Chapter 2
Introduction to the Theory of Dirichlet Forms

A Dirichlet form is a generalization of the energy form $f \mapsto \int_\Omega |\nabla f|^2 d\lambda$ introduced in the 1840s especially by William Thomson (Lord Kelvin) (cf. Temple [351] Chap. 15) in order to solve by minimization the problem without second member $\Delta f = 0$ in the open set Ω (Dirichlet principle). Riemann adopted the expression Dirichlet form [314]. The generalization now known as a Dirichlet form keeps the notion in the same relationship with the semigroup as the energy form holds with the heat semigroup.

On non-local non-symmetric general Dirichlet forms see Dellacherie–Meyer [122] p. 128 *and seq*, and Ma–Röckner [253]. On symmetric Dirichlet forms, local or non-local, on a locally compact space the reference is Fukushima [168, 170].

We will need only local symmetric Dirichlet forms with carré du champ but possibly in infinite dimension. Our reference is Bouleau–Hirsch [79]. To be local means that the form satisfies some algebraic properties which, when the space is topological, amount to saying that the associated Markov process has continuous sample paths. To possess a carré du champ operator is a regularity property which expresses that the domain of the generator of the semigroup contains a dense algebra. Local Dirichlet forms with carré du champ—as the historical energy form—satisfy a functional calculus useful in many questions (see Bouleau [63–71], Chorro [101], Scotti [331–334], Scotti–LyVath [335], Regis [310], Bavouzet–Messaoud [39], Bavouzet et al. [38] for applications to error theory).

Since the theory of Dirichlet forms is based on operator theory, we start by presenting without proofs some background from functional analysis.

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2.1 Unbounded Operators, Semigroups and Closed Forms

Let H be a real separable Hilbert space equipped with its scalar product $\langle \cdot, \cdot \rangle_H$ and norm

$$\forall u \in H, \ \| u \|_H = \langle u, u \rangle_H^{1/2}.$$

Let us mention that most of the notions introduced in this chapter are valid on a Banach space but since Dirichlet forms are defined on L^2-spaces we restrict ourselves to Hilbert spaces.

2.1.1 Self-adjoint Operators

Definition 2.1 A pair $(\mathcal{D}(a), a)$ is called a (linear) operator on H if $\mathcal{D}(a) \subset H$ is a dense linear space and $a : \mathcal{D}(a) \to H$ is a linear map. We shall often denote it simply a and $\mathcal{D}(a)$ is called the domain of a.

An operator $(\mathcal{D}(a), a)$ is closed if its graph

$$G(a) = \{(x, a[x]); \ x \in \mathcal{D}(a)\}$$

closed with respect to the graph-norm $\|(x, y)\|_{G(a)} = \|x\|_H + \|y\|_H$.

In other words, if $(x_n)_n$ is a sequence in $\mathcal{D}(a)$ which converges to x in H and such that $(a(x_n))_n$ is a Cauchy sequence in H then x belongs to $\mathcal{D}(a)$ and $a(x) = \lim_{n \to +\infty} a(x_n)$.

We now turn to the notion of symmetric operators:

Definition 2.2 An operator a is symmetric if

$$\forall x, y \in \mathcal{D}(a), \ \langle a[x], y \rangle_H = \langle x, a[y] \rangle_H.$$

Moreover, a is said to be self-adjoint if it is equal to its adjoint operator ($a = a^*$ with standard notation) which implies that a is symmetric and closed.

In the sequel, we shall consider *non-negative* (resp. *non-positive*) operators where in a natural way we say that an operator $(\mathcal{D}(a), a)$ is *non-negative* (resp. *non-positive*): if

$$\forall x \in H, \ \langle a[x], x \rangle_H \ \geqslant \ 0 \ \text{ (resp. } \leqslant 0).$$

2.1.2 Semigroup and Resolvent Associated to a Non-negative Self-adjoint Operator

The semigroup associated to an operator shall play an important role in our construction.

Definition 2.3 A symmetric strongly continuous contraction semigroup on H is a family $(p_t)_{t\,\geqslant\,0}$ of everywhere defined symmetric operators such that

1. $p_0 = I$ where I denotes the identity map from H into H,
2. $\forall s, t > 0,\ p_s \circ p_t = p_{t+s}$,
3. $\forall x \in H, \lim_{s\to 0} p_s x = x$,
4. $\forall x \in H, \|p_t x\|_H \leqslant \|x\|_H$.

Thus the strong continuity property is included in the definition. In the next proposition, we give the relationship between semigroups and self-adjoint operators.

Proposition 2.4 *(1) Let $(p_t)_{t\,\geqslant\,0}$ be a symmetric strongly continuous contraction semigroup on H. Define*

$$\mathbf{c} = \{x \in H : \lim_{t\downarrow 0} \uparrow \frac{1}{t}\langle x - p_t x, x\rangle_H < +\infty\}$$

and let us set

$$\forall x \in \mathbf{c} \quad a[x] = \lim_{t\downarrow 0} \frac{p_t x - x}{t}.$$

Then a is a non-positive self-adjoint operator with $\mathcal{D}(a) = \mathbf{c}$.

It is called the (infinitesimal) generator *of* $(p_t)_{t\,\geqslant\,0}$.

(2) Conversely, if $(\mathcal{D}(a), a)$ is a non-positive self-adjoint operator then it is the generator of the symmetric strongly continuous contraction semigroup defined by:

$$\forall t \geqslant 0,\ p_t = e^{ta}.$$

Remark 2.5 The rigorous definition of e^{ta} involves the notion of resolution of identity that we do not recall here. For more details, see for example [170], Chap. 1. □

The semigroup associated to a self-adjoint operator has remarkable properties, let us recall some of them.

Proposition 2.6 *Let $(p_t)_{t\,\geqslant\,0}$ be a symmetric strongly continuous contraction semigroup with generator a. Then*

1. $\forall x \in H, \forall t > 0$, $p_t x \in \mathcal{D}(a)$ *and* $p_t x - x = \int_0^t p_s a[x]\, ds$.
2. $\forall x \in \mathcal{D}(a)\,, \forall t > 0$, $a[p_t x] = p_t a[x]$.

3. *$\forall x \in H$, the map $t \to p_t x$ is differentiable on $]0, +\infty[$ with continuous derivative and*

$$\forall t > 0, \ \frac{d}{ds} p_s x|_{s=t} = a[p_t x] = p_t a[x].$$

Moreover if $x \in \mathcal{D}(a)$ the map is differentiable at $t = 0$.

For some arguments, we will need the *resolvent family*, $(R_\lambda)_{\lambda>0}$ associated to a given symmetric strongly continuous contraction semigroup (p_t). Let us recall the definition:

$$\forall \lambda > 0, \ \ R_\lambda = \int_0^\infty e^{-\lambda t} p_t \, dt.$$

Then, it is a *strongly continuous resolvent* on H in the sense that for any $u \in H$, $\lambda R_\lambda u$ tends to u strongly in H when $\lambda \to \infty$. Moreover, we have the spectral representation

$$\forall \lambda > 0, \ R_\lambda = (\lambda I - a)^{-1}$$

which makes sense since for any $\lambda > 0$ the operator $(\lambda I - a)$ is invertible on H.

2.1.3 Closed Forms

Definition 2.7 A symmetric closed form is a quadratic form e defined on a dense subspace $\mathbf{d} \subset H$ which is non-negative ($\forall x \in \mathbf{d}, \ e(x) \geqslant 0$) and such that $\mathbf{d}$ equipped with the norm

$$\|x\|_{\mathbf{d}} = \left(\|x\|_H^2 + e(x)\right)^{1/2}$$

is a Hilbert space. We denote by $e(x, y)$ the associated symmetric bilinear form:

$$\forall x, y \in \mathbf{d}, \ e(x, y) = \frac{1}{4}(e(x + y) - e(x - y)).$$

There is a one to one correspondence between the set of symmetric closed forms and the set of self-adjoint non-positive operators given by the following proposition:

Proposition 2.8 *(1) Let $(\mathbf{d}, e)$ a symmetric closed form then its generator a is defined by*

$$\mathcal{D}(a) = \{x \in H; \ \exists y \in H \ \forall z \in H \ e(x, y) = -\langle z, y\rangle_H\}$$

and

$$a[x] = z.$$

$(\mathcal{D}(a), a)$ is a non-positive self-adjoint operator.

(2) Conversely, if $(\mathcal{D}(a), a)$ *is a non-positive self-adjoint operator then it is the generator of the symmetric closed form* $(\mathbf{d}, e)$ *defined by*

$$\mathbf{d} = \mathcal{D}(\sqrt{-a}) \text{ and } \forall x, y \in \mathbf{d},\ e(x, y) = \langle \sqrt{-a}[x], \sqrt{-a}[y] \rangle_H.$$

Remark 2.9 Here again, the rigorous definition of the square root of a non-negative operator involves the notion of resolution of identity (see [170] Chap. 1). □

To construct a Dirichlet form which is a symmetric closed form having specific properties as we shall see in the next section, usually we consider a *pre-Dirichlet form* i.e. a symmetric non-negative bilinear form. The last step consists in proving that it admits an *extension* which is closed. This leads to the following definition:

Definition 2.10 A (not necessarily closed) non-negative symmetric bilinear form on H, e, defined on a dense subspace $\mathcal{D}(e) \subset H$ is said to be closable if there exists a symmetric closed form $(\tilde{\mathbf{d}}, \tilde{e})$ extending $(\mathbf{d}, e)$ in the sense that

$$\mathbf{d} \subset \tilde{\mathbf{d}} \text{ and } \forall x \in \mathbf{d},\ e(x) = \tilde{e}(x).$$

Remark 2.11 Let us remark that the form $(\tilde{\mathbf{d}}, \tilde{e})$ extending $(\mathbf{d}, e)$ is not unique in general. □

The next propositions give a criterium ensuring closability of a given quadratic form and an important application of this criterium: the Friedrichs extension of any symmetric non-negative operator (see [79] Proposition 1.3.2 and Example 1.3.4).

Proposition 2.12 *A non-negative symmetric bilinear form e defined on a dense subset* $\mathbf{d} \subset H$ *is closable if and only if, whenever a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbf{d}$ *satisfies*

$$\lim_{n\to+\infty} x_n = 0 \text{ and } \lim_{n,m\to+\infty} e(x_n - x_m) = 0$$

then $\lim_{n\to+\infty} e(x_n) = 0$. *Under this assumption,* $(\mathbf{d}, e)$ *admits a* smallest closed extension $(\tilde{\mathbf{d}}, \tilde{e})$ *called the* closure of $(\mathbf{d}, e)$ *and* $\mathbf{d}(e)$ *is dense in* $\tilde{\mathbf{d}}$ *endowed with the Hilbertian structure.*

Corollary 2.13 *Let* $(\mathcal{D}(l), l)$ *be a symmetric non-negative operator, it is naturally associated to the symmetric bilinear form* $(\mathcal{D}(l), e)$ *defined by*

$$\forall x, y \in \mathcal{D}(l),\ e(x, y) = \langle l[x], y \rangle_H.$$

Then $(\mathcal{D}(l), e)$ *is closable so it admits a smallest closed extension* $(\tilde{\mathbf{d}}, \tilde{e})$ *and if we denote by a the generator of* $(\tilde{\mathbf{d}}, \tilde{e})$, $-a$ *is a positive self-adjoint operator which is an extension of* $(\mathcal{D}(l), l)$ *called the* Friedrichs extension *of* l.

2.2 Dirichlet Forms

For this section, we mainly refer to Bouleau–Hirsch [79].

Let $(X, \mathcal{X}, \nu)$ be a measure space equipped with a σ-finite measure ν.

2.2.1 Definition and Fundamental Relationships

Definition 2.14 A Dirichlet form $(\mathbf{d}, e)$ is a closed form on $L^2(\nu)$ such that

$$f \in \mathbf{d} \Longrightarrow f \wedge 1 \in \mathbf{d} \text{ and } e(f \wedge 1) \leqslant e(f).$$

One of the main properties of Dirichlet forms is that contractions operate in the following sense:

Proposition 2.15 *Let $(\mathbf{d}, e)$ be a Dirichlet form on $L^2(\nu)$, $l \in \mathbb{N}^*$, $f = (f_1, \cdots, f_l) \in \mathbf{d}^l$ and $F : \mathbb{R}^l \to \mathbb{R}$ a normal contraction i.e.*

$$F(0) = 0 \textit{ and } |F(x) - F(y)| \leqslant |x - y|.$$

Then $F(f) \in \mathbf{d}$ and

$$(e(F(f)))^{1/2} \leqslant \sum_{i=1}^{l} (e(f_i))^{1/2}.$$

Proposition 2.16 *Let $(\mathbf{d}, e)$, a Dirichlet form, its generator a is a* Dirichlet operator *i.e. a is a non-positive self-adjoint operator with domain $\mathcal{D}(a)$ which satisfies*

$$\forall f \in \mathcal{D}(a),\ \left(a[f], (f-1)^+\right)_{L^2} \leqslant 0.$$

There is a one to one correspondence between Dirichlet forms and Dirichlet operators.

Proposition 2.17 *Let $(p_t)_{t \geqslant 0}$ the strongly continuous contraction semigroup generated by a Dirichlet operator a, $p_t = e^{ta}$, then $(p_t)_{t \geqslant 0}$ is a symmetric strongly continuous contraction semigroup on $L^2(X, \nu)$ and is sub-Markovian i.e.*

$$\forall t \geqslant 0\, \forall f \in L^2(X, \nu),\ 0 \leqslant f \leqslant 1 \Rightarrow 0 \leqslant p_t f \leqslant 1.$$

Moreover, there is a one-to-one correspondence between Dirichlet operators and sub-Markovian strongly continuous contraction semigroups.

Let us summarize these correspondences by the following diagram:

$$\begin{tikzcd}
(\mathbf{d}, e) \arrow[r, shift left] \arrow[d, shift left] & (\mathcal{D}(a), a) \arrow[l, shift left] \arrow[dl, shift left] \\
(p_t)_{t \geqslant 0} \arrow[u, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

We shall only consider *local* Dirichlet forms admitting a *carré du champ* and even a *gradient*, so we recall these notions:

Proposition 2.18 *Let* $(\mathbf{d}, e)$ *be a Dirichlet form on* $L^2(\nu)$*. The following propositions are equivalent:*

1. $\forall F, G \in C_0^\infty(\mathbb{R})$,

$$supp\ F \bigcap supp\ G = \emptyset \Rightarrow e(F(f) - F(0), G(f) - G(0)) = 0.$$

2. $\forall f \in \mathbf{d},\ \mathcal{E}(|f+1| - 1) = \mathcal{E}(f)$.

In this case, we say that $(\mathbf{d}, e)$ *is local.*

Moreover, if ν *is finite and* $1 \in \mathbf{d}$*, then locality is equivalent to any of the following*

1. $e(1) = 0$ *and* $\forall f \in \mathbf{d}$*,* $e(|f|) = e(f)$.
2. $e(1) = 0$ *and* $\forall f, g \in \mathbf{d}$*,* $fg = 0 \Rightarrow e(f, g) = 0$.

2.2.2 *Carré du Champ, Gradient and the (EID) Property*

Definition 2.19 Let $(\mathbf{d}, e)$ be a local Dirichlet form. We say that it admits *a carré du champ*, if there exists a unique positive, symmetric and continuous bilinear form, γ, from $\mathbf{d} \times \mathbf{d}$ into $L^1(\nu)$ such that

$$\forall f, g \in \mathbf{d},\ e(f, g) = \frac{1}{2} \int \gamma[f, g]\, d\nu.$$

From now on, we consider a given Dirichlet form $(\mathbf{d}, e)$ which is local and possesses a carré du champ operator γ.

We suppose

$$\exists k_n \in \mathbf{d} \text{ and } A_n \in \mathcal{X}, A_n \uparrow X, \text{ and } k_n = 1 \text{ on } A_n, \text{ with } \gamma[k_n] = 0 \text{ on } A_n. \quad (2.1)$$

Using the terminology which will be introduced below in Sects. 4.6 and 5.3.1 (see also Bouleau–Hirsch [79] pp. 44–45) this means $1 \in \mathbf{d}_{loc}$ and $\gamma[1] = 0$. This hypothesis simplifies many technicalities.

Then we have the functional calculus of class $\mathcal{C}^1 \cap Lip$:

$$\forall f, g \in \mathbf{d}^n,\ \forall F, G \text{ of class } \mathcal{C}^1 \cap Lip \text{ on } \mathbb{R}^n$$

$$\gamma[F(f), G(g)] = \sum_{ij} \partial_i F(f) \partial_j G(g) \gamma[f_i, g_j]. \tag{2.2}$$

We will always write $\gamma[f]$ for $\gamma[f, f]$ and $e(f)$ for $e(f, f)$. (Fitzsimmons [159] showed that functions which operate are necessarily locally Lipschitz, actually Lipschitz as soon as $\mathbf{d}$ contains non-bounded functions.)

If $f = (f_i)_{1 \leqslant i \leqslant d}$ and $g = (g_j)_{1 \leqslant j \leqslant d}$ are $\mathbb{R}^d$-valued functions belonging to $\mathbf{d}^d$ ($d \in \mathbb{N}^*$) we denote by $\gamma[f, g^t]$ the $d \times d$ matrix:

$$\gamma[f, g^t] = (\gamma[f_i, g_j])_{1 \leqslant i,j \leqslant d}.$$

The space $\mathbf{d}$ equipped with the norm $(\|.\|^2_{L^2(\nu)} + e(.,.))^{\frac{1}{2}}$ is a Hilbert space that we assume to be separable. Then, there exists a linear operator called *gradient*, denoted D, which has the properties of a derivation (see Bouleau–Hirsch [79] ex. 5.9 p. 242).

More precisely, there exists a separable Hilbert space H and a continuous linear map D from $\mathbf{d}$ into $L^2(X, \nu; H)$ such that

- $\forall u \in \mathbf{d}$, $\|Du\|^2_H = \gamma[u]$.
- If $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz then $\forall u \in \mathbf{d},\ D(F \circ u) = (F' \circ u)Du$, where F' is the Lebesgue derivative of F almost everywhere defined.

Moreover, it is possible to substitute the functional calculus (2.2) for the bilinear operator γ with a functional calculus for the gradient operator D:

- If F is $\mathcal{C}^1$ and Lipschitz from $\mathbb{R}^d$ into $\mathbb{R}$ then

$$\forall u = (u_1, \cdots, u_d) \in \mathbf{d}^d,\ D(F \circ u) = \sum_{i=1}^{d} (\partial_i F \circ u) Du_i. \tag{2.3}$$

Remark 2.20 Since all the separable infinite dimensional Hilbert spaces are in one-to-one correspondence, the choice of the Hilbert space H plays no role. We'll take advantage of this remark by choosing an appropriate space H on what we will call the *bottom space* (see Sect. 4.3). □

Remark 2.21 In formula (2.3) if F does not vanish at zero, the function $F \circ u$ is not in $L^2(\nu)$ if ν has an infinite mass. Nevertheless we still write (2.3) in that case thanks to the convention $D(1) = 0$ (by the hypothesis $1 \in \mathbf{d}_{loc}$ and $\gamma[1] = 0$). Equivalently, this means that we adopt the convention:

$$D(F \circ u) = D(F \circ u - F(0)).$$ □

Starting from D regarded as a linear continuous operator from $\mathbf{d}$ into $L^2(\nu; H)$, we define a *divergence* operator by duality (cf. Malliavin [251], Bouleau–Hirsch [79], Nualart [273], Bouleau [63])

$$dom\,\delta = \{v \in L^2(\nu; H) : \exists c > 0 \quad |\nu(\langle Du, v\rangle_H)| \leqslant c\|u\|_{L^2(\nu)} \quad \forall u \in \mathbf{d}\}$$

and for $u \in dom\,\delta$

$$\langle \delta v, u\rangle_{L^2(\nu)} = \langle v, Du\rangle_{L^2(\nu,H)}\ \forall u \in \mathbf{d}.$$

This differential calculus gives rise to an integration by parts formula as in classical finite dimensional case or the Malliavin calculus. We have the equality for $u \in \mathbf{d}$, $v \in dom\ \delta$ and for φ Lipschitz

$$\int \varphi'(u)\langle Du, v\rangle_H d\nu = \int \varphi(u)\delta v d\nu. \tag{2.4}$$

See for instance [63] Chaps. V–VIII and [66] for applications of such formulas.

But Dirichlet forms do possess particular features allowing to show the existence of density without using integration by parts formulae (cf. Bouleau–Hirsch [78, 79], Denis [126]).

To understand precisely what this property is, let us explain it with a very basic example:

Example: Take $X = [0, 1]$, $\mathcal{X} = \mathcal{B}([0, 1])$, $\nu = dx$ the Lebesgue measure and for $\mathbf{d}$ the usual Sobolev space $H^1([0, 1])$

$$\mathbf{d} = \{f \in L^2([0, 1]);\ f' \in L^2([0, 1])\}$$

and consider the energy form:

$$\forall f \in \mathbf{d},\ e(f) = \int_0^1 |f'(x)|^2\, dx.$$

Then it is clear that $(\mathbf{d}, e)$ is a local Dirichlet form admitting a carré du champ: $\gamma[f] = |f'(x)|^2$ and for gradient D taking values in $\mathbb{R}$:

$$\forall f \in \mathbf{d},\ Df(x) = f'(x).$$

It is easy to verify that the domain of the divergence operator is $H_0^1([0, 1])$ the subset of functions in $\mathbf{d}$ vanishing at 0 and 1 and that

$$\forall v \in H_0^1([0, 1]),\ \delta v = -v'.$$

Indeed we have thanks to the integration by parts formula:

$$\forall u \in \mathbf{d}\, \forall v \in H_0^1([0,1]),\ \int_0^1 u'(x)v(x)\,dx = -\int_0^1 u(x)v'(x)dx.$$

Remark 2.22 Other examples are given in the context of *error structures* which are particular cases of Dirichlet structures, namely local Dirichlet forms admitting a carré du champ on a probability space, see Appendix A.2. □

To understand the kind of generalization we have in mind, consider each function in $L^2([0,1])$ as a random variable defined on the "probability space" $([0,1], \mathcal{B}([0,1]), P)$ with $P = dx$. We have the following criterion of absolute continuity of the "law" of any random variable F:

Proposition 2.23 *Let $F \in \mathbf{d}$ such that $\frac{1}{F'}$ belongs to dom $\delta = H_0^1([0,1])$. Then the law of F is absolutely continuous w.r.t. the Lebesgue measure. Moreover, its density is bounded, continuous and given by the following formula:*

$$\forall x \in \mathbb{R},\ p(u) = E[\mathbf{1}_{\{F>u\}}\delta\left(\frac{1}{F'}\right)] = \int_0^1 \mathbf{1}_{\{F(y)>u\}}\frac{F''(y)}{|F'(y)|^2}\,dy.$$

Proof We give the proof inspired from a more general case (see [273] p. 87 or [60]). Let ψ be a continuous function on $\mathbb{R}$ with compact support and define

$$\varphi(x) = \int_{-\infty}^{x} \psi(z)\,dz.$$

Then it is obvious that $\varphi(F)$ belongs to $\mathbf{d}$ and that $D\varphi(F) = \psi(F)F'$ so that

$$E[\psi(F)] = \int_0^1 \frac{D\varphi(F)(x)}{F'(x)}\,dx = \int_0^1 \varphi(F(x))\frac{F''(x)}{|F'(x)|^2}\,dx.$$

Take $a \leqslant b$, by approximating $\mathbf{1}_{[a,b]}$ by a sequence of smooth functions, it is clear that the previous equality holds for $\psi = \mathbf{1}_{[a,b]}$ and this yields by Fubini's theorem:

$$\begin{aligned} P(F \in [a,b]) &= E[\int_{-\infty}^{F} \mathbf{1}_{[a,b]}(z)\,dz\frac{F''}{|F'|^2}] \\ &= \int_a^b E[\mathbf{1}_{F>u}\frac{F''}{|F'|^2}]\,du. \end{aligned}$$

The result now easily follows. □

Remark 2.24 Let us remark that the conditions imply the invertibility of $\gamma[F]$. □

One of the main achievement of the theory of Dirichlet forms is the generalization of this property to very general cases encompassing infinite dimensional spaces as the Wiener space as we'll see in the next subsection. This is *energy image density property* or (EID) for local Dirichlet forms.

For any integer $d \geqslant 1$, let $\mathcal{B}(\mathbb{R}^d)$ be the Borelian σ-field of $\mathbb{R}^d$ and λ_d be the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For measurable f, $f_*\nu$ denotes the image of the measure ν by f.

Definition 2.25 The Dirichlet structure $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ is said to satisfy (EID) if for all d and all function U with values in $\mathbb{R}^d$ whose components are in the domain of the form

$$U_*[(\det\gamma[U, U^t]) \cdot \nu] \ll \lambda_d$$

where det denotes the determinant.

This property is true for any local Dirichlet structure with carré du champ when $d = 1$ (cf. Bouleau [60] Theorem 5 and Corollary 6). It has been conjectured in 1986 (Bouleau–Hirsch [78] p. 251) that (EID) be true for any local Dirichlet structure with carré du champ. This has been proved for the Wiener space equipped with the Ornstein–Uhlenbeck form and for some other Dirichlet structures by Bouleau–Hirsch (cf. [79] Chap. II Sect. 5 and Chap. V Example 2.2.4) but since this conjecture has presently neither been proved nor refuted in full generality, (EID) has to be established in each particular framework.

For the Poisson space, it has been proved by Agnès Coquio [105] when the intensity measure is the Lebesgue measure on an open set and the form is associated with the Laplacian operator and we have obtained it under rather general assumptions ([76] Sect. 2 Theorem 2 and Sect. 4) based on a criterion of Albeverio and Röckner [8] and an argument of Song [346].

(EID) on the Wiener space is now a very frequently used tool (cf. recently Sardanyons [325], Nualart–Sardanyons [275], Nualart [274], Chighouby [99], Cass–Friz–Victoir [93], N'Zi–Ouknine [278], Kusuoka [221]) and extending its use to the case of processes with jumps is among the aims of the present monograph.

2.3 The Ornstein–Uhlenbeck Structure on the Wiener Space

Since there are some analogies but also some differences between the Dirichlet structure on the Poisson space we shall construct and the Ornstein–Uhlenbeck structure on the Wiener space, we briefly recall it.

Let $d \in \mathbb{N}^*$ and $\Omega = C_0(\mathbb{R}^+; \mathbb{R}^d)$ the canonical Wiener space i.e. the set of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}^+$ vanishing at 0, m the Wiener measure, $\mathcal{F}$ the Borelian σ-field completed with m-null sets and $(B_t)_{t \geqslant 0}$ the canonical coordinates process which is a d-dimensional Brownian motion under m.

We present here a construction based on the chaos, see Sect. A.6 for another construction of the Ornstein–Uhlenbeck structure based on the fact that the Wiener space may be viewed as the infinite product of finite dimensional Gaussian spaces.

For all $n \in \mathbb{N}^*$ we denote by $\mathbb{R}_{d,n}$ the n-times tensor product $\mathbb{R}_{d,n} = \underbrace{\mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d}_{n\ times}$ and by $\hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})$ the set of symmetric functions in $L^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})$. Let us remark that in a natural way, $\hat{L}^2(\mathbb{R}_+^{n+1}; \mathbb{R}_{d,n+1})$ is identified with a subspace of $L^2(\mathbb{R}_+^n; \mathbb{R}_{d,n} \otimes L^2(\mathbb{R}_+; \mathbb{R}^d))$.

Moreover we put

$$\Delta_n = \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n;\ 0 < t_1 < \cdots < t_n\}$$

and for $f_n \in \hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})$ we define

$$I_n(f_n) = n! \int_{\Delta_n} f_n(s) d^{(n)} dB_s,$$

where $\int_{\Delta_n} f_n(s) d^{(n)} dB_s$, denotes the n-iterated Itô integral of f_n. We have:

$$E[|I_n(f_n)|^2] = n! \| f_n \|^2_{L^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})}.$$

The set

$$\Pi_n = \{ \int_{\Delta_n} f_n(s) d^{(n)} dB_s;\ f_n \in \hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n}) \}$$

is called the *nth Wiener chaos*. With the convention $\Pi_0 = \mathbb{R}$ we have

Proposition 2.26 *The space $L^2(m)$ is the Hilbert sum of the Wiener chaos:*

$$L^2(m) = \bigotimes_{n=0}^{+\infty} \Pi_n.$$

As a consequence if $F \in L^2(m)$ there exists a sequence $(f_n)_{n \geqslant 1}$ of functions such that for all $n \in \mathbb{N}^*$, f_n belongs to $\hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})$ and $F = E[F] + \sum_{n=1}^{+\infty} I_n(f_n)$. Moreover

$$E[F^2] = (E[F])^2 + \sum_{n=1}^{+\infty} n! \| f_n \|^2_{\hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})}.$$

This chaos decomposition permits us to easily define the Ornstein–Uhlenbeck structure on the Wiener space.

Definition 2.27 We denote by $\mathbb{D}$ the subspace of elements $F = E[F] + \sum_{n=1}^{+\infty} I_n(f_n)$ in $L^2(m)$ such that

$$\sum_{n=1}^{+\infty} n n! \| f_n \|^2_{\hat{L}^2(\mathbb{R}_+^n; \mathbb{R}_{d,n})} < +\infty.$$

If $F = E[F] + \sum_{n=1}^{+\infty} I_n(f_n)$ belongs to $\mathbb{D}$ we set $\mathcal{E}(F) = \dfrac{1}{2}\sum_{n=1}^{+\infty} nn!\|f_n\|^2_{\hat{L}^2(\mathbb{R}_+^n;\mathbb{R}_{d,n})}$.

It is easy to check that $(\mathbb{D}, \mathcal{E})$ is a quadratic closed form. Let A be its generator, called the *Ornstein–Uhlenbeck operator* then $F = E[F] + \sum_{n=1}^{+\infty} I_n(f_n)$ belongs to $\mathcal{D}(A)$ if and only if

$$\sum_{n=1}^{+\infty} n^2 n!\|f_n\|^2_{\hat{L}^2(\mathbb{R}_+^n;\mathbb{R}_{d,n})} < +\infty,$$

and then

$$A[F] = -\frac{1}{2}\sum_{n=1}^{+\infty} nI_n(f_n).$$

The associated Ornstein–Uhlenbeck semigroup $(P_t)_{t\geqslant 0}$ follows immediately: if $F = E[F] + \sum_{n=1}^{+\infty} I_n(f_n)$ is in $L^2(m)$, then

$$\forall t \geqslant 0,\ P_t(F) = E[F] + \sum_{n=1}^{+\infty} e^{-\frac{nt}{2}} I_n(f_n).$$

Finally, we introduce one gradient, D, which takes values in $H = L^2(\mathbb{R}_+;\mathbb{R}^d)$ defined by

$$\forall F = E[F] + \sum_{1}^{+\infty} I_n(f_n),\ D_t F = \sum_{1}^{+\infty} I_n(f_n(\cdot, t)),$$

with the convention $I_0(f_1(t)) = f_1(t)$.

Let us note that the expression $I_n(f(\cdot, t))$ means that we fix the last variable t and take the $n-1$-iterated Itô integral w.r.t. the $n-1$ first variables (one has to remember that f_n is a symmetric function).

Theorem 2.28 *$(\mathbb{D}, \mathcal{E})$ is a local Dirichlet form admitting a carré du champ, Γ given by:*

$$\forall F \in \mathbb{D},\ \Gamma[F] = \|D.F\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}.$$

Moreover it satisfies the (EID) property.

To prove that it is a Dirichlet form, it is better to give the expression of D on smooth functions. To this end, we set for any $h \in L^2(\mathbb{R}_+;\mathbb{R}^d)$, $\mathring{h} = \int_0^{+\infty} h_s dB_s$. And we have the next proposition from which we can deduce that contractions operate on this structure.

Proposition 2.29 *Let $n \in \mathbb{N}^*$, $h_1, \cdots, h_n$ be functions in $L^2(\mathbb{R}_+; \mathbb{R}^d)$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a function of class C^1 with bounded derivatives. Then $F = \varphi(\overset{\circ}{h}_1, \cdots, \overset{\circ}{h}_n)$ belongs to $\mathbb{D}$ and*

$$D_t F = \sum_{i=1}^{n} \partial_i \varphi(\overset{\circ}{h}_1, \cdots, \overset{\circ}{h}_n) h_i(t).$$

Moreover, the set of such functions F is dense in $\mathbb{D}$.

Another approach consists in proving that the Ornstein–Uhlenbeck semigroup is Markovian (this will be our approach in Sect. 4.4). To this end, the *Mehler formula* immediately yields:

$$P_t f(\omega) = \int_\Omega f\left(e^{-\frac{t}{2}}\omega + \sqrt{1 - e^{-t}\omega'}\right) dm(\omega') \tag{2.5}$$

We end this short introduction to the theory of Dirichlet forms by introducing a new gradient due to Feyel and De La Pradelle which illustrates Remark 2.20.

Consider $(\hat{\Omega}, \hat{m}, (\hat{B}_t)_{t \geqslant 0})$ a copy of the Wiener space $(\Omega, m, (B_t)_{t \geqslant 0})$, the product space $\Omega \times \hat{\Omega}$ is equipped with the product measure $m \times \hat{m}$ and we define:

$$\forall F \in \mathbb{D},\ F'(w, \hat{w}) = \int_0^{+\infty} D_t F(w) d\hat{B}_t(\hat{w}),$$

then clearly F' belongs to $L^2(\Omega \times \hat{\Omega}, m \times \hat{m})$ that we identify with $L^2(\Omega, m; L^2(\hat{\Omega}, \hat{m}))$ and it defines a gradient for the Ornstein–Uhlenbeck structure on $L^2(m)$ taking values in $L^2(\hat{\Omega}, \hat{m})$. Indeed, we have thanks to the isometry property of the stochastic integral with respect to $\hat{B}$:

$$\hat{\mathbb{E}}[|F'(w, \cdot)|^2] = \|F'(w, \cdot)\|^2_{L^2(\hat{\Omega}, \hat{m})} = \|D_{\cdot} F(w)\|^2_{L^2(\mathbb{R}_+, dx)} = \Gamma[F](w).$$

Although less popular than the preceding one, this other gradient often yields simpler calculations (see for example [79] Chap. IV or [125]).

2.4 Sufficient Conditions for (EID) Property

In this part we give sufficient conditions for a Dirichlet structure to fulfill (EID) property. These conditions are put first for finite dimensional cases and will be extended to the infinite dimensional setting of Poisson measures in Sect. 4.6.

2.4.1 A Sufficient Condition on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$

A Generalization of Hamza's Condition

Given $r \in \mathbb{N}^*$, for any $\mathcal{B}(\mathbb{R}^r)$-measurable function $u : \mathbb{R}^r \to \mathbb{R}$, all $i \in \{1, \cdots, r\}$ and all $\bar{x} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r) \in \mathbb{R}^{r-1}$, we consider $u_{\bar{x}}^{(i)} : \mathbb{R} \to \mathbb{R}$ the function defined by

$$\forall s \in \mathbb{R}\, ,\, u_{\bar{x}}^{(i)}(s) = u((\bar{x}, s)_i),$$

where $(\bar{x}, s)_i = (x_1, \cdots x_{i-1}, s, x_{i+1}, \cdots, x_r)$.

Conversely if $x = (x_1, \cdots, x_r)$ belongs to $\mathbb{R}^r$ we set $x^i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r)$.

Then following standard notation, for any $\mathcal{B}(\mathbb{R})$ measurable function $\rho : \mathbb{R} \to \mathbb{R}_+$, we denote by $R(\rho)$ the largest open set on which ρ^{-1} is locally integrable.

Finally, we are given $k : \mathbb{R}^r \to \mathbb{R}_+$ a Borel function and $\xi = (\xi_{ij})_{1 \leqslant i,j \leqslant r}$ an $\mathbb{R}^{r\times r}$-valued and symmetric Borel function.

We make the following assumptions which generalize Hamza's condition (cf. Fukushima–Oshima–Takeda [170] Chap. 3 Sect. 3.1 (3°) p. 105):

Hypotheses (HG)

1. For any $i \in \{1, \cdots, r\}$ and λ_{r-1}-almost all $\bar{x} \in \{y \in \mathbb{R}^{r-1} : \int_{\mathbb{R}} k_y^{(i)}(s)\, ds > 0\}$, $k_{\bar{x}}^{(i)} = 0$, λ_1-a.e. on $\mathbb{R} \setminus R(k_{\bar{x}}^{(i)})$.
2. There exists an open set $O \subset \mathbb{R}^r$ such that $\lambda_r(\mathbb{R}^r \setminus O) = 0$ and ξ is locally elliptic on O in the sense that for any compact subset K, in O, there exists a positive constant c_K such that

$$\forall x \in K,\ \forall c \in \mathbb{R}^r\ \sum_{i,j=1}^{r} \xi_{ij}(x) c_i c_j \ \geqslant\ c_K |c|^2.$$

Following Albeverio–Röckner, Theorems 3.2 and 5.3 in [8] and also Röckner–Wielens Sect. 4 in [316], we consider **d** the set of $\mathcal{B}(\mathbb{R}^r)$-measurable functions u in $L^2(kdx)$, such that for any $i \in \{1, \cdots, r\}$, and λ_{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u_{\bar{x}}^{(i)}$ has an absolutely continuous version $\tilde{u}_{\bar{x}}^{(i)}$ on $R(k_{\bar{x}}^{(i)})$ (defined λ_1-a.e.) and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \in L^1(kdx)$, where

$$\frac{\partial u}{\partial x_i} = \frac{d\tilde{u}_{\bar{x}}^{(i)}}{ds}.$$

Sometimes, we will simply denote $\frac{\partial}{\partial x_i}$ by ∂_i.

And we consider the following bilinear form on $\mathbf{d}$:

$$\forall u, v \in \mathbf{d},\ e(u, v) = \frac{1}{2}\int_{\mathbb{R}^r} \sum_{i,j} \xi_{ij}(x)\partial_i u(x)\partial_j v(x)k(x)\,dx.$$

We have

Proposition 2.30 $(\mathbf{d}, e)$ *is a local Dirichlet form on* $L^2(kdx)$ *which admits a carré du champ operator* γ *given by*

$$\forall u, v \in \mathbf{d},\ \gamma[u, v] = \sum_{i,j} \xi_{ij}\partial_i u \partial_j v.$$

Proof All is clear except the fact that e be a closed form on $\mathbf{d}$. To prove it, let us consider a sequence $(u_n)_{n\in\mathbb{N}^*}$ of elements in $\mathbf{d}$ which converges to u in $L^2(kdx)$ and such that $\lim_{n,m\to+\infty} e(u_n - u_m) = 0$. Let $W \subset O$, an open subset whose closure satisfies $\bar{W} \subset O$ and such that $\bar{W}$ is compact.

Let $\mathbf{d}_W$ be the set of $\mathcal{B}(\mathbb{R}^r)$-measurable functions u in $L^2(\mathbf{1}_W \times k\,dx)$, such that for any $i \in \{1, \cdots, r\}$, and λ_{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u^{(i)}_{\bar{x}}$ has an absolutely continuous version $\tilde{u}^{(i)}_{\bar{x}}$ on $R((\mathbf{1}_W \times k)^{(i)}_{\bar{x}})$ and such that $\sum_{i,j} \xi_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \in L^1(\mathbf{1}_W \times k\,dx)$, equipped with the bilinear form

$$\forall u, v \in \mathbf{d}_W, e_W[u, v] = \frac{1}{2}\int_W \sum_i \partial_i u(x)\partial_i v(x)k(x)\,dx = \frac{1}{2}\int_W \nabla u(x)\cdot\nabla v(x)k(x)\,dx.$$

One can easily verify, since W is an open set, that for all $\bar{x} \in \mathbb{R}^{r-1}$

$$S^i_{\bar{x}}(W) \cap R(k^{(i)}_{\bar{x}}) \subset R((\mathbf{1}_W \times k)^{(i)}_{\bar{x}}), \tag{2.6}$$

where $S^i_{\bar{x}}(W)$ is the open set $\{s \in \mathbb{R} : (\bar{x}, s)_i \in W\}$.

Then it is clear that the function $\mathbf{1}_W \times k$ satisfies property 1 of (HG) and as a consequence of Theorems 3.2 and 5.3 in [8], $(\mathbf{d}_W, e_W)$ is a Dirichlet form on $L^2(\mathbf{1}_W \times kdx)$.

We have for all $n, m \in \mathbb{N}$

$$e_W(u_n - u_m) = \frac{1}{2}\int_W |\nabla u_n(x) - \nabla u_m(x)|^2\, k(x)dx \leqslant \frac{1}{c_{\bar{W}}} e(u_n - u_m),$$

as $(\mathbf{d}, e_W)$ is a closed form, we conclude that u belongs to $\mathbf{d}_W$.

Consider now an exhaustive sequence (W_m), of relatively compact open sets in O such that for all $m \in \mathbb{N}$, $\bar{W}_m \subset W_{m+1} \subset O$. We have that for all m, u belongs to $\mathbf{d}_{W_m}$ hence by Theorems 3.2 and 5.3 in [8], for all $i \in \{1, \cdots, r\}$, and λ_{r-1}-almost all $\bar{x} \in \mathbb{R}^{r-1}$, $u^{(i)}_{\bar{x}}$ has an absolutely continuous version on $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)^{(i)}_{\bar{x}})$. Using relation (2.6), we have

$$S^i_{\bar{x}}(O) \cap R(k^{(i)}_{\bar{x}}) = \bigcup_{m=1}^{+\infty} S^i_{\bar{x}}(W_m) \cap R(k^{(i)}_{\bar{x}}) \subset \bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)^{(i)}_{\bar{x}}).$$

As $\lambda_r(\mathbb{R}^r \setminus O) = 0$, we get that for almost all $\bar{x} \in \mathbb{R}^{r-1}$, $\bigcup_{m=1}^{+\infty} R((\mathbf{1}_{W_m} \times k)^{(i)}_{\bar{x}}) = R(k^{(i)}_{\bar{x}})$ λ_1-a.e. Moreover, by a diagonal extraction, we have that a subsequence of (∇u_n) converges kdx-a.e. to ∇u, so by Fatou's Lemma, we conclude that $u \in \mathbf{d}$ and then $\lim_{n\to+\infty} e[u_n - u] = 0$, which is the desired result. □

Theorem 2.31 (EID) property *: the structure* $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r), k\,dx, \mathbf{d}, \gamma)$ *satisfies*

$$\forall d \in \mathbb{N}^* \; \forall u \in \mathbf{d}^d \quad u_*[(\det \gamma[u, u^t]) \cdot kdx] \ll \lambda_d.$$

Proof Let us mention that a proof was given by S. Song in [346] Theorem 16, in the more general case of *classical Dirichlet forms*. Following his ideas, we present here a shorter proof.

The proof is based on the *co-area formula* stated by H. Federer in [148], see the Appendix B.

We first introduce the subset $A \subset \mathbb{R}^r$:

$$A = \{x \in \mathbb{R}^r : \; x_i \in R(k^{(i)}_{x^i}) \; i = 1, \cdots, r\}.$$

As a consequence of property 1 of (HG), $\int_{A^c} k(x)dx = 0$.

Let $u = (u_1, \cdots, u_d) \in \mathbf{d}^d$. We follow the notation and definitions introduced by Bouleau–Hirsch in [79] Chap. II Sect. 5.1 and recalled in the Appendix B. It is based on the notion of *approximate derivative*.

Thanks to Theorem 3.2 in [8] and Stepanoff's Theorem (see Theorem B.3 in the Appendix B), it is clear that for almost all $a \in A$, the *approximate derivatives* $\mathrm{ap}\frac{\partial u}{\partial x_i}$ exist for $i = 1, \cdots, r$ and if we set: $Ju = \left[\det\left(\left(\sum_{k=1}^r \partial_k u_i \partial_k u_j\right)_{1 \leqslant i,j \leqslant d}\right)\right]^{1/2}$, this is equal kdx-a.e. to the determinant of the *approximate Jacobian matrix* of u. Then, by Theorem 3.1.4 in [148], u is *approximately differentiable* at almost all points a in A.

We denote by $\mathcal{H}^{r-d}$ the $(r-d)$-dimensional Hausdorff measure on $\mathbb{R}^r$.

As a consequence of Theorems 3.1.8, 3.1.16 and Lemma 3.1.7 in [148], for all $n \in \mathbb{N}^*$, there exists a map $u^n : \mathbb{R}^r \to \mathbb{R}^d$ of class $\mathcal{C}^1$ such that

$$\lambda_r(A \setminus \{x : u(x) = u^n(x)\}) \leqslant \frac{1}{n}$$

and

$$\forall a \in \{x : u(x) = u^n(x)\}, \; \mathrm{ap}\frac{\partial u}{\partial x_i}(a) = \mathrm{ap}\frac{\partial u^n}{\partial x_i}(a), \; i = 1, \cdots, r.$$

Assume first that $d \leqslant r$. Let B be a Borelian set in $\mathbb{R}^d$ such that $\lambda_r(B) = 0$. Thanks to the co-area formula we obtain

$$\begin{aligned}
\int_{\mathbb{R}^r} \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx &= \int_A \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{A\cap\{u=u^n\}} \mathbf{1}_B(u(x)) Ju(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{A\cap\{u=u^n\}} \mathbf{1}_B(u^n(x)) Ju^n(x) k(x)\, dx \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^r} \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A\cap\{u=u^n\}}(x) \mathbf{1}_B(u^n(x)) k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda_r(y) \\
&= \lim_{n\to+\infty} \int_{\mathbb{R}^r} \mathbf{1}_B(y) \left(\int_{(u^n)^{-1}(y)} \mathbf{1}_{A\cap\{u=u^n\}}(x) k(x) d\mathcal{H}^{r-d}(x) \right) d\lambda_r(y) \\
&= 0
\end{aligned}$$

So that, $u_*(Ju \cdot k dx) \ll \lambda_d$.

We have the equalities

$$Ju = \left[\det\left(Du \cdot (Du)^t\right)\right]^{1/2} \text{ and } \gamma(u) = Du \cdot \xi \cdot Du^t,$$

where Du is the $d \times r$ matrix: $\left(\dfrac{\partial u_i}{\partial x_k}\right)_{1 \leqslant i \leqslant d; 1 \leqslant k \leqslant r}$.

From the fact that $\xi(x)$ is symmetric and positive definite on O and $\lambda_r(\mathbb{R}^r \setminus O) = 0$, we deduce

$$\{x \in A;\ Ju(x) > 0\} = \{x \in A;\ \det(\gamma(u, u^t)(x)) > 0\}\ a.e.,$$

and this ends the proof in this case.

Now, if $d > r$, $\det(\gamma(u, u^t)) = 0$ and the result is trivial. $\square$

Application

Lévy processes with infinitely many jumps play an important role and intervene in many examples in this book. Intuitively, we want to "derivate" only with respect to small jumps. That is why we need to "localize" the previous example. To this end, as above we are given $k : \mathbb{R}^r \to \mathbb{R}_+$ a Borel function and $\xi = (\xi_{ij})_{1 \leqslant i,j \leqslant r}$ an $\mathbb{R}^{r\times r}$-valued and symmetric Borel function. We also consider W a fixed open set which represents the set on which we are going to "derivate". We make the following assumptions:

Hypotheses (HG')

1. There exists a positive and continuous function $\psi : W \to \mathbb{R}_+$ such that

$$k \geqslant \psi > 0 \text{ on } W.$$

2. There exists an open set $O \subset \mathbb{R}^r$ such that $\lambda_r(\mathbb{R}^r \backslash O) = 0$ and ξ is locally elliptic on O.

Put $k' = k \times \mathbf{1}_W$. Then, using the fact that ψ is positive and continuous on W, it appears that k' satisfies hypotheses (HG). Hence, on $L^2(k'dx)$ we can define as above the local Dirichlet form $(\mathbf{d}', e')$ with carré du champ operator γ' given by

$$\forall u, v \in \mathbf{d},\ \ \gamma'[u, v] = \sum_{i,j} \xi_{ij} \partial_i u \partial_j v = \mathbf{1}_W \times \sum_{i,j} \xi_{ij} \partial_i u \partial_j v \ \ k'dx - a.e.$$

We now define on $L^2(kdx)$ the following form $(\mathbf{d}, e)$ such that

$$\mathbf{d} = \{u \in L^2(kdx);\ \mathbf{1}_W \times u \in \mathbf{d}'\}$$

and

$$\forall (u, v) \in \mathbf{d}^2, e(u, v) = e'(\mathbf{1}_W \times u, \mathbf{1}_W \times v) = \int_W \sum_{i,j} \xi_{ij}(x) \partial_i u(x) \partial_j v(x)\, k(x) dx.$$

The following proposition is a consequence of Propositions 2.12, 2.30 and Theorem 2.31.

Proposition 2.32 $(\mathbf{d}, e)$ *is a local Dirichlet form on* $L^2(kdx)$ *which admits a carré du champ operator* γ *given by*

$$\forall u, v \in \mathbf{d},\ \ \gamma[u, v] = \mathbf{1}_W \times \sum_{i,j} \xi_{ij} \partial_i u \partial_j v.$$

Moreover, it satisfies (EID).

2.4.2 The Case of a Product Structure

We consider a sequence of functions ξ^i and k_i, $i \in \mathbb{N}^*$, k_i being non-negative Borel functions such that $\int_{\mathbb{R}^r} k_i(x)\, dx = 1$. We assume that for all $i \in \mathbb{N}^*$, ξ^i and k_i satisfy hypotheses (HG) so that, we can construct, as for k in the previous subsection, the Dirichlet form $(\mathbf{d}_i, e_i)$ on $L^2(\mathbb{R}^r, k_i dx)$ associated to the carré du champ operator γ_i given by:

$$\forall u, v \in \mathbf{d}_i,\ \ \gamma_i[u, v] = \sum_{k,l} \xi^i_{kl} \partial_k u \partial_l v.$$

We now consider the product Dirichlet form $(\tilde{\mathbf{d}}, \tilde{e}) = \prod_{i=1}^{+\infty} (\mathbf{d}_i, e_i)$ defined on the product space $\left((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(R^r))^{\mathbb{N}^*}\right)$ equipped with the product probability $\Lambda = \prod_{i=1}^{+\infty} k_i dx$. We denote by $(X_n)_{n \in \mathbb{N}^*}$ the coordinates maps on $(\mathbb{R}^r)^{\mathbb{N}^*}$.

Let us recall that $U = F(X_1, X_2, \cdots, X_n, \cdots)$ belongs to $\tilde{\mathbf{d}}$ if and only if:

1. U belongs to $L^2((\mathbb{R}^r)^{\mathbb{N}^*}, (\mathcal{B}(\mathbb{R}^r))^{\mathbb{N}^*}, \Lambda)$.
2. For all $k \in \mathbb{N}^*$ and Λ-almost all $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots)$ in $(\mathbb{R}^r)^{\mathbb{N}^*}$, $F(x_1, \cdots, x_{k-1}, \cdot, x_{k+1}, \cdots)$ belongs to $\mathbf{d}_k$.
3. $\tilde{e}(U) = \sum_k \int_{(\mathbb{R}^r)^{\mathbb{N}^*}} e_k(F(X_1(x), \cdots, X_{k-1}(x), \cdot, X_{k+1}(x), \cdots))\, \Lambda(dx) < +\infty$.

Where as usual, the form e_k acts only on the kth coordinate.

It is also well known that $(\tilde{\mathbf{d}}, \tilde{e})$ admits a carré du champ $\tilde{\gamma}$ given by

$$\tilde{\gamma}[U] = \sum_k \gamma_k[F(X_1, \cdots, X_{k-1}, \cdot, X_{k+1}, \cdots)](X_k).$$

To prove that (EID) is satisfied by this structure, we first prove that it is satisfied for a finite product. So, for all $n \in \mathbb{N}^*$, we consider $(\tilde{\mathbf{d}}_n, \tilde{e}_n) = \prod_{i=1}^n (\mathbf{d}_i, e_i)$ defined on the product space $((\mathbb{R}^r)^n, (\mathcal{B}(R^r))^n)$ equipped with the product probability $\Lambda_n = \prod_{i=1}^n k_i dx$. By restriction, we keep the same notation as the one introduced for the infinite product. We know that this structure admits a carré du champ operator $\tilde{\gamma}_n$ given by $\tilde{\gamma}_n = \sum_{i=1}^n \gamma_i$.

Lemma 2.33 *For all $n \in \mathbb{N}^*$, the Dirichlet structure $(\tilde{\mathbf{d}}_n, \tilde{e}_n)$ satisfies (EID):*

$$\forall d \in \mathbb{N}^* \; \forall U \in (\tilde{\mathbf{d}}_n)^d \;\; U_*[(\det \tilde{\gamma}_n[U, U^t]) \cdot \Lambda_n] \ll \lambda_d.$$

Proof The proof consists in remarking that this is nothing but a particular case of Theorem 2.31 on $\mathbb{R}^{nd}$, ξ being replaced by Ξ, the diagonal matrix of the ξ^i, and the density being the product density. □

As a consequence of Chap. V Proposition 2.2.3. in Bouleau–Hirsch [79], we have

Theorem 2.34 *The product Dirichlet structure $(\tilde{\mathbf{d}}, \tilde{e})$ satisfies* (EID)*:*

$$\forall d \in \mathbb{N}^* \; \forall U \in \tilde{\mathbf{d}}^d \;\; U_*[(\det \tilde{\gamma}[U, U^t]) \cdot \Lambda] \ll \lambda_d.$$

2.4.3 The Case of Structures Obtained by Injective Images

The next result could be extended to more general images (see Bouleau–Hirsch [79] Chap. V Sect. 1.3 p. 196 *et seq.*). We give the statement in the most useful form for Poisson measures and processes with independent increments.

Let $(\mathbb{R}^p \backslash \{0\}, \mathcal{B}(\mathbb{R}^p \backslash \{0\}), \nu, \mathbf{d}, \gamma)$ be a Dirichlet structure on $\mathbb{R}^p \backslash \{0\}$ satisfying (EID). Thus ν is σ-finite, γ is the carré du champ operator and the Dirichlet form is $e[u] = 1/2 \int \gamma[u] d\nu$.

Let $U : \mathbb{R}^p \backslash \{0\} \mapsto \mathbb{R}^q \backslash \{0\}$ be an injective map such that $U \in \mathbf{d}^q$. Then $U_*\nu$ is σ-finite. If we put

$$\begin{aligned}\mathbf{d}_U &= \{\varphi \in L^2(U_*\nu) : \varphi \circ U \in \mathbf{d}\}\\ e_U(\varphi) &= e(\varphi \circ U)\\ \gamma_U[\varphi] &= \tfrac{d\,U_*(\gamma[\varphi \circ U].\nu)}{d\,U_*\nu}\end{aligned}$$

we have

Proposition 2.35 *The term* $(\mathbb{R}^q\backslash\{0\}, \mathcal{B}(\mathbb{R}^q\backslash\{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ *is a Dirichlet structure satisfying* (EID).

Proof (a) That $(\mathbb{R}^q\backslash\{0\}, \mathcal{B}(\mathbb{R}^q\backslash\{0\}), U_*\nu, \mathbf{d}_U, \gamma_U)$ is a Dirichlet structure is true in general and does not use the injectivity of U (cf. the case ν finite in Bouleau–Hirsch [79] Chap. V Sect. 1 p. 186 *et seq.*).
(b) By the injectivity of U, we see that for $\varphi \in \mathbf{d}_U$

$$(\gamma_U[\varphi]) \circ U = \gamma[\varphi \circ U] \quad \nu\text{-a.s.}$$

so that if $f \in (\mathbf{d}_U)^r$

$$f_*[\det \gamma_U[f] \cdot U_*\nu] = (f \circ U)_*[\det \gamma[f \circ U] \cdot \nu]$$

which proves (EID) for the image structure. □

Remark 2.36 Applying this result yields examples of Dirichlet structures on $\mathbb{R}^n$ satisfying (EID) whose measures are carried by a (Lipschitzian) curve in $\mathbb{R}^n$ or, under some hypotheses, a countable union of such curves, and therefore without density (see for example the second example in Sect. 8.3.3 devoted to stochastic Lévy area). □

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