum_{r=0}^{t-1} y(r), \quad t \in \mathbb{N}_0;$
(iv) $y(t) = 2^{t-a} + \sum_{r=a}^{t-1} 4^{t-r-1} y(r), \quad t \in \mathbb{N}_a.$

2.14. Use Laplace transforms to solve each of the following:

(i) $y(t) = 3^t + \sum_{m=0}^{t-1} 3^{k-m-1} y_m, \quad t \in \mathbb{N}_0;$
(ii) $y(t) = 3^t + \sum_{m=0}^{t-1} 4^{k-m-1} y_m, \quad t \in \mathbb{N}_0.$

2.15. Show that

(i) $\Delta_a^{-\nu} f(a+\nu) = f(a);$
(ii) $\Delta_a^{-\nu} f(a+\nu+1) = \nu f(a) + f(a+1).$

2.16. Complete the proof of Theorem 2.27.

2.17. Work each of the following:

(i) Use the definition of the ν-th fractional sum (Definition 2.25) to find $\Delta_a^{-\frac{1}{3}} 1;$
(ii) Use the definition of the fractional difference (Definition 2.29) and part (2.32) to find $\Delta_a^{\frac{2}{3}} 1.$

2.18. Show that the following hold:

(i) $\Delta_{a+\mu}^{-\nu} (t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}} (t-a)^{\underline{\mu+\nu}}, \quad t \in \mathbb{N}_{a+\mu+\nu};$
(ii) $\Delta_{a+\mu}^{\nu} (t-a)^{\underline{\mu}} = \mu^{\underline{\nu}} (t-a)^{\underline{\mu-\nu}}, \quad t \in \mathbb{N}_{a+\mu+N-\nu}.$

2.19. Verify that (2.12) holds.

2.20. Show that $h_\mu(t, t-\mu+k) = 0$ for $k \in \mathbb{N}_1$, $\mu - k + 1 \notin \{0, -1, -2, \cdots\}$.

2.21. Evaluate each of the following using Theorem 2.38 and Theorem 2.40

(i) $\Delta_{\frac{3}{2}}^{-1}(t-1)^{\underline{\frac{1}{2}}}, \quad t \in \mathbb{N}_{\frac{5}{2}}$;
(ii) $\Delta_4^{-.7}(t-1.7)^{\underline{2.3}}, \quad t \in \mathbb{N}_{4.7}$;
(iii) $\Delta_{5.5}^{.5}(t-3)^{\underline{2.5}}, \quad t \in \mathbb{N}_5$;
(iv) $\Delta_3^{\frac{1}{2}} t(t-1)(t-2), \quad t \in \mathbb{N}_{\frac{5}{2}}$.

2.22. Prove that part (ii) of Theorem 2.42, follows from Theorem 2.40.

2.23. Prove (2.26).

2.24. Solve each of the following IVPs:

(i) $\Delta_{-0.3}^{2.7}x(t) = t^{\underline{2}}, \quad t \in \mathbb{N}_0$
$x(-0.3) = x(0.7) = x(1.7) = 0$;
(ii) $\Delta_{-0.4}^{1.6}x(t) = t^{\underline{4}}, \quad t \in \mathbb{N}_0$
$x(-0.4) = x(0.6) = 0$;
(iii) $\Delta_{-0.1}^{0.9}x(t) = t^{\underline{5}}, \quad t \in \mathbb{N}_0$
$x(-0.1) = 0$.

2.25. Use Theorems 2.54 and 2.58 to show that $\mathcal{L}_a\{h_1(t,a)\} = \frac{1}{s^2}$. Evaluate the convolution product $1 * 1$ and show directly (do not use the convolution theorem) that $\mathcal{L}_a\{1 * 1\}(s) = \mathcal{L}_a\{1\}(s)\ \mathcal{L}_a\{1\}(s)$.

2.26. Assume $p \in \mathcal{R}$ and $p \neq 0$. Using the definition of the convolution product (Definition 2.59), find

$$[h_1(t,a) * e_p(t,a)](t).$$

2.27. Assume $p, q \in \mathcal{R}$ and $p \neq q$. Using the definition of the convolution product (Definition 2.59), find

$$[e_p(t,a) * e_q(t,a)](t).$$

2.28. For N a positive integer, use the definition of the Laplace transform to prove that (2.4) holds (that is, (2.52) holds when $\nu = N$).

Springer

http://www.springer.com/978-3-319-25560-6

Discrete Fractional Calculus
Goodrich, C.; Peterson, A.C.
2015, XIII, 556 p., Hardcover
ISBN: 978-3-319-25560-6