Chapter 2
Ordering of Fuzzy Numbers

Abstract This chapter describes different methods for comparing and ordering fuzzy numbers. Theoretically, fuzzy numbers can only be partially ordered, and hence cannot be compared. However, in practical applications, such as decision making, scheduling, market analysis or optimisation with fuzzy uncertainties, the comparison of fuzzy numbers becomes crucial.

Theoretically, fuzzy numbers can only be partially ordered, and hence cannot be compared. However, when they are used in practical applications, e.g., when a decision must be made among alternatives or an optimal value of an objective function must be found, the comparison of fuzzy numbers becomes crucial.

There are numerous approaches to the ordering relation between fuzzy numbers [1–6] qualitative, quantitative and based on α–cuts. Jain [7] and Dubois and Prade [4] were the first who considered this problem. Some methods to rank fuzzy numbers were reviewed by Bortolan and Degani [8]. Detyniecki and Yager [3] proposed the α-weighted valuations of fuzzy numbers. Hong and Kim [9] proposed an easy way to compute the min and max operation for fuzzy numbers. Asady and Zendehnam [10] proposed the ranking fuzzy numbers by distance minimisation method. Comparison of various ranking methods for fuzzy numbers with the possibility of ranking the crisp numbers was described by Thorani et al. [11]. The problem of comparing of fuzzy numbers was also considered by Allahviranloo et al. [12]. They proposed a method based on the centroid point of a fuzzy number and its area. Sevastjanov and Róg [13] developed a probability-based comparison of fuzzy numbers. The probabilistic approach was also considered in [14]. The large number of fuzzy ordering methods can be justified by the fact that different methods can be useful for different purposes. For example, problems involving ranking, prioritising or choosing between large number of alternatives will benefit from methods that assign to fuzzy numbers crisp values thus reducing the fuzzy ordering problem to ordering of real numbers.

An overview of selected approaches to ordering (ranking) of fuzzy numbers is presented below. The presented approaches can be generally divided into two groups. The first group consists of methods which enable two fuzzy numbers to be compared. Included in this group are such methods as probabilistic approach, centroid point approach or radius of gyration approach. To order a set of fuzzy numbers using

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these methods, some dedicated procedures are required. The second group consists of methods, which assign to a fuzzy number a crisp value. These are methods such as Yager ranking index based approach, defuzzification approach or weighted average. The methods from the second group can be directly used to order a set of fuzzy numbers, by employing one of the several methods for ordering (sorting) real numbers. All the above mentioned methods are compared using an example of ordering four triangular fuzzy numbers.

2.1 Probabilistic Approach

The *probabilistic* (also known as *probability degree-based* or *probability-based*) approach to ordering fuzzy numbers is based on the α–cuts representation of fuzzy numbers. The α–cuts based orderings are so attractive, because they can be used regardless the type of the membership function. Moreover, each α-level is an interval, so the powerful tools of interval arithmetic [15] can be employed to solve the problem of fuzzy ordering [13].

Let $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$ be two closed and compact intervals. The possibility degree-based ranking method which is shown in Table 2.1 was proposed by Jiang et al. [16]. The non-overlapping cases are omitted as they are obvious.

A similar, but slightly extended approach to ordering of intervals was proposed in [13]. Let the real values $a \in \boldsymbol{a}$ and $b \in \boldsymbol{b}$ be given. They can be considered as two independent uniform random variables. If $\boldsymbol{a}$ and $\boldsymbol{b}$ overlaps, then some disjoint subintervals can be distinguished. The fall of random variables a and b in the subintervals $[a_1, b_1]$, $[b_1, a_2]$, $[a_2, b_2]$ may be treated as a set of independent random events.

Let the events $H_k : a \in \boldsymbol{a}_i, b \in \boldsymbol{b}_j$ be defined for $k = 1, \ldots, n$, where $\boldsymbol{a}_i$ and $\boldsymbol{b}_j$ are certain subintervals of intervals $\boldsymbol{a}$ and $\boldsymbol{b}$ in accordance with $\boldsymbol{a} = \bigcup_i \boldsymbol{a}_i$ and $\boldsymbol{b} = \bigcup_i \boldsymbol{b}_i$ ($n = 4$ for the case depicted in Fig. 2.1) [13]. Let $P(H_k)$ be the probability of event H_k, and $P(\boldsymbol{b} > \boldsymbol{a}|H_k)$ be the conditional probability of $\boldsymbol{b} > \boldsymbol{a}$ given H_k. Hence, the composite probability may be expressed as follows:

$$P(\boldsymbol{b} > \boldsymbol{a}) = \sum_{k=1}^{n} P(H_k) P(\boldsymbol{b} > \boldsymbol{a}|H_k). \tag{2.1}$$

Table 2.1 Cases of interval comparison proposed by Jiang [16]

Case	$P(\boldsymbol{b} \leqslant \boldsymbol{a})$
1. $b_1 \geqslant a_1 \wedge b_2 \geqslant a_2 \wedge b_1 \leqslant a_2$	$\frac{a_1-b_1}{b_2-b_1} \cdot \frac{a_2-a_1}{b_2-b_1}$
2. $a_1 \geqslant b_1 \wedge a_2 \leqslant b_2$	$\frac{a_1-b_1}{b_2-b_1} \cdot \frac{1}{2} \frac{a_2-a_1}{b_2-b_1}$
3. $a_1 \geqslant b_1 \wedge a_2 \geqslant b_2 \wedge a_1 \leqslant b_2$	$\frac{a_1-b_1}{b_2-b_1} + \frac{b_2-a_1}{b_2-b_1} \cdot \frac{a_2-b_2}{a_2-a_1} + \frac{1}{2} \frac{b_2-a_1}{b_2-b_1} \cdot \frac{b_1-a_1}{a_2-a_1}$
4. $b_1 \geqslant a_1 \wedge b_2 \leqslant a_2$	$\frac{a_2-b_2}{a_2-a_1} + \frac{1}{2} \frac{b_2-b_1}{a_2-a_1}$

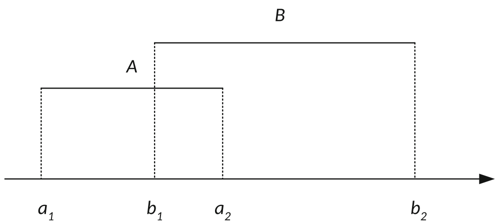

Fig. 2.1 Example of overlapping intervals

The resulting formula for the case of overlapping intervals is as follows:

$$P(\boldsymbol{b} > \boldsymbol{a}) = 1 - \tfrac{1}{2}\tfrac{(a_2-b_1)^2}{(a_2-a_1)(b_2-b_1)}. \tag{2.2}$$

The results obtained in [13] for all possible interval overlapping are shown in Table 2.2.

The above possibility degree-based methods have the following features:

1. $0 \leqslant P(\boldsymbol{a} \leqslant \boldsymbol{b}) \leqslant 1$.
2. If $P(\boldsymbol{b} \leqslant \boldsymbol{a}) \leqslant \alpha$, then $P(\boldsymbol{a} \leqslant \boldsymbol{b}) \leqslant 1 - \alpha$.
3. If $P(\boldsymbol{b} \leqslant \boldsymbol{a}) = P(\boldsymbol{a} \leqslant \boldsymbol{b})$, then $\boldsymbol{a} \equiv \boldsymbol{b}$.

It follows from 2 and 3 that if $\boldsymbol{a} \equiv \boldsymbol{b}$, then $P(\boldsymbol{b} \leqslant \boldsymbol{a}) = P(\boldsymbol{a} \leqslant \boldsymbol{b}) = 0.5$.

This approach can be treated as a framework for elaboration of constructive methods of interval comparison in various special situations. Some aspects of the interval comparison and ordering group of intervals, based on this approach, is presented, e.g., in [17].

Now, let $\tilde{A}$ and $\tilde{B}$ be some arbitrary fuzzy numbers, and let $\tilde{A}^{\alpha} = \{x \mid \mu_A(x) \geqslant \alpha\}$ and $\tilde{B}^{\alpha} = \{x \mid \mu_B(x) \geqslant \alpha\}$ be their respective α–cuts. Since $\tilde{A}^{\alpha}$ and $\tilde{B}^{\alpha}$ are intervals, probability $P^{\alpha}(\tilde{B}^{\alpha} > \tilde{A}^{\alpha})$ for each pair $\tilde{A}^{\alpha}$ and $\tilde{B}^{\alpha}$ can be calculated in the way described in the previous section. The set of probabilities P^{α}, $\alpha \in (0, 1]$, may be treated as the support of the fuzzy subset [13]:

$$P(\tilde{A} > \tilde{B}) = \{\alpha \mid P^{\alpha}(\tilde{B}^{\alpha} > \tilde{A}^{\alpha})\}, \tag{2.3}$$

Table 2.2 Typical cases of interval comparison [13]

Case	$P(a > b)$	$P(a = b)$
1. $a_1 > b_1 \wedge a_1 < b_2 \wedge a_1 = a_2$	$\frac{b_2-a_1}{b_2-b_1}$	0
2. $b_1 > a_1 \wedge b_1 < a_2 \wedge b_1 = b_2$	$\frac{b_1-a_1}{a_2-a_1}$	0
3. $b_1 \geqslant a_1 \wedge b_2 \leqslant a_2$	$\frac{b_1-a_1}{a_2-a_1} + \frac{1}{2}\frac{a_2-a_1}{b_2-b_1}$	$\frac{b_2-b_1}{a_2-a_1}$
4. $a_1 \geqslant b_1 \wedge a_2 \leqslant b_2$	$\frac{b_2-a_2}{b_2-b_1} + \frac{1}{2}\frac{a_2-a_1}{b_2-b_1}$	$\frac{a_2-a_1}{b_2-b_1}$
5. $b_1 \geqslant a_1 \wedge b_2 \geqslant a_2 \wedge b_1 \leqslant a_2$	$1 - \frac{1}{2}\frac{(a_2-b_1)^2}{(a_2-a_1)(b_2-b_1)}$	$\frac{(a_2-b_1)^2}{(a_2-a_1)(b_2-b_1)}$
6. $a_1 \geqslant b_1 \wedge a_2 \geqslant b_2 \wedge a_1 \leqslant b_2$	$1 - \frac{1}{2}\frac{(b_2-a_1)^2}{(a_2-a_1)(b_2-b_1)}$	$\frac{(b_2-a_1)^2}{(a_2-a_1)(b_2-b_1)}$

where the values of α may be considered as grades of membership of the fuzzy number $P(\tilde{A} > \tilde{B})$. In this way, the fuzzy subset $P(\tilde{A} = \tilde{B})$ may also be easily created.

In the case of triangular or trapezoidal fuzzy number comparison, the obtained results may be interpreted as a fuzzy number [13]. Nevertheless, in practice, real number indices are needed for fuzzy numbers ordering. For this purpose, some characteristic numbers of a fuzzy set [18] could be used. It seems, however, more natural to substitute the obtained discrete set I of α-levels with a real number:

$$\overline{P}(\tilde{B} > \tilde{A}) = \sum_{\alpha \in I} \alpha P^{\alpha}(\tilde{B}^{\alpha} > \tilde{A}^{\alpha}) / \sum_{\alpha \in I} \alpha. \qquad (2.4)$$

The equation (2.4) emphasises that the contribution of the α-level to the overall probability estimation is increasing with an increase in its number. Of course, as proposed in [3], the set of complementary parametrised functions of α can be applied in the equation (2.4) instead of α.

Example 2.1 Let the following four triangular fuzzy numbers, depicted in Fig. 2.2, be given [19]:

$$\tilde{A}_1 = (0.12, 0.19, 0.29), \ \tilde{A}_2 = (0.22, 0.32, 0.48),$$

$$\tilde{A}_3 = (0.11, 0.15, 0.23), \ \tilde{A}_4 = (0.21, 0.33, 0.49).$$

The results of pairwise comparison of these numbers using the probability degree-based approach are presented in Table 2.3. This gives the following order: $\tilde{A}_3 < \tilde{A}_1 < \tilde{A}_2 < \tilde{A}_4$.

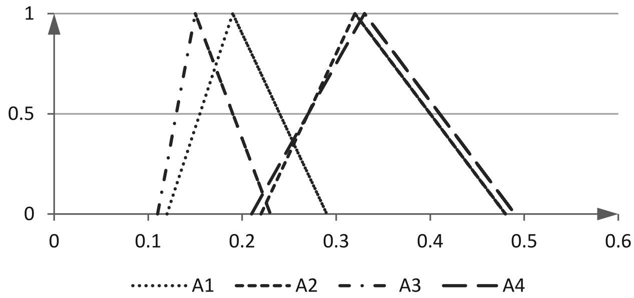

Fig. 2.2 Exemplary triangular fuzzy numbers

Table 2.3 The results of comparison of triangular fuzzy numbers $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$

Pair	$\overline{P}$	Comparison
$\tilde{A}_1, \tilde{A}_2$	0.999	$\tilde{A}_1 < \tilde{A}_2$
$\tilde{A}_1, \tilde{A}_3$	0.059	$\tilde{A}_1 > \tilde{A}_3$
$\tilde{A}_1, \tilde{A}_4$	0.998	$\tilde{A}_1 < \tilde{A}_4$
$\tilde{A}_2, \tilde{A}_3$	3.00E-08	$\tilde{A}_2 > \tilde{A}_3$
$\tilde{A}_2, \tilde{A}_4$	0.6823	$\tilde{A}_2 < \tilde{A}_4$
$\tilde{A}_3, \tilde{A}_4$	1	$\tilde{A}_3 < \tilde{A}_4$

2.2 Defuzzification Approach

Fuzzy numbers can also be ranked using the *defuzzification* methods. A defuzzification is the process of producing a real (crisp) value corresponding to a fuzzy number. In order to rank fuzzy numbers using the defuzzification approach, the fuzzy numbers are first defuzzified and then, the obtained crisp numbers are ordered using the order relation of real numbers. There are several defuzzification methods, among them:

- Centre of area (*COA*) or centre of gravity (*COG*):

$$COA(\tilde{A}) = COG(\tilde{A}) = \frac{\int_{x_{\min}}^{x_{\max}} x\mu_{\tilde{A}}(x)dx}{\int_{x_{\min}}^{x_{\max}} \mu_{\tilde{A}}(x)dx}.$$

- First of maxima (*FOM*):

$$FOM(\tilde{A}) = \min \ker(\tilde{A}).$$

- Middle of maxima (*MOM*):

$$MOM(\tilde{A}) = \frac{\min \ker(\tilde{A}) + \max \ker(\tilde{A})}{2}.$$

- Last of maxima (*LOM*):

$$LOM(\tilde{A}) = \max \ker(\tilde{A}).$$

In the case of triangular fuzzy numbers of the form $\tilde{A} = (a, b, c)$:

$$FOM(\tilde{A}) = MOM(\tilde{A}) = LOM(\tilde{A}) = c.$$

The results of ordering of fuzzy numbers from Example 2.1 using the abovementioned defuzzification methods are the same as the result from Example 2.1. The values of *FOM*, *MOM* and *LOM* are obvious, whereas $COG(\tilde{A}_1) = 0.2$, $COG(\tilde{A}_2) = 0.34$, $COG(\tilde{A}_3) = 0.1633$, $COG(\tilde{A}_4) = 0.3452$.

2.3 Centroid-Point Approach

A fuzzy number $\tilde{A}$ can be identified with an ordered pair of continuous real functions defined on the interval [0, 1], i.e., $\tilde{A} = (f_{\tilde{A}}, g_{\tilde{A}})$, where $f_{\tilde{A}}, g_{\tilde{A}} : [0, 1] \rightarrow \Re$ are continuous functions. Functions $f_{\tilde{A}}$ and $g_{\tilde{A}}$ are called, respectively, the *up* and *down*-parts of a fuzzy number $\tilde{A}$.

The continuity of the functions f and g implies that their images are bounded intervals (see Fig. 2.3a) denoted, respectively, as UP and $DOWN$. If, additionally, the $f_{\tilde{A}}$ and $g_{\tilde{A}}$ functions are monotone, and thus invertible, the following membership can be defined:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^{-1}(x), \ x \in [f_{\tilde{A}}(0), f_{\tilde{A}}(1)] = [l_{\tilde{A}}, 1_{\tilde{A}}^{-}], \\ g_{\tilde{A}}^{-1}(x), \ x \in [g_{\tilde{A}}(1), g_{\tilde{A}}(0)] = [1_{\tilde{A}}^{+}, p_{\tilde{A}}], \\ 1, \qquad x = [1_{\tilde{A}}^{-}, 1_{\tilde{A}}^{+}], \end{cases} \tag{2.5}$$

if $f_{\tilde{A}}$ is increasing and $g_{\tilde{A}}$ is decreasing, and $f_{\tilde{A}} \leqslant g_{\tilde{A}}$ for all $y \in [0, 1]$. The obtained membership function $\mu_{\tilde{A}}(x), x \in \Re$ represents a mathematical object which resembles a convex fuzzy number in the classical sense.

Definition 2.1 Let $\tilde{A} = (a, b, c, d)$. Then, the centroid (centre of gravity) point of $\tilde{A}$ is obtained as follows [20]:

$$COGP(\tilde{A}) = (\overline{x}_0(\tilde{A}), \overline{y}_0(\tilde{A})), \tag{2.6}$$

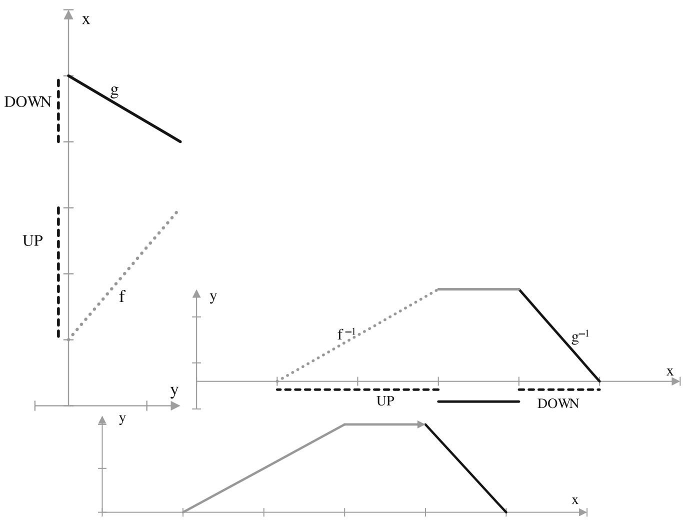

Fig. 2.3 An ordered fuzzy number (**a**), an ordered fuzzy number presented as a fuzzy number in a classical sense (**b**), and a simplified mark denoting the order of inverted functions (**c**)

where

$$\begin{cases} \overline{x}_0(\tilde{A}) = \dfrac{\int_a^b x f_{\tilde{A}}^{-1}(x)dx + \int_b^c x dx + \int_c^d x g_{\tilde{A}}^{-1}(x)dx}{\int_a^b f_{\tilde{A}}^{-1}(x)dx + \int_b^c dx + \int_c^d g_{\tilde{A}}^{-1}(x)dx}, \\ \overline{y}_0(\tilde{A}) = \dfrac{\int_0^1 y f_{\tilde{A}}(y)dy + \int_0^1 y g_{\tilde{A}}(y)dy}{\int_0^1 f_{\tilde{A}}(y)dy + \int_0^1 g_{\tilde{A}}(y)dy}. \end{cases} \tag{2.7}$$

In the case of trapezoidal fuzzy numbers, the above formula takes the form:

$$\begin{cases} \overline{x}_0(\tilde{A}) = \frac{1}{3}\left[a + b + c + d - \frac{cd-ab}{(c+d)-(a+b)}\right], \\ \overline{y}_0(\tilde{A}) = \frac{1}{3}\left[1 + \frac{c-b}{(c+d)-(a+b)}\right]. \end{cases} \tag{2.8}$$

Based on a centroid point, two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are compared using the following rules [21]:

$$\begin{aligned} &\text{If } \overline{x}_0(\tilde{A}) > \overline{x}_0(\tilde{B}), \text{ Then } \tilde{A} > \tilde{B}. \\ &\text{If } \overline{x}_0(\tilde{A}) < \overline{x}_0(\tilde{B}), \text{ Then } \tilde{A} < \tilde{B}. \\ &\text{If } \overline{x}_0(\tilde{A}) = \overline{x}_0(\tilde{B}), \text{ Then} \\ &\quad \text{If } \overline{y}_0(\tilde{A}) > \overline{y}_0(\tilde{B}), \text{ Then } \tilde{A} > \tilde{B}. \\ &\quad \text{Else If } \overline{y}_0(\tilde{A}) < \overline{y}_0(\tilde{B}), \text{ Then } \tilde{A} < \tilde{B}. \\ &\quad \text{Else} \tilde{A} = \tilde{B}. \end{aligned} \tag{2.9}$$

For the fuzzy numbers from Example 2.1 the following centroid points were obtained (Table 2.4):

This gives the ordering: $\tilde{A}_3 < \tilde{A}_1 < \tilde{A}_2 < \tilde{A}_4$. The method is rather simple, but it requires a pairwise comparison of fuzzy numbers to be ordered.

Table 2.4 The considered fuzzy numbers and their centroid points

Fuzzy number	$\overline{x}_0$	$\overline{y}_0$
$\tilde{A}_1 = (0.12, 0.19, 0.29)$	0.343	0.333
$\tilde{A}_2 = (0.22, 0.32, 0.48)$	0.34	0.333
$\tilde{A}_3 = (0.11, 0.15, 0.23)$	0.2	0.333
$\tilde{A}_4 = (0.21, 0.33, 0.49)$	0.163	0.333

Table 2.5 Yager index for the fuzzy numbers from Example 2.1

Fuzzy number	Yager index
$\tilde{A}_1 = (0.12, 0.19, 0.29)$	0.1317
$\tilde{A}_2 = (0.22, 0.32, 0.48)$	0.2233
$\tilde{A}_3 = (0.11, 0.15, 0.23)$	0.1067
$\tilde{A}_4 = (0.21, 0.33, 0.49)$	0.2267

2.4 Yager Ranking Index Approach

In [22], Yager proposed the following index to ordering fuzzy numbers:

$$Y(\tilde{A}) = \tfrac{1}{2} \int_0^1 (f_{\tilde{A}}(y) + g_{\tilde{A}}(y))dy.$$

For example, given two triangular fuzzy numbers $\tilde{A}_1 = (35, 50, 61)$ and $\tilde{A}_2 = (30, 41, 49)$ the Yager's values are $Y(\tilde{A}_1) = 49$ and $Y(\tilde{A}_2) = 40.25$. Thus, $\tilde{A}_2$ is smaller than $\tilde{A}_1$ in the context of Yager index.

For the fuzzy numbers from Example 2.1, the Yager index takes the values presented in Table 2.5. They yield exactly the same order as the one obtained using the possibilistic approach.

2.5 Degree of Possibility Approach

The ordering of fuzzy numbers using priority approach is based on the research presented in [19].

Definition 2.2 Given two convex fuzzy numbers $\tilde{A}$ and $\tilde{B}$ the degree of possibility of $\tilde{A} > \tilde{B}$ is defined as

$$V(\tilde{A} > \tilde{B}) = \sup_{x \geq y}\{min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}\}. \tag{2.10}$$

Thus, if there exists a pair (x, y) such that $x \geqslant y$ and $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(y) = 1$, then the degree of possibility $V(\tilde{A} > \tilde{B}) = 1$. Since $\tilde{A}$ and $\tilde{B}$ are convex fuzzy numbers, the following holds [19]:

$$\begin{aligned} &V(\tilde{A} > \tilde{B}) = 1 \text{ iff } \sup \ker(\tilde{A}) \geqslant \inf \ker(\tilde{B}), \\ &V(\tilde{B} \geqslant \tilde{A}) = \text{hgt}(\tilde{A} \cap \tilde{B}) = \mu_{\tilde{A}}(d), \end{aligned} \tag{2.11}$$

where d is the x-coordinate of the highest intersection point between $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$. When $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$, then

$$\text{hgt}(\tilde{A} \cap \tilde{B}) = \frac{a_1 - b_3}{(b_2 - b_3) - (a_2 - a_1)}. \tag{2.12}$$

To compare $\tilde{A}$ and $\tilde{B}$ both values $V(\tilde{A} \geqslant \tilde{B})$ and $V(\tilde{B} \geqslant \tilde{A})$ are needed.

Now, the degree of possibility for a convex fuzzy number $\tilde{A}$ to be greater than k convex fuzzy numbers $\tilde{A}_i$ $(i = 1, \ldots, k)$ is given by

$$\begin{aligned} V(\tilde{A} \geq \tilde{A}_1, \ldots, \tilde{A}_k) &= V[(\tilde{A} \geq \tilde{A}_1) \wedge (\tilde{A} \geq \tilde{A}_2) \wedge \cdots \wedge (\tilde{A} \geqslant \tilde{A}_k)] \\ &= \min V(\tilde{A} \geqslant \tilde{A}_i), i = 1, \ldots, k. \end{aligned} \tag{2.13}$$

For the triangular fuzzy numbers from Example 2.1, the following values of the respective degrees of possibility are obtained.

$$\begin{aligned} &V(\tilde{A}_1 \geqslant \tilde{A}_2, \tilde{A}_3, \tilde{A}_4) = 0.35, \\ &V(\tilde{A}_2 \geqslant \tilde{A}_1, \tilde{A}_3, \tilde{A}_4) = 0.96, \\ &V(\tilde{A}_3 \geqslant \tilde{A}_1, \tilde{A}_2, \tilde{A}_4) = 0.06, \\ &V(\tilde{A}_4 \geqslant \tilde{A}_1, \tilde{A}_2, \tilde{A}_3) = 1. \end{aligned}$$

This gives exactly the same order as the one obtained using the probabilistic approach and Yager index.

2.6 Weighted Averaging Approach Based on α–cuts

This section describes the ordering of *LR*-type fuzzy numbers associated with defuzzification of parametrically represented fuzzy numbers [23]. In the case of *LR*-type fuzzy numbers, the parametric representation (1.8) can be written in the following form:

$$\tilde{A} = \bigcap_{\alpha \in [0,1]} (\alpha, [L_{\tilde{A}}^{-1}(\alpha), R_{\tilde{A}}^{-1}(\alpha)]), \tag{2.14}$$

where $L_{\tilde{A}}^{-1}, R_{\tilde{A}}^{-1} : [0, 1] \to \Re$ are inverse functions of the respective shape functions of an *LR*-type fuzzy number $\tilde{A}$.

Definition 2.3 ([24]) Let $\tilde{A} \in \mathcal{F}(\Re)_{LR}$. The weighted averaging based on α–cuts representation of a fuzzy number $\tilde{A}$ is defined by:

$$I(\tilde{A}) = \int_0^1 (c_L L_{\tilde{A}}^{-1}(\alpha) + c_R R_{\tilde{A}}^{-1}(\alpha)) p(\alpha) d\alpha, \tag{2.15}$$

where c_L, c_R are, respectively, the optimism and pessimism parameters, $p(\alpha)$ is a distribution function of the importance of the α–cuts.

The c_L, c_R parameters and the function $p(\alpha)$ satisfy the conditions:

$$c_L > 0, c_R > 0, c_L + c_R = 1,$$

$$p : [0, 1] \to \Re_+, \quad \int_0^1 p(\alpha)d\alpha = 1.$$

The function $p(\alpha)$ is also called the weighted averaging parameter. Following [23], it is assumed that

$$p(\alpha) = (k + 1)\alpha^k,$$

where $k > 0$ is a parameter.

Theorem 2.1 ([24]) *Let $\tilde{A} = (a, b, \alpha, \beta)_{LR}$ and assume that the distribution of the function of the importance of the degrees have the form of relation (2.15). Then, the following formula is valid for weighted averaging:*

$$I(\tilde{A}) = c_L\left(\beta - \frac{k+1}{k+2}(\beta - \alpha)\right) + c_R\left(a - \frac{k+1}{k+2}(b - a)\right). \tag{2.16}$$

The value $I(\tilde{A})$ is a crisp value used to rank fuzzy numbers. The greater this value is, the greater is the fuzzy number. Moreover, $I(\tilde{A}) = I(\tilde{A}) = I(\tilde{B})$ if and only if $\tilde{A} = \tilde{B}$.

Example 2.2 In this example it is assumed that $p(\alpha) = 2$ ($k = 1$), and the "optimism/pessimism" coefficients are 0.5. Now, let the following three sets of trapezoidal fuzzy numbers and a set of triangular fuzzy numbers be given (see Fig. 2.4):

Set 1: $\tilde{A}_1 = (0.5, 0.5, 0.1, 0.5)$, $\tilde{A}_2 = (0.7, 0.7, 0.3, 0.3)$, $\tilde{A}_3 = (0.9, 0.9, 0.5, 0.1)$;
Set 2: $\tilde{A}_1 = (0.4, 0.7, 0.4, 0.1)$, $\tilde{A}_2 = (0.5, 0.5, 0.3, 0.4)$, $\tilde{A}_3 = (0.6, 0.6, 0.5, 0.2)$;
Set 3: $\tilde{A}_1 = (0.5, 0.5, 0.2, 0.2)$, $\tilde{A}_2 = (0.5, 0.8, 0.2, 0.1)$, $\tilde{A}_3 = (0.5, 0.5, 0.2, 0.4)$;
Set 4: $\tilde{A}_1 = (0.12, 0.19, 0.29)$, $\tilde{A}_2 = (0.22, 0.32, 0.48)$, $\tilde{A}_3 = (0.11, 0.15, 0.23)$, $\tilde{A}_4 = (0.21, 0.33, 0.49)$.

The ranking index values obtained for the set 1 are $I(\tilde{A}_1) = 0.37$, $I(\tilde{A}_2) = 0.50$, $I(\tilde{A}_3) = 0.63$. This gives the following order $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$. For the set 2, the ranking index values are $I(\tilde{A}_1) = 0.45$, $I(\tilde{A}_2) = 0.42$, $I(\tilde{A}_3) = 0.50$, which gives the order $\tilde{A}_2 < \tilde{A}_1 < \tilde{A}_3$. For the set 3, the ranking index values are $I(\tilde{A}_1) = 0.35$, $I(\tilde{A}_2) = 0.43$, $I(\tilde{A}_3) = 0.38$, which gives the order $\tilde{A}_1 < \tilde{A}_3 < \tilde{A}_2$. Finally, for the set 4 which is the same as in Example 2.1, the ranking index values are $I(\tilde{A}_1) = 0.14$, $I(\tilde{A}_2) = 0.22$, $I(\tilde{A}_3) = 0.10$, $I(\tilde{A}_4) = 0.23$. This gives exactly the same order as the one obtained using the previous approaches.

The α–cuts based approach is the most time consuming as it requires the pairwise comparison of all fuzzy numbers to be ordered. Also, the procedure of computing the possibility degree is more complicated that the computation of ranking indices in the two latter approaches.

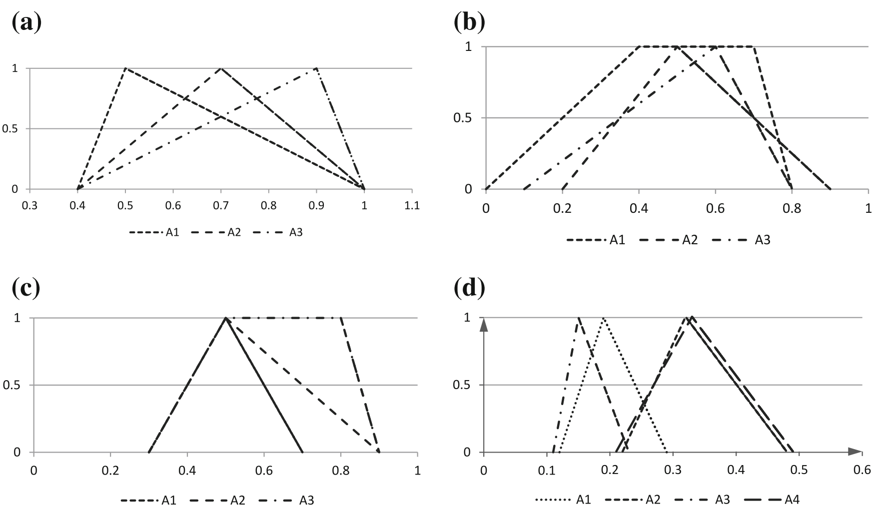

Fig. 2.4 Set 1 (**a**); Set 2 (**b**), Set 3 (**c**); Set 4 (**d**)

2.7 Two-Dimensional Radius of Gyration Approach

The two-dimensional radius of gyration (ROG) or *gyradius* is a concept in mechanics [25]:

$$r_g = \sqrt{I/A} \tag{2.17}$$

where I is the *second moment of area* (see Fig. 2.5) and A is the total cross-sectional area. The second moment of area of an arbitrary shape with respect to an arbitrary axis Z is defined and is computed by:

$$I_Z = \int_A r^2 dA, \tag{2.18}$$

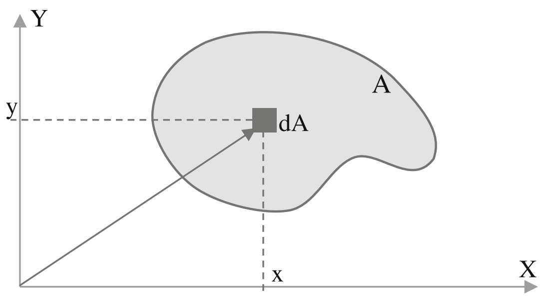

Fig. 2.5 A scheme of how the second moment of area is calculated for an arbitrary shape with respect to the Z axis; r is the radial distance to the element dA, with projections x and y on the axes

where dA is a differential area of the arbitrary shape and r is a distance from the axis Z to dA.

For example, when the desired reference axis is the X-axis, the second moment of area, I_x can be computed in Cartesian coordinates as:

$$I_x = \iint_A y^2 dxdy \tag{2.19}$$

The ROG point $(r_x^{\tilde{A}}, r_y^{\tilde{A}})$ for a fuzzy number $\tilde{A}$ is provided as [25]:

$$r_x^{\tilde{A}} = \sqrt{I_x/A}, \tag{2.20}$$

$$r_y^{\tilde{A}} = \sqrt{I_y/A}, \tag{2.21}$$

where I_x is the second moment of area with respect to x, and I_y is the second moment of area with respect to y. It is assumed that the mass density at each point of the area equals 1.

In the case of a trapezoidal fuzzy number, the second moment of area can be calculated in the following way. First, the trapezoidal area of a fuzzy number is divided onto three areas A_1, A_2, A_3 (see Fig. 2.6). It is known that for an area made up of a number of simple shapes, the second moment of area is the sum of the second moments of each of the individual areas about the desired axis [25]:

$$\begin{aligned} I_x &= I_x^{A_1} + I_x^{A_2} + I_x^{A_3} \\ I_y &= I_y^{A_1} + I_y^{A_2} + I_y^{A_3} \end{aligned} \tag{2.22}$$

The respective moments of inertia of the areas are given by [25]:

$$I_x^{A_1} = \int_{A_1} y^2 dA = \int_0^1 y^2(b-q)(1-y)dy = \frac{b-a}{12} \tag{2.23}$$

$$I_y^{A_1} = \int_{A_1} x^2 dA = \frac{(b-a)^3}{4} + \frac{(b-a)a^2}{2} + \frac{2(b-a)^2 a}{3} \tag{2.24}$$

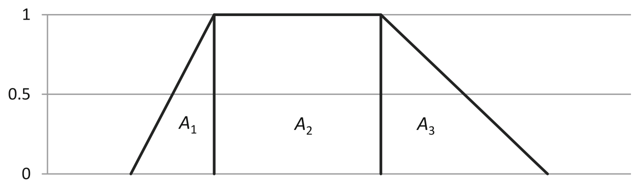

Fig. 2.6 A division of a trapezoid into three parts A_1, A_2 and A_3

$$I_x^{A_2} = \frac{(c-b)}{3} \tag{2.25}$$

$$I_y^{A_2} = \frac{(c-b)^3}{3} + (c-b)b^2 + (c-b)^2 b \tag{2.26}$$

$$I_x^{A_3} = \frac{(d-c)}{12} \tag{2.27}$$

$$I_y^{A_3} = \frac{(d-c)^3}{12} + \frac{(d-c)c^2}{2} + \frac{(d-c)^2 c}{3} \tag{2.28}$$

Then, the ROG point of a trapezoidal fuzzy number is calculated as [25]:

$$\begin{aligned} r_x^{\tilde{A}} &= \sqrt{\tfrac{I_x^{A_1}+I_x^{A_2}+I_x^{A_3}}{((c-b)+(d-a))/2}}, \\ r_y^{\tilde{A}} &= \sqrt{\tfrac{I_y^{A_1}+I_y^{A_2}+I_y^{A_3}}{((c-b)+(d-a))/2}}. \end{aligned} \tag{2.29}$$

For a crisp number a, the ROG point is defined by [25]:

$$r_x^a = \sqrt{3}/3, \quad r_y^a = a. \tag{2.30}$$

The ROG point $(r_x^{\tilde{A}}, r_y^{\tilde{A}})$ is used to define the index [25]

$$S(\tilde{A}) = r_x^{\tilde{A}} \cdot r_y^{\tilde{A}} \tag{2.31}$$

which is used to compare fuzzy numbers. The larger is the index, the greater is a fuzzy number. Thus, given two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the following holds:

$$\begin{aligned} &\text{If } S(\tilde{A}) > S(\tilde{B}) \text{ Then } \tilde{A} > \tilde{B}, \\ &\text{If } S(\tilde{A}) < S(\tilde{B}) \text{ Then } \tilde{A} < \tilde{B}, \\ &\text{If } S(\tilde{A}) = S(\tilde{B}) \text{ Then } \tilde{A} = \tilde{B}. \end{aligned} \tag{2.32}$$

Example 2.3 The values of the index S obtained for the fuzzy numbers from Example 2.1 are presented in Table 2.6.

This gives exactly the same ordering as those obtained using previously described approaches.

Table 2.6 Yager index for the fuzzy numbers from Example 2.1

Fuzzy number	S
$\tilde{A}_1 = (0.12, 0.19, 0.29)$	0.8167
$\tilde{A}_2 = (0.22, 0.32, 0.48)$	1.2484
$\tilde{A}_3 = (0.11, 0.15, 0.23)$	0.4410
$\tilde{A}_4 = (0.21, 0.33, 0.49)$	0.4217

2.8 Fuzzy Maximising-Minimising Points Approach

The ordering of fuzzy numbers using fuzzy maximising-minimising points is based on the centre of gravity point, defined in the previous section, left and right spreads and the distance between fuzzy numbers.

Definition 2.4 The distance between two arbitrary fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined by:

$$d(\tilde{A}, \tilde{B}) = \left[\int_0^1 (f_{\tilde{A}}(y) - f_{\tilde{B}}(y))^2 dy + \int_0^1 (g_{\tilde{A}}(y) - g_{\tilde{B}}(y))^2 dy\right] \tag{2.33}$$

The fuzzy minimising-maximising points are obtained using the method from [26]. Let the fuzzy numbers $\tilde{A}_i$, $i = 1, 2, \ldots, n$ be given and let $\tilde{M}$ denote the fuzzy maximising point and $\tilde{m}$ the fuzzy minimising point. The *COGP* of the minimising and maximising points are computed as follows:

$$COGP(\tilde{M}) = \left(\max_{i=1,2,\ldots,n}\{\overline{x}_0(\tilde{A}_i)\}, \max_{i=1,2,\ldots n}\{\overline{y}_0(\tilde{A}_i)\}\right)$$

$$COGP(\tilde{m}) = \left(\min_{i=1,2,\ldots,n}\{\overline{x}_0(\tilde{A}_i)\}, \min_{i=1,2,\ldots n}\{\overline{y}_0(\tilde{A}_i)\}\right)$$

The left and right spreads of $\tilde{M}$ and $\tilde{m}$ are computed analogously:

$$L_{\tilde{M}} = \max_{i=1,2,\ldots,n}\{L_{\tilde{A}_i}\}, \quad R_{\tilde{M}} = \max_{i=1,2,\ldots,n}\{R_{\tilde{A}_i}\},$$

$$L_{\tilde{m}} = \min_{i=1,2,\ldots n}\{L_{\tilde{A}_i}\}, \quad R_{\tilde{m}} = \min_{i=1,2,\ldots n}\{R_{\tilde{A}_i}\}.$$

Now, given $COGP(\tilde{M})$, $L_{\tilde{M}}$, $R_{\tilde{M}}$ and $COGP(\tilde{m})$ $L_{\tilde{m}}$, $R_{\tilde{m}}$, the goal is to uniquely determine, respectively, $\tilde{M}$ and $\tilde{m}$.

In general case, an unknown fuzzy number $\tilde{A}$ can be uniquely determined based on its centroid point $(\overline{x}_0(\tilde{A}), \overline{y}_0(\tilde{A}))$ and left L and right R spreads by solving the following system of nonlinear equations [26]:

$$\begin{cases} b - a = L \\ d - c = R \\ \frac{1}{3}\left[a + b + c + d - \frac{cd-ab}{(c+d)-(a+b)}\right] = \overline{x}_0(\tilde{A}) \\ \frac{1}{3}\left[1 + \frac{c-b}{(c+d)-(a+b)}\right] = \overline{y}_0(\tilde{A}) \end{cases}$$

The fuzzy ranking using fuzzy minimising-maximising points uses the following relative closeness coefficient:

$$D(\tilde{A}) = \gamma(\tilde{A}) \cdot \frac{D^L_{\tilde{A}}}{1 + D^R_{\tilde{A}}}, \tag{2.34}$$

where

$$D^L_{\tilde{A}} = d(\tilde{A}, \tilde{m}),$$

$$D^R_{\tilde{A}} = d(\tilde{A}, \tilde{M}),$$

and

$$\gamma(\tilde{A}) = \begin{cases} 1, & \text{if } \int_0^1 \{f_{\tilde{A}}(y) + g_{\tilde{A}}(y)\}dy \geqslant 0, \\ -1, & \text{if } \int_0^1 \{f_{\tilde{A}}(y) + g_{\tilde{A}}(y)\}dy < 0. \end{cases}$$

The ranking rules for fuzzy numbers $\tilde{A}_i$, $i = 1, 2, \ldots, n$, are the following [26]:

$$\begin{array}{l} A_i < A_j \text{ iff } D(A_i) < D(A_j) \\ A_i > A_j \text{ iff } D(A_i) > D(A_j) \\ A_i \approx A_j \text{ iff } D(A_i) = D(A_j) \end{array}$$

Example 2.4 Consider the fuzzy numbers from Example 2.1. The corresponding fuzzy maximising and minimising points are depicted in Fig. 2.7.

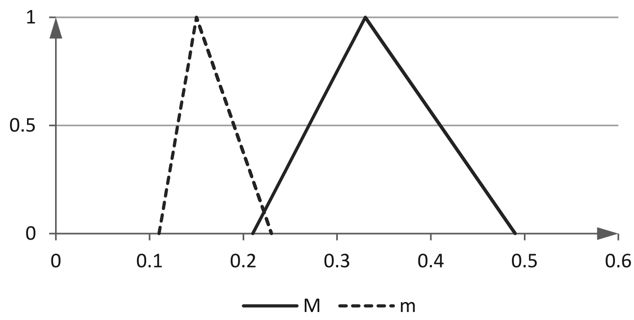

Fig. 2.7 Fuzzy maximising $\tilde{M}$ and minimising $\tilde{m}$ points

The following values of the relative closeness coefficient were obtained: $D(A_1) = 0.0067$, $D(A_2) = 0.0395$, $D(A_3) = 0.0015$, $D(A_4) = 0.0468$, which gives exactly the same ordering as those obtained using previously described approaches.

The method is quite complicated compared to other presented methods. It is also time consuming and requires to use additional tools, such as numerical integration and solving systems of nonlinear equations. It also requires the pairwise comparison of fuzzy numbers to be ordered.

2.9 Area Based Approach

For a fuzzy number $\tilde{A} = (f_{\tilde{A}}, g_{\tilde{A}})$, the following values are defined [27]:

$$\begin{aligned} \tilde{A}_l &= m + \tfrac{1}{2} H_l, \\ \tilde{A}_u &= m + \tfrac{1}{2} H_u \end{aligned} \tag{2.35}$$

where $m = \frac{1}{2}(f_{\tilde{A}}^{-1}(1) + g_{\tilde{A}}^{-1}(1))$, and H_l, H_u are defined as follows:

$$\begin{aligned} H_l &= \frac{\int_0^1 f_{\tilde{A}}(y)dy}{\int_0^1 f_{\tilde{A}}(y)dy + \int_0^1 g_{\tilde{A}}(y)dy}, \\ H_u &= \frac{\int_0^1 g_{\tilde{A}}(y)dy}{\int_0^1 f_{\tilde{A}}(y)dy + \int_0^1 g_{\tilde{A}}(y)dy} \end{aligned} \tag{2.36}$$

Now, for given two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the following values are defined [27]:

$$\overline{R}(\tilde{A}, \tilde{B}) = \tilde{A}_u - \tilde{B}_u, \quad \underline{R}(\tilde{A}, \tilde{B}) = \tilde{A}_l - \tilde{B}_l \tag{2.37}$$

They are used to determine the comparison rules:

$$\begin{aligned} \underline{R}(\tilde{B}, \tilde{A}) > \overline{R}(\tilde{A}, \tilde{B}) \text{ iff } \tilde{A} < \tilde{B}, \\ \underline{R}(\tilde{B}, \tilde{A}) = \overline{R}(\tilde{A}, \tilde{B}) \text{ iff } \tilde{A} \approx \tilde{B}. \end{aligned} \tag{2.38}$$

It follows from the definition of $\overline{R}(\tilde{A}, \tilde{B})$ and $\underline{R}(\tilde{B}, \tilde{A})$ that

$$\begin{aligned} \overline{R}(\tilde{A}, \tilde{B}) &= -\overline{R}(\tilde{B}, \tilde{A}) \\ \underline{R}(\tilde{A}, \tilde{B}) &= -\underline{R}(\tilde{B}, \tilde{A}) \end{aligned}$$

Thus, the comparison rules (2.38) can be written in the following form:

$$\begin{aligned} -\underline{R}(\tilde{A}, \tilde{B}) > \overline{R}(\tilde{A}, \tilde{B}) \text{ iff } \tilde{A} < \tilde{B}, \\ -\underline{R}(\tilde{A}, \tilde{B}) = \overline{R}(\tilde{A}, \tilde{B}) \text{ iff } \tilde{A} = \tilde{B}. \end{aligned} \tag{2.39}$$

Table 2.7 The values of $\underline{R}$, $\overline{R}$ and the pairwise comparison results

$\underline{R}$	$\overline{R}$	Comparison
$\underline{R}(\tilde{A}_1, \tilde{A}_2) = -0.07558$	$\underline{R}(\tilde{A}_1, \tilde{A}_2) = -0.05442$	$\tilde{A}_1 < \tilde{A}_2$
$\underline{R}(\tilde{A}_1, \tilde{A}_3) = 0.006155$	$\underline{R}(\tilde{A}_1, \tilde{A}_3) = 0.033845$	$\tilde{A}_1 > \tilde{A}_3$
$\underline{R}(\tilde{A}_1, \tilde{A}_4) = -0.074654$	$\underline{R}(\tilde{A}_1, \tilde{A}_4) = -0.065346$	$\tilde{A}_1 < \tilde{A}_4$
$\underline{R}(\tilde{A}_2, \tilde{A}_3) = 0.081735$	$\underline{R}(\tilde{A}_2, \tilde{A}_3) = 0.088265$	$\tilde{A}_2 > \tilde{A}_3$
$\underline{R}(\tilde{A}_2, \tilde{A}_4) = 0.000926$	$\underline{R}(\tilde{A}_2, \tilde{A}_4) = -0.010926$	$\tilde{A}_2 > \tilde{A}_4$
$\underline{R}(\tilde{A}_3, \tilde{A}_4) = -0.080809$	$\underline{R}(\tilde{A}_3, \tilde{A}_4) = -0.099191$	$\tilde{A}_3 > \tilde{A}_4$

Example 2.5 Consider the fuzzy numbers from Example 2.1. The values of $\overline{R}$, $\underline{R}$ and the pairwise comparison results are presented in Table 2.7.

This gives exactly the same order as the one obtained using the methods described so far.

2.10 Left and Right Dominance Approach

The ordering of fuzzy numbers based on the left and right dominance was proposed in [28]. This approach uses left and right bounds of selected α–cuts of fuzzy numbers to be compared.

Definition 2.5 The left $D^L_{i,j}$ and right $D^R_{i,j}$ dominance of a fuzzy number $\tilde{A}_i$ over a fuzzy number $\tilde{A}_j$ is defined as the average difference of the left and right bounds of $\tilde{A}_i$ and $\tilde{A}_j$ at some α-levels:

$$D^L_{ij} = \frac{1}{n+1} \sum\nolimits_{k=0}^{n} (l_{ik} - l_{jk}) \tag{2.40}$$

$$D^R_{ij} = \frac{1}{n+1} \sum\nolimits_{k=0}^{n} (r_{ik} - r_{jk}) \tag{2.41}$$

where n is the numbers of α–cuts, l_{ik}, r_{ik} are, respectively, left and right spreads of a fuzzy number $\tilde{A}_i$ at the α_k-level.

It is assumed that α-levels are spread uniformly, i.e., k/n, $k = 1, 2, \ldots, n$. The values $D^L_{i,j}$ and $D^R_{i,j}$ approximate the area difference of $\tilde{A}_i$ over $\tilde{A}_j$ according to the membership axis to the, respectively, left and right membership function as $n \to \infty$ [28]. The total dominance of $\tilde{A}_i$ over $\tilde{A}_j$ with the index of optimism $\beta \in [0, 1]$ is defined as follows.

Definition 2.6 The total dominance $D^T_{i,j}$ of a fuzzy number $\tilde{A}_i$ over a fuzzy number $\tilde{A}_j$ with optimism index $\beta \in [0, 1]$ is defined as a convex combinations of left $D^L_{i,j}$

Table 2.8 The values of total dominance

Total dominance index	Value
$D^T_{12}(0.5)$	−0.1375
$D^T_{13}(0.5)$	0.0375
$D^T_{14}(0.5)$	−0.1425
$D^T_{23}(0.5)$	0.1750
$D^T_{24}(0.5)$	−0.005
$D^T_{34}(0.5)$	−0.1800

and right $D^R_{i,j}$ dominance:

$$D^T_{ij}(\beta) = \beta D^L_{ij} + (1-\beta) D^R_{ij} \tag{2.42}$$

The index of optimism is used to reflect a decision maker's degree of optimism [28].

The total dominance index is used to define the rules of comparison of two fuzzy numbers. They are the following:

$$\begin{array}{l} \text{If } D^T_{ij} < 0 \text{ Then } \tilde{A}_i < \tilde{A}_j, \\ \text{If } D^T_{ij} > 0 \text{ Then } \tilde{A}_i > \tilde{A}_j, \\ \text{If } D^T_{ij} = 0 \text{ Then } \tilde{A}_i = \tilde{A}_j. \end{array}$$

Example 2.6 Consider the fuzzy numbers from Example 2.1. The obtained values of the total dominance with $\beta = 0.5$ and $n = 5$ are summarised in Table 2.8.

This gives the ordering $\tilde{A}_3 < \tilde{A}_1 < \tilde{A}_2 < \tilde{A}_4$, which exactly the same as the one obtained using the previous methods.

2.11 An α-weighted Valuations Approach

Approaches to the ranking of fuzzy numbers based upon the idea of associating with a fuzzy number a scalar value, i.e., its valuation, was developed by Yager [22]. Later, Yager and Filev [29] improved this valuation method by the transformation of a fuzzy subset into an associated probability distribution. They introduced a family of parametric valuation functions. The problem of ranking fuzzy numbers using valuation methods was also considered by Detyniecki and Yager [3].

A generalised formula for a class of valuation functions has the form:

$$Val(\tilde{A}) = \frac{\int_0^1 Ave(\tilde{A}^\alpha) f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha} \tag{2.43}$$

where f is a mapping $f : [0, 1] \to [0, 1]$. In [29], Yager and Filev proposed two complementary families of parametric valuation functions. The one is an increasing family:

$$f : [0, 1] \ni \alpha \to \alpha^q \in [0, 1], \quad q > 0,$$

and the other one is decreasing

$$f : [0, 1] \ni \alpha \to (1-\alpha)^q \in [0, 1], \quad q > 0.$$

Some interesting properties of this two families of functions can be found in [29]. One of them is that increasing family emphasises the higher α-levels, whereas the decreasing family emphasises lower α-levels, which causes that these two families can produce two opposite orderings.

In order to calculate the valuation for a given fuzzy number $\tilde{A}$, the value of

$$Ave(\tilde{A}^\alpha) = \frac{\inf(\tilde{A}_\alpha) + \sup(\tilde{A}_\alpha)}{2}$$

must be first computed. In the case of trapezoidal fuzzy numbers

$$Ave(\tilde{A}^\alpha) = \frac{a + (b-a)\alpha + d - (d-c)\alpha}{2} = \frac{b+c}{2}\alpha + \frac{a+d}{2}(1-\alpha).$$

Then, the valuation formula (2.43) takes the form:

$$Val(\tilde{A}) = \frac{\frac{1}{2}\int_0^1 ((b+c)\alpha + (a+d)(1-\alpha)) f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}. \tag{2.44}$$

which can be simplified to:

$$Val(\tilde{A}) = \frac{b+c}{2} w + \frac{a+d}{2}(1-w),$$

where

$$w = \frac{\int_0^1 \alpha f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}.$$

For the increase case

$$w = \frac{q+1}{q+2},$$

and for the decrease case

$$w = \frac{1}{q+2}.$$

Table 2.9 The results of comparison of $\tilde{A}_1$ and $\tilde{A}_2$ using increasing valuation function with different values of the parameter q

q	$Val(\tilde{A}_1)$	$Val(\tilde{A}_2)$	Comparison
0	6	7	$<$
1	6.33	6.67	$<$
2	6.5	6.5	$=$
3	6.6	6.4	$>$
∞	7	6	$>$

Example 2.7 Consider the following triangular fuzzy numbers: $\tilde{A}_1 = (1, 7, 9)$, $\tilde{A}_2 = (4, 6, 12)$, $\tilde{A}_3 = (5, 8, 9)$, $\tilde{A}_4 = (2, 9, 10)$. The results of comparison of $\tilde{A}_1$ and $\tilde{A}_2$, obtained using an increasing functions with different values of q, are presented in Table 2.9. The results of ordering obtained using increasing and decreasing functions with $q = 2$ are given in Table 2.11. Finally, Table 2.10 presents the ordering of fuzzy numbers from Example 2.1 obtained using decreasing and increasing valuation functions with different value of the parameter q.

The results show that for $q = 2$, the ordering of fuzzy numbers is exactly the same as the one obtained using other considered methods.

Table 2.10 The results of comparison of exemplary fuzzy numbers using increasing and decreasing valuation functions with different values of the parameter q

	$Val(\cdot)$ (increasing)	$Val(\cdot)$ (decreasing)	Order (increasing)	Order (decreasing)
$q = 0$				
$\tilde{A}_1$	6.0	6.0	$\tilde{A}_4 = \tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$	$\tilde{A}_4 = \tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$
$\tilde{A}_2$	7.0	7.0		
$\tilde{A}_3$	7.5	7.5		
$\tilde{A}_4$	7.5	7.5		
$q = 1$				
$\tilde{A}_1$	6.33	5.67	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_2 > \tilde{A}_1$	$\tilde{A}_3 = \tilde{A}_2 > \tilde{A}_4 > \tilde{A}_1$
$\tilde{A}_2$	6.67	7.33		
$\tilde{A}_3$	7.67	7.33		
$\tilde{A}_4$	8.00	7.00		
$q = 2$				
$\tilde{A}_1$	6.5	5.5	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_2 = \tilde{A}_1$	$\tilde{A}_2 > \tilde{A}_3 > \tilde{A}_4 > \tilde{A}_1$
$\tilde{A}_2$	6.5	7.5		
$\tilde{A}_3$	7.75	7.25		
$\tilde{A}_4$	8.25	6.75		
$q = \infty$				
$\tilde{A}_1$	7.0	5.0	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$	$\tilde{A}_2 > \tilde{A}_3 > \tilde{A}_4 > \tilde{A}_1$
$\tilde{A}_2$	6.0	8.0		
$\tilde{A}_3$	8.0	7.0		
$\tilde{A}_4$	9.0	6.0		

Table 2.11 The results of comparison of exemplary fuzzy numbers using increasing and decreasing valuation functions with different values of the parameter q

	$Val(\cdot)$ (increasing)	$Val(\cdot)$ (decreasing)	Order (increasing)	Order (decreasing)
$q = 0$				
$\tilde{A}_1$	0.2	0.2	$\tilde{A}_4 = \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$	$\tilde{A}_4 = \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$
$\tilde{A}_2$	0.34	0.34		
$\tilde{A}_3$	0.16	0.16		
$\tilde{A}_4$	0.34	0.34		
$q = 1$				
$\tilde{A}_1$	0.2	0.2	$\tilde{A}_4 > \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$	$\tilde{A}_4 = \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$
$\tilde{A}_2$	0.33	0.34		
$\tilde{A}_3$	0.16	0.16		
$\tilde{A}_4$	0.34	0.34		
$q = 2$				
$\tilde{A}_1$	0.19	0.2	$\tilde{A}_4 > \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$	$\tilde{A}_4 > \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$
$\tilde{A}_2$	0.33	0.34		
$\tilde{A}_3$	0.16	0.17		
$\tilde{A}_4$	0.34	0.35		
$q = \infty$				
$\tilde{A}_1$	0.19	0.2	$\tilde{A}_4 = \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$	$\tilde{A}_4 = \tilde{A}_2 > \tilde{A}_1 > \tilde{A}_3$
$\tilde{A}_2$	0.33	0.35		
$\tilde{A}_3$	0.15	0.17		
$\tilde{A}_4$	0.33	0.35		

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