Chapter 2
Variance Contracts: Fixed Income Security Design

2.1 Introduction

Variance swaps are contracts in which the seller pays the amount by which the realized variance of some variable of interest exceeds a threshold predetermined at contract origination. The pricing of variance swaps on equities is well-understood. Decades' worth of financial theory suggests that these contracts can be cast in a "model-free" fashion—the only ingredient that is required for pricing is the price of at-the-money (ATM) and out-of-the-money (OTM) options referencing the stock or stock index of interest.

This model-free methodology still relies on some assumptions, such as absence of arbitrage in frictionless markets. Nevertheless, it is an appealing methodology, which the Chicago Board Options Exchange adopted with a change in the definition of its VIX index in September 2003 to incorporate financial theory developed after the seminal efforts that led to the initial launch of the index in 1993 (see Whaley 1993).

This chapter develops theoretical foundations for variance swaps in the fixed income space. As in the equity case, we seek contract designs that admit model-free pricing. The designs need to be internally consistent in that the contracts' value collapses to a constant, the Black (1976) implied variance, under the hypothetical circumstance of markets with constant uncertainty. The next chapters are self-contained but rely on much of the analysis in this chapter. We shall identify situations where fixed income volatility can be priced in this fashion, and point to cases where it cannot be due to the complex nature of the assets underlying the markets we study.

For example, it is well known that the prices of options on S&P futures can be used to construct an equity-like VIX index. In contrast, Chap. 4 explains that aggregating the prices of options expiring strictly before the expiry of their underlying bond futures (a practically relevant case) results in a model-dependent government bond volatility index; that is, a model-dependent bias arises once a model-free expression is utilized to approximate the true index. A similar bias comes into

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existence when constructing basis point volatility indexes for rates in time-deposit markets (see, again, Chap. 4).

To fully understand the nature of these biases, we start by focusing on ideal situations where these biases do not arise in the first place. We shall establish a connection between this issue and the theory of *market numéraires*. Market numéraires are conceptual tools that allow us to deal with the various complexities arising whilst evaluating interest rate derivatives. It is well known that absence of arbitrage in frictionless markets is equivalent to the property that asset prices, once rescaled by the money market account, are martingales under the risk-neutral probability. While this result is a powerful tool for the analysis of equity derivatives, numéraires other than the money market account are useful when pricing interest rate derivatives, as initially explained by Geman (1989), Jamshidian (1989), and Geman et al. (1995) and, then, further developed by Jamshidian (1997) and Schönbucher (2003, Chap. 7).

The connection we make in this chapter is that the price of fixed income variance swaps is model-free only once the relevant payoffs are rescaled by the numéraire appropriate for each market of interest. We show how to incorporate different market numéraires into the early theory of "spanning contracts" developed over the years in the equity case by Neuberger (1994), Dumas (1995), Demeterfi et al. (1999a, 1999b), Bakshi and Madan (2000), Britten-Jones and Neuberger (2000), and Carr and Madan (2001), among others.

There are additional complications arising in fixed income markets, such as the notion of basis point volatility, which does not arise in equity markets. Basis point volatility is actually not a mere matter of quoting convention, and highlights a fundamental difference between interest rate and equity volatility. The concept of basis point volatility naturally arises because absolute changes describe risk more effectively than relative changes in the context of yields and spreads. A rate increase from 10 bps to 15 bps shares the same percentage change as one from 100 bps to 150 bps, but, all else equal and accounting for convexity, the latter is a nearly tenfold P&L and risk event. In this basic example, it is more useful for rates traders to know whether a position is likely to experience 5 bps moves or 50 bps moves over a given horizon, and a basis point formulation of the problem addresses this by model-free pricing of a variance swap on arithmetic changes in the fixed income instrument of interest instead of logarithmic changes as in the more standard case of equity variance swaps.

Dealing with basis point variance contracts in a context with random interest rates and numéraires leads to issues that have not been considered in the equity literature. We shall deal with these complications through an insight, a linkage between a class of spanning contracts known as "quadratic" to a notion of basis point variability. This insights allows us to price basis point volatility in a model-free and consistent fashion with the notion of numéraire prevailing in each market of interest. The ensuing fixed income volatility indexes in this and the following chapters originate from these dedicated contract designs.

The plan of this chapter is as follows. The next section provides definitions of the risks we wish to price, and the notions of market numéraires needed to achieve this purpose. Section 2.3 introduces dedicated contract designs leading to model-free pricing, and Sect. 2.4 deals with indexes constructed upon these contracts.

Section 2.5 develops further properties of basis point variance swaps, contains indications on how to implement them in the presence of shrinking maturities, and provides estimates of variance risk-premiums based on CBOE's SRVIX index for interest rate swap market volatility. Section 2.6 unveils theoretical properties of basis point and percentage volatility indexes, and compares them in cases in which a limited number of options are available for calculating these indexes. Section 2.7 extends the analysis to markets with discontinuities. Appendix A provides technical details omitted from the main text.

The reader who is not interested in the unified theory may skip the present chapter and directly access the subsequent chapters, in which variance swaps and accompanying model-free volatility indexes are dealt with in a self-contained fashion for various fixed income asset classes.

2.2 Market Numéraires and Volatilities

Consider a forward starting agreement, originated at time t, with a payoff Π_T at time T equal to

$$\Pi_T \equiv N_T \times (X_T - K), \tag{2.1}$$

where both X_T and N_T are measurable with respect to the information set at time T, $\mathbb{F}_T$, and K is chosen at t, so that the value of the contract is zero at inception. Let Q denote the risk-neutral probability, and $\mathbb{E}_t(\cdot)$ the expectation under Q, taken conditionally on $\mathbb{F}_t$, and let r_t denote the short-term rate at t. We assume that N_τ is the price of a tradeable asset for each $\tau \in [t, T]$, and that it is strictly positive.

It is well known (e.g., Mele 2014, Chap. 4) that under regularity conditions, there exist (i) a probability Q^N, and (ii) a martingale process X_τ under Q^N that clears the agreement, i.e. $X_t = K$, so that the value of Π_T is zero at t. Accordingly, we refer to X_τ as the *forward risk process*, and N_T as the value of a *market numéraire* at T, so that any asset price process S_t normalized by N_t is a martingale under Q^N,

$$\frac{S_t}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{S_T}{N_T}\right),$$

where $\mathbb{E}_t^{Q^N}$ denotes the conditional expectation under Q^N. We call Q^N the *market numéraire probability*. The money market account is the standard notion of numéraire, for which Q^N collapses to the risk-neutral probability.

Examples 2.1 In Chap. 3, N_t is the present value of an annuity of one dollar, Q^N is the *annuity, or swap, probability*, and X_t is the forward swap rate; accordingly, the payoff Π_T in Eq. (2.1) is that of a forward starting swap. In Chap. 4, X_t can be either the forward price of a coupon bearing bond or the forward price of time deposits such as Eurodollars; in both cases, N_t is the price of a zero coupon bond expiring at time T, so that $N_T = 1$, and Q^N is the *forward probability*. Finally, in

Chap. 5, N_t is the present value of a defaultable annuity of one dollar, Q^N is the *survival contingent probability* and, finally, X_t is the loss-adjusted forward default swap index, so that the payoff Π_T in Eq. (2.1) is that of an index default swap.

It is the volatility of X_τ that we are interested in pricing. Unless otherwise stated, we assume that X_τ is a strictly positive diffusion process with stochastic volatility. Section 2.7 contains extensions to jump-diffusions. Let W_τ denote a multidimensional Wiener process under Q^N. Since X_τ is strictly positive, there exists a process σ_τ adapted to W_τ, such that

$$\frac{dX_\tau}{X_\tau} = \sigma_\tau \cdot dW_\tau, \quad \tau \in [t, T]. \tag{2.2}$$

We consider two notions of realized variance. One, based on *arithmetic*, or *basis point* (BP henceforth), changes of X_t in Eq. (2.2), and another based on the *logarithmic*, or *percentage*, changes of X_t. Accordingly, let $V^{\mathrm{bp}}(t, T)$ and $V(t, T)$ denote the realized BP variance and percentage variance in the time interval $[t, T]$,

$$V^{\mathrm{bp}}(t, T) \equiv \int_t^T X_\tau^2 \|\sigma_\tau\|^2 d\tau \quad \text{and} \quad V(t, T) \equiv \int_t^T \|\sigma_\tau\|^2 d\tau. \tag{2.3}$$

While the concept of percentage variance is widely known and used in equity markets, we also consider pricing BP variance to match the fixed income market practice of quoting implied volatilities both in percentage and basis point terms. The aim of the next section is to search for variance swap contract designs, based on $V^{\mathrm{bp}}(t, T)$ and $V(t, T)$, for which the fair value may be expressed in a model-free fashion in a sense to be made precise below. Indexes of expected volatility can then be formulated based on these variance contract designs. We shall return to the definition and properties of $V^{\mathrm{bp}}(t, T)$ in Sect. 2.5.

2.3 Interest Rate Variance Swaps

The risks we study in this book are spanned by interest rate derivatives with payoffs such as those in Eq. (2.1). These payoffs have two components: (i) the forward risk, X_τ at $\tau = T$, which we want to price the volatility of, and (ii) the market numéraire, N_T, which links to the very nature of the derivative involved in the market of interest. We aim to design variance swaps corresponding to the two variances in Eqs. (2.3) so that component (ii) does not affect model-free pricing of these contracts.

2.3.1 Contracts and Model-Free Pricing

We introduce a class of forward contracts for which we replace the standard unit notional with a stochastic notional, as in the following definition:

Definition 2.1 (Forward Contract with Stochastic Multiplier) A forward contract with stochastic multiplier is a contract originated at time t, which promises to pay the following payoff at $T > t$: $\Phi_T \equiv Y_T \times (\Psi(\{X_s\}_{s\in[t,T]}) - K_Y)$, where $\Psi(\cdot)$ is a functional of the entire path of X_t in Eq. (2.2), Y_τ is $\mathbb{F}_\tau$-measurable for $\tau \in [t, T]$, and the strike K_Y is set so that the value of the contract is zero at inception. Y_T is referred to as stochastic multiplier of the security design.

We want to express the fair value, K_Y, as the conditional expectation of the payoff, $\Psi(\cdot)$, taken under an appropriate probability. If interest rates were constant or deterministic, this probability would be the risk-neutral probability once we assume $Y_T \equiv 1$. In the general case, which is relevant to problems arising in fixed income markets, the appropriate probability is obtained once we impose the usual condition that $\mathbb{E}_t(e^{-\int_t^T r_u du}\Phi_T) = 0$, yielding the expression recorded in the next proposition, given without proof. We shall refer to this probability as the *forward multiplier probability*.

Proposition 2.1 (Forward Multiplier Probability) *The fair value of the strike in the forward contract of Definition 2.1 is*

$$K_Y = \mathbb{E}_t^{Q^Y}\left(\Psi\left(\{X_s\}_{s\in[t,T]}\right)\right), \tag{2.4}$$

where $E_t^{Q^Y}(\cdot)$ denotes the time t conditional expectation under Q^Y, and the Radon–Nikodym derivative of Q^Y with respect to Q is

$$\left.\frac{dQ^Y}{dQ}\right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_u du} Y_T}{\mathbb{E}_t(e^{-\int_t^T r_u du} Y_T)}.$$

The probability Q^Y is the forward multiplier probability.

Proposition 2.1 contains a simple but general result, yet our motivation lies in the pricing of interest rate volatility based on $V^{\mathrm{bp}}(t, T)$ and $V(t, T)$ in (2.3). Accordingly, we now only consider the two cases, $\Psi(\cdot) = V^{\mathrm{bp}}(t, T)$ and $\Psi(\cdot) = V(t, T)$.

Our next step is to identify necessary and sufficient conditions such that K_Y in Eq. (2.4) is model-free and begin with a definition of "model-free" pricing for our context.

Definition 2.2 (Model-Free Pricing) The strike price K_Y in Eq. (2.4) is model-free if we can find a numéraire with value N_τ such that X_τ is a martingale under Q^N as in Eq. (2.2), and a stochastic multiplier Y_T such that K_Y equals the value of a portfolio of European call and options with strike K, say $\mathrm{Call}_t(K)$ and $\mathrm{Put}_t(K)$, where:

$$\frac{\mathrm{Call}_t(K)}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{\max\{\Pi_T, 0\}}{N_T}\right), \qquad \frac{\mathrm{Put}_t(K)}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{\max\{-\Pi_T, 0\}}{N_T}\right), \tag{2.5}$$

and Π_T is as in Eq. (2.1).

Absence of arbitrage in frictionless markets implies that there exists a numéraire such that option prices can be expressed as in Eq. (2.5). The additional requirement of the previous definition is that we need to find a stochastic multiplier Y_T such that the value of a variance swap is model-free. We emphasize that our definition of "model-free" pricing does not rely on the replicability of a variance swap. We only require that the value of the variance swap equals the market value of a portfolio of tradeable securities. Section 2.3.3 below contains a more detailed discussion of these issues.

The question arises as to how the two expectations in (2.4) and (2.5) relate to each other. We have:

Proposition 2.2 (Model-Free Contracts) *The fair value of K_Y in the forward contract of Definition 2.1 is model-free if and only if the Radon–Nikodym derivative of the forward multiplier probability Q^Y with respect to the market numéraire probability Q^N is uncorrelated with $V^{\text{bp}}(t,T)$ and $V(t,T)$. For $\Psi(\cdot) = V^{\text{bp}}(t,T)$, it is given by*:

$$K_Y = \mathrm{V}_t^{\text{bp}} \equiv \frac{2}{N_t}\left(\int_0^{X_t} \mathrm{Put}_t(K)dK + \int_{X_t}^{\infty} \mathrm{Call}_t(K)dK\right) \quad (\textit{Basis Point pricing}); \tag{2.6}$$

for $\Psi(\cdot) = V(t,T)$, it is given by:

$$K_Y = \mathrm{V}_t \equiv \frac{2}{N_t}\left(\int_0^{X_t} \frac{\mathrm{Put}_t(K)}{K^2}dK + \int_{X_t}^{\infty} \frac{\mathrm{Call}_t(K)}{K^2}dK\right) \quad (\textit{Percentage pricing}). \tag{2.7}$$

The previous proposition is proven in Appendix A.1. It generalizes Mele and Obayashi (2012), who provide a model-free expression for interest rate variance swaps in the interest rate swap space. The focus of Proposition 2.2 is wider.

First, it provides guidance for model-free pricing of variance swaps regarding other fixed income securities. Notably, it suggests choices for the random multiplier Y_T for each market of interest where different numéraires arise (see, e.g., the previous Examples 2.1).[1]

Second, Proposition 2.2 identifies both necessary and sufficient conditions under which interest rate variance swaps can be priced in a model-free fashion. The most intuitive case arises when the stochastic multiplier of Definition 2.1 coincides with the value of the market numéraire, $Y_T = N_T$, in which case the Radon–Nikodym derivative of Q^Y against Q^N is obviously constant and equal to one, and uncorrelated with the realized variance, $V^{\text{bp}}(t,T)$ and $V(t,T)$. Intuitively, tilting an interest rate variance swap through the market numéraire, N_T, leads to a market space where both the strike K_Y and the price of all available options are expectations under Q^N,

[1] Proposition 2.2 also covers the standard equity case with constant interest rates. In this case, N_t is the price of a zero-coupon bond expiring at T, i.e. $e^{-\bar{r}(T-t)}$, where $\bar{r}$ denotes the constant interest rate.

with no additional information required to price the contract. The market numéraire is indeed the benchmark in this book. Note, however, that Proposition 2.2 points to a larger set of stochastic multipliers. For example, in Appendix A.2, we show that the following stochastic multiplier is also in the set of those identified by Proposition 2.2, $Y_T = N_T \epsilon_T$, where ϵ_T is any $\mathbb{F}_T$-measurable random variable satisfying $\mathrm{cov}^{Q^N}(V^{\mathrm{bp}}(t, T), \epsilon_T) = 0$.

Finally, we check the internal consistency of the contract design, namely that the variance strikes in Proposition 2.2 collapse to a constant, assuming uncertainty is constant. The notion of uncertainty depends on the assumptions we make regarding the data generating process in Eq. (2.2). Consider the highly idealized case of a constant basis point variance, in which case the risk X_τ could now take on negative values, according to the following Gaussian, or "Bachelier market," model

$$dX_\tau = \sigma_{\mathrm{n}} \cdot dW_\tau, \tag{2.8}$$

for some vector of constants σ_{n}. This assumption is the obvious counterpart to that of a constant percentage volatility underlying the standard Black–Scholes market for equity options. In Appendix A.3, we show that in this case, the variance strike in Eq. (2.6) collapses to

$$K_Y = \frac{2}{N_t}\left(\int_{-\infty}^{X_t} \mathrm{Put}_t(K)dK + \int_{X_t}^{\infty} \mathrm{Call}_t(K)dK\right) = \|\sigma_{\mathrm{n}}\|^2(T-t). \tag{2.9}$$

One can verify that an analogous result holds in the percentage case in Eq. (2.7), using results in Carr and Lee (2009).[2]

Note, finally, that this result relies on the Gaussian assumption in Eq. (2.8). Alternatively, consider a Black–Scholes market, viz

$$\frac{dX_\tau}{X_\tau} = \sigma_{\mathrm{bs}} \cdot dW_\tau, \tag{2.10}$$

for some vector of constants σ_{bs}. In this case, an index of expected annualized BP volatility (not variance) is easily seen to equal,

$$\sqrt{\frac{K_Y}{T-t}} = X_t \sqrt{\frac{e^{\|\sigma_{\mathrm{bs}}\|(T-t)} - 1}{T-t}}. \tag{2.11}$$

Equation (2.11) reveals the intuitive property that expected BP volatility is the product of the forward, X_t, times a pure volatility component. In Sect. 2.4 (see

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[2]The assumption in Eq. (2.8) that basis point volatility is constant is quite stylized, and is only made for the purpose of neatly illustrating the differences between basis point and percentage volatility. In general, the variance of forward risks in fixed income markets is time-varying. For example, Vasicek (1977) predicts that the basis point volatility of government bond forward prices is time-varying, albeit deterministically (see Chap. 4), $\sigma_{v,\mathrm{n}}^2(\tau)$ say, so that Eq. (2.9) would read, $K_Y = \int_t^T \sigma_{v,\mathrm{n}}^2(\tau)d\tau$.

Proposition 2.3), we generalize the previous formula to the more general case in which X_τ has stochastic volatility.

We now turn to explaining two important features of the variance swap strikes K_Y in Proposition 2.2: (i) their connection to contracts with payoffs linked to the realization of the forward risk at time T, X_T, and hedging issues arising therefrom (Sects. 2.3.2 and 2.3.3); and (ii) the weighting schemes applying to the OTM options, which differ, according to the concept of variance involved—the basis point variance, in Eq. (2.6), and the percentage variance, in Eq. (2.7) (Sect. 2.3.4).

2.3.2 *Log Versus Quadratic Contracts*

The first part of Proposition 2.2 hinges upon a key insight, namely that the price of a BP variance swap relates to that of a "quadratic contract," one with a payoff equal to $N_T X_T^2$ and fair value $N_t \mathbb{E}_t^{Q^N}(X_T^2)$. To establish this link, note, heuristically, that by Itô's lemma,

$$V^{\mathrm{bp}}(t,T) = X_T^2 - X_t^2 - 2\int_t^T X_\tau dX_\tau, \tag{2.12}$$

so that, by the martingale property of X_τ under Q^N,

$$\mathbb{E}_t^{Q^N}\big(V^{\mathrm{bp}}(t,T)\big) = \mathbb{E}_t^{Q^N}\big(X_T^2 - X_t^2\big). \tag{2.13}$$

That is, up to an affine transformation, the BP variance of the forward risk and the quadratic contract on X_T have the same value as claimed.

Therefore, to hedge against BP variance swaps, we need quadratic contracts instead of log-contracts (Neuberger 1994), i.e. those with a payoff equal to $N_T \ln X_T$. To illustrate, set for simplicity $N_\tau \equiv 1$ for all τ, an assumption we relax in each of the next chapters (and in Sect. 2.3.3 below) whilst dealing with the market numéraires of interest. The payoff of a quadratic contract can be approximated by the sum of (i) the payoff of two forwards, and (ii) the payoff of two portfolios comprising OTM options and one ATM option,

$$\begin{aligned} X_T^2 - X_o^2 \approx{}& 2X_o(X_T - X_o) \\ &+ 2\Bigg(\sum_{j:K_j<X_o}(K_j - X_T)^+ + \sum_{j:K_j\geq X_o}(X_T - K_j)^+\Bigg)\Delta K \equiv \hat{P}_T^{\mathrm{q}}, \end{aligned} \tag{2.14}$$

where $X_o \equiv X_t$ is the forward and ΔK is the interval between the strikes K_j. The previous approximation turns into an equality when we consider a continuum of options (see Eq. (A.4) in Appendix A.1). In contrast, the payoff of a logarithmic

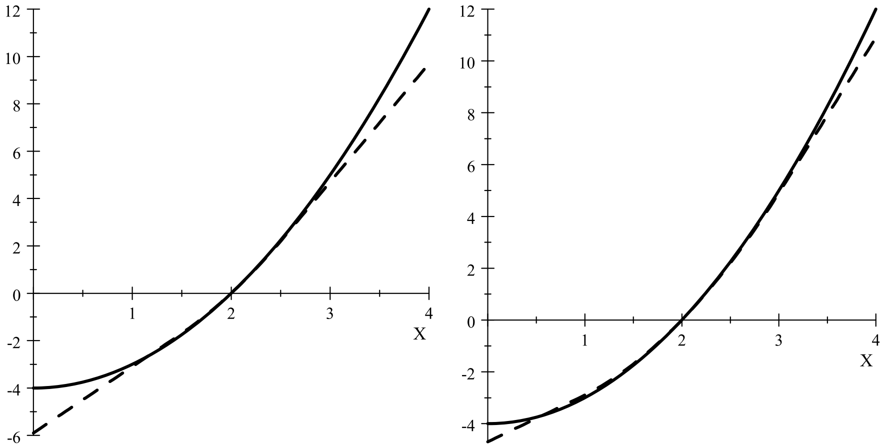

Fig. 2.1 Hedging quadratic contracts with options. In both panels, the *solid line* depicts the terminal value of a quadratic contract, $X^2 - X_o^2$, with $X_o = 2$, and the *dashed line* depicts that of a replicating portfolio, $\hat{P}_T^{\mathrm{q}}$ in Eq. (2.14), comprising: (i) two forwards struck at $X_o = 2$; and (ii) two additional equally weighted portfolios, with $\Delta K = \frac{1}{10}$, each including one ATM option and a number of OTM put and call options. The *dashed line* in the left-hand panel is obtained with a total of 5 puts and 5 calls, and the right-hand panel is with a total of 10 puts and 10 calls

contract can be approximated as

$$\begin{aligned}\ln \frac{X_T}{X_o} &\approx \frac{1}{X_o}(X_T - X_o) \\ &\quad - \left(\sum_{j:K_j<X_o} \frac{1}{K_j^2}(K_j - X_T)^+ + \sum_{j:K_j \geq X_o} \frac{1}{K_j^2}(X_T - K_j)^+ \right) \Delta K \equiv \hat{P}_T. \end{aligned} \tag{2.15}$$

Not only do the portfolio weightings in $\hat{P}_T^{\mathrm{q}}$ and $\hat{P}_T$ differ, in that we require (i) $2X_o$ forward contracts in Eq. (2.14) and $\frac{1}{X_o}$ in Eq. (2.15), and (ii) ΔK options in Eq. (2.14) and $\frac{\Delta K}{K_j^2}$ in Eq. (2.15). We also require to go *long* the option portfolio in the case of the quadratic contract, and *short* the option portfolio in the log-contract case.[3]

The portfolio payoff $\hat{P}_T$ has been known to approximate the log-contract payoff quite closely since Demeterfi et al. (1999a, 1999b). Figure 2.1 depicts the quadratic contract payoff and the portfolio payoff $\hat{P}_T^{\mathrm{q}}$, assuming that the forward,

[3]Note, however, that for the percentage variance contract, we have, $\mathbb{E}_t^{Q^N}(V(t,T)) = -2\mathbb{E}_t^{Q^N}(\ln \frac{X_T}{X_t})$, so that the option positions have the same sign both when it comes to hedge the basis point and the percentage realized variance, as further clarified in the next chapters. Still, the forward positions have opposite signs as Eq. (2.14) and Eq. (2.15) reveal.

$X_o = 2$, interpreted as an interest rate (say, e.g., a forward swap rate), and that the equally weighted option portfolio has 10 (left panel) and, then, 20 (right panel) out-of-the-money options, with equidistant strikes and distance $\Delta K = \frac{1}{10}$, such that $\min_j K_j = 1.5$ and $\max_j K_j = 2.5$ (left panel) and $\min_j K_j = 1$ and $\max_j K_j = 3$ (right panel). Naturally, the quality of the approximation of the portfolio payoff $\hat{P}_T^{\mathrm{q}}$ to the quadratic contract improves as we increase the number of options in the portfolio. Yet even in the case with fewer options, depicted in the left-hand side of Fig. 2.1, the approximation is still remarkably accurate over a wide range of values of X_T around the forward, $X_o = 2$.

Note that $\hat{P}_T^{\mathrm{q}}$ only aims to hedge the payoff of the quadratic contract. To hedge the BP variance, $V^{\mathrm{bp}}(t, T)$, we also need to hedge the additional term $2\int_t^T X_\tau dX_\tau$ in Eq. (2.12). The next chapters contain details on these issues for each market and numéraire of interest. We provide preliminary intuition on these details in Sect. 2.3.3 below.

A final issue pertains to the reasons for the constant weights in Eq. (2.6). These weightings follow by spanning arguments that generalize the theory in Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, Eq. (1)) (see, also, Lee 2010) to the case of general numéraires. Section 2.3.4 provides further results and intuition on the origins of these weightings.

2.3.3 Hedging

While our derivations rely on the assumption that the forward risk is a continuous-time process, we can illustrate the main issues arising in our context while relying on a simple discrete-time example. Define the realized annualized variance over n trading days as

$$\begin{aligned} \mathrm{var}_{X,n} &\equiv \frac{N}{n}\sum_{t=1}^{n}(X_t - X_{t-1})^2 \\ &= \frac{N}{n}\left(X_n^2 - X_0^2\right) - 2\frac{N}{n}\sum_{t=1}^{n} X_{t-1}(X_t - X_{t-1}), \end{aligned} \tag{2.16}$$

where N denotes the number of trading days over the year. Equation (2.16) is the discrete-time counterpart to Eq. (2.12) and its first term can be expanded just as in Eq. (2.14). In particular, we have:

$$\mathrm{var}_{X,n} = \frac{N}{n}2X_0(X_n - X_0) + \frac{N}{n}(X_n - X_0)^2 - 2\frac{N}{n}\sum_{t=1}^{n} X_{t-1}(X_t - X_{t-1}), \tag{2.17}$$

where the second term can be replicated through a static position in out-of-the-money options and one at-the-money options as explained. Carr and Corso (2001) consider the variance of price changes rather than returns based on this approach, while only focusing on the second and third term in Eq. (2.17), and markets with constant interest rates.[4] Our work in this book is distinct as we explicitly take into account the first term in Eq. (2.17) and, importantly, consider markets in which interest rates are obviously random. Random interest rates naturally lead to random numéraires, as opposed to the assumption in this section that $N_t \equiv 1$; rescaling by N_t is crucial to model-free evaluation of variance contracts as Eqs. (2.6) and (2.7) reveal.

To summarize, the realized variance, $\text{var}_{X,n}$, cannot be replicated when the market numéraire is a random process. We shall see that within the continuous time setting of the following chapters, we can, instead, replicate the realized variance rescaled by N_T, provided the forward risk X_t is actually traded (or could be replicated), and under additional conditions applying to the specific fixed income securities traded in each market of interest.

To illustrate the reasons $\text{var}_{X,n}$ cannot be replicated in the presence of random numéraires, let us set up the arguments we would use to replicate the last term in Eq. (2.17) in a hypothetical case in which the numéraire is *deterministic* but not necessarily constant. Suppose we are long an amount θ_t of a forward starting agreement at time t and strike K_t such that the value of this agreement at t right after the trade is $\pi_{t^+} \equiv \theta_t N_t (X_t - K_t)$ and the value at $t+1$ before the trade is $\pi_{t+1} \equiv \theta_t N_{t+1}(X_{t+1} - K_t)$. Choosing the strike K_t that clears π_{t^+} delivers the forward risk, $K_t = X_t$, such that $\pi_{t+1} = \theta_t N_{t+1}(X_{t+1} - X_t)$. Therefore, the portfolio strategy

$$\theta_{t-1} = \frac{X_{t-1}}{N_t}, \tag{2.18}$$

implies that its overall value at the end of the period is

$$\sum_{t=1}^{n} \pi_t = \sum_{t=1}^{n} X_{t-1}(X_t - X_{t-1}),$$

indicating that the last term in Eq. (2.17) could be replicated.[5] Of course, the crucial assumption underlying the strategy θ_t in Eq. (2.18) is that the market numéraire is deterministic.

If N_t is random and only measurable with respect to the information set at time t, a replication argument based on Eq. (2.18) breaks down. Note that Carr and Corso (2001) deal with markets with constant interest rates and where N_t is just the money market account, such that replication of $\text{var}_{X,n}$ is possible in their setup. Naturally, hedging requires not only dealing with the third term in Eq. (2.17) but also with the first and the second. Still, Proposition 2.2 establishes that in a continuous time

[4] Martin (2013) has also recently considered the same setup in Carr and Corso (2001) assuming constant interest rates.

[5] Note that this replication argument hinges upon the forward starting agreement in Eq. (2.1), not the underlying risk X_t, as the latter is not necessarily traded.

setting, the price of interest rate variance swaps is model-free once the realized variance is re-scaled by the value of an appropriate market numéraire.

In some cases, the realized variance can be replicated by incorporating the numéraire into the replicating portfolio strategies. Replication is possible through replication of the forward risk X_t, rather than by trading the forward agreement through Eq. (2.18). Yet X_t may not be traded in situations of interest. For example, in Chap. 3, X_t is the forward swap rate, which cannot be perfectly hedged (it is not traded). However, we show that a model-free expression for the fair value of a variance swap is available under certain conditions, even when the risk X_t cannot be replicated.

This point underlies our notion of *model-free contracts* in Definition 2.2, which is less stringent than one requiring that a variance swap must be perfectly hedged. We simply require that the fair value of a variance swap equals the value of a portfolio of tradeable securities rescaled by the value of a market numéraire. Regarding the previous interest rate swap example, Chap. 3 shows that while the underlying risk (the forward swap rate) is not traded, variance swaps are priced in a model-free fashion in terms of Definition 2.2. Finally, note that there are additional instances of markets in which the underlying risks cannot be perfectly hedged; for example, the underlying risks can exhibit jumps as in the credit markets studied in Chap. 5.

When hedging is difficult, modeling assumptions are important. Merener (2012) develops an approach for hedging against interest rate swap volatility by relying on a set of assumptions regarding the yield curve. He shows how to replicate the third term in (2.17) (with X_t being the forward swap rate) under the model assumptions. He utilizes positions in forward-starting interest rate swaps in which gains are reinvested in LIBOR accounts at the end of each trading period. His approach does not focus on model-free indexes of expected interest rate volatility, but is an alternative to our hedging strategies of interest rate variance swaps on interest rate swaps explained in Chap. 3. We now turn to provide intuition regarding the option weights in Eq. (2.6).

2.3.4 Constant Gamma Exposure

The weighting in the option portfolio leading to BP expected variance differs from that underlying percentage variance, which is the standard scheme underlying the CBOE VIX index for equity volatility. A basic intuition behind the uniform weighting in Eq. (2.6) is that the instantaneous *realized* BP variance simply equals that of the logarithmic changes of the forward risk rescaled by the squared forward risk, $\|\sigma_\tau\|^2 \times X_\tau^2$. The implied BP variance shares a similar property: comparing Eq. (2.6) with Eq. (2.7) reveals that for the BP variance, each option price i carries the same weight as $\frac{dK_i}{K_i^2}$ (i.e., the contribution of each option price to the implied percentage variance), rescaled by the squared strike K_i^2, i.e. $\frac{dK_i}{K_i^2} \times K_i^2$.

We can illustrate the proportional weighting in Eq. (2.6) from a different angle. In their derivation of the fair value of equity variance swaps, Demeterfi et al. (1999a, 1999b) develop an intuitive approach relying on the Black and Scholes (1973) market. They explain that a portfolio of options has a vega that is insensitive to changes in the stock price only when the options are weighted inversely proportional to the squared strike. This property obviously holds in our context with the general market numéraire (see Eq. (2.7)). We develop a similar approach to gain intuition regarding the uniform weightings in Eq. (2.6).

Assume a Gaussian market, i.e. one where the forward risk X_τ in Eq. (2.2) is the solution to Eq. (2.8), and denote the price of an option (be it a put or a call) at time t when the forward risk is X with $\mathcal{O}_t(X, K, T, \sigma_{\mathrm{n}})$. We create a portfolio with a continuum of these options having the same maturity, and denote the portfolio weights with $\omega(K)$, which are taken to be independent of X. The value of this portfolio is

$$\pi_t(X_t, T, \sigma_{\mathrm{n}}) \equiv \int \omega(K)\mathcal{O}_t(X_t, K, T, \sigma_{\mathrm{n}})dK. \tag{2.19}$$

We require that the vega of the portfolio, defined as $\nu_t(X, T, \sigma) \equiv \frac{\partial \pi_t(X,T,\sigma)}{\partial \sigma}$, be insensitive to changes in the forward risk,

$$\frac{\partial \nu_t(X, T, \sigma)}{\partial X} = 0. \tag{2.20}$$

In Appendix A.3, we show that under regularity conditions, a portfolio of an ATM and all of the OTM options, has vega independent of X if and only if the weightings are independent of K, consistently with Eq. (2.9),

$$\text{Eq. (2.20) holds true} \quad \Longleftrightarrow \quad \omega(K) = \text{const.} \tag{2.21}$$

That is, a portfolio aiming to replicate the BP volatility, σ_{n}, which is immune to changes in the underlying forward swap rate, is an equally weighted portfolio of out-of-the-money and at-the-money options. In Appendix A.3, we also explain that under the same conditions, the gamma exposure of the options portfolio is constant across different realizations of the forward risk X_t, a result that Bibkov and Misra (2012) illustrate numerically in the case of CBOE's SRVIX index based on the next chapter.

2.4 Implied Volatility Indexes

2.4.1 Model-Free Indexes

Model-free indexes of expected volatility related to the market numéraire N_t follow from Proposition 2.2 in a natural fashion. We define the two indexes,

$$\mathrm{VX}_t^j(T) \equiv \sqrt{(T-t)^{-1}\mathrm{V}_t^j}, \quad j \in \{\mathrm{bp}, \mathrm{p}\}, \tag{2.22}$$

where $\mathrm{V}_t^{\mathrm{bp}}$ is the strike K_Y for the basis point variance in Eq. (2.6), and $\mathrm{V}_t^{\mathrm{p}}$ is the strike K_Y for the percentage in Eq. (2.7).

2.4.2 *Comparisons to Model-Based Log-Normal and Normal Implied Volatility*

2.4.2.1 Skews

The indexes of percentage and basis point volatility in Eq. (2.22) link to the fair value of dedicated interest rate variance swaps in a model-free fashion. As such, they generalize special instances of markets we now describe.

Consider the price of a European call option:

$$\mathrm{Call}_t(K) = \mathbb{E}_t\left(e^{-\int_t^T r_\tau d\tau} N_T (X_T - K)^+\right) = N_t \mathbb{E}_t^{Q^N} (X_T - K)^+, \tag{2.23}$$

where the second equality follows by the usual change of probability.

We consider two benchmark option pricing models that are derived from specific assumptions on the dynamics of the forward risk.

The first benchmark relies on the Black–Scholes assumption in Eq. (2.10) that the percentage volatility is constant and equal to σ_{bs} (a scalar, say). In this market, the expression for the expectation under the numéraire probability in Eq. (2.23) is given by the Black (1976) formula:

$$\mathbb{E}_t^{Q^N} (X_T - K)^+ = X_t \Phi(d_t) - K\Phi(d_t - \sqrt{T-t}\sigma_{\mathrm{bs}}),$$

where

$$d_t \equiv \frac{\ln \frac{X_t}{K} + \frac{1}{2}(T-t)\sigma_{\mathrm{bs}}^2}{\sqrt{T-t}\sigma_{\mathrm{bs}}},$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution. The *log-normal* skew is defined as the mapping $K \longmapsto \mathrm{IV}(K)$ where $\mathrm{IV}(K)$ denotes the value of σ_{bs} such that the option price implied by Black's model coincides with the market price.

The second benchmark relies on the assumption of a Gaussian–Bachelier market with constant basis point volatility σ_{n} (a scalar, say) (see Eq. (2.8)). In this market, the expectation in the second equality of Eq. (2.23) is

$$\mathbb{E}_t^{Q^N} (X_T - K)^+ = (X_t - K)\Phi(\delta_t) + \frac{\sigma_{\mathrm{n}}\sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{1}{2}\delta_t^2}, \tag{2.24}$$

where

$$\delta_t \equiv \frac{X_t - K}{\sigma_{\mathrm{n}}\sqrt{T-t}}.$$

The *normal* skew is defined as the mapping $K \longmapsto \mathrm{IV}^{\mathrm{bp}}(K)$, where $\mathrm{IV}^{\mathrm{bp}}(K)$ is the value of σ_{n} such that the option price in this Gaussian market equals the market price.

Clearly, the log-normal skew is flat at σ_{bs} when the forward risk is as in Eq. (2.10) and the normal skew is flat at σ_{n} should the forward risk be as in Eq. (2.8). In the former case, the index $\mathrm{VX}_t^{\mathrm{p}}(T)$ in Eq. (2.22) equals σ_{bs} and in the latter, the index $\mathrm{VX}_t^{\mathrm{bp}}(T)$ in Eq. (2.22) collapses to σ_{n}, as explained in Sect. 2.3.1.

2.4.2.2 Estimating Expected Volatility from ATM Implied Volatilities

It is well known (see Brenner and Subrahmanyam 1988) that the price of ATM options are approximately linear in volatility. For example, consider the price of an ATM call option in the Black–Scholes market:

$$
\begin{aligned}
\frac{\mathrm{Call}_t^{\mathrm{bs}}(K)|_{K=X_t}}{N_t} &= X_t\left(2\Phi\left(\frac{1}{2}\sqrt{T-t}\sigma_{\mathrm{bs}}\right)-1\right)\\
&\approx X_t\left(2\left(\Phi(0)+\phi(0)\frac{1}{2}\sqrt{T-t}\sigma_{\mathrm{bs}}\right)-1\right)\\
&= X_t\frac{1}{\sqrt{2\pi}}\sqrt{T-t}\cdot\sigma_{\mathrm{bs}},
\end{aligned}
\tag{2.25}
$$

with obvious notation. One can easily recover σ_{bs} from the previous formula. Similarly, in Bachelier's market,

$$
\frac{\mathrm{Call}_t^{\mathrm{n}}(K)|_{K=X_t}}{N_t} = \frac{1}{\sqrt{2\pi}}\sqrt{T-t}\cdot\sigma_{\mathrm{n}},
$$

where the L.H.S. of this equation is the expectation of the L.H.S. of Eq. (2.24) evaluated at $K = X_t$.

Carr and Wu (2006, Appendix A) rely on these insights and provide an estimator for the expected variance under the risk-neutral probability, assuming that the instantaneous changes in the standard deviation of a forward risk are independent of those of the forward risk, just as in the seminal paper of Hull and White (1987): in terms of Eq. (2.2), the instantaneous standard deviation, σ_τ (taken to be a scalar), is independent of the entire path of X_τ, up to maturity. Under this assumption, and in terms of the model of this chapter, their results imply that the price of an ATM option, $\mathrm{Call}_t^{\mathrm{sv}}(\cdot)$ is given by

$$
\frac{\mathrm{Call}_t^{\mathrm{sv}}(K)|_{K=X_t}}{N_t} \approx \frac{X_t}{\sqrt{2\pi}}\mathbb{E}_t^{Q^N}\left(\sqrt{V(t,T)}\right),
\tag{2.26}
$$

where $V(t,T)$ denotes the percentage variance, defined as in the second of Eqs. (2.3). Note also that by Eq. (2.25),

$$
\frac{\mathrm{Call}_t^{\$}(K)|_{K=X_t}}{N_t} \approx \frac{X_t}{\sqrt{2\pi}}\sqrt{T-t}\cdot\mathrm{IV}(X_t),
\tag{2.27}
$$

where $\mathrm{Call}_t^{\$}(K)$ denotes the market call price, and $\mathrm{IV}(X_t)$ is the implied volatility for strike $K = X_t$. Therefore, setting $\mathrm{Call}_t(K) = \mathrm{Call}_t^{\mathrm{sv}}(K)$, and then comparing Eq. (2.26) and Eq. (2.27), leaves:

$$
\mathbb{E}_t^{Q^N}\left(\sqrt{\frac{V(t,T)}{T-t}}\right) \approx \mathrm{IV}(X_t).
$$

The authors explain that the approximation is accurate. The assumption underlying this approximation is that the forward risk volatility is independent of the forward risk path over the life of the option.

2.4.3 Index Decompositions

Let $\sigma(X, K)$ denote Black's implied volatility, defined as usual as $\sigma(\cdot,\cdot)$: $\text{Call}_t(K) = \mathcal{C}_t(X_t, K, \sigma(X_t, K))$, where $\mathcal{C}_t(X, K, \sigma)$ is the price of a call given by the Black (1976) formula when the forward risk is X, the strike is K and the volatility σ_τ in Eq. (2.2) is constant and equal to σ. The next proposition provides basic properties of the indexes in Eq. (2.22), which rely on a standard assumption on implied volatility.

Proposition 2.3 (Index Skew Factors) *Suppose the implied volatility surface has the sticky delta property, or that implied volatilities are homogeneous of degree zero in X and K, i.e. $\sigma(X, K) = \sigma(\lambda X, \lambda K)$, for any constant $\lambda > 0$. Then, (i) there exists a function $\xi(t, T)$ independent of X, such that $\mathrm{V}_t^{\mathrm{bp}}$ in Eq. (2.6) can be written as $\mathrm{V}_t^{\mathrm{bp}} = X_t^2 \times \xi(t, T)$, and (ii) $\mathrm{V}_t^{\mathrm{p}}$ in Eq. (2.7) is independent of X_t.*

Appendix A.4 provides a proof of this proposition. A well-known example of models for which the zero homogeneity property of $\sigma(X, K)$ holds are diffusion processes such that the volatility σ_τ in Eq. (2.2) is a Markov process (see, e.g., Renault 1997), as is the case for the celebrated Heston (1993) model. For example, in the special case of a Black–Scholes market (e.g., Eq. (2.10)), expected volatility is as in Eq. (2.11), consistently with the previous Proposition 2.3. More generally, this proposition provides novel characterizations of the indexes in Eq. (2.22), which are potentially useful in empirical analyses as discussed below. For example,

$$\mathrm{VX}_t^{\mathrm{bp}}(T) = X_t \times \sqrt{(T-t)^{-1}\xi(t, T)} \equiv X_t \times \hat{\xi}(t, T). \tag{2.28}$$

Proposition 2.3 tells us that under mild conditions, the function $\hat{\xi}(t, T)$ in Eq. (2.28) is independent of the forward X_t, and can be time-varying, subsuming movements in fundamental uncertainty unrelated to the level of X_t, a "skew factor." Therefore, a BP volatility index moves linearly with the forward, holding uncertainty constant. In contrast, a percentage volatility index, such as VIX, should not change with the forward risk, provided uncertainty remains the same. In this sense, the rescaled skew factor, $\hat{\xi}(t, T)$, shares similarities (and orders of magnitude) with a percentage index.

Mele et al. (2015a) perform an empirical analysis of CBOE's SRVIX, the interest rate swap BP volatility index developed in the next chapter. They document that this index is at times led by movements of the forward swap rate, and at other times by uncertainty (see Fig. 2.2). For example, the surge in expected BP rate volatility over

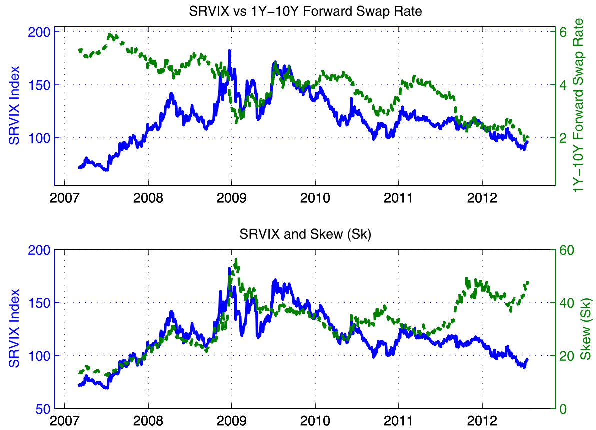

Fig. 2.2 *Top panel*: The CBOE SRVIX index relating to the 1Y-10Y forward swap rate, and the 1Y-10Y forward swap rate. *Bottom panel*: The CBOE SRVIX index relating to the 1Y-10Y forward swap rate, and the skew factor, $\hat{\xi}(t, T)$ in Eq. (2.28)

the global financial crisis in 2008 is led by uncertainty, whereas the decline in the same volatility over 2012 is partially explained by the extraordinarily low interest rate climate. Mele and Obayashi (2015) provide additional empirical properties of this index.

2.5 Implementing Basis Point Variance Swaps

The realized variance $V^{\text{bp}}(t, T)$ in Eq. (2.3) is a notion that captures the up and down movements the forward risk (e.g., the forward swap rate) might experience over a given period. It measures the dispersion of interest rate changes as the sum of the dispersions occurring over each trading period. It is the relevant notion in the context of volatility trading and risk management for its potential to track episodes of sustained and prolonged uncertainty. In Sect. 2.5.1, we provide one additional definition of realized variance, and in Sect. 2.5.2 we provide estimates of variance risk-premiums based on these two notions of realized variance.

2.5.1 Incremental Versus Point-to-Point Realized Variance

2.5.1.1 Basis Point

Mele et al. (2015a) consider an additional definition of realized basis point volatility

$$V^{\text{bp}}_{\text{p-t-p}}(t, T) \equiv \sqrt{\frac{(X_T - X_t)^2}{T - t}}, \tag{2.29}$$

which they label "point-to-point" basis point volatility. While $V^{\text{bp}}(t, T)$ in Eq. (2.3) is "incremental" in nature, point-to-point volatility captures the dispersion of changes in the forward risk over two distinct points in time. For example, regarding swap markets, point-to-point volatility measures the distance of the future forward rate from the current one, thereby ignoring anything that occurs during the trading period—it may take a small value even after a prolonged period of market turbulence.

While incremental and point-to-point basis point realized variance are obviously not the same, they have the same expectation under the market probability. Consider, for example, basis point variance contracts in swap markets, in which the relevant probability is the annuity. Thus, while the fair values of basis point variance swaps in Sect. 2.3 correctly track the expected variance relevant for trading purposes (i.e. $V^{\text{bp}}(t,T)$ in Eq. (2.3)), they can also be interpreted in terms of numéraire-adjusted expected dispersion of the relevant risk, X_T.

The claim that the expectation of the point-to-point variance under the market probability is the same as that of $V^{\text{bp}}(t,T)$ follows by the so-called isometry property of Itô integrals (e.g., Øksendal 1998; p. 26), i.e. the second of the next equalities:

$$\begin{aligned}\mathbb{E}_t^{Q^N}(X_T - X_t)^2 &= \mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau \sigma_\tau \cdot dW_\tau\right)^2 = \mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2 \|\sigma_\tau\|^2 d\tau\right) \\ &= \mathbb{E}_t^{Q^N}\left(V^{\text{bp}}(t,T)\right),\end{aligned} \tag{2.30}$$

where the first equality follows by Eq. (2.2), and the last is the definition of the incremental basis point variance in Eq. (2.3).

Note, then, that we may define a new variance contract, delivering the following payoff,

$$\Pi_T^* = N_T \times \left((X_T - X_t)^2 - K^*\right),$$

where the fair value, K^*, is:

$$K^* = \frac{1}{N_t}\mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T (X_T - X_t)^2\right) = \mathbb{E}_t^{Q^N}(X_T - X_t)^2 = K_Y,$$

and the last equality follows by Eq. (2.30), with K_Y defined as in Eq. (2.6) of Proposition 2.2. That is, the fair values of point-to-point and basis point variance swaps are the same. Mele et al. (2015a) rely on this property and calculate approximate confidence bands for the forward swap rate forecasts, based on the CBOE SRVIX, reproduced in Fig. 2.3.[6]

2.5.1.2 Percentage

The previous equivalence property—two distinct variance contracts with the same price—does not hold in the case of percentage variance contracts. Consider a contract with payoff referenced to the variance of the cumulative log-return on an asset

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[6]The forward swap rate is a martingale under the swap market probability but is not necessarily Gaussian. Therefore, the bands in Fig. 2.3 are approximate.

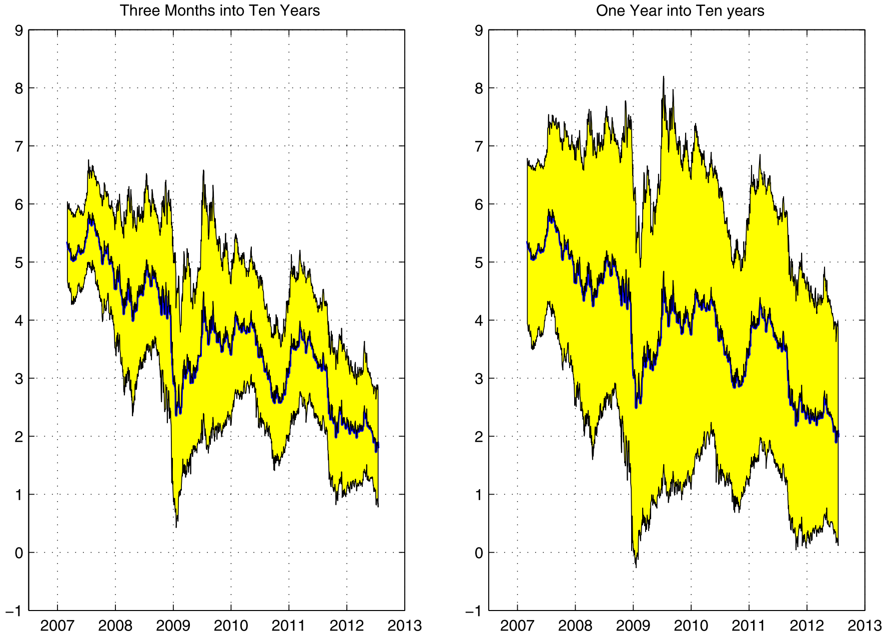


Fig. 2.3 Forward swap rate ∓ 1.96 times the CBOE-SRVIX volatility index for the 3m-10Y forward swap rate (*left panel*) and 1Y-10Y forward swap rate (*right panel*)

price with process X_τ solution to Eq. (2.2), $(\ln \frac{X_T}{X_t})^2$. Such a contract links to the second moment of the cumulative return $\ln \frac{X_T}{X_t}$, rather than the percentage variance. By Itô's lemma, its expectation equals,

$$\mathbb{E}_t^{Q^N}\left[\left(\ln \frac{X_T}{X_t}\right)^2\right] = \mathbb{E}_t^{Q^N}\left[2\int_t^T \ln\left(\frac{X_\tau}{X_t}\right)\frac{dX_\tau}{X_\tau} + \int_t^T \left(1 - \ln \frac{X_\tau}{X_t}\right)\left(\frac{dX_\tau}{X_\tau}\right)^2\right],$$

and can be expressed in a model-free format as

$$\mathbb{E}_t^{Q^N}\left[\left(\ln \frac{X_T}{X_t}\right)^2\right] = \frac{2}{N_t}\left(\int_0^{X_t} \frac{1}{K^2}\left(1 - \ln \frac{K}{X_t}\right)\text{Put}_t(K)dK + \int_{X_t}^{\infty} \frac{1}{K^2}\left(1 - \ln \frac{K}{X_t}\right)\text{Call}_t(K)dK\right), \tag{2.31}$$

with the usual notation.[7]

[7]The expression in Eq. (2.31) was first derived by Bakshi et al. (2003) in the equity case and constant interest rate $\bar{r}$ (i.e. for $N_\tau = e^{-\bar{r}(T-\tau)}$), in their attempt to determine model-free measures of skewness.

A percentage variance swap based on a cumulative, point-to-point notion is unusual at the time of writing—the standard notion one typically relies on is the incremental realized variance, $\int_t^T (\frac{dX_\tau}{X_\tau})^2$, rather than the point-to-point realized variance, $(\ln \frac{X_T}{X_t})^2$. However, the fair value of *basis point* variance swaps based on the previous two notions coincide.

2.5.2 Volatility Risk Premiums

Volatility risk premiums measure how much investors are willing to pay to hedge against volatility rising above a given threshold. Roughly, they are the difference between the expectation of future volatility under the physical and market probabilities. While the literature on equity volatility risk premiums is large (see, e.g., Bollerslev et al. 2009; Carr and Wu 2008; or Corradi et al. 2013, and references therein), relatively little is known about volatility risk premiums in fixed income markets. Mele et al. (2015a) undertake an empirical study pertaining to swap markets based on the CBOE SRVIX, and here we extend some of their findings by calculating variance risk premiums for both incremental and point-to-point formulations of basis point volatility.

We use realized variance as a proxy for the expectation of future realized variance under the physical probability. Accordingly, we define the incremental and point-to-point variance risk premiums as follows:

$$\pi_{t+S}^{\text{incr}} \equiv \text{SRVIX}_t^2(S) - \text{Vol}_{t+S}^2 \quad \text{and} \quad \pi_{t+S}^{\text{p-t-p}} \equiv \text{SRVIX}_t^2(S) - \text{Vol}_{\text{p-t-p},t+S}^2,$$

where $\text{SRVIX}_t(S)$ denotes the CBOE SRVIX for tenor equal to 10 years, maturity S expressed in fraction of years, and the two realized volatilities, Vol_t and $\text{Vol}_{\text{p-t-p},t}$, are defined below. Note the following important interpretation of π_{t+S}^{incr} and $\pi_{t+S}^{\text{p-t-p}}$: they are P&Ls at time $t+S$ of a short position in a variance swap contract originated at time t.[8]

The realized basis point (incremental) volatility in Eq. (2.3) is estimated as the annualized "quadratic variation" of the daily changes in the forward swap rate,

$$\text{Vol}_t \equiv \sqrt{\frac{251}{21 \cdot n} \sum_{i=1}^{21 \cdot n} \Delta F_{t+1-i}^2},$$

where ΔF_t denotes the change at t of the forward swap rate for a n-month forward starting swap with 10-year tenor.

[8]In the equity literature, one usually defines a variance risk premium as the difference between the expectation of future realized variance under the risk-neutral and the physical probabilities (see, e.g., Bollerslev et al. 2009). Our notion of variance risk premium is consistent with the purpose of defining payoffs that have zero value under the market probability, as is the case with π_{t+S}^{incr}. The expectation of π_{t+S}^{incr} under the physical probability *is* the variance risk premium as usually defined in the literature, although we shall keep on referring to π_{t+S}^{incr} as variance risk premium.

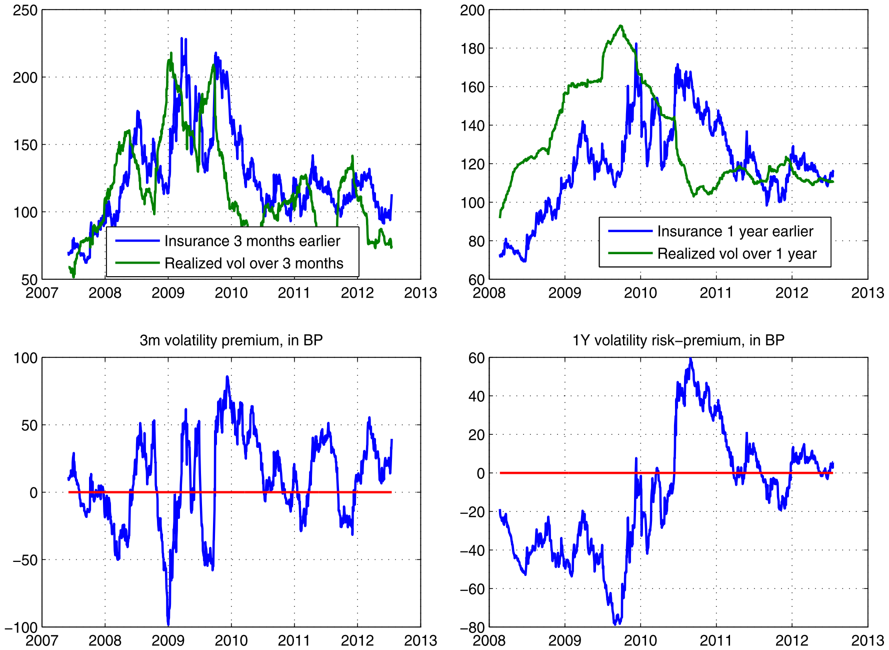


Fig. 2.4 Price of insurance in the interest rate swap volatility space (CBOE SRVIX), realized basis point *incremental* volatility, and swap volatility premiums for three month (*left panels*) and one year (*right panels*) horizons

Estimating asset price volatility is one of the most important and studied topics in financial econometrics (Engle 2004). Andersen et al. (2010) and Aït-Sahalia and Jacod (2014) survey the literature on realized variance and related measurement methods. One complication arising whilst dealing with fixed income market volatility is that the squared changes of variables of interest, such as forward swap rates, may be unobservable due to the lack of constant maturity contracts. To calculate realized volatility, we follow Mele et al. (2015a), and estimate missing forward swap rates through linear interpolation.

The annualized realized point-to-point volatility in Eq. (2.29) is simply

$$\mathrm{Vol}_{\text{p-t-p},t+S} \equiv \sqrt{\frac{(R_{t+S} - F_t(S))^2}{S}},$$

where $F_t(S)$ denotes the forward swap rate at t for a forward starting swap at $t + S$ with ten year tenor, and $R_{t+S} = F_{t+S}(0)$ is the spot swap rate at time $t + S$.

Figures 2.4 and 2.5 plot the price of volatility (SRVIX) and realized volatility, along with *volatility* risk premiums defined as $\hat{\pi}^{\text{incr}}_{t+S} \equiv \text{SRVIX}_t(S) - \text{Vol}_{t+S}$ and $\hat{\pi}^{\text{p-t-p}}_{t+S} \equiv \text{SRVIX}_t(S) - \text{Vol}_{\text{p-t-p},t+S}$. For comparison, Fig. 2.6 depicts equity counterparts, in percentage, utilizing data on the VIX and the three-month horizon

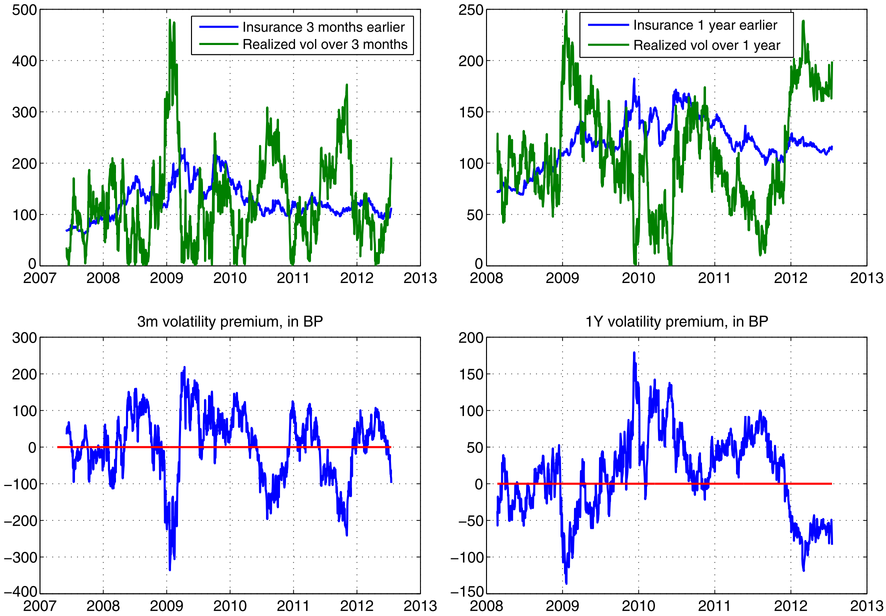


Fig. 2.5 Price of insurance in the interest rate swap volatility space (CBOE SRVIX), realized basis point *point-to-point* volatility, and swap volatility premiums for three month (*left panels*) and one year (*right panels*) horizons

VXV.[9] Finally, Fig. 2.7 provides volatility risk premiums in the Treasury space based on the CBOE/CBOT TYVIX index (see Chap. 4) and realized volatility of the underlying 10-year Treasury Note future price.

Note that the persistence of the volatility risk premium increases with the horizon length in both interest rate and equity cases. Figures 2.4 and 2.6 suggest that at a three month horizon, volatility risk premiums in swap and equity markets display similar persistence properties, although they do not quite react in the same way in times of distress. Mele et al. (2015a) provide further analysis of these issues in swap and equity markets.

2.6 Skew Shifts and the Dynamics of Volatility Indexes

In practice, there are only a finite number options for calculating a volatility index. This section analyzes theoretical properties of the index in the presence of unavoidable truncations. It also develops a few basic numerical experiments that illustrate these properties.

[9]Fornari (2010) documents early estimates of volatility risk premiums in the interest rate swap space. His estimates regard percentage volatility, not basis point as in this section, and rely on proxies for model-free implied volatility based on the standard equity methodology instead of the interest rate methodology, which was subsequent to his work.

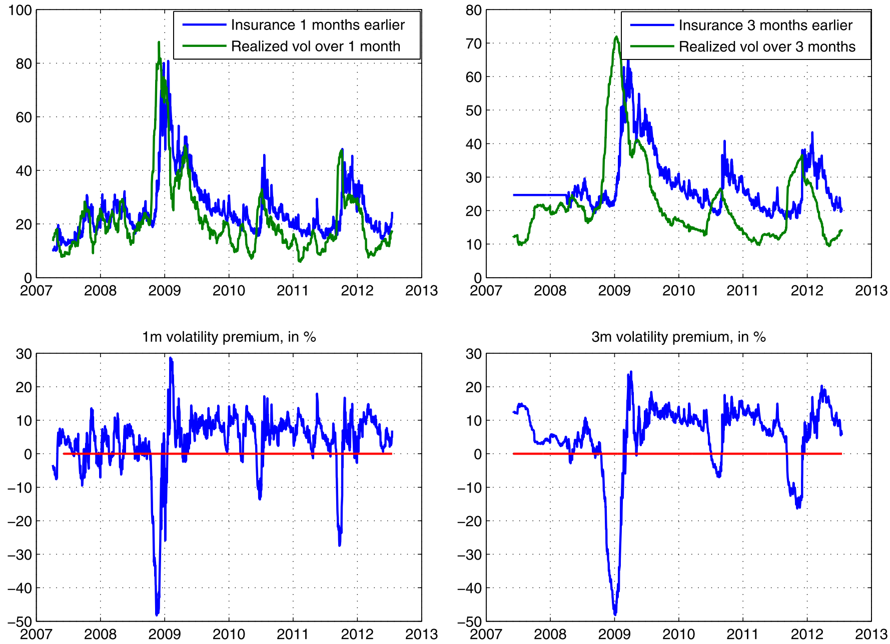
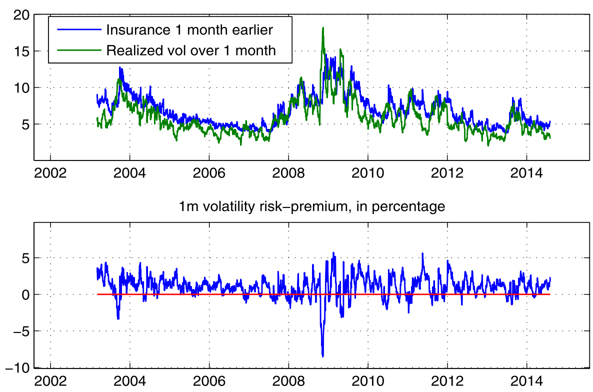


Fig. 2.6 Price of insurance in the equity volatility space (CBOE VIX and VXV), realized percentage volatility, and equity volatility premiums for one month (*left panels*) and three month (*right panels*) horizons

Fig. 2.7 Price of insurance in the government bond volatility space (CBOE/CBOT TYVIX), realized percentage volatility, and volatility premiums for one month horizons. The TYVIX is referenced to one month volatility in the 10-year Treasury Note Future price



2.6.1 Truncations

Consider the following approximations to the expressions for V^{bp} and V in Eq. (2.6) and Eq. (2.7),

$$\mathrm{V}_{\ell}^{\mathrm{bp}} \equiv \frac{2}{N}\left(\int_{X-\ell}^{X} \mathrm{Put}\big(X, K, \sigma(X, K)\big) dK + \int_{X}^{X+\ell} \mathrm{Call}\big(X, K, \sigma(X, K)\big) dK\right), \tag{2.32}$$

and

$$\mathrm{V}_\ell \equiv \frac{2}{N}\left(\int_{X-\ell}^{X} \frac{\mathrm{Put}(X, K, \sigma(X, K))}{K^2} dK + \int_{X}^{X+\ell} \frac{\mathrm{Call}(X, K, \sigma(X, K))}{K^2} dK\right), \tag{2.33}$$

where $\ell \in (0, X)$ is a constant, and $\sigma(X, K)$ is Black's implied volatility introduced in Sect. 2.4.

The indexes in Eqs. (2.32) and (2.33) are calculated with a strip of options centered at X over a range equal to 2ℓ. In Appendix A.5, we show that for all $\ell \in (0, X)$:

$$\begin{aligned}\frac{\partial \mathrm{V}_\ell^{\mathrm{bp}}}{\partial X} = \frac{2}{X}\mathrm{V}_\ell^{\mathrm{bp}} - \frac{2\ell}{NX}\big(&\mathrm{Put}\big(X, K, \sigma(X, K)\big)\big|_{K=X-\ell} \\ &+ \mathrm{Call}\big(X, K, \sigma(X, K)\big)\big|_{K=X+\ell}\big),\end{aligned} \tag{2.34}$$

and

$$\frac{\partial \mathrm{V}_\ell}{\partial X} = -\frac{2\ell}{NX}\left(\frac{\mathrm{Put}(X, K, \sigma(X, K))|_{K=X-\ell}}{(X-\ell)^2} + \frac{\mathrm{Call}(X, K, \sigma(X, K))|_{K=X+\ell}}{(X+\ell)^2}\right). \tag{2.35}$$

According to Proposition 2.3 in Sect. 2.4, the theoretical percentage index is unresponsive to movements in the forward. Instead, Eq. (2.35) shows that its approximation based on a finite strip of option prices, V_ℓ in Eq. (2.33), moves inversely with X: that is, V_ℓ moves even if the skew remains the same. We now discuss these issues in detail.

2.6.1.1 The Effects on the BP Volatility Index

How does the weighting scheme affect the index behavior in the presence of approximations? It is instructive to analyze the basis point index first. Consider a market in which uncertainty is constant but the forward X increases from X_0 to X_1, say, with $X_1 - X_0 < \ell$. The usable strip of option prices is then re-centered towards the right tail of the available strike distribution, with (i) a new set of call prices entering into the index calculations (those with strikes between $X_0 + \ell$ and $X_1 + \ell$), (ii) a set of call prices leaving the index basis (those with strikes between X_0 and X_1), (iii) a new set of put prices entering into the index (those with strikes between X_0 and X_1), and, finally, (iv) a set of put prices leaving the index basis (those with strikes between $X_0 - \ell$ and $X_1 - \ell$). The index value changes due to the options entering and leaving the index as described *and* due to the change in value of the options remaining in the index. Marginally, when $X_1 - X_0$ is small, we have that

$$\begin{aligned}\frac{N}{2}\frac{\partial \mathrm{V}_\ell^{\mathrm{bp}}}{\partial X} = &\overbrace{\mathrm{Put}\big(X, K, \sigma(X, K)\big)\big|_{K=X}}^{\text{new puts (iii)}} - \overbrace{\mathrm{Put}\big(X, K, \sigma(X, K)\big)\big|_{K=X-\ell}}^{\text{old puts (iv)}} \\ &+ \int_{X-\ell}^{X} \partial_X \mathrm{Put}\big(X, K, \sigma(X, K)\big) dK\end{aligned}$$

$$+\underbrace{\mathrm{Call}\big(X,K,\sigma(X,K)\big)\big|_{K=X+\ell}}_{\text{new calls (i)}}-\underbrace{\mathrm{Call}\big(X,K,\sigma(X,K)\big)\big|_{K=X}}_{\text{old calls (ii)}}$$

$$+\int_X^{X+\ell}\partial_X\mathrm{Call}\big(X,K,\sigma(X,K)\big)dK, \tag{2.36}$$

where ∂_X denotes the total derivative, e.g. $\partial_X\mathrm{Put}(X,K,\sigma(X,K)) = \mathrm{Put}_X(X,K,\sigma(X,K))+\mathrm{Put}_\sigma(X,K,\sigma(X,K))\sigma_X(X,K)$, and subscripts denote partial derivatives. Naturally, the two sets in (ii) and (iii) collapse to ATM call and put prices; therefore, their combined effect is zero.

In Appendix A.5, we show that if the skew has the sticky delta property, Eq. (2.36) is indeed consistent with Eq. (2.34). Equation (2.34) shows that the sign of $\frac{\partial \mathrm{V}_\ell^{\mathrm{bp}}}{\partial X}$ depends on two terms. The first is always positive, and is intuitively so because $\mathrm{V}_\ell^{\mathrm{bp}}$ is proportional to X^2 just like its theoretical counterpart (see Proposition 2.3 in Sect. 2.4). Consistent with this intuition, in Appendix A.5 we show that the approximating BP variance index, $\mathrm{VX}_{\ell,t}^{\mathrm{bp}}(T)$, is

$$\mathrm{VX}_{\ell,t}^{\mathrm{bp}}(T)=X\times\sqrt{(T-t)^{-1}\xi_{\ell,X}}, \tag{2.37}$$

for a function $\xi_{\ell,X}$ of the forward risk (see Eq. (A.12) in Appendix A.5). Instead, the second term in Eq. (2.34) is negative, although it is likely significantly less than the first term in absolute value.

2.6.1.2 The Percentage Index

The percentage index behaves quite differently, as $\frac{\partial \mathrm{V}_\ell}{\partial X}$ in Eq. (2.35) is *always* negative. That is, a percentage volatility index (such as the VIX or TYVIX (see Chap. 4)) can change (driven by movements of X) even while uncertainty is constant. Naturally, a drop in the market level may well be accompanied by increased uncertainty, but the analysis of this section identifies an additional, mechanical effect, arising from the index reliance on a moving window of out-of-the-money option prices. We now develop numerical experiments and illustrate these properties based on realistic market data inputs.

2.6.2 Numerical Experiments and Interpretation of Actual Index Behavior

We consider numerical experiments in which we take as given a log-normal skew, representing hypothetical conditions in the swaption market as of December 21, 2011, and summarized by Table 2.1. Table 2.1 also provides the values of the percentage and BP volatility indexes, $\mathrm{VX}_{\ell,t}(T)$ and $\mathrm{VX}_{\ell,t}^{\mathrm{bp}}(T)$, and the value of the BP ATM implied volatility.

Table 2.1 Black's skew and volatility indexes

	atm								
K	0.305	1.305	1.805	2.055	2.305	2.555	2.805	3.305	4.305
Black Vol	79.7	52.9	47.5	45.7	44.4	43.5	42.9	42.2	41.7
$\mathrm{VX}_{\ell,t}(T) = 51.2536$, $\mathrm{VX}^{\mathrm{bp}}_{\ell,t}(T) = 108.3121$, BP ATM $= 102.3420$									

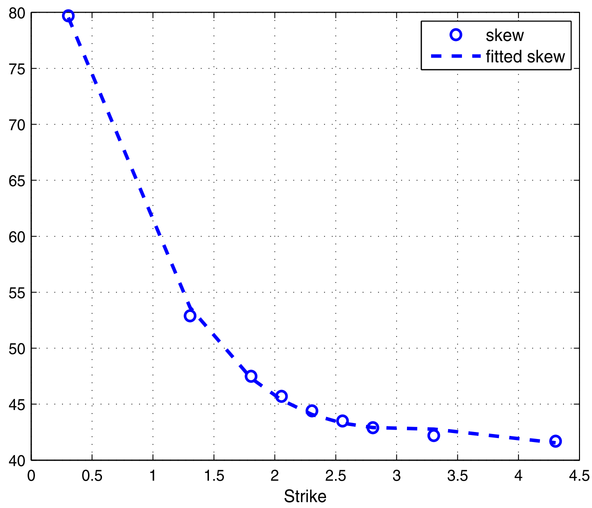


Fig. 2.8 The Black's skew in Table 2.1 (*circles*) along with the cubic lines fit (*dashed line*)

We fit the skew in Table 2.1 with a cubic spline, obtaining the results depicted in Fig. 2.8 where the fitted skew in Fig. 2.8 is denoted by $\hat{\sigma}_K$, and is a continuous function of the strike swap rate K.

In the first experiment, we fix uncertainty and vary the forward rate from 2.10 to 3. For each value of the forward X, we select a set of strikes centered at the forward swap rate X and the same range and coarseness as those in Table 2.1, and calculate a discretized version of Eqs. (2.32) and (2.33) using as an input the fitted skew $\hat{\sigma}_K$.

The right panel of Fig. 2.9 shows that the approximating percentage volatility index is inversely related to the forward, even though its theoretical value is not, as established by Proposition 2.3. The left panel of Fig. 2.9 shows, instead, that the approximating index $\mathrm{VX}^{\mathrm{bp}}_{\ell,t}(T)$ in Eq. (2.37) increases roughly linearly with X, as the theoretical value predicts in Eq. (2.28). The left hand panel also depicts the BP ATM volatility, obtained as,

$$\text{BP ATM} \equiv 100 \times (44.4 \cdot X). \tag{2.38}$$

Note that when uncertainty is fixed, the relative magnitude of the BP volatility index vis-à-vis the BP ATM volatility could change. In this example, the BP volatility index is lower than the ATM only when the forward swap rate is relatively high.

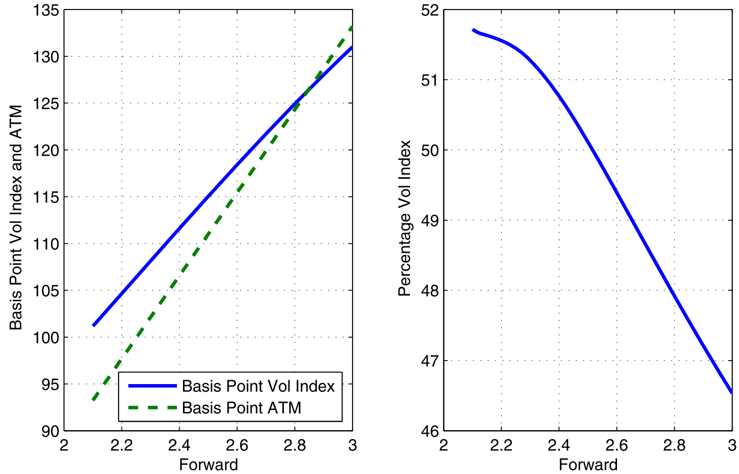


Fig. 2.9 *Left panel*: A Basis Point expected volatility index, calculated as in Eq. (2.32), compared to the Basis Point ATM volatility, depicted as a function of the forward. *Right panel*: A percentage expected volatility index, calculated as in Eq. (2.33). The indexes in the left and right panel are calculated using a strip of swaption prices and the skew in Table 2.1

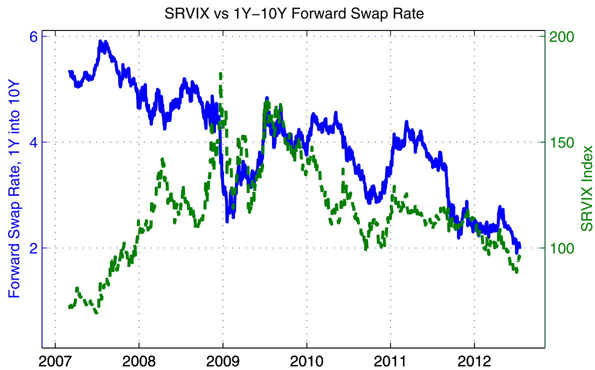


Fig. 2.10 The forward swap rate for 1 year maturity and 10 year tenor (1Y-10Y) versus the CBOE SRVIX index regarding expected volatility of the 1Y-10Y forward rate

We emphasize that this exercise is one of comparative statics. We would expect that uncertainty also changes while the forward swap rate changes.

Accordingly, we consider an additional experiment in which we vary uncertainty, assuming that the swaption market experiences a parallel and positive shift in the skew of 35 percentage points, which is a realistic figure over periods of stress such as those experienced during the 2007–2009 crisis (see Sect. 3.7 in Chap. 3).

A negative change in the forward swap rate coupled with increased uncertainty can be interpreted as the result of an aggressive monetary policy action aimed at stabilizing market expectations and liquidity conditions. An historical instance of these events occurred in the last months of 2008 that followed Lehman Brothers' collapse, which culminated with a spike in both the CBOE SRVIX index (an upward spike) and the forward swap rate (a downward spike) (see Fig. 2.10, which reproduces the top panel of Fig. 2.2).

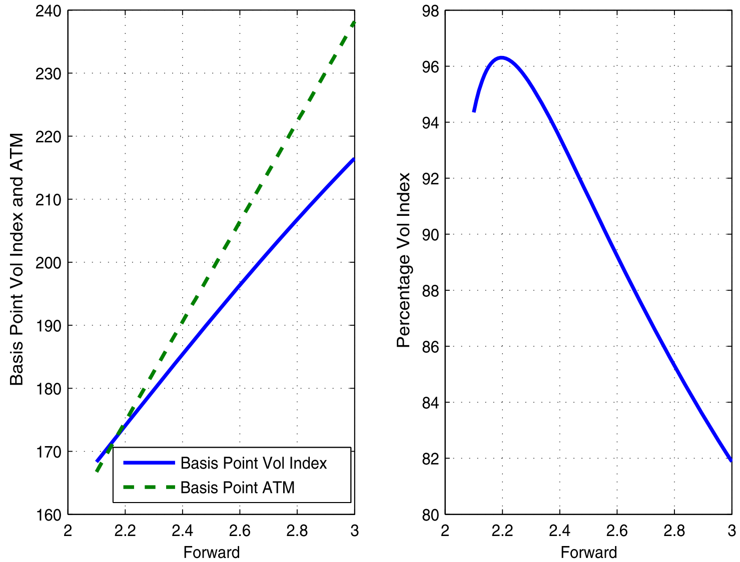


Fig. 2.11 *Left panel*: A Basis Point expected volatility index, calculated as in Eq. (2.32), compared to the Basis Point ATM volatility, depicted as a function of the forward. *Right panel*: A percentage expected volatility index, calculated as in Eq. (2.33). The indexes in the left and right panel are calculated using a strip of swaption prices and the skew in Table 2.1 increased by a positive parallel shift of 35 percentage points

Finally, an increase in the forward swap rate associated with increased uncertainty can be interpreted as the result of decreased risk appetite, a credit crunch, or a combination of the two, such as during some periods in the early part of 2009.

Figure 2.11 illustrates that for each level of the forward, the expected volatility indexes increase after a positive parallel shift in the skew underlying the calculations in Fig. 2.9. Interestingly, the BP volatility index increases less than the ATM for most of the possible range of variation of the forward. An increase in the skew and, hence, an increase in the ATM implied volatility from a value of 44.4 to a value of 79.4, leads to a higher slope of 79.5 in the BP ATM of Eq. (2.38).

In contrast, Figs. 2.9 and 2.11 reveal that the BP expected volatility increases simply through a positive parallel shift for each X. This property is to be expected because, as Fig. 2.9 suggests in this example, $\mathrm{VX}_{\ell,t}^{\mathrm{bp}}(T)$ in Eq. (2.37) increases roughly linearly with X; that is, the term $\xi_{\ell,X}$ moves primarily through changes in uncertainty. The behavior of the percentage index is quite different as it is always higher than the ATM volatility in the examples depicted in Fig. 2.11 (right panel). Figure 2.12 depicts the spread of the expected volatility indexes versus their ATM counterparts. The experiments in this section predict that during periods of distress, a percentage volatility *spread* (index minus ATM) peaks up as can be seen in the right panel of Fig. 2.12. In contrast, assuming interest rates are relatively unresponsive, a BP interest rate volatility can even change sign.

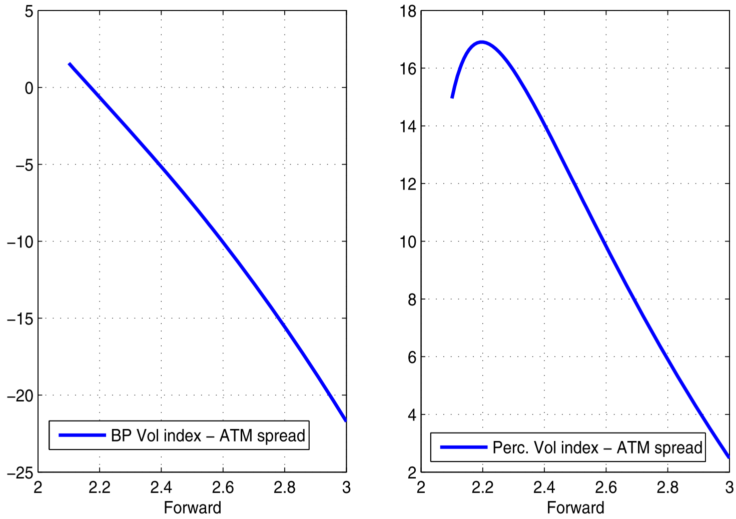


Fig. 2.12 *Left panel*: The spread between the Basis Point volatility index calculated as in Eq. (2.32) and Basis Point ATM volatility, depicted as a function of the forward. *Right panel*: The spread between the percentage volatility index calculated as in Eq. (2.33) and ATM percentage volatility, depicted as a function of the forward

2.7 Jumps

This section examines pricing and indexing of expected volatility in markets where the forward risk is a jump-diffusion process with stochastic volatility:

$$\frac{dX_\tau}{X_\tau} = -\left(\mathbb{E}_\tau^{Q^N}\left(e^{j_\tau} - 1\right)\eta_\tau\right)d\tau + \sigma_\tau \cdot dW_\tau + \left(e^{j_\tau} - 1\right)dJ_\tau, \quad \tau \in [t, T], \tag{2.39}$$

where σ_τ is a diffusion component, adapted to W_τ, J_τ is a Cox process under Q^N with intensity equal to η_τ, and j_τ is the logarithmic jump size.[10] By applying Itô's lemma for jump-diffusion processes to Eq. (2.39), we have that

$$d \ln X_\tau = (\cdots)d\tau + \sigma_\tau \cdot dW_\tau + j_\tau dJ_\tau, \tag{2.40}$$

for a drift function given in Appendix A.6 (see Eq. (A.18)).

Next, define the realized variance of the *arithmetic changes* of the forward risk over $[t, T]$, as

$$V_J^{\text{bp}}(t, T) \equiv \int_t^T X_\tau^2 \|\sigma_\tau\|^2 d\tau + \int_t^T X_\tau^2 \left(e^{j_\tau} - 1\right)^2 dJ_\tau \tag{2.41}$$

[10]See, e.g., Jacod and Shiryaev (1987, pp. 142–146), for a succinct discussion of jump-diffusion processes.

and the realized variance of the *logarithmic changes* of the forward risk over $[t, T]$ as

$$V_J(t,T) \equiv \int_t^T \|\sigma_\tau\|^2 d\tau + \int_t^T j_\tau^2 dJ_\tau. \tag{2.42}$$

The definitions in Eqs. (2.41) and (2.42) generalize those in Eqs. (2.3). In Appendix A.6, we show the remarkable property that the fair value K_Y of the *basis point* variance swaps in Proposition 2.2 is resilient to the presence of jumps, in that $K_{J,Y} = K_Y$ where $K_{J,Y}$ denotes the fair value of the contract in a market with jumps and K_Y is as in Eq. (2.6). Accordingly, the basis point index of expected volatility, $\mathrm{VX}_{Jt}^{\mathrm{bp}}(T)$, is the same as that we derived in the absence of jumps:

$$\mathrm{VX}_{Jt}^{\mathrm{bp}}(T) = \mathrm{VX}_t^{\mathrm{bp}}(T), \tag{2.43}$$

where $\mathrm{VX}_t^{\mathrm{bp}}(T)$ is as in Eq. (2.22).

In contrast, the fair value of a *percentage* variance swap is

$$K_{J,Y} \equiv K_Y - 2\mathbb{E}_t^{Q^N}\left[\int_t^T \left(e^{j_\tau} - 1 - j_\tau - \frac{1}{2} j_\tau^2\right) dJ_\tau\right], \tag{2.44}$$

where K_Y is as in Eq. (2.7) of Proposition 2.2. Suppose, for example, that the distribution of jumps is skewed towards negative values. The fair value $K_{J,Y}$ should then be higher than it would be in the absence of jumps.

To illustrate, assume that the distribution of jumps collapses to a single point, $\bar{j} < 0$ say, and that the jump intensity equals some positive constant $\bar{\eta}$, in which case the fair value in Eq. (2.44) collapses to, $K_{J,Y} = K_J + 2(T-t) \cdot \bar{\eta}\mathcal{J}$, where $\mathcal{J} \equiv -(e^{\bar{j}} - 1 - \bar{j} - \frac{1}{2}\bar{j}^2) > 0$. In this example, the percentage volatility index is

$$\mathrm{VX}_{Jt}^{\mathrm{p}}(T) = \sqrt{\mathrm{VX}_t^{\mathrm{p}}(T) + 2\bar{\eta}\mathcal{J}},$$

where $\mathrm{VX}_t^{\mathrm{p}}(T)$ is as in Eq. (2.22).

Remark 2.4 Carr and Wu (2008) derive an expression for a *percentage* variance swap strike incorporating information about jumps, which Eq. (2.44) generalizes to general market numéraires. Mele and Obayashi (2012) derive the "jumps irrelevance result" in Eq. (2.43) in the context of interest rate swap markets. This result is extended to general numéraires in this chapter.

Remark 2.5 Consider the definition of $V_J(t,T)$ in Eq. (2.42),

$$V_J(t,T) \equiv \underbrace{\int_t^T \|\sigma_\tau\|^2 d\tau}_{\equiv \mathcal{V}(t,T)} + \underbrace{\int_t^T j_\tau^2 dJ_\tau}_{\equiv \mathcal{J}(t,T)}.$$

In the statistics literature, one usually refers to $V_J(t,T)$ as the *total variation* of $\ln X_\tau$ over the interval of time $[t,T]$ (an analogous definition can be provided regarding $V_J^{\text{bp}}(t,T)$). Thus, $\mathcal{V}(t,T)$ is the contribution to the total variation of $\ln X_\tau$ due to its continuous component, whereas $\mathcal{J}(t,T)$ is the jump contribution. In this book, we shall still refer to $V_J(t,T)$ as *variance* rather than *variation* to keep the presentation simple. Aït-Sahalia and Jacod (2014) summarize the state of the art regarding filtering methods for both the continuous and jump contributions of a process in a general context and in the presence of market microstructure noise.

Appendix A: Appendix on Security Design and Volatility Indexing

A.1 *Proof of Proposition 2.2*

We begin with the following preliminary result, from which Eqs. (2.6)–(2.7) follow for $Y_t = N_t$. Note that the arguments in this proof rely on spanning arguments similar to those utilized by Bakshi and Madan (2000) and Carr and Madan (2001) in the equity case, although centered around the notion of a market numéraire. We then provide the proof of the proposition with general stochastic multipliers.

Lemma A.1 *We have*:

$$\mathbb{E}_t^{Q^N}\left(V^{\text{bp}}(t,T)\right) = \frac{2}{N_t}\left(\int_0^{X_t} \text{Put}_t(K)dK + \int_{X_t}^{\infty} \text{Call}_t(K)dK\right), \tag{A.1}$$

and

$$\mathbb{E}_t^{Q^N}\left(V(t,T)\right) = \frac{2}{N_t}\left(\int_0^{X_t} \frac{\text{Put}_t(K)}{K^2}dK + \int_{X_t}^{\infty} \frac{\text{Call}_t(K)}{K^2}dK\right). \tag{A.2}$$

Proof We provide the proof of Eq. (A.1), as that of Eq. (A.2) follows as a special case of the arguments leading to Eq. (A.19) and Eq. (A.20) in Appendix A.5 regarding the jump-diffusion case. By Itô's lemma,

$$\mathbb{E}_t^{Q^N}\left(V^{\text{bp}}(t,T)\right) = \mathbb{E}_t^{Q^N}\left(X_T^2 - X_t^2\right). \tag{A.3}$$

Moreover, by a Taylor expansion with remainder,

$$X_T^2 - X_t^2 = 2X_t(X_T - X_t) + 2\left(\int_0^{X_t} (K - X_T)^+ dK + \int_{X_t}^{\infty} (X_T - K)^+ dK\right). \tag{A.4}$$

Multiplying both sides of the previous equation by $e^{-\int_t^T r_u du} N_T$, and taking expectation under the risk-neutral probability leaves

$$\begin{aligned}
\mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T\left(X_T^2 - X_t^2\right)\right) &= 2X_t \mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T (X_T - X_t)\right) \\
&\quad + 2\left(\int_0^{X_t} \mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T (K - X_T)^+\right) dK\right. \\
&\quad \left. + \int_{X_t}^{\infty} \mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T (X_T - K)^+\right) dK\right) \\
&= 2\left(\int_0^{X_t} \mathrm{Put}_t(K) dK + \int_{X_t}^{\infty} \mathrm{Call}_t(K) dK\right),
\end{aligned} \tag{A.5}$$

where the last line follows by a change of probability, from Q to Q^N,

$$\left.\frac{dQ^N}{dQ}\right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_u du} N_T}{N_t},$$

the martingale property of X_τ under Q^N, and the expressions for $\mathrm{Put}_t(K)$ and $\mathrm{Call}_t(K)$ in Definition 2.2. By the assumption that N_τ is the price of a traded asset, and $N_\tau > 0$, dQ^N integrates to one. Similarly, by a change of probability,

$$\mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T\left(X_T^2 - X_t^2\right)\right) = N_t \mathbb{E}_t^{Q^N}\left(X_T^2 - X_t^2\right). \tag{A.6}$$

Combining Eqs. (A.5) and (A.6) with Eq. (A.3) yields Eq. (A.1). □

Next, we prove the claims of Proposition 2.2 regarding basis point variance, $V^{\mathrm{bp}}(t,T)$; those for percentage variance $V(t,T)$ follow through a mere change in notation. We only prove the "only if" part, as the "if" part is trivial from the derivation of the proof to follow. Consider the Radon–Nikodym derivative of Q^Y against Q^N,

$$\zeta_T \equiv \left.\frac{dQ^Y}{dQ^N}\right|_{\mathbb{F}_T},$$

and suppose on the contrary that there exists a stochastic multiplier Y_T such that

$$\mathrm{cov}^{Q^N}\left(V^{\mathrm{bp}}(t,T), \zeta_T\right) \neq 0,$$

and that at the same time,

$$\mathbb{E}_t^{Q^Y}\left(V^{\mathrm{bp}}(t,T)\right) = \mathbb{E}_t^{Q^N}\left(V^{\mathrm{bp}}(t,T)\right).$$

In this case, Eqs. (A.3) and (A.5) would imply that the conclusions of Lemma A.1 hold. However, we also have:

$$\mathbb{E}_t^{Q^Y}\left(V^{\mathrm{bp}}(t,T)\right) = \mathbb{E}_t^{Q^N}\left(V^{\mathrm{bp}}(t,T)\right) + \mathrm{cov}^{Q^N}\left(V^{\mathrm{bp}}(t,T), \zeta_T\right).$$

Then, $\mathrm{cov}^{Q^N}(V^{\mathrm{bp}}(t,T), \zeta_T) = 0$, a contradiction. Proposition 2.2 follows by Proposition 2.1 in the main text and Lemma A.1. □

A.2 A Stochastic Multiplier Beyond the Market NumÉraire

Consider the following stochastic multiplier, $Y_T = N_T \epsilon_T$, where ϵ_T is $\mathbb{F}_T$-measurable, and such that $\mathrm{cov}^{Q^N}(V^{\mathrm{bp}}(t,T), \epsilon_T) = 0$. We have,

$$\zeta_T \equiv \left.\frac{dQ^Y}{dQ^N}\right|_{\mathbb{F}_T} = c_t \epsilon_T, \quad \text{with } c_t \equiv \frac{\mathbb{E}_t(e^{-\int_t^T r_u du} N_T)}{\mathbb{E}_t(e^{-\int_t^T r_u du} N_T \epsilon_T)}.$$

Heuristically,

$$\zeta_T = \left.\left(\frac{dQ^Y}{dQ} : \frac{dQ^N}{dQ}\right)\right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_u du} Y_T}{\mathbb{E}_t(e^{-\int_t^T r_u du} Y_T)} : \frac{e^{-\int_t^T r_u du} N_T}{\mathbb{E}_t(e^{-\int_t^T r_u du} N_T)} = \epsilon_T c_t.$$

Next, we claim that

$$\mathrm{cov}^{Q^N}\left(V^{\mathrm{bp}}(t,T), \zeta_T\right) = c_t \cdot \mathrm{cov}^{Q^N}\left(V^{\mathrm{bp}}(t,T), \epsilon_T\right) = 0,$$

as in the class of multipliers identified by Proposition 2.2. Indeed, we have:

$$\mathbb{E}_t^{Q^Y}\left(V^{\mathrm{bp}}(t,T)\right) = c_t \mathbb{E}_t^{Q^N}(\epsilon_T) \mathbb{E}_t^{Q^N}\left(V^{\mathrm{bp}}(t,T)\right) = \mathbb{E}_t^{Q^N}\left(V^{\mathrm{bp}}(t,T)\right),$$

where the first equality follows by the fact that ϵ_T is uncorrelated with $V^{\mathrm{bp}}(t,T)$, and the second follows by the definition of c_t,

$$\mathbb{E}_t^{Q^N}(\epsilon_T) = \frac{1}{N_t} \mathbb{E}_t\left(e^{-\int_t^T r_u du} N_T \epsilon_T\right) = \frac{1}{c_t}.$$

A.3 Vega and Gamma in Gaussian Markets

We first show Eq. (2.9) ("Constant volatility"), then prove that the statement in (2.21) is true ("Constant vega") and, finally, validate our claims regarding the implications on gamma of option portfolios ("Constant gamma exposure"). In what follows, we assume that the portfolio weightings and their first order derivative are integrable with respect to the Gaussian distribution, (i) $E(|\omega(\tilde{y})|) < \infty$ and $E(\int_{-\infty}^{0} |\omega(u)u|\Phi(u)du) < \infty$, and (ii) $E(|\omega'(\tilde{y})|) < \infty$, where $\tilde{y}$ is Gaussian. The first two conditions are necessary for the existence of an abstract volatility index based on at-the-money and out-of-the-money options in a Gaussian market (see Eqs. (A.7) and (A.8) below). The second is a regularity condition needed while dealing with boundedness of the sensitivity of vega with respect to the forward risk, X_t.

We shall rely on the following result. Consider the market in Sects. 2.2 and 2.3. If the forward risk X_t is a solution to Eq. (2.8), the prices of put and call options are given by the "Bachelier formulae" (see Eq. (2.24) in the main text):

$$\begin{aligned} \mathcal{O}_t^{\mathrm{P}}(X_t, K, T, \hat{\sigma}_{\mathrm{n}}) &\equiv N_t \cdot \mathcal{Z}_t^{\mathrm{P}}(X_t, K, T, \hat{\sigma}_{\mathrm{n}}), \\ \mathcal{O}_t^{\mathrm{C}}(X_t, K, T, \hat{\sigma}_{\mathrm{n}}) &\equiv N_t \cdot \mathcal{Z}_t^{\mathrm{C}}(X_t, K, T, \hat{\sigma}_{\mathrm{n}}), \end{aligned} \tag{A.7}$$

where $\hat{\sigma}_{\mathrm{n}} \equiv \sqrt{\|\sigma_{\mathrm{n}}\|^2}$,

$$\mathcal{Z}_t^{\mathrm{P}}(X, K, T, \sigma) = (K - X)\Phi\left(\frac{K - X}{\sigma\sqrt{T - t}}\right) + \sigma\sqrt{T - t}\phi\left(\frac{X - K}{\sigma\sqrt{T - t}}\right),$$

$$\mathcal{Z}_t^{\mathrm{C}}(X, K, T, \sigma) = (X - K)\Phi\left(\frac{X - K}{\sigma\sqrt{T - t}}\right) + \sigma\sqrt{T - t}\phi\left(\frac{X - K}{\sigma\sqrt{T - t}}\right),$$

and ϕ denotes the standard normal density.

CONSTANT VOLATILITY. Plugging Eq. (A.7) into Eq. (2.6) leaves, after substituting the expressions $\mathrm{Put}_t(K) \equiv \mathcal{O}_t^{\mathrm{P}}(X, K, T, \hat{\sigma}_{\mathrm{n}})$ and $\mathrm{Call}_t(K) \equiv \mathcal{O}_t^{\mathrm{C}}(X, K, T, \hat{\sigma}_{\mathrm{n}})$, and extending the left limit of the first integral in Eq. (2.6) to $-\infty$,

$$\begin{aligned} \frac{1}{2} K_Y &= \sigma^2(T - t) + \int_{-\infty}^{X} (K - X)\Phi\left(\frac{K - X}{\sigma\sqrt{T - t}}\right) dK \\ &\quad + \int_{X}^{\infty} (X - K)\Phi\left(\frac{X - K}{\sigma\sqrt{T - t}}\right) dK \\ &= \sigma^2(T - t)\left[1 + 2\int_{-\infty}^{0} u\Phi(u)du\right] \\ &= \sigma^2(T - t)\left[1 + 2\left(-\frac{1}{2}\int_{-\infty}^{0} u^2\phi(u)du\right)\right] \\ &= \frac{1}{2}\sigma^2(T - t), \end{aligned}$$

where the first equality follows by the property of the normal distribution that, $\int_{-\infty}^{\infty} \phi(\frac{X-K}{\nu})dK = \nu$ for $\nu > 0$, and the second holds by a change in variables, the third by an integration by parts, and the fourth by a basic property of the standard normal distribution.

CONSTANT VEGA. We prove that in Gaussian markets, a portfolio with all out-of-the-money and at-the-money European options has constant vega if and only if these options are equally weighted. The value of the portfolio we consider is a special case of Eq. (2.19),

$$\pi_t(X, T, \sigma) = N_t\left(\int_{-\infty}^{X} \omega(K)\mathcal{Z}_t^{\mathrm{P}}(X, K, T, \sigma)dK + \int_{X}^{\infty} \omega(K)\mathcal{Z}_t^{\mathrm{C}}(X, K, T, \sigma)dK\right), \tag{A.8}$$

where $\mathcal{Z}_t^{\mathrm{P}}$ and $\mathcal{Z}_t^{\mathrm{C}}$ are as in Eqs. (A.7).

Note that the vega of a put is the same as the vega of a call, and equals $N_t\nu_t^{\mathcal{O}}(X, K, T, \sigma)$, where:

$$\nu_t^{\mathcal{O}}(X, K, T, \sigma) \equiv \frac{\partial \mathcal{Z}_t^{\mathrm{P}}(X, K, T, \sigma)}{\partial \sigma} = \frac{\partial \mathcal{Z}_t^{\mathrm{C}}(X, K, T, \sigma)}{\partial \sigma} = \sqrt{T-t}\,\phi\left(\frac{X-K}{\sigma\sqrt{T-t}}\right),$$

so that the vega of the portfolio is:

$$\nu_t(X, T, \sigma) \equiv \frac{\partial \pi_t(X, T, \sigma)}{\partial \sigma} = N_t\sqrt{T-t}\int \omega(K)\phi\left(\frac{X-K}{\sigma\sqrt{T-t}}\right)dK. \tag{A.9}$$

As for the "if" part in (2.21), let $\omega(K) = \text{const.}$, so that by Eq. (A.9), and the fact that the Gaussian density ϕ integrates to one, we have that the vega is independent of X:

$$\nu_t(X, T, \sigma) = N_t\sqrt{T-t} \cdot \text{const.}$$

As for the "only if" part, let us differentiate $\nu_t(X, T, \sigma)$ in Eq. (A.9) with respect to X:

$$\frac{\partial \nu_t(X, T, \sigma)}{\partial X} = -\frac{N_t}{\sigma^2\sqrt{T-t}}\int \omega(K)\phi\left(\frac{X-K}{\sigma\sqrt{T-t}}\right)(X-K)dK.$$

We claim that the constant weighting is the only function ω independent of X, and such that $\frac{\partial \nu_t(X,T,\sigma)}{\partial X} = 0$. Suppose not, and note that $\frac{\partial \nu_t(X,T,\sigma)}{\partial X}$ is zero if and only if,

$$X\int \omega(K)\phi\left(\frac{X-K}{\sigma\sqrt{T-t}}\right)dK = \int K\omega(K)\phi\left(\frac{X-K}{\sigma\sqrt{T-t}}\right)dK.$$

Let us moreover define a random variable $\tilde{y} \sim N(\mu, \sigma^2(T-t))$. In terms of $\tilde{y}$, the previous equality is, by Stein's Lemma,

$$\mu E\big[\omega(\tilde{y})\big] = E\big[\tilde{y}\omega(\tilde{y})\big] = \mu E\big[\omega(\tilde{y})\big] + \mathrm{cov}\big[\tilde{y}, \omega(\tilde{y})\big] = \mu E\big[\omega(\tilde{y})\big] + E\big[\omega'(\tilde{y})\big]\sigma^2(T-t),$$

which is a contradiction unless $\omega(\cdot)$ is constant.

Remark A.1 By Proposition 2.2, and previous results in this appendix, the normalized value of the portfolio in Eq. (A.8) *is* σ^2, in the special case $\omega(K) = 2$,

$$\sqrt{\frac{\pi_t(X_t, T, \sigma)}{N_t}} = \sigma\sqrt{T-t}.$$

We illustrate this fact numerically. We set $X_t = 5\ \%$, $T - t = 1$ (one year) and $\sigma = 150$ bps, and approximate the integral in Eq. (A.8),

$$\begin{aligned} &100^2 \times \sqrt{\frac{\hat{\pi}_t(X_t, T, \sigma)}{N_t}} \\ &\approx 100^2 \times \sqrt{2\left(\sum_{i:K_i<X_t} \mathcal{Z}_t^{\mathrm{P}}(X_t, K_i, T, \sigma) + \sum_{i:K_i \geq X_t} \mathcal{Z}_t^{\mathrm{C}}(X_t, K_i, T, \sigma)\right)\Delta K} \\ &\approx 150.3907, \end{aligned}$$

where $\min_i\{K_i\} = 0$, $\max_i\{K_i\} = 10\ \%$, $\Delta K = 0.0001$.

CONSTANT GAMMA EXPOSURE. Our claim of a constant gamma exposure in a Gaussian market follows because the option price in Eq. (A.7) satisfies,

$$\frac{\partial^2 \mathcal{O}_t^{\mathrm{U}}(X, K, T, \sigma)}{\partial X^2} = N_t \frac{1}{\sigma(T-t)} \nu_t^{\mathcal{O}}(X, K, T, \sigma),$$

where $\mathcal{O}_t^{\mathrm{U}}(X, K, T, \sigma)$, $\mathrm{U} \in \{\mathrm{P}, \mathrm{C}\}$ denotes an out-of-the-money option price (see Eq. (A.7)). Therefore, we have

$$\begin{aligned} \frac{\partial^2 \pi_t(X, T, \sigma)}{\partial X^2} &= \int \omega(K) \frac{\partial^2 \mathcal{O}_t^{\mathrm{U}}(X, K, T, \sigma)}{\partial X^2} dK \\ &= \frac{1}{\sigma(T-t)} \int \omega(K) \frac{\partial \mathcal{O}_t^{\mathrm{U}}(X, K, T, \sigma)}{\partial \sigma} dK \\ &= \frac{1}{\sigma(T-t)} \frac{\partial \pi_t(X, T, \sigma)}{\partial \sigma}. \end{aligned} \tag{A.10}$$

The L.H.S. of this equation is independent of X if and only if the R.H.S. is. That is, the gamma exposure of the portfolio is constant if and only if the portfolio vega is independent of X, as claimed in the main text.

A.4 Proof of Proposition 2.3

For simplicity, we suppress the dependence of all the variables and functions on t and T. Assuming the zero homogeneity assumption is satisfied by the implied volatility $\sigma_K \equiv \sigma(X, K)$, we have that the two Black pricers, $\mathcal{P}(X, K, \sigma_K) \equiv \text{Put}(K)$ and $\mathcal{C}(X, K, \sigma_K) \equiv \text{Call}(K)$, collapse to

$$\mathcal{P}(X, K, \sigma_K) = K\varphi_{\mathrm{p}}^1(u) - X\varphi_{\mathrm{p}}^2(u), \qquad \mathcal{C}(X, K, \sigma_K) = X\varphi_{\mathrm{c}}^1(u) - K\varphi_{\mathrm{c}}^2(u), \tag{A.11}$$

for some functions $\varphi_{\mathrm{p}}^i(x)$ and $\varphi_{\mathrm{c}}^i(x)$ of the moneyness $u \equiv \ln(\frac{K}{X})$. Substituting the two expressions in Eqs. (A.11) into Eq. (2.6) of Proposition 2.2, and making the change of variable $K \mapsto u$, leaves,

$$\mathrm{V}^{\mathrm{bp}} = X^2 \cdot \xi,$$
$$\xi \equiv \frac{2}{N}\left(\int_{-\infty}^{0} \left(e^u\varphi_{\mathrm{p}}^1(u) - \varphi_{\mathrm{p}}^2(u)\right)e^u du + \int_0^{\infty} \left(\varphi_{\mathrm{c}}^1(u) - e^u\varphi_{\mathrm{c}}^2(u)\right)e^u du\right),$$

where N is the market numéraire at t, and the function ξ is independent of X, establishing Part (i) of the proposition. Part (ii) is similar. Substituting $\mathcal{P}(X, K, \sigma_K)$ and $\mathcal{C}(X, K, \sigma_K)$ in Eq. (A.11) into Eq. (2.7) of Proposition 2.2, and by changing the variable of integration to $u = \ln\frac{K}{X}$, leaves

$$\mathrm{V} = \frac{2}{N}\left(\int_{-\infty}^{0} \left(\varphi_{\mathrm{p}}^1(u) - e^{-u}\varphi_{\mathrm{p}}^2(u)\right)du + \int_0^{\infty} \left(e^{-u}\varphi_{\mathrm{c}}^1(u) - \varphi_{\mathrm{c}}^2(u)\right)du\right),$$

which is independent of X. $\square$

A.5 Approximating Indexes

We derive Eq. (2.34) and Eq. (2.35). We begin with Eq. (2.35). Substituting Eq. (A.11) into Eq. (2.33) yields, for $\ell \in (0, X)$,

$$\mathrm{V}_\ell = \frac{2}{N}\left(\int_{\ln(\frac{X-\ell}{X})}^{0} \left(\varphi_{\mathrm{p}}^1(u) - e^{-u}\varphi_{\mathrm{p}}^2(u)\right)du + \int_0^{\ln(\frac{X+\ell}{X})} \left(e^{-u}\varphi_{\mathrm{c}}^1(u) - \varphi_{\mathrm{c}}^2(u)\right)du\right),$$

so that,

$$\frac{\partial \mathrm{V}_\ell}{\partial X} = \frac{2}{N}\Bigg(-\left(\varphi_{\mathrm{p}}^1(u) - e^{-u}\varphi_{\mathrm{p}}^2(u)\right)\Big|_{u=\ln(\frac{X-\ell}{X})} \cdot \frac{\ell}{X(X-\ell)}$$
$$+ \left(e^{-u}\varphi_{\mathrm{c}}^1(u) - \varphi_{\mathrm{c}}^2(u)\right)\Big|_{u=\ln(\frac{X+\ell}{X})} \cdot \frac{-\ell}{X(X+\ell)}\Bigg).$$

Utilizing the expressions in Eq. (A.11) delivers Eq. (2.35). The proof of Eq. (2.34) proceeds similarly: substitute Eq. (A.11) into Eq. (2.32) to obtain, for $\ell \in (0, X)$,

$$\mathrm{V}_\ell^{\mathrm{BP}} = X^2 \times \xi_{\ell,X}, \tag{A.12}$$

where,

$$\xi_{\ell,X} \equiv \frac{2}{N}\left(\int_{\ln(\frac{X-\ell}{X})}^{0} \left(e^u\varphi_{\mathrm{p}}^1(u) - \varphi_{\mathrm{p}}^2(u)\right)e^u du + \int_0^{\ln(\frac{X+\ell}{X})} \left(\varphi_{\mathrm{c}}^1(u) - e^u\varphi_{\mathrm{c}}^2(u)\right)e^u du\right).$$

We have

$$\begin{aligned}
\frac{\partial \xi_{\ell,X}}{\partial X} &= \frac{2}{N}\Bigg(-\left(e^u\varphi_{\mathrm{p}}^1(u) - \varphi_{\mathrm{p}}^2(u)\right)e^u\Big|_{u=\ln(\frac{X-\ell}{X})} \cdot \frac{\ell}{X(X-\ell)} \\
&\quad + \left(\varphi_{\mathrm{c}}^1(u) - e^u\varphi_{\mathrm{c}}^2(u)\right)\Big|_{u=\ln(\frac{X+\ell}{X})} \cdot \frac{-\ell}{X(X+\ell)}\Bigg) \\
&= -\frac{2\ell}{NX^3}\left(\mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X-\ell} + \mathrm{Call}\big(X,K,\sigma(X,K)\big)\big|_{K=X+\ell}\right),
\end{aligned}$$

where the second equality follows by the expressions in Eq. (A.11). Equation (2.34) follows by straightforward differentiation of Eq. (A.12).

Next we show that Eqs. (2.34) and (2.36) are mutually consistent. We have

$$\begin{aligned}
&\frac{\partial}{\partial X}\int_{X-\ell}^{X} \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK \\
&\quad = \mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X} - \mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X-\ell} \\
&\qquad + \int_{X-\ell}^{X} \partial_X \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK \\
&\quad = \mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X} - \mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X-\ell} \\
&\qquad + \frac{1}{X}\int_{X-\ell}^{X} \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK \\
&\qquad - \frac{1}{X}\int_{X-\ell}^{X} K\cdot\partial_K \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK,
\end{aligned} \tag{A.13}$$

where the second equality follows by the assumption the implied volatilities are homogenous of degree zero in (X,K), so that $\mathrm{Put}(\cdot) = X\cdot\partial_X\mathrm{Put}(\cdot) + K\cdot\partial_K\mathrm{Put}(\cdot)$. An integration by parts of the last term in Eq. (A.13) produces

$$\begin{aligned}
&\int_{X-\ell}^{X} K\cdot\partial_K \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK \\
&\quad = X\cdot\mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X} - (X-\ell)\cdot\mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X-\ell} \\
&\qquad - \int_{X-\ell}^{X} \mathrm{Put}\big(X,K,\sigma(X,K)\big)dK.
\end{aligned}$$

Substituting this term into Eq. (A.13) leaves

$$\begin{aligned}&\frac{\partial}{\partial X}\int_{X-\ell}^{X}\mathrm{Put}\big(X,K,\sigma(X,K)\big)dK\\&\quad=\frac{2}{X}\int_{X-\ell}^{X}\mathrm{Put}\big(X,K,\sigma(X,K)\big)dK-\frac{\ell}{X}\mathrm{Put}\big(X,K,\sigma(X,K)\big)\big|_{K=X-\ell}.\end{aligned}\tag{A.14}$$

Similarly,

$$\begin{aligned}&\frac{\partial}{\partial X}\int_{X}^{X+\ell}\mathrm{Call}\big(X,K,\sigma(X,K)\big)dK\\&\quad=\frac{2}{X}\int_{X}^{X+\ell}\mathrm{Call}\big(X,K,\sigma(X,K)\big)dK-\frac{\ell}{X}\mathrm{Call}\big(X,K,\sigma(X,K)\big)\big|_{K=X+\ell}.\end{aligned}\tag{A.15}$$

Equation (2.34) now follows by taking derivatives with respect to X in Eq. (2.32), using Eqs. (A.14)–(A.15), and rearranging terms.

A.6 Jumps

We derive the expression for the fair value of K_Y in Proposition 2.2 under the assumption that the forward risk X_t is a solution to the jump-diffusion process in Eq. (2.39).

BASIS POINT. Apply Itô's lemma for jump-diffusion processes to Eq. (2.39), obtaining,

$$\begin{aligned}\frac{dX_\tau^2}{X_\tau^2}&=-2\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau+2\sigma_\tau\cdot dW_\tau+\|\sigma_\tau\|^2d\tau+\big(e^{2j_\tau}-1\big)dJ_\tau\\&=-2\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau+2\big(e^{j_\tau}-1\big)dJ_\tau+2\sigma_\tau\cdot dW_\tau+\|\sigma_\tau\|^2d\tau\\&\quad+\big(e^{j_\tau}-1\big)^2dJ_\tau.\end{aligned}$$

By integrating, taking expectations under Q^N, and using the definition of basis point variance, $V_J^{\mathrm{bp}}(t,T)$ in Eq. (2.41), leaves:

$$\begin{aligned}&\mathbb{E}_t^{Q^N}\big(X_T^2-X_t^2\big)\\&\quad=\underbrace{-2\mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau\right)+2\mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2\big(e^{j_\tau}-1\big)dJ_\tau\right)}_{=0}\\&\qquad+\underbrace{2\mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2\sigma_\tau^2\cdot dW_\tau\right)}_{=0}+\mathbb{E}_t^{Q^N}\big[V_J^{\mathrm{bp}}(t,T)\big],\end{aligned}\tag{A.16}$$

where the first term is zero as,

$$\mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2\big(e^{j_\tau}-1\big)dJ_\tau\right)=\mathbb{E}_t^{Q^N}\left(\int_t^T \mathbb{E}_\tau^{Q^N}\big(X_\tau^2\big(e^{j_\tau}-1\big)dJ_\tau\big)\right)$$
$$=\mathbb{E}_t^{Q^N}\left(\int_t^T X_\tau^2\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau\right). \quad (A.17)$$

Comparing Eq. (A.16) with Eqs. (A.5) and (A.6) and then (2.6) leads to the conclusions of the main text, and in particular to Eq. (2.43).

PERCENTAGE. First, apply Itô's lemma to Eq. (2.39), obtaining Eq. (2.40), viz

$$\begin{aligned} d\ln X_\tau &= -\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau-\frac{1}{2}\|\sigma_\tau\|^2 d\tau+\sigma_\tau\cdot dW_\tau+j_\tau dJ_\tau \\ &= -\frac{1}{2}\big(\|\sigma_\tau\|^2 d\tau+j_\tau^2 dJ_\tau\big)+\sigma_\tau\cdot dW_\tau-\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau \\ &\quad + j_\tau dJ_\tau+\frac{1}{2}j_\tau^2 dJ_\tau, \end{aligned} \quad (A.18)$$

so that, by the definition of $V_J(t,T)$ in Eq. (2.42) and (by arguments similar to those leading to Eq. (A.17)),

$$\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)dJ_\tau=\big(\mathbb{E}_\tau^{Q^N}\big(e^{j_\tau}-1\big)\eta_\tau\big)d\tau,$$

we obtain,

$$-2\mathbb{E}_t^{Q^N}\left(\ln\frac{X_T}{X_t}\right)-2\mathbb{E}_t^{Q^N}\left[\int_t^T\left(e^{j_\tau}-1-j_\tau-\frac{1}{2}j_\tau^2\right)dJ_\tau\right]=\mathbb{E}_t^{Q^N}\big(V_J(t,T)\big). \quad (A.19)$$

Next, consider the standard Taylor's expansion with remainder,

$$\ln\frac{X_T}{X_t}=\frac{1}{X_t}(X_T-X_t)-\left(\int_0^{X_t}\frac{1}{K^2}(K-X_T)^+dK+\int_{X_t}^\infty\frac{1}{K^2}(X_T-K)^+dK\right).$$

Taking the expectation under Q^N yields,

$$\mathbb{E}_t^{Q^N}\left(\ln\frac{X_T}{X_t}\right)=-\frac{1}{N_t}\left(\int_0^{X_t}\frac{1}{K^2}\text{Put}_t(K)dK+\int_{X_t}^\infty\frac{1}{K^2}\text{Call}_t(K)dK\right), \quad (A.20)$$

where we have made use of the martingale property of X_τ under Q^N, and the expressions for $\text{Put}_t(K)$ and $\text{Call}_t(K)$ in Definition 2.2. Combining Eq. (A.19) and Eq. (A.20), and using Proposition 2.1, leaves the expression for $K_{J,Y}\equiv\mathbb{E}_t^{Q^N}[V_J(t,T)]$ in Eq. (2.44) of the main text.

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