Chapter 2
Function Spaces

2.1 L^p, C^α, BMO, $L^{p,\lambda}, L_w^{p,\lambda}$

Listed here are several classical function spaces defined with some of their properties useful in the sequel. Generally, Ω will be a bounded domain in $\mathbb{R}^n$, and when it matters, with smooth boundary $\partial\Omega$ or at least a boundary of type A (as noted in the Introduction). L^p denotes the usual Lebesgue space of p^{th} power integrable functions on $\mathbb{R}^n$ or respectively on $\Omega, L^p(\Omega)$; $1 \leq p < \infty$, and $||f||_{L^p}$ or $||f||_{L^p(\Omega)}$. On the other hand, weak - $L^p(L_w^p)$ consists of those f for which

$$\sup_{t>0}\left[t^p\ \mathscr{L}_n\ (\{x \in \mathbb{R}^n : |f(x)| > t\})\right]^{1/p} \equiv ||f||_{L_w^p} < \infty,$$

and the corresponding $L_w^p(\Omega).L^\infty$ is the essentially bounded functions on R^n, $L^\infty(\Omega)$ those on Ω. Whereas, C^α is the usual space of Hölder continuous functions on $\mathbb{R}^n$ with exponent $\alpha \in (0, 1)$ and normed by

$$||f||_{C^\alpha} = ||f||_{L^\infty} + [f]_\alpha < \infty,$$

where,

$$[f]_\alpha = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

and the corresponding $C^\alpha(\Omega)$. We shall also have some need of the John-Nirenberg space of functions of bounded mean oscillation on a fixed cube $Q_0 \subset \mathbb{R}^n$ or on $\mathbb{R}^n$ itself, i.e., $f \in \mathrm{BMO}(Q_0)$ if

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$$\sup_{Q \subset Q_0} \fint_Q |f(x) - f_Q| dx = [f]_{*,Q_0} < \infty,$$

where Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes; $f_Q =$ the integral average of f over Q. For BMO on $\mathbb{R}^n$, just replace Q_0 by $\mathbb{R}^n$. And in particular, we have the celebrated J-N Lemma:

Lemma 2.1. If $[f]_{*,Q_0} < \infty$, then there exist two constants c_1 and c_2 such that

$$\mathscr{L}_n(\{x \in Q : |f(x) - f_Q| > \lambda\}) \leq c_1 e^{-c_2 \lambda / [f]_{*,Q_0}} \cdot |Q|$$

for all cubes $Q \subset Q_0$ and any $\lambda > 0$. Here $|Q| = \mathscr{L}_n(Q)$.

(For this, see either [St2] or [To]). It thus follows that $L^\infty(Q_0) \subset \mathrm{BMO}(Q_0) \subset L^p(Q_0)$ for all $p \geq 1$.

2.2 The Campanato scale $\mathscr{L}^{p,\lambda}$, $\mathscr{L}^{p,\lambda}(Q_0)$

For $-p < \lambda \leq n$, set

$$\mathscr{L}^{p,\lambda}(Q_0) = \{f : ||f||_{L^p(Q_0)} + [f]_{p,\lambda;Q_0} < \infty\}$$

where

$$[f]_{p,\lambda;Q_0} = \sup_{Q \subset Q_0} \left(|Q|^{-\lambda/n} \fint_Q |f - f_Q|^p \, dx \right)^{1/p}.$$

As mentioned, this scale includes many of the classical function spaces of Harmonic Analysis, notably $L^p(Q_0), L^{p,\lambda}(Q_0), \mathrm{BMO}(Q_0),$ and $C^\alpha(Q_0)$ - as well as versions over $\mathbb{R}^n$. We will not give a complete proof of (1.6) even just over Q_0, but refer the reader to [Tr] for missing details. But because these notes are about Morrey Spaces, it seems appropriate to at least prove:

$$\mathscr{L}_0^{p,\lambda}(Q_0) = L^{p,\lambda}(Q_0) \tag{2.1}$$

with equivalence of norms: $0 < \lambda \leq n, \ 1 < p < \infty$. In fact this is a consequence of the following iteration lemma:

Lemma 2.2. Let $\varphi(r)$ be a non-negative function on $(0, R]$ and suppose there are numbers $\beta, \gamma > 0$ and $K > 1$ such that

$$\varphi(\rho) \leq K \left(\frac{\rho}{r} \right)^\beta \varphi(r) + K \rho^\gamma$$

whenever $\frac{r}{s} \leq \rho < r$ and $r \leq R$, for some $s > 1$ and $R < \infty$. Then for any $0 < \epsilon < \beta - r$,

$$\varphi(\rho) \leq K\left(\frac{\rho}{r}\right)^{\beta-\epsilon} \varphi(r) + KC\rho^{\gamma} \tag{2.2}$$

for some constant C depending only on $K, \ \beta, \ \gamma$. In particular, (2.2) holds for all $0 < \rho < r = R$. This Lemma is a special case of Lemma 1.18 of [Tr].

Applying this lemma to the estimate below gives the inclusion $\mathscr{L}^{p,\lambda}(Q_0) \subset L^{p,\lambda}(Q_0)$. The reverse is obvious. So let $Q = Q_\rho$ = a cube with edge length ρ, then

$$\begin{aligned}\varphi(\rho) = \int_{Q_\rho} |u|^p &\leq 2^\rho \int_Q |u - u_Q|^p + 2^\rho \int_Q |u_Q|^p \\ &\leq 2^p \rho^{n-\lambda} [u]^p{}_{p,\lambda;Q_0} + 2^\rho \frac{\rho^n}{r^n} \int_{Q_r} |u|^p.\end{aligned}$$

Then with Lemma 2.2 and $\gamma = n - \lambda < n - \epsilon$ for some $\epsilon > 0$, and $\beta = n$, it follows that

$$\varphi(\rho) \leq C\rho^{n-\lambda}\left([u]^p_{p,\lambda;Q_0} + ||u||^p_{L^p(Q_0)}\right)$$

for all $0 < \rho < R$.

For the case $\lambda = 0$, applying the J-N Lemma gives $1 < p < \infty$

$$\mathrm{BMO}(Q_0) = \left\{f : \sup_{Q\subset Q_0} \left(⨍_Q |f - f_Q|^p \, dx\right)^{1/p} < \infty\right\},$$

for any $Q_0 \subset \mathbb{R}^n$.

Finally, when $-p < \lambda < 0$, we remark that

$$\mathscr{L}^{p,\lambda}(Q_0) \equiv C^\alpha(Q_0), \alpha \equiv -\lambda/p > 0, \tag{2.3}$$

was the consequence of independent investigations by S. Campanato [Ca] and N.G. Meyers [M2]. See [To], VIII.5. The proof in this case is a bit more delicate than that given above for $0 < \lambda \leq n$, but is repeated in several sources, namely in both [Tr] and [G].

2.3 Sobolev Spaces $W^{m,p}(\Omega), G_\alpha(L^p), I_\alpha(L^p)$

The space $W^{m,p}(\Omega)$ consists of those weakly differentiable functions $u(x)$ on Ω for which

$$\int_\Omega |D^m\, u|^p\, dx + \int_\Omega |u|^p\, dx \equiv ||u||^p_{W^{m,p}(\Omega)} < \infty,$$

where $m =$ positive integer, $D^m\, u =$ the set of all m^{th} order derivatives of u on Ω. But for us, it will be essential to have a potential theoretic version of $W^{m,p}(\mathbb{R}^n)$. Here we refer to [AH] with $G_\alpha(x)$ the Bessel potential operator, $\alpha > 0, \hat{G_\alpha}(\zeta) =$ (Fourier transform of G_α) $= (1 + |\zeta|^2)^{-\alpha/2}, \zeta \in \mathbb{R}^n$. So when $\alpha = m$, a positive integer, then $W^{m,p}(\mathbb{R}^n) = G_\alpha(L^p(\mathbb{R}^n))$, and with $u(x) = G_m f,\ ||u||_{W^{m,p}(\mathbb{R}^n)} \sim ||f||_{L^p(\mathbb{R}^n)}$, with equivalent norms; $1 < p < \infty$.

Now if $I_\alpha(x) = |x|^{\alpha-n}$, $0 < \alpha < n$, and $f \in L^p(\mathbb{R}^n)$, $I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy$, the Riesz potential of f of order α (which is finite a.e. when $\alpha p < n$), then

$$||f||_{L^p(\mathbb{R}^n)} + ||I_\alpha f||_{L^p(\mathbb{R}^n)}$$

is an equivalent Sobolev norm, $\alpha = m < n, 1 < p < n/\alpha$. This is the result of the fact that the Calderon-Zymund singular integrals are bounded on $L^p(\mathbb{R}^n),\ 1 < p < \infty$. See [St2] and [To].

2.4 Morrey-Sobolev Spaces $I_\alpha(L^{p,\lambda})$

From the above, we shall denote by $I_\alpha(L^{p,\lambda}(\mathbb{R}^n))$ a Morrey-Sobolev space, $0 < \lambda < n,\ 1 < p < n/\alpha,\ 0 < \alpha < n$. And often we will write $I_\alpha f$ when f has compact support, and then from the results latter of Chapter 8, we set

$$||u||_{W^{m,p;\lambda}} = ||u||_{L^{p,\lambda}(\mathbb{R}^n)} + ||f||_{L^{p,\lambda}(\mathbb{R}^n)},$$

when $u(x) = I_m f$; i.e., u and its m-th order derivatives belong to the Morrey Space $L^{p,\lambda}(\mathbb{R}^n)$.

2.5 Dense/non-dense subspaces, Zorko Spaces, $VL^{p,\lambda}$, VMO

It is well known that class $C_0{}^\infty(\Omega) = C^\infty$ functions on Ω with compact support in Ω is dense in $W^{m,p}$ and in $G_\alpha(L^p)$ when $\Omega = \mathbb{R}^n$. What we seek here is the density subspaces for $L^{p,\lambda}(\Omega)$. For this we turn to Zorko [Z].

Our first observation is: there are $f \in L^{p,\lambda}$ that cannot be approximated even by continuous functions in the norm $||\cdot||_{L^p} + ||\cdot||_{L^{p,\lambda}}$. In fact Zorko shows that $f_{x_0}(x) = |x - x_0|^{-\lambda/p},\ x_0 \in \Omega$, is one, for every x_0. In fact

$$\begin{aligned}\int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p dx &\geq 2^{-p}\int_{|x-x_0|<\rho} |f_{x_0}(x)|^p\ dx - \int_{|x-x_0|<\rho} |g(x)|^p\ dx \\ &\geq \frac{2^{-p}w_{n-1}}{n-\lambda}\rho^{n-\lambda} - ||g||_{L^\infty(B(x_0,\rho))}\cdot\frac{w_{n-1}}{n}\rho^n \\ &= w_{n-1}\rho^{n-\lambda}\left(\frac{2^p}{n-\lambda} - ||g||^p_{L^\infty(B(x_0,\rho))}\cdot\rho^\lambda\right).\end{aligned}$$

So for $\rho \leq \rho_0 =$ sufficiently small,

$$\rho^{\lambda-n}\int_{|x-x_0|<\rho} |f_{x_0}(x) - g(x)|^p dx \geq c_0 > 0.$$

This motivates us to set

$$VL^{p,\lambda}(\mathbb{R}^n) = \left\{f \in L^{p,\lambda} : ||f(\cdot - y) - f(\cdot)||_{L^{p,\lambda}} \to 0,\ |y| \to 0\right\}.$$

Here the V stands for "vanishing" honoring Sarason's VMO = Vanishing mean oscillation function space; see [To]. We also at times will refer to this space as the Zorko subspace of $L^{p,\lambda}$, the Morrey Space on $\mathbb{R}^n$. We now note

Theorem 2.3. If $f \in VL^{p,\lambda}$, then f can be approximated by $C_0^\infty(\mathbb{R}^n)$ in the norm

$$||\cdot||_{\mathscr{L}^{p,\lambda}} = ||\cdot||_{L^p(\mathbb{R}^n)} + ||\cdot||_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Proof. Let $\varphi \in C_0^\infty(B(0,1))^+$ with $\int \varphi(x)dx = 1$, then upon setting $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon),\ \epsilon > 0$, and $\varphi_\epsilon * f$ the usual convolution, we get

$$\left(\int_{B(x_0,\rho)} |\varphi_\epsilon * f - f|^p\ dx\right)^{1/p} \leq \int \varphi_\epsilon(y)\left\{\int_{B(x_0,\rho)} |f(x-y) - f(x)|^p\ dx\right\}^{1/p} dy$$

hence

$$\left(\rho^{\lambda-n}\int_{B(x_0,\rho)} |\varphi_\epsilon * f - f|^p\ dx\right)^{1/p} \leq \int \varphi_\epsilon(y)||f(\cdot - y) - f(\cdot)||_{L^{p,\lambda}}\ dy$$

or

$$||\varphi_\epsilon * f - f||_{L^{p,\lambda}} \leq \sup_{|y|<\epsilon} ||f(\cdot - y) - f(\cdot)||_{L^{p,\lambda}}.$$

$\square$

This may seem a bit disconcerting, but as it turns out, all is not lost, for we have:

Theorem 2.4. If $f \in L^{p,\lambda}$, then $f \in VL^{p,\mu}$ for all $\mu > \lambda,\ 0 < \lambda < n$.

Proof. Again by mollifying f as above, we can write

$$\begin{aligned} r^{\mu-n} &\int_{B(x_0,r)} |f - f * \varphi_\epsilon|^p \\ &\leq \left(r^{\lambda-n} \int_{B(x_0,r)} |f - f * \varphi_\epsilon|^p \, dx\right)^{1/q} \left(\int_{B(x_0,r)} |f - f * \varphi_\epsilon|^p \, dx\right)^{1/q'} \end{aligned}$$

where $\lambda < \mu < n,\ q = (n - \lambda)/(n - \mu)$, and $q' = q/(q - 1)$. The first factor is bounded by $C\ ||f||_{L^{p,\lambda}}$ and the second factor tends to zero as $\epsilon \to 0$ due to the density of smooth functions in the L^p spaces. □

2.6 Note

1. Extensions of the idea of a Morrey Space and the study of various operators (classical or not) on these spaces are numerous. We collect a few of these that have caught our attention in the Bibliography under "Generalized Morrey Spaces and some applications."

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