Chapter 2
Order Relations and Ordering Cones

In this chapter, first, we give an introduction to order relations and cone properties. Then we present a detailed overview of solution concepts in vector-valued as well as set-valued optimization. We introduce and discuss the following solution concepts for set-valued optimization problems:

- solution concepts based on vector approach,
- solution concepts based on set approach,
- solution concepts based on lattice structure.

Furthermore, we present the embedding approach by Kuroiwa and show how it is possible to transform a set-valued optimization problem into a vector optimization problem using this embedding approach. Solution concepts for set-valued optimization problems with respect to abstract preference relations and for set-valued problems with variable order structure are studied. Moreover, we introduce approximate solutions of set-valued optimization problems. Finally, relationships between different solution concepts are studied.

2.1 Order Relations

In this section, our objective is to study some useful order relations. We begin by recalling that given a nonempty set M, by $M \times M$ we represent the set of ordered pairs of elements of M, that is,

$$M \times M := \{(x_1, x_2) \mid x_1, x_2 \in M\}.$$

The following definition gives the notion of an order relation.

Definition 2.1.1. Let M be a nonempty set and let $\mathscr{R}$ be a nonempty subset of $M \times M$. Then $\mathscr{R}$ is called an **order relation** (or a **binary relation**) on M and the

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pair $(M, \mathscr{R})$ is called a set M with **order relation** $\mathscr{R}$. The containment $(x_1, x_2) \in \mathscr{R}$ will be denoted by $x_1 \mathscr{R} x_2$. The order relation $\mathscr{R}$ is called:

(a) **reflexive** if for every $x \in M$, we have $x\mathscr{R}x$;
(b) **transitive** if for all $x_1, x_2, x_3 \in M$, the relations $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_3$ imply that $x_1\mathscr{R}x_3$;
(c) **antisymmetric** if for all $x_1, x_2 \in M$, the relations $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_1$ imply that $x_1 = x_2$.

Moreover, an order relation $\mathscr{R}$ is called a **preorder** on M if $\mathscr{R}$ is transitive, a **quasiorder** if $\mathscr{R}$ is reflexive and transitive and a **partial order** on M if $\mathscr{R}$ is reflexive, transitive, and antisymmetric. In all the three cases, the containment $(x_1, x_2) \in \mathscr{R}$ is denoted by $x_1 \leq_{\mathscr{R}} x_2$, or simply by $x_1 \leq x_2$ if there is no risk of confusion. The binary relation $\mathscr{R}$ is called a **linear** or **total order** if $\mathscr{R}$ is a partial order and any two elements of M are **comparable**, that is

(d) for all $x_1, x_2 \in M$ either $x_1 \leq_{\mathscr{R}} x_2$ or $x_2 \leq_{\mathscr{R}} x_1$.

Furthermore, if each nonempty subset M' of M has a first element x' (meaning that $x' \in M'$ and $x' \leq_{\mathscr{R}} x \ \forall\, x \in M'$), then M is called **well-ordered**.

We recall Zermelo's theorem: For every nonempty set M there exists a partial order $\mathscr{R}$ on M such that $(M, \mathscr{R})$ is well-ordered.

An illustrative example of a relation is $\Delta_M := \{(x, x) \mid x \in M\}$ which is reflexive, transitive, and antisymmetric, but it satisfies (d) only when M is a singleton.

We recall that the **inverse** of the relation $\mathscr{R} \subset M \times M$ is the relation

$$\mathscr{R}^{-1} := \{(x_1, x_2) \in M \times M \mid (x_2, x_1) \in \mathscr{R}\},$$

and if $\mathscr{S}$ is a relation on M, then the **composition** of $\mathscr{R}$ and $\mathscr{S}$ is the relation

$$\mathscr{S} \circ \mathscr{R} := \{(x_1, x_3) \mid \exists\, x_2 \in M \mid (x_1, x_2) \in \mathscr{R},\ (x_2, x_3) \in \mathscr{S}\}.$$

Using these two notations, the conditions (a), (b), (c), and (d) are equivalent to $\Delta_M \subset \mathscr{R}$, $\mathscr{R} \circ \mathscr{R} \subset \mathscr{R}$, $\mathscr{R} \cap \mathscr{R}^{-1} \subset \Delta_M$ and $\mathscr{R} \cup \mathscr{R}^{-1} = M \times M$, respectively.

Definition 2.1.2. Let $\mathscr{R}$ be an order relation on the nonempty set M and let $M_0 \subset M$ be nonempty. An element $x_0 \in M_0$ is called a **maximal (minimal) element** of M_0 **relative to** $\mathscr{R}$ if for every $x \in M_0$,

$$x_0\mathscr{R}x \Rightarrow x\mathscr{R}x_0 \qquad (x\mathscr{R}x_0 \Rightarrow x_0\mathscr{R}x). \tag{2.1}$$

The collection of all maximal (minimal) elements of M_0 with respect to (w.r.t. for short) $\mathscr{R}$ is denoted by $\mathrm{Max}(M_0, \mathscr{R})$ $(\mathrm{Min}(M_0, \mathscr{R}))$.

Note that x_0 is a maximal element of M_0 w.r.t. $\mathscr{R}$ if and only if x_0 is a minimal element of M_0 w.r.t. $\mathscr{R}^{-1}$, and hence $\mathrm{Max}(M_0, \mathscr{R}) = \mathrm{Min}(M_0, \mathscr{R}^{-1})$.

Remark 2.1.3. 1. If the order relation $\mathscr{R}$ in Definition 2.1.2 is antisymmetric, then $x_0 \in M_0$ is maximal (minimal) if and only if for every $x \in M_0$

$$x_0 \mathscr{R} x \Rightarrow x = x_0 \quad (x \mathscr{R} x_0 \Rightarrow x_0 = x). \tag{2.2}$$

2. If $\mathscr{R}$ is an order relation on M and $\emptyset \neq M_0 \subset M$, then $\mathscr{R}_0 := \mathscr{R} \cap (M_0 \times M_0)$ is an order relation on M_0. In such a situation, the set M_0 will always be endowed with the order structure $\mathscr{R}_0$ if not stated explicitly otherwise. If $\mathscr{R}$ is a preorder (partial order, linear order) on M, then $\mathscr{R}_0$ is a preorder (partial order, linear order) on M_0. Therefore, x_0 is a maximal (minimal) element of M_0 relative to $\mathscr{R}$ iff x_0 is a maximal (minimal) element of M_0 relative to $\mathscr{R}_0$.

In the following, we give some examples to illustrate the above notions.

Example 2.1.4. (1) Assume that X is a nonempty set and $M := \mathscr{P}(X)$ represents the collection of subsets of X. Then the order relation $\mathscr{R} := \{(A, B) \in M \times M \mid A \subset B\}$ is a partial order on M. However, if X contains at least two elements, then $\mathscr{R}$ is not a linear order.

(2) Assume that $\mathbb{N}$ is the set of non-negative integers and

$$\mathscr{R}_{\mathbb{N}} := \{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} \mid \exists\, p \in \mathbb{N} \,:\, n_2 = n_1 + p\}.$$

Then $\mathbb{N}$ is well-ordered by $\mathscr{R}_{\mathbb{N}}$. Note that $\mathscr{R}_{\mathbb{N}}$ defines the usual order relation on $\mathbb{N}$, and $n_1 \mathscr{R}_{\mathbb{N}} n_2$ will always be denoted by $n_1 \le n_2$ or, equivalently, $n_2 \ge n_1$.

(3) Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+ := [0, \infty[$ be the set of non-negative real numbers. The usual order relation on $\mathbb{R}$ is defined by

$$\mathscr{R}_1 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \mid \exists\, y \in \mathbb{R}_+ \,:\, x_2 = x_1 + y\}.$$

Then $\mathscr{R}_1$ is a linear order on $\mathbb{R}$, but $\mathbb{R}$ is not well-ordered by $\mathscr{R}_1$. In the following, the fact $x_1 \mathscr{R}_1 x_2$ will always be denoted by $x_1 \le x_2$ or, equivalently, $x_2 \ge x_1$.

(4) Given $n \in \mathbb{N}$, $n \ge 2$, we consider the binary relation $\mathscr{R}_n$ on $\mathbb{R}^n$ defined by

$$\mathscr{R}_n := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall\, i \in \overline{1, n} \,:\, x_i \le y_i\},$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $\overline{1, n} := \{i \in \mathbb{N} \mid 1 \le i \le n\}$. Then $\mathscr{R}_n$ is a partial order on $\mathbb{R}^n$, but $\mathscr{R}_n$ is not a linear order. For example, the elements e_1 and e_2 are not comparable w.r.t. $\mathscr{R}_n$, where $e_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$. As usual, by e_i we denote the vector whose entries are all 0 except the ith one, which is 1).

Remark 2.1.5. Every well-ordered subset W of $\mathbb{R}$ (equipped with its usual partial order defined above) is at most countable. Indeed, every element $y \in W$, except the greatest element w of W (provided that it exists), has a successor $s(y) \in W$. Clearly, if $y, y' \in W$, $y < y'$, then $s(y) \le y'$. Therefore, fixing $q_y \in \mathbb{Q}$ such that

$y < q_y < s(y)$ for $y \in W \setminus \{w\}$, we get an injective function from $W \setminus \{w\}$ into $\mathbb{Q}$, and so W is at most countable.

We emphasize that even when $\mathscr{R}$ is a partial order on M, a nonempty subset M_0 of M may have zero, one, or several maximal elements, but if $\mathscr{R}$ is a linear order, then every subset has at most one maximal (minimal) element.

Definition 2.1.6. Let $\emptyset \neq M_0 \subset M$ and let $\mathscr{R}$ be an order relation on M. Then:

1. M_0 is **lower (upper) bounded** (w.r.t. $\mathscr{R}$) if there exists $a \in M$ such that $a\mathscr{R}x$ ($x\mathscr{R}a$) for every $x \in M_0$. In this case, the element a is called a **lower (upper) bound** of M_0 (w.r.t. $\mathscr{R}$).
2. If, moreover, $\mathscr{R}$ is a partial order, we say that $a \in M$ is the **infimum (supremum)** of M_0 if a is a lower (upper) bound of M_0 and for any lower (upper) bound a' of M_0 we have that $a'\mathscr{R}a$ ($a\mathscr{R}a'$).

In set-valued optimization, the existence of maximal elements w.r.t. order relations is an important problem. For this, the following **Zorn's lemma** (or Zorn's axiom) plays a crucial role.

Axiom 2.1.7 (Zorn) *Let $(M, \leq)$ be a reflexively preordered set. If every nonempty totally ordered subset of M is upper bounded, then M has maximal elements.*

We recall that given a linear space X, a nonempty set $M \subset X$ is **affine** (or a **linear manifold**) if $\lambda x + (1-\lambda)y \in M$ for all $x, y \in M$ and $\lambda \in \mathbb{R}$. A nonempty set C of X is called **convex** if $[x, y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\} \subset C$ for all $x, y \in C$. By convention the empty set $\emptyset$ is considered to be affine and convex. It is obvious that a linear subspace is affine and an affine set is convex. Moreover, any intersection of linear subspaces, affine sets, or convex sets is a linear subspace, an affine set, or a convex set, respectively. These properties allow us to introduce the **linear hull**, the **affine hull**, and the **convex hull** of a nonempty set $A \subset X$ as being, respectively,

$$\operatorname{lin} A := \bigcap \{Y \subset X \mid A \subset Y,\ Y \text{ linear subspace}\},$$

$$\operatorname{aff} A := \bigcap \{M \subset X \mid A \subset M,\ M \text{ linear manifold}\},$$

$$\operatorname{conv} A := \bigcap \{C \subset X \mid A \subset C,\ C \text{ convex set}\}.$$

Clearly, for $X = \mathbb{R}^n$ and $\mathscr{R} = \mathscr{R}_n$ (from Example 2.1.4 (4)), we have

$$\forall\, x_1, x_2 \in X,\ \forall\, \lambda \in \mathbb{R} \ :\ x_1\mathscr{R}x_2,\ 0 \leq \lambda \Rightarrow \lambda x_1\mathscr{R}\lambda x_2, \tag{2.3}$$

$$\forall\, x_1, x_2, x \in X \ :\ x_1\mathscr{R}x_2 \Rightarrow (x_1 + x)\mathscr{R}(x_2 + x). \tag{2.4}$$

It is easy to find examples of relations satisfying (2.4). In fact, a nonempty relation $\mathscr{R}$ on the linear space X satisfies (2.4) if and only if there exists $\emptyset \neq D \subset X$ such that $\mathscr{R} = \mathscr{R}_D$, where

$$\mathscr{R}_D := \{(x_1, x_2) \in X \times X \mid x_2 - x_1 \in D\}.$$

Moreover, $\mathscr{R}_D$ is reflexive if and only if $0 \in D$, and $\mathscr{R}_D$ is transitive if and only if $D + D \subset D$.

Definition 2.1.8. Let $\mathscr{R}$ be an order relation on the linear space X; we say that $\mathscr{R}$ is **compatible** with the linear structure of X if (2.3) and (2.4) hold.

In linear spaces, a large number of relations $\mathscr{R}$ can be defined by cones which are compatible with the linear structure of the space. For this we first give the following:

Definition 2.1.9. A nonempty set $C \subset X$ is a **cone** if for every $x \in C$ and for every $\lambda \in \mathbb{R}_+$ we have $\lambda x \in C$. Clearly, if C is a cone, then $0 \in C$. The cone C is called

(a) **convex** if for all $x_1, x_2 \in C$ we have $x_1 + x_2 \in C$,
(b) **nontrivial** or **proper** if $C \neq \{0\}$ and $C \neq X$,
(c) **reproducing** if $C - C = X$,
(d) **pointed** if $C \cap (-C) = \{0\}$.

Clearly, the cone C satisfies condition (b) in the definition above iff, C is a convex set.

In the following, we collect a few examples of cones.

Example 2.1.10. (1) Let

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall\, i \in \overline{1, n}\} = \{x \in \mathbb{R}^n \mid (0, x) \in \mathscr{R}_n\}. \tag{2.5}$$

$\mathbb{R}^n_+$ is obviously a cone in the linear space $\mathbb{R}^n$, which fulfills all the conditions of Definition 2.1.9.

(2) Let $C[0, 1]$ be the linear space of all real functions defined and continuous on the interval $[0, 1] \subset \mathbb{R}$. Addition and multiplication by scalars are defined, as usual, by

$$(x + y)(t) = x(t) + y(t), \quad (\lambda x)(t) = \lambda x(t) \quad \forall\, t \in [0, 1]$$

for $x, y \in C[0, 1]$ and $\lambda \in \mathbb{R}$. Then

$$C_+[0, 1] := \{x \in C[0, 1] \mid x(t) \geq 0 \ \forall\, t \in [0, 1]\} \tag{2.6}$$

is a convex, nontrivial, pointed, and reproducing cone in $C[0, 1]$. Note that the set

$$Q := \{x \in C_+[0, 1] \mid x \text{ is nondecreasing}\} \tag{2.7}$$

is also a convex, nontrivial, and pointed cone in the space $C[0, 1]$, but it doesn't satisfy condition (c) from Definition 2.1.9: $Q - Q$ is the proper linear subspace of all functions with bounded variation of $C[0, 1]$.

(3) Consider the set $C \subset \mathbb{R}^n$ defined by

$$\begin{aligned} C := \{x = (x_1, \dots, x_n)^T \in \mathbb{R}^n \mid x_1 > 0,\ \text{or} \\ x_1 = 0,\ x_2 > 0,\ \text{or} \\ \dots \\ x_1 = \dots = x_{n-1} = 0,\ x_n > 0,\ \text{or} \\ x = 0\}. \end{aligned}$$

Then the cone C satisfies all the conditions of Definition 2.1.9.

In the following result, we characterize compatibility between linear and order relations:

Theorem 2.1.11. *Let X be a linear space and let C be a cone in X. Then the relation*

$$\mathscr{R}_C := \{(x_1, x_2) \in X \times X \mid x_2 - x_1 \in C\} \tag{2.8}$$

is reflexive and satisfies (2.3) and (2.4). Moreover, C is convex if and only if $\mathscr{R}_C$ is transitive, and, respectively, C is pointed if and only if $\mathscr{R}_C$ is antisymmetric. Conversely, if $\mathscr{R}$ is a reflexive relation on X satisfying (2.3) and (2.4), then $C := \{x \in X \mid 0\mathscr{R}x\}$ is a cone and $\mathscr{R} = \mathscr{R}_C$.

Proof. See [214, Theorem 2.1.13]. □

The above result shows that when $\emptyset \neq C \subset X$, the relation $\mathscr{R}_C$ defined by (2.8) is a reflexive preorder iff C is a convex cone, and $\mathscr{R}_C$ is a partial order iff C is a pointed convex cone.

We note that $\mathscr{R}_{\mathbb{R}^n_+} = \mathscr{R}_n$ (defined in Example 2.1.4 (4)), while the relation $\mathscr{R}_C$ with $C \subset \mathbb{R}^n$ defined in Example 2.1.10 (3) is a linear order, called the **lexicographic order** on $\mathbb{R}^n$.

Let Y be a linear topological space, partially ordered by a proper pointed convex closed cone $C \subset Y$.

Denote this order by "$\leq_C$". Its ordering relation is described by

$$y_1 \leq_C y_2 \quad \text{if and only if} \quad y_2 - y_1 \in C \quad \text{for all } y_1, y_2 \in Y. \tag{2.9}$$

In the sequel, we omit the subscript C as no confusion occurs.

As usual, we denote by

$$C^+ := \{y^* \in Y^* \mid y^*(y) \geq 0\ \forall y \in C\}$$

the continuous positive dual cone of C, and by

$$C^\# := \{y^* \in C^+ \mid y^*(y) > 0\ \forall\, y \in C \setminus \{0\}\}$$

the quasi-interior of C^+.

We recall that the **interior** and the **closure** of the subset A of the topological space (X, τ) are defined, respectively, by

$$\operatorname{int} A := \bigcup\{D \subset X \mid D \subset A,\ D \text{ open}\},$$
$$\operatorname{cl} A := \overline{A} := \bigcap\{B \subset X \mid A \subset B,\ B \text{ closed}\}.$$

Clearly, $\operatorname{int} A$ is open and $\operatorname{cl} A$ is closed.

2.2 Cone Properties Related to the Topology and the Order

We discuss now the connections between **topology** and **order**. Unlike the notion of an ordered linear space (i.e., a linear space equipped with a compatible reflexive preorder), the notion of an ordered topological linear space does not demand for any direct relation between the order and the involved topology. However, because a compatible reflexive preorder on a linear space is defined by a convex cone, it is customary to ask that the cone defining the order be closed, have nonempty interior, or be normal. Before introducing the notion of a normal cone, we recall that a nonempty set A of the linear space X is **full** with respect to the convex cone $C \subset X$ if $A = [A]_C$, where

$$[A]_C := (A + C) \cap (A - C).$$

Note that $[A]_C$ is full w.r.t. C for every set $A \subset X$.

Definition 2.2.1. Let (X, τ) be a t.v.s. and let $C \subset X$ be a convex cone. Then C is called **normal** (relative to τ) if the origin $0 \in X$ has a neighborhood base formed by full sets w.r.t. C.

In the next result we give several characterizations of normal cones. We are using the notation $\mathcal{N}_X$ for the set of balanced neighborhoods of $0 \in X$ in the t.v.s. (X, τ).

Theorem 2.2.2. *Let (X, τ) be a topological linear space and let $C \subset X$ be a convex cone. Then the following statements are equivalent:*

(i) *C is normal,*
(ii) $\forall\, V \in \mathcal{N}_X,\ \exists\, U \in \mathcal{N}_X\ :\ [U]_C \subset V$,
(iii) *for all nets $(x_i)_{i\in I}, (y_i)_{i\in I} \subset X$ such that $0 \leq_C x_i \leq_C y_i$ for every $i \in I$ one has $(y_i) \to 0 \Rightarrow (x_i) \to 0$,*
(iv) *$\operatorname{cl} C$ is normal.*

Proof. See [214, Theorem 2.1.22]. □

The following corollary is immediate.

Corollary 2.2.3. *Let (X, τ) be a Hausdorff t.v.s. and let $C \subset X$ be a convex cone. If C is normal, then* $\operatorname{cl} C$ *is pointed, and so C is pointed, too.*

Let (X, τ) be a Hausdorff t.v.s. partially ordered by the convex cone C. We say that a net $(x_i)_{i \in I} \subset X$ is nonincreasing if

$$\forall\, i, j \in I \;:\; j \succeq i \Rightarrow x_j \leq_C x_i. \tag{2.10}$$

Given $\emptyset \neq A \subset X$, we say that A is **lower bounded with respect to** C if A is lower bounded with respect to $\mathscr{R}_C$ (see Definition 2.1.6). Similarly, $a \in X$ is a lower bound (infimum) of A w.r.t. C if a is so for $\mathscr{R}_C$. Hence $a \in X$ is a lower bound of A w.r.t. C if $a \leq_C x$ for every $x \in A$. An element a is the infimum of A w.r.t. C if a is a lower bound and for any lower bound a' of A we have that $a' \leq_C a$. The infimum of A w.r.t. C will be denoted by $\inf_C A$ when it exists.

Proposition 2.2.4. *Let (X, τ) be a Hausdorff t.v.s. partially ordered by the closed convex cone C. If the net $(x_i)_{i \in I} \subset X$ is nonincreasing and convergent to $x \in X$, then $\{x_i \mid i \in I\}$ is bounded below and $x = \inf\{x_i \mid i \in I\}$.*

Proof. See [214, Proposition 2.1.24]. □

We emphasize that in ordered topological linear spaces, the classical result concerning the bounded monotone sequences is not generally true. We consider the linear space ℓ^∞ of all bounded sequences $x = (x^k)_{k \geq 1} \subset \mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x^k| \mid k = 1, 2, \ldots\}$. In ℓ^∞ we consider the "usual" partial order generated by the cone $\ell^\infty_+ := \{x \in l^\infty \mid x^k \geq 0 \;\forall\, k \geq 1\}$; ℓ^∞_+ is a pointed closed convex cone (even reproducing and with nonempty interior).

Example 2.2.5 (Peressini [475, p. 91]). The sequence $\{x_n\} \subset \ell^\infty$, defined by (for n fixed)

$$x_n^k = \begin{cases} -1 & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

is nonincreasing w.r.t. C, and $\inf\{x_n \mid n \geq 1\} = e' := -e$ where $e = (1, 1, 1, \ldots) \in \ell^\infty$. But $\|x_n - e'\| = 1$ for every $n \geq 1$. Consequently, $\{x_n\}_{n \geq 1}$ does not converge to its infimum.

We also recall that a cone C that partially orders a Hausdorff linear topological space (X, τ) is said to be **Daniell** if any nonincreasing net having a lower bound τ-converges to its infimum (see Jahn [292, p.29], Luc [402, p. 47], Borwein [67]).

In the following, let us recall some useful notions of cones which play an important role in proving existence results for solutions of optimization problems in infinite dimensional spaces.

Definition 2.2.6. Let Y be a Hausdorff topological vector space and $C \subset Y$ a proper convex cone.

Fig. 2.1 Cone properties

$$
\begin{tikzcd}
C \text{ well-based} \arrow[d, Leftrightarrow, "Y \text{ normed}"] & & C \text{ pointed} \arrow[ll, Rightarrow, "C=\operatorname{cl} C,\, Y=\mathbb{R}^n"'] \\
C \text{ nuclear} \arrow[rr, Rightarrow] \arrow[d, Rightarrow, "C \text{ complete}"] & & C \text{ normal} \arrow[u, Rightarrow] \\
C \text{ Daniell} & & C \text{ compact base.} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

(i) C is **based** if there exists a nonempty convex subset B of C such that $C = \mathbb{R}_+ B$ (where $\mathbb{R}_+ B := \{\lambda b \mid b \in B \text{ and } \lambda \geq 0\}$) and $0 \notin \operatorname{cl} B$; the set B is called a **base** for C.
(ii) C is called **well-based** if C has a bounded base.
(iii) Let the topology of Y be defined by a family $\mathscr{P}$ of seminorms. C is called **supernormal** or **nuclear** if for each $p \in \mathscr{P}$ there exists $y^* \in Y^*$, such that $p(y) \leq \langle y, y^* \rangle$ for all $y \in C$; it holds $y^* \in C^+$ in this case.
(iv) C is said to be **Daniell** if any nonincreasing net having a lower bound converges to its infimum.
(iv) C is said to be **regular** if any decreasing (increasing) net which has a lower bound (upper bound) is convergent.

In Fig. 2.1 we give an overview of such additional cone properties and corresponding relations for the case that Y is a Banach space, C a proper and convex cone in Y.

The following result gives useful information for cones with bases:

Theorem 2.2.7. *Let X be a Hausdorff locally convex space and $C \subset X$ a proper convex cone. Then C has a base if and only if $C^\# \neq \emptyset$.*

Proof. See [214, Theorem 2.2.12]. □

In the following, we collect a few examples of Daniell cones.

Example 2.2.8. 1. We recall that if $(x_\alpha)_{\alpha \in A}$ is a net which is increasing (decreasing) in a topological vector space (Y, τ) ordered by a closed convex cone C and if $\overline{x}$ is a cluster point of (x_α), then $\overline{x} = \sup_{\alpha \in A} x_\alpha$ ($\overline{x} = \inf_{\alpha \in A} x_\alpha$) (see Peressini [475, Proposition 3.1]). Therefore, any regular cone is Daniell.
2. If $(Y, ||\cdot||)$ is a Banach lattice, that is, Y is a Banach space, vector lattice and the norm is absolute, i.e., $||x|| = ||\,|x|\,||$ for any $x \in Y$, then the cone $Y_+ = \{y \in Y \mid y \geq 0\}$ is Daniell if Y has weakly compact intervals.
3. A convex cone with a weakly compact base is a Daniell cone.

The following result connects some useful cones.

Proposition 2.2.9 (Isac [280]). *Let $(Y, \mathscr{P})$ be a Hausdorff locally convex space and $C \subset Y$ a proper convex cone. Then*

$$C \text{ well-based} \implies C \text{ nuclear} \implies C \text{ normal}.$$

If Y is a normed space, then

$$C \text{ nuclear} \implies C \text{ well-based.}$$

Remark 2.2.10. Among the classical Banach spaces their usual positive cones are well-based only in l^1 and $L^1(\Omega)$ (but l^1 is not an Asplund space (see Definition 3.5.3)).

Let Y be a topological vector space over $\mathbb{R}$. Assume (Y, C) is simultaneously a vector lattice with the lattice operations $x \mapsto x^+$, $x \mapsto x^-$, $x \mapsto |x|$, $(x, y) \mapsto \sup\{x, y\}$ and $(x, y) \mapsto \inf\{x, y\}$.

Definition 2.2.11. A set $A \subset Y$ is called **solid**, if $x \in A$ and $|y| \leq |x|$ implies $y \in A$. The space Y is called **locally solid**, if it possesses a neighborhood of 0 consisting of solid sets.

Lemma 2.2.12. *The following properties are equivalent:*

(i) *Y is locally solid.*
(ii) *C is normal, and the lattice operations are continuous.*

In order to derive optimality conditions or duality statements in general spaces (cf. Chaps. 8, 12), the ordering cone is often required to have a nonempty interior. Therefore, in the following, we give some examples of convex cones with nonempty interior.

Example 2.2.13. 1. Any closed convex cone C in the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ such that C is self-adjoint (i.e., $C = C^+$) has a nonempty interior.

2. Consider the space of continuous functions $C[a, b]$ with the norm $||x|| = \sup\{|x(t)| \mid t \in [a, b]\}$. Then the cone of positive functions in $C[a, b]$

$$C[a, b]_+ := \{x \in C[a, b] \mid \forall t \in [a, b] \ : \ x(t) \geq 0\}$$

has a nonempty interior.

3. Let $Y = l^2(\mathbb{N}^*, \mathbb{R})$ with the well-known structure of a Hilbert space. The convex cone

$$C_{l^2} := \{x = \{x_i\}_{i \geq 1} \mid x_1 \geq 0 \text{ and } \sum_{i=2}^{\infty} x_i^2 \leq x_1^2\}$$

has a nonempty interior

$$\text{int } C_{l^2} := \{x = \{x_i\}_{i \geq 1} \mid x_1 > 0 \text{ and } \sum_{i=2}^{\infty} x_i^2 < x_1^2\}.$$

4. Let l^∞ be the space of bounded sequences of real numbers, equipped with the norm $||x|| = \sup_{n \in \mathbb{N}}\{|x_n|\}$. The cone

$$l^{\infty}{}_{+} := \{x = \{x_n\}_{n\in\mathbb{N}} \mid x_n \geq 0 \text{ for any } n \in \mathbb{N}\}$$

has a nonempty interior (cf. Peressini [475], p. 186).

5. Let $C^1[a,b]$ be the real vector space formed by all real continuously differentiable functions defined on $[a,b]$ $(a,b \in \mathbb{R}, a < b)$, equipped with the norm

$$||f||_1 := \{\int_a^b (f(t))^2 \,\mathrm{d}\,t + \int_a^b (f'(t))^2 \,\mathrm{d}\,t\}^{1/2}$$

for any $f \in C^1[a,b]$. Using a Sobolev's imbedding theorem, we can show that the natural ordering cone

$$C^1[a,b]_+ := \{f \in C^1[a,b] \mid f \geq 0\}$$

has a nonempty interior. The proof is based on some technical details (cf. da Silva [532]).

6. About the locally convex spaces, we put in evidence the following result. If (Y,τ) is a real locally convex space, then for every closed convex pointed cone $C \subset Y$, with nonempty interior, there exists a continuous norm $||\cdot||$ on Y such that C has a nonempty interior in the normed space $(Y, ||\cdot||)$.

Proof. Take $y_0 \in \operatorname{int} C$ and $A := (y_0 - C) \cap (C - y_0)$. Then A is a closed convex and balanced set with $0 \in \operatorname{int} A$ such that the Minkowski functional $p_A : Y \to \mathbb{R}$ defined by

$$p_A(y) := \inf\{t > 0 \mid y \in tA\}$$

is a seminorm. Because $\operatorname{int} A = \operatorname{core} A = \{y \in Y \mid p_A(y) < 1\} \subset A = \{y \in Y \mid p_A(y) \leq 1\}$ (see Proposition 6.2.1), p_A is also continuous. Take $y \in Y$ with $p_A(y) = 0$. Then $y \in n^{-1}A$ for every $n \geq 1$, whence $n^{-1}y_0 \pm y \in C$ for such n. It follows that $\pm y \in \operatorname{cl} C = C$, and so $y = 0$. Hence $\|\cdot\| := p_A$ is a norm and $A = B_{(Y,\|\cdot\|)}$, and so $y_0 \in \operatorname{int}_{\|\cdot\|} C$. □

Finally, we give an example of a normed (vector) space (n.v.s.) where the natural ordering cone has a nonempty interior as well as the Daniell property.

Example 2.2.14 (see Jahn [293]). Consider the real linear space $L^{\infty}(\Omega)$ of all (equivalence classes of) essentially bounded functions $f : \Omega \to \mathbb{R}$ $(\emptyset \neq \Omega \subset \mathbb{R}^n)$ measurable with the norm $||\cdot||_{L^{\infty}(\Omega)}$ given by

$$||f||_{L^{\infty}(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}\{|f(x)|\} \text{ for all } f \in L^{\infty}(\Omega).$$

The ordering cone

$$L^{\infty}(\Omega)_+ := \{f \in L^{\infty}(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}$$

has a nonempty interior and is weak* Daniell.

2.3 Convexity Notions for Sets and Set-Valued Maps

Throughout this section X, Y are real topological vector spaces.

Definition 2.3.1. Let $A \subseteq X$ be a nonempty set. We say that A is **α-convex**, where $\alpha \in]0,1[$, if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$. The set A is **mid-convex** if A is $\frac{1}{2}$-convex. The set A is **nearly convex** if A is α-convex for some $\alpha \in]0,1[$. The set A is **closely convex** if $\operatorname{cl} A$ is convex. The empty set is α-convex for all $\alpha \in]0,1[$ and closely convex (and so nearly convex).

Of course, A is convex if and only if A is α-convex for every $\alpha \in]0,1[$. Moreover, if $T : X \to Y$ is a linear operator and $A \subseteq X$, $B \subseteq Y$ are α-convex (nearly convex, convex), then $T(A)$ and $T^{-1}(B)$ are α-convex (nearly convex, convex), too.

Some properties of nearly convex sets are mentioned in the next result (see [214, Proposition 2.4.3, Corollary 2.4.4]).

Proposition 2.3.2. *Let $A \subseteq X$ be a nonempty nearly convex set. Then*

(i) $\operatorname{cl} A$ *is convex.*
(ii) *If $x \in \operatorname{icr} A$ and $y \in A$, then $[x, y] \subseteq A$. Moreover, if $x \in \operatorname{int} A$ and $y \in A$, then $[x, y[\subseteq \operatorname{int} A$.*
(iii) *If $\operatorname{int} A \neq \emptyset$, then $\operatorname{int} A$ is convex and $\operatorname{icr} A = \operatorname{int} A$.*
(iv) *If A is open or closed, then A is convex.*

Definition 2.3.3. Let $C \subseteq Y$ be a convex cone. We say that $A \subseteq Y$ is **C-α-convex** if $A + C$ is α-convex; A is **nearly C-convex** if $A + C$ is nearly convex; A is **closely C-convex** if $A + C$ is closely convex. Moreover, A is **closely c-C-convex** (nearly C-subconvexlike in [601]) if $\operatorname{cl}(\mathbb{P}(A + C))$ is convex; A is **ic-C-convex** (see [517]) if $\operatorname{int}(\mathbb{P}(A + C))$ is convex and $\mathbb{P}(A + C) \subseteq \operatorname{cl}(\operatorname{int}(\mathbb{P}(A + C)))$.

The next result, stated essentially in [79, Lemma 2.5], proves to be useful in the following sections.

Lemma 2.3.4. *Assume that $C \subseteq Y$ is a convex cone with $\operatorname{int} C \neq \emptyset$ and let $A \subseteq Y$. Then*

$$\operatorname{cl}(A + C) = \operatorname{cl}(\operatorname{cl} A + C) = \operatorname{cl}(A + \operatorname{int} C), \tag{2.11}$$

$$A + \operatorname{int} C = \operatorname{cl} A + \operatorname{int} C = \operatorname{int}(A + C) = \operatorname{int}(\operatorname{cl} A + C) = \operatorname{int}(\operatorname{cl}(A + C)). \tag{2.12}$$

Therefore, $\operatorname{cl}(A + C)$ *is convex iff* $A + \operatorname{int} C$ *is convex.*

Proof. The equalities in (2.11) follow immediately from the known relation

$$\operatorname{cl}(A + B) = \operatorname{cl}(\operatorname{cl} A + B) = \operatorname{cl}(\operatorname{cl} A + \operatorname{cl} B), \tag{2.13}$$

valid for all subsets $A, B \subseteq Y$, and the fact that $\operatorname{cl} C = \operatorname{cl}(\operatorname{int} C)$.

Note that $A + \operatorname{int} C$ is open being the union $\cup_{a\in A}(a + \operatorname{int} C)$ of open sets. The inclusions $A + \operatorname{int} C \subseteq \operatorname{cl} A + \operatorname{int} C$ and $\operatorname{int}(A + C) \subseteq \operatorname{int}(\operatorname{cl}(A + C))$ are obvious. Take $y \in \operatorname{cl} A$ and $k \in \operatorname{int} C$. Since $k - \operatorname{int} C \in \mathscr{N}_Y(0)$, we have that $A \cap (y + k - \operatorname{int} C) \neq \emptyset$, whence $y + k \in A + \operatorname{int} C$; hence $\operatorname{cl} A + \operatorname{int} C \subseteq A + \operatorname{int} C$, and so the first equality in (2.12) is true.

The inclusion $A + \operatorname{int} C \subseteq \operatorname{int}(A + C)$ is obvious because $A + \operatorname{int} C$ is open. Fix $k^0 \in \operatorname{int} C$ and take $y \in \operatorname{int}(A + C)$. Then there exists $\alpha > 0$ such that $y - \alpha k^0 \in A + C$, whence $y \in A + C + \operatorname{int} C = A + \operatorname{int} C$. It follows that $\operatorname{int}(A + C) \subseteq A + \operatorname{int} C$, and so the second equality in (2.12) is true. The third equality in (2.12) follows immediately from the first two equalities.

Clearly, $\operatorname{cl}(A + C) = \operatorname{cl}(A + C) + C$; using the first three equalities in (2.12) we get

$$\begin{aligned}\operatorname{int}(\operatorname{cl}(A + C)) &= \operatorname{int}(\operatorname{cl}(A + C) + C) = \operatorname{cl}(A + C) + \operatorname{int} C \\ &= (A + C) + \operatorname{int} C = A + \operatorname{int} C.\end{aligned}$$

If $\operatorname{cl}(A + C)$ is convex then $A + \operatorname{int} C = \operatorname{int}(\operatorname{cl}(A + C))$ is convex. Conversely, if $A + \operatorname{int} C$ is convex, then $\operatorname{cl}(A + C) = \operatorname{cl}(A + \operatorname{int} C)$ is convex. The proof is complete. □

It is worth observing that

$$\mathbb{P}(A + C) = \mathbb{P}A + C, \quad \mathbb{P}(A + \operatorname{int} C) = \mathbb{P}A + \operatorname{int} C. \tag{2.14}$$

Moreover, if $\operatorname{int} C \neq \emptyset$, using (2.14) and (2.12) we get

$$\operatorname{int}(\mathbb{P}(A + C)) = \mathbb{P}A + \operatorname{int} C = \operatorname{int}(\operatorname{cl}(\mathbb{P}(A + C))). \tag{2.15}$$

In the next result we establish some relationships between the C-convexity notions above.

Proposition 2.3.5. *Let $A \subseteq Y$. The following assertions hold:*

(i) *Let $\alpha \in]0, 1[$. Then A is C-α-convex iff $\alpha A + (1 - \alpha)A \subseteq A + C$.*
(ii) *A is closely C-convex iff $\lambda A + (1 - \lambda)A \subseteq \operatorname{cl}(A + C)$ for all $\lambda \in]0, 1[$.*
(iii) *A is closely C-convex and* $\operatorname{int} C \neq \emptyset$ *iff*

$$\exists k \in \operatorname{int} C,\ \forall \alpha > 0,\ \forall \lambda \in]0, 1[: \alpha k + \lambda A + (1 - \lambda)A \subseteq A + C. \tag{2.16}$$

(iv) *If A is nearly C-convex then A is closely C-convex.*
(v) *If A is closely C-convex then A is closely c-C-convex.*
(vi) *A is ic-C-convex iff A is closely c-C-convex and*

$$\operatorname{int}(\operatorname{cl}(\mathbb{P}(A + C))) = \operatorname{int}(\mathbb{P}(A + C)) \neq \emptyset. \tag{2.17}$$

(vii) *Assume that* $\operatorname{int} C \neq \emptyset$. *Then A is ic-C-convex iff A is closely c-C-convex.*

Proof. All the assertions a clearly true if A is empty. Therefore, we assume that $A \neq \emptyset$.

(i) The assertion is (almost) evident.

(ii) Assume that A is closely C-convex and take $\lambda \in]0,1[$. Since $\mathrm{cl}(A+C)$ is convex we get

$$\lambda A + (1-\lambda)A \subseteq \lambda \,\mathrm{cl}(A+C) + (1-\lambda)\,\mathrm{cl}(A+C) = \mathrm{cl}(A+C).$$

Conversely, assume that $\lambda A + (1-\lambda)A \subseteq \mathrm{cl}(A+C)$ for all $\lambda \in]0,1[$. Taking $\lambda \in]0,1[$, we have that $\lambda(A+C) + (1-\lambda)(A+C) \subseteq \mathrm{cl}(A+C) + C = \mathrm{cl}(A+C)$. Using (2.13) we get $\lambda\,\mathrm{cl}(A+C) + (1-\lambda)\,\mathrm{cl}(A+C) \subseteq \mathrm{cl}(A+C)$, and so $\mathrm{cl}(A+C)$ is convex.

(iii) Assume that (2.16) holds; then clearly $\mathrm{int}\, C \neq \emptyset$. Take $y_1, y_2 \in A$ and $\lambda \in]0,1[$. Then $n^{-1}k + \lambda y_1 + (1-\lambda)y_2 \in A + C$ for every $n \in \mathbb{N}^*$; taking the limit we get $\lambda y_1 + (1-\lambda)y_2 \in \mathrm{cl}(A+C)$, and so A is closely C-convex.

Assume now that A is closely C-convex and $\mathrm{int}\, C \neq \emptyset$. Consider $k \in \mathrm{int}\, C$. Take $\alpha > 0$, $y_1, y_2 \in A$ and $\lambda \in]0,1[$. Then $\lambda y_1 + (1-\lambda)y_2 \in \mathrm{cl}(A+C)$. Then using (2.12),

$$\begin{aligned} \alpha k + \lambda y_1 + (1-\lambda)y_2 &\in \mathrm{cl}(A+C) \\ &\quad + \mathrm{int}\, C = (A+C) + \mathrm{int}\, C = A + \mathrm{int}\, C \subseteq A + C. \end{aligned}$$

(iv) Assume that A is nearly C-convex. Then, using (i), $A+C$ is nearly convex. Then, by Proposition 2.3.2 (i) we obtain that $\mathrm{cl}(A+C)$ is convex, that is A is closely C-convex.

(v) Clearly, $\mathrm{cl}\,(\mathbb{P}(A+C)) = \mathrm{cl}\,(\mathbb{P}(\mathrm{cl}(A+C)))$. Since $\mathrm{cl}(A+C)$ is convex, from the preceding relation we obtain that $\mathrm{cl}\,(\mathbb{P}(A+C))$ is convex, that is A is c-C-convex.

(vi) Assume that A is ic-C-convex. From the definition of the ic-C-convexity we have that $B := \mathrm{int}\,(\mathbb{P}(A+C))$ is nonempty and convex, and $\mathbb{P}(A+C) \subseteq \mathrm{cl}\, B$. It follows that $\mathrm{int}(\mathrm{cl}\, B) = B \subseteq \mathbb{P}(A+C)$ and $\mathrm{cl}(\mathbb{P}(A+C)) \supseteq \mathrm{cl}\, B$, and so $\mathrm{cl}(\mathbb{P}(A+C)) = \mathrm{cl}\, B$ is convex. Therefore, A is closely c-C-convex. Moreover, since B is open, convex and nonempty, we have that $\mathrm{int}\,(\mathrm{cl}(\mathbb{P}(A+C))) = \mathrm{int}(\mathrm{cl}\, B) = B$. Therefore, (2.17) holds.

Assume now that A is closely c-C-convex and (2.17) holds. Then $C := \mathrm{cl}(\mathbb{P}(A+C))$ is convex $\mathrm{int}\, C = \mathrm{int}\,(\mathbb{P}(A+C)) =: B \neq \emptyset$. Then clearly B is convex and $\mathrm{cl}\, B = \mathrm{cl}\,(\mathrm{int}\, C) = C \supseteq \mathbb{P}(A+C)$. Hence A is ic-C-convex.

(vii) Let $\mathrm{int}\, C \neq \emptyset$. Then (2.15) holds and $\mathrm{int}\,(\mathbb{P}(A+C)) \neq \emptyset$. The conclusion follows using (vi). The proof is complete. □

Note that Proposition 2.3.5 (vii) is stated in [601, Theorem 3.1], while the fact that A is closely c-C-convex if A is ic-C-convex in Proposition 2.3.5 (vi) is proved in [601, Theorem 3.2]

Let $F : X \rightrightarrows Y$. We say that F is **α-convex (mid-convex, nearly convex, convex)** if graph F is α-convex (mid-convex, nearly convex, convex). It is obvious that if F is α-convex (mid-convex, nearly convex, convex), so are dom F, Im F, and $F(x)$ for every $x \in X$. It is easy to see that F is α-convex if and only if

$$\forall\, x, x' \in \operatorname{dom} F : \alpha F(x) + (1-\alpha)F(x') \subseteq F\left(\alpha x + (1-\alpha)x'\right);$$

in the relation above $x, x' \in \operatorname{dom} F$ can be replaced by $x, x' \in X$.

To $F : X \rightrightarrows Y$ we associate the set-valued maps cl F, conv F, $\overline{\operatorname{conv}} F : X \rightrightarrows Y$ defined by

$$\begin{aligned}(\operatorname{cl} F)(x) &:= \operatorname{cl}[F(x)], \ (\operatorname{conv} F)(x) := \operatorname{conv}[F(x)],\\ (\overline{\operatorname{conv}} F)(x) &:= \overline{\operatorname{conv}}[F(x)] \ (x \in X).\end{aligned}$$

It is almost obvious that cl F, conv F and $\overline{\operatorname{conv}} F$ are α-convex (mid-convex, nearly convex, convex) if F is α-convex (mid-convex, nearly convex, convex).

To $F : X \rightrightarrows Y$ and $y^* \in Y^*$, where Y^* is the topological dual of Y, we also associate

$$\phi_{y^*} := \phi_{y^*}^F : X \to \overline{\mathbb{R}}, \quad \phi_{y^*}(x) := \inf\{\langle y, y^*\rangle \mid y \in F(x)\} \quad (x \in X), \tag{2.18}$$

where, as usual, $\inf\emptyset := +\infty$; then $\operatorname{dom}\phi_{y^*} = \operatorname{dom} F$ for every $y^* \in Y^*$ and $\phi_0 = \iota_{\operatorname{dom} F}$. Clearly, $\phi_{y^*}^F = \phi_{y^*}^{\operatorname{cl} F} = \phi_{y^*}^{\operatorname{conv} F} = \phi_{y^*}^{\overline{\operatorname{conv}} F}$ for every $y^* \in Y^*$. The function ϕ_{y^*} (but with sup instead of inf) was introduced in [138], and used (for example) in [422, 423, 518], too.

Proposition 2.3.6. *Let $F : X \rightrightarrows Y$.*

(i) *If F is convex then ϕ_{y^*} is convex for every $y^* \in Y^*$.*
(ii) *Assume that Y is a locally convex space. If ϕ_{y^*} is convex for every $y^* \in Y^*$ then $\overline{\operatorname{conv}} F$ is convex.*

Proof. (i) Consider $x, x' \in \operatorname{dom}\phi_{y^*}$ and $\alpha \in \,]0,1[$. Take $\gamma, \gamma' \in \mathbb{R}$ such that $\phi_{y^*}(x) < \gamma$, $\phi_{y^*}(x') < \gamma'$. Then there exist $y \in F(x)$, $y' \in F(x')$ such that $\langle y, y^*\rangle < \gamma$, $\langle y', y^*\rangle < \gamma'$. Then $\alpha y + (1-\alpha)y' \in F\,(\alpha x + (1-\alpha)x')$, and so

$$\begin{aligned}\phi_{y^*}\left(\alpha x + (1-\alpha)x'\right) &\le \left\langle \alpha y + (1-\alpha)y', y^*\right\rangle = \alpha\,\langle y, y^*\rangle + (1-\alpha)\left\langle y', y^*\right\rangle\\ &< \alpha\gamma + (1-\alpha)\gamma'.\end{aligned}$$

Letting $\gamma \to \phi_{y^*}(x)$, $\gamma' \to \phi_{y^*}(x')$ we get $\phi_{y^*}(\alpha x + (1-\alpha)x') \le \alpha\phi_{y^*}(x) + (1-\alpha)\phi_{y^*}(x')$. Hence ϕ_{y^*} is convex.

(ii) Since $\phi_{y^*}^F = \phi_{y^*}^{\overline{\operatorname{conv}} F}$ for every $y^* \in Y^*$, we may (and do) assume that $F = \overline{\operatorname{conv}} F$. We have that $\operatorname{dom} F = \operatorname{dom}\phi_0$ is convex, $\phi_0 = \iota_{\operatorname{dom} F}$ being convex. Assume that F is not convex. Then there exist $x, x' \in \operatorname{dom} F$, $y \in F(x)$, $y' \in$

$F(x')$ and $\alpha \in]0,1[$ such that $z := \alpha y + (1-\alpha)y' \notin F(\alpha x + (1-\alpha)x') =: A$. Since A is a nonempty closed convex set, there exists $y^* \in Y^*$ such that $\langle z, y^*\rangle < \inf\{\langle v, y^*\rangle \mid v \in A\} = \phi_{y^*}(\alpha x + (1-\alpha)x')$. Since

$$\alpha\phi_{y^*}(x) + (1-\alpha)\phi_{y^*}(x') \leq \alpha\,\langle y, y^*\rangle + (1-\alpha)\left\langle y', y^*\right\rangle = \langle z, y^*\rangle$$
$$< \phi_{y^*}\left(\alpha x + (1-\alpha)x'\right),$$

we get the contradiction that ϕ_{y^*} is not convex. Hence F is convex. □

Let $C \subseteq Y$ be a convex cone. We say that F is **C-α-convex (C-mid-convex, C-nearly convex, C-convex)** if the set-valued map

$$F_C : X \rightrightarrows Y, \quad F_C(x) := F(x) + C,$$

is α-convex (mid-convex, nearly convex, convex). Of course, F is C-α-convex if and only if

$$\forall\, x, x' \in \operatorname{dom} F : \alpha F(x) + (1-\alpha)F(x') \subseteq F\left(\alpha x + (1-\alpha)x'\right) + C.$$

Note that sometimes $\operatorname{graph} F_C$ is denoted by $\operatorname{epi}_C F$, or simply $\operatorname{epi} F$, and is called the **epigraph** of F.

Corollary 2.3.7. *Let $F : X \rightrightarrows Y$ and $C \subset Y$ be a convex cone.*

(i) *If F is C-convex then ϕ_{y^*} is convex for every $y^* \in C^+$.*
(ii) *Assume that Y is a locally convex space and ϕ_{y^*} is convex for every $y^* \in C^+$. Then $\overline{\operatorname{conv}} F_C$ is convex; in particular, if $F(x) + C$ is closed and convex for every $x \in X$, then F is C-convex.*

Proof. Of course, F is C-convex if and only if F_C is convex. Let us set $\tilde{\phi}_{y^*} := \phi_{y^*}^{\overline{\operatorname{conv}} F}$. Note that $\tilde{\phi}_{y^*} = \phi_{y^*}^F = \phi_{y^*}$ for $y^* \in C^+$, while for $y^* \in Y^* \setminus C^+$, $\tilde{\phi}_{y^*}(x) = +\infty$ for $x \in \operatorname{dom} F$ and $\tilde{\phi}_{y^*}(x) = -\infty$ for $x \notin \operatorname{dom} F$. Hence $\tilde{\phi}_{y^*}$ is convex for every $y^* \in Y^*$ if and only if ϕ_{y^*} is convex for every $y^* \in C^+$. The conclusion follows applying Proposition 2.3.6 to F_C. □

Of course, Proposition 2.3.6 can be obtained from Corollary 2.3.7 taking $C = \{0\}$. Corollary 2.3.7 can be found, essentially, in [138, Proposition 1.6] and [518, Lemma 3].

The **sublevel set of F of height y** (w.r.t. C) is the set

$$\operatorname{lev}_F(y) := \{x \in X \mid F(x) \cap (y - C) \neq \emptyset\};$$

when $\operatorname{int} C \neq \emptyset$ we also consider the **strict sublevel set of F of height y** (w.r.t. C) defined by

$$\operatorname{lev}_F^<(y) := \{x \in X \mid F(x) \cap (y - \operatorname{int} C) \neq \emptyset\}.$$

In this way we get the **sublevel** and **strict sublevel set-valued maps** lev_F, $\mathrm{lev}_F^{<} : Y \rightrightarrows X$.

We say that F is **C-α-quasiconvex (C-mid-quasiconvex, C-nearly quasiconvex, C-quasiconvex)** if for every $z \in Y$ the sublevel set $\mathrm{lev}_F(z)$ is α-convex (mid-convex, nearly convex, convex). An equivalent definition of C-α-quasiconvexity is that

$$\forall\, x, x' \in \operatorname{dom} F : (F(x) + C) \cap (F(x') + C) \subseteq F\left(\alpha x + (1-\alpha)x'\right) + C.$$

Notice that F is C-α-quasiconvex whenever

$$\begin{aligned} \forall x, x' \in \operatorname{dom} F : F(x) &\subseteq F\left(\alpha x + (1-\alpha)x'\right) + C \text{ or} \\ F(x') &\subseteq F\left(\alpha x + (1-\alpha)x'\right) + C. \end{aligned}$$

Note also that F is C-α-quasiconvex (C-mid-quasiconvex, C-nearly quasiconvex, C-quasiconvex) whenever F is C-α-convex (C-mid-convex, C-nearly convex, C-convex).

The set-valued map F is C-**convexlike** if

$$\begin{aligned} \forall x_1, x_2 &\in X,\ \forall y_1 \in F(x_1),\ \forall y_2 \in F(x_2),\ \forall \lambda \in]0,1[, \\ \exists x_3 &\in X : \lambda y_1 + (1-\lambda)y_2 \in F(x_3) + C, \end{aligned}$$

or, equivalently, $F(X) + C$ is convex, that is $F(X)$ is C-convex. Of course, if F is C-convex then F is C-convexlike.

Li and Chen [387] (see also [602]) say that F is C-**subconvexlike** if

$$\exists k \in \operatorname{int} C,\ \forall \alpha > 0,\ \forall x, x' \in X,\ \forall \lambda \in]0,1[: \alpha k + \lambda F(x) + (1-\lambda)F(x') \subseteq F(X) + C.$$

Using Proposition 2.3.5 (iii), F is C-subconvexlike iff $\operatorname{int} C \neq \emptyset$ and $F(X)$ is closely C-convex.

We say that $f : X \to Y^\bullet$ is **C-α-convex (C-mid-convex, C-nearly convex, C-convex, C-α-quasiconvex, C-mid-quasiconvex, C-nearly quasiconvex, C-quasiconvex)** if the set-valued map $F_{f,C}$ is C-α-convex (C-mid-convex, C-nearly convex, C-convex, C-α-quasiconvex, C-mid-quasiconvex, C-nearly quasiconvex, C-quasiconvex); in particular, f is C-convex if and only if

$$\forall\, x, x' \in X,\ \forall\, \alpha \in [0,1] : f\left(\alpha x + (1-\alpha)x'\right) \leq_C \alpha f(x) + (1-\alpha) f(x').$$

If f is C-α-convex (C-mid-convex, C-nearly convex, C-convex), then $\operatorname{dom} f$ is so, and f is C-α-quasiconvex (C-mid-quasiconvex, C-nearly quasiconvex, C-quasiconvex).

2.4 Solution Concepts in Vector Optimization

In this section, we first recall concepts of Pareto minimal points, weakly and properly minimal points, then we introduce the concept of Q-minimal points and establish relations among them.

Unless otherwise mentioned, in the following we consider a linear topological space Y, partially ordered by a proper pointed convex closed cone C and a nonempty set $A \subset Y$.

We introduce the following sets of *Pareto minimal points* (*Pareto maximal points*, respectively) of A with respect to C:

Definition 2.4.1 (Pareto Minimal (Maximal) Points). Consider

$$\mathrm{Min}(A, C) := \{\overline{y} \in A \mid A \cap (\overline{y} - C) = \{\overline{y}\}\}. \tag{2.19}$$

An element $\overline{y} \in \mathrm{Min}(A, C)$ is called a **Pareto minimal point** of A with respect to C.

Furthermore, consider

$$\mathrm{Max}(A, C) := \{\overline{y} \in A \mid A \cap (\overline{y} + C) = \{\overline{y}\}\}. \tag{2.20}$$

An element $\overline{y} \in \mathrm{Max}(A, C)$ is called a **Pareto maximal point** of A with respect to C.

Moreover, in order to describe weak minimality we will study the following solution concept in Y. Many solution procedures for vector optimization problems generate weakly minimal elements.

Definition 2.4.2 (Weakly Minimal (Maximal) Points). Suppose that $\mathrm{int}\, C \neq \emptyset$. Consider

$$\mathrm{WMin}(A, C) := \{\overline{y} \in A \mid A \cap (\overline{y} - \mathrm{int}\, C) = \emptyset\}. \tag{2.21}$$

An element $\overline{y} \in \mathrm{WMin}(A, C)$ is called a **weakly minimal point** of A with respect to C. Furthermore, consider

$$\mathrm{WMax}(A, C) := \{\overline{y} \in A \mid A \cap (\overline{y} + \mathrm{int}\, C) = \emptyset\}. \tag{2.22}$$

An element $\overline{y} \in \mathrm{WMax}(A, C)$ is called a **weakly maximal point** of A with respect to C.

Moreover, we introduce the concept of strongly minimal points:

Definition 2.4.3. Consider

$$\mathrm{StrMin}(A, C) := \{\overline{y} \in A \mid A \subseteq \overline{y} + C\}. \tag{2.23}$$

An element $\overline{y} \in \mathrm{StrMin}(A, C)$ is called a **strong minimal point** of A with respect to C.

In the following, we introduce different concepts of properly minimal points. Properly minimal points are important in the proofs of many theoretical assertions because corresponding scalarizing functionals (see Sect. 5.1) are strictly C-monotone. The first concept for proper minimality in the following definition (cf. Ha [228]) is based on scalarization by means of (strictly C-monotone) functionals $y^* \in C^\#$.

Definition 2.4.4 (Properly Minimal Points).

(a) Suppose that $C^\# \neq \emptyset$ and consider

$$\text{S-PMin}(A, C) := \{\overline{y} \in A \mid \exists y^* \in C^\#,\ \ \forall y \in A:\ y^*(\overline{y}) \leq y^*(y)\}. \tag{2.24}$$

An element $\overline{y} \in \text{S-PMin}(A, C)$ is called a **S-properly minimal point** of A w.r.t. C.

(b) Let

$$\text{Hu-PMin}(A, C) := \{\overline{y} \in A \mid (\mathrm{cl\ conv\ cone}[(A - \overline{y}) \cup C]) \cap (-C) = \{0\}\}.$$

An element $\overline{y} \in \text{Hu-PMin}(A, C)$ is called a **Hurwicz properly minimal point** of A w.r.t. C.

(c) Assume that Y is a n.v.s.; $\overline{y}$ is a **Hartley properly minimal point** of A w.r.t. C ($\overline{y} \in \text{Ha-PMin}(A, C)$) if $\overline{y} \in \mathrm{Min}(A, C)$ and there exists a constant $M > 0$ such that, whenever there is $\lambda \in C^+$ with $\lambda(y - \overline{y}) > 0$ for some $y \in A$, one can find $\mu \in C^+$ with

$$\lambda(y - \overline{y})/\|\lambda\| \leq -M(\mu(y - \overline{y})/\|\mu\|).$$

(d) Consider

$$\text{Be-PMin}(A, C) := \{\overline{y} \in A \mid \mathrm{cl\ cone}[(A - \overline{y}) + C] \cap (-C) = \{0\}\}.$$

$\overline{y} \in \text{Be-PMin}(A, C)$ is called a **Benson properly minimal point** of A w.r.t. C.

(e) Consider

$$\text{Bo-PMin}(A, C) := \{\overline{y} \in A \mid \mathrm{cl\ cone}(A - \overline{y}) \cap (-C) = \{0\}\}.$$

$\overline{y} \in \text{Bo-PMin}(A, C)$ is called a **Borwein properly minimal point** of A w.r.t. C.

(f) Consider

GHe-PMin$(A, C) := \{\overline{y} \in A \mid \exists$ proper convex pointed cone D with $C \setminus \{0\} \subset \operatorname{int} D$ such that $(A - \overline{y}) \cap (-\operatorname{int} D) = \emptyset\}$.
$\overline{y} \in$ GHe-PMin(A, C) is called a **Henig global properly minimal point** of A w.r.t. C.

(g) Suppose that Y is a n.v.s., C has a base Θ and consider

He-PMin$(A, C) := \{\overline{y} \in A \mid \exists\ \varepsilon > 0$ such that $\operatorname{cl} \operatorname{cone}(A - \overline{y}) \cap (-\Theta + \varepsilon B_Y) = \emptyset\}$.

$\overline{y} \in$ He-PMin(A, C) is called a **Henig properly minimal point** of A w.r.t. to C.

(h) Assume that Y is a n.v.s. Consider

Sup-PMin$(A, C) := \{\overline{y} \in A \mid \exists\ \rho > 0$ such that $\operatorname{cl} \operatorname{cone}(A - \overline{y}) \cap (B_Y - C) \subseteq \rho B_Y\}$.

$\overline{y} \in$ Sup-PMin(A, C) is called a **super efficient point** of A w.r.t. C.

For the notions of minimal points in Definition 2.4.4, we refer to [292, 293, 402] and [228]. The concepts of Henig proper minimality and Henig global proper minimality have been presented in [242]. The above definition of Henig properly minimal points can be found in [71, 627]; see also [228, 242, 618, 619]. For an equivalent definition of Henig properly minimal points by means of a functional from $C^{\#}$ the reader is referred to [619]. We note that positive proper minimality has been introduced by Hurwicz [11], and super efficiency has been introduced by Borwein and Zhuang [71]. We refer the reader to [219] for a survey and materials on proper efficiency.

In the sequel, when speaking of weakly minimal points (resp. S-properly minimal points) we mean that $\operatorname{int} C$ (resp. $C^{\#}$) is nonempty, when speaking of Henig minimal points we mean that C has a base Θ and when speaking that C has a bounded base we mean that Θ is bounded.

Let $B \subset Y$ be a convex set such that $0 \notin \operatorname{cl} B$ (that is B is a base for cone B); we set

$$\mathcal{N}_Y^B := \mathcal{N}^B := \{V \in \mathcal{N}_Y \mid V \text{ convex}, V \cap B = \emptyset\}; \tag{2.25}$$

clearly, $\mathcal{N}_Y^B \neq \emptyset$. For $V \in \mathcal{N}^B$ we set

$$P_V^B := \operatorname{cone}(B + V);$$

then P_V^B is a proper convex cone with $\operatorname{int} P_V^B = \mathbb{P}(B + \operatorname{int} V) \neq \emptyset$.

Similar to the case of normed vector spaces, if Θ is a base of C, we set

$$\text{He-PMin}(A,\Theta) := \{y \in A \mid \exists V \in \mathcal{N}_Y^{\Theta} : [\operatorname{cl}\operatorname{cone}(A-y)] \cap (V-\Theta) = \emptyset\}.$$

Observe that

$$\text{He-PMin}(A,\Theta) = \{y \in A \mid \exists V \in \mathcal{N}_Y^{\Theta} : (A-y) \cap \left(-P_V^{\Theta}\right) = \{0\}\} \tag{2.26}$$

$$= \{y \in A \mid \exists V \in \mathcal{N}_Y^{\Theta} : (A-y) \cap \left(-\operatorname{int} P_V^{\Theta}\right) = \emptyset\}. \tag{2.27}$$

Indeed, $[\operatorname{cl}\operatorname{cone}(A-y)] \cap (V-\Theta) = \emptyset \Rightarrow (A-y) \cap \operatorname{cone}(V-\Theta) = \{0\}$ and $(A-y) \cap \operatorname{cone}(V-\Theta) = \{0\} \Rightarrow [\operatorname{cl}\operatorname{cone}(A-y)] \cap (\operatorname{int} V - \Theta) = \emptyset$.

Moreover, the super efficiency in the case in which Y is a locally convex space is defined by

$\text{Sup-PMin}(A,C) := \{y \in A \mid \forall V \in \mathcal{N}_Y,\ \exists U \in \mathcal{N}_Y : \operatorname{cl}\operatorname{cone}(A-y) \cap (U-C) \subseteq V\}$.

The following two Propositions 2.4.5 and 2.4.6 are shown under weaker assumptions concerning the cone $C \subset Y$ (Y a linear topological space), namely that C is a proper convex cone, for the corresponding solution concepts.

Proposition 2.4.5. *Let $A \subset Y$ be nonempty. Then*

(i) $\text{StrMin}(A,C) \subseteq \bigcap\left\{\arg\min_A y^* \mid y^* \in C^+ \setminus 0\right\}$, *with equality if Y is a l.c.s. and C is closed.*
(ii) $\text{S-PMin}(A,C) = \bigcup\left\{\arg\min_A y^* \mid y^* \in C^{\#}\right\} = \text{S-PMin}(A+C,C)$.
(iii) *If* $\operatorname{int} C \neq \emptyset$, *then* $\text{Min}(A,C) \subseteq \text{WMin}(A,C)$ *and*

$$\text{WMin}(A,C) = A \cap \text{WMin}(A+C,C) = A \cap \operatorname{bd}(A+C) \tag{2.28}$$

$$\supseteq \bigcup\left\{\arg\min_A y^* \mid y^* \in C^+ \setminus \{0\}\right\}, \tag{2.29}$$

with equality if A is closely C-convex.
(iv) *We have that* $\text{GHe-PMin}(A,C) \subseteq \text{Min}(A,C)$ *and*

$$\text{GHe-PMin}(A,C) = \bigcup\{\text{WMin}(A,D) \mid D \in \mathcal{D}_C\}$$

$$= \text{GHe-PMin}(A+C,C) \supseteq \text{S-PMin}(A,C) \tag{2.30}$$

with equality if A is closely C-convex, where

$$\mathcal{D}_C := \{D \subset Y \mid D \textit{ proper pointed convex cone with } C \setminus \{0\} \subset \operatorname{int} D\}. \tag{2.31}$$

(v) *Assume that Θ is a base of C. Then*

$$\text{He-PMin}(A,\Theta) = \bigcup\{\text{WMin}(A,D) \mid D \in \mathcal{D}_\Theta\} = \text{He-PMin}(A+C,\Theta) \tag{2.32}$$

$$\supseteq \bigcup \{\arg\min_A y^* \mid y^* \in Y^*,\ \inf y^*(\Theta) > 0\}, \tag{2.33}$$

with equality if A is closely C-convex, where

$$\mathscr{D}_\Theta := \{P_V^\Theta \mid V \in \mathscr{N}_Y^\Theta\}. \tag{2.34}$$

(vi) *If Θ is a base of C then $\mathscr{D}_\Theta \subseteq \mathscr{D}_C$; consequently,* He-PMin$(A,\Theta) \subseteq$ GHe-PMin(A,C).

Proof. (i) Let $\overline{y} \in$ StrMin(A,C), that is $\overline{y} \in A \subseteq \overline{y} + C$. Then clearly StrMin$(A,C) = \{\overline{y}\}$ and the inclusion holds. Assume that Y is a l.c.s. and C is closed, and take $\overline{y} \in \bigcap\{\arg\min_A y^* \mid y^* \in C^+\}$. Then $\overline{y} \in A$ and $\langle y - \overline{y}, y^*\rangle \geq 0$ for all $y \in A$ and $y^* \in C^+$. By the bipolar theorem we get $y - \overline{y} \in C^{++} = \operatorname{cl} C = C$, whence $A \subseteq \overline{y} + C$.

(ii) The first equality is given by the definition of S-PMin(A,C). The inclusion S-PMin$(A,C) \subseteq$ S-PMin$(A+C,C)$ is obvious. If $\overline{y} \in$ S-PMin$(A+C,C)$, then $\overline{y} = \overline{a} + \overline{c}$ for some $\overline{a} \in A$, $\overline{c} \in C$, and there exists $\overline{y}^* \in C^\#$ such that $\langle \overline{a} + \overline{c}, \overline{y}^*\rangle \leq \langle y, \overline{y}^*\rangle$ for all $y \in A + C$. In particular $\langle \overline{a} + \overline{c}, \overline{y}^*\rangle \leq \langle \overline{a}, \overline{y}^*\rangle$, whence $\langle \overline{c}, \overline{y}^*\rangle \leq 0$. It follows that $\overline{c} = 0$, and so $\overline{y} = \overline{a} \in A$. Consequently, $\overline{y} \in$ S-PMin(A,C).

(iii) Assume that $\operatorname{int} C \neq \emptyset$. Taking into account that $\operatorname{int} C \subseteq C \setminus \{0\}$, the inclusion Min$(A,C) \subseteq$ WMin(A,C) follows.
For $y \in Y$, using Lemma 2.3.4, we have

$$(A-y)\cap(-\operatorname{int} C) = \emptyset \Leftrightarrow y \notin A+\operatorname{int} C \Leftrightarrow y \notin A+C+\operatorname{int} C \Leftrightarrow y \notin \operatorname{int}(A+C).$$

It follows that

$$\begin{aligned} \overline{y} \in \mathrm{WMin}(A,C) &\Leftrightarrow [\overline{y} \in A,\ (A + C - \overline{y}) \cap (-\operatorname{int} C) = \emptyset] \\ &\Leftrightarrow \overline{y} \in A \cap \mathrm{WMin}(A,C) \\ &\Leftrightarrow [\overline{y} \in A,\ \overline{y} \notin \operatorname{int}(A+C)] \Leftrightarrow \overline{y} \in A \cap \operatorname{bd}(A+C). \end{aligned}$$

Hence the equalities in (2.28) hold.

Take $\overline{y} \in \arg\min_A y^*$ for some $y^* \in C^+ \setminus \{0\}$. Because $\operatorname{int} C \subseteq \{y \in Y \mid \langle y, y^*\rangle > 0\}$, we obtain that $(A - \overline{y}) \cap (-\operatorname{int} C) = \emptyset$, and so $\overline{y} \in$ WMin(A,C). Hence the inclusion in (2.28) holds.

Assume now that A is closely C-convex and $\overline{y} \in$ WMin(A,C) [$\subseteq$ WMin$(A+C,C)$]. Then $(A+C) \cap (\overline{y} - \operatorname{int} C) = \emptyset$; it follows that $\operatorname{cl}(A+C) \cap (\overline{y} - \operatorname{int} C) = \emptyset$. Because $\operatorname{cl}(A+C)$ is convex, by a separation theorem (see [293, Theorem 3.16]) there exists $y^* \in Y^* \setminus \{0\}$ such that $\langle y + v, y^*\rangle \geq \langle \overline{y} - v', y^*\rangle$ for all $y \in A$ and $v, v' \in C$. It follows that $y^* \in C^+$ and $\langle y - \overline{y}, y^*\rangle \geq 0$ for every $y \in A$, and so $y^* \in C^+ \setminus \{0\}$ and $\overline{y} \in \arg\min_A y^*$.

(iv) The first equality in (2.30) is just the definition of GHe-PMin(A, C).

Take $\overline{y} \in$ GHe-PMin(A, C); then $\overline{y} \in$ WMin(A, D) for some $D \in \mathscr{D}_C$, and so $(A - \overline{y}) \cap (-\operatorname{int} D) = \emptyset$. Because $C + \operatorname{int} D = \operatorname{int} D$ (for every $D \in \mathscr{D}_C$), we clearly have that GHe-PMin$(A, C) \subseteq$ GHe-PMin$(A + C, C)$.

Take $\overline{y} \in$ GHe-PMin$(A + C, C)$; then $\overline{y} = y + v$ for some $y \in A$, $v \in C$. Assuming that $v \neq 0$, we get the contradiction $v \in (C \setminus \{0\}) \cap (A + C - \overline{y}) \subseteq (-\operatorname{int} D) \cap (A + C - \overline{y}) = \emptyset$. Hence $\overline{y} = y \in A$.

Take $\overline{y} \in$ S-PMin(A, C). Then there exists $y^* \in C^\#$ with $0 \leq \langle y - \overline{y}, y^* \rangle \leq \langle y + v - \overline{y}, y^* \rangle$ for all $y \in A$, $v \in C$. Take $D := \{0\} \cup \{y \in Y \mid \langle y, y^* \rangle > 0\}$. Then D is a pointed convex cone with $\operatorname{int} D = \{y \in Y \mid \langle y, y^* \rangle > 0\}$. It follows that $(A + C - \overline{y}) \cap (-\operatorname{int} D) = \emptyset$, and so $\overline{y} \in$ GHe-PMin(A, C). Hence S-PMin$(A, C) \subseteq$ GHe-PMin(A, C).

Assume that A is closely C-convex and take $\overline{y} \in$ GHe-PMin(A, C). Then there exists $D \in \mathscr{D}_C$ such that $(A - \overline{y}) \cap (-\operatorname{int} D) = \emptyset$. It follows that $\overline{y} \in A \cap$ WMin(A, D). Since A is closely C-convex and $C \subseteq D$, A is closely D-convex. From (iii) we get $y^* \in D^+ \setminus \{0\}$ such that $\overline{y} \in \arg\min_A y^*$. Because $D^+ \setminus \{0\} \subseteq C^\#$, we obtain that $\overline{y} \in$ S-PMin(A, C).

(v) The first equality in (2.32) is given in (2.27). For the equality He-PMin$(A, \Theta) =$ He-PMin$(A + C, \Theta)$ use a similar argument to that used in (iv) (possibly taking into account that $\mathscr{D}_\Theta \subseteq \mathscr{D}_C$).

Take $\overline{y} \in \arg\min_A y^*$ for some $y^* \in Y^*$ with $2\gamma := \inf y^*(\Theta) > 0$ and set $V := \{y \in Y \mid |\langle y, y^* \rangle| < \gamma\} \in \mathscr{N}_Y^\Theta$. Clearly, $D := P_V^\Theta \in \mathscr{D}_\Theta$. Because $y^* \in D^+ \setminus \{0\}$, from (iii) we get $\overline{y} \in$ WMin(A, D), and so $\overline{y} \in$ He-PMin(A, Θ). Hence the inclusion in (2.33) holds.

The proof of the equality in (2.33) for A closely C-convex is similar to the proof of the corresponding equality in (2.30).

(vi) The arguments used at the beginning of the proof of (v) show that $\mathscr{D}_\Theta \subseteq \mathscr{D}_C$.

□

Proposition 2.4.6. *Let $A \subset Y$ be nonempty.*

(i) *One has*

$$\text{S-PMin}(A, C) \subseteq \text{Hu-PMin}(A + C, C) \subseteq \text{Hu-PMin}(A, C) \subseteq \text{Be-PMin}(A, C) = \text{Be-PMin}(A + C, C) \tag{2.35}$$

with Hu-PMin$(A, C) =$ Be-PMin(A, C) *if A is closely C-convex, and*

$$\text{Be-PMin}(A, C) = \text{Bo-PMin}(A + C, C) \subseteq \text{Bo-PMin}(A, C) \subseteq \text{Min}(A, C), \tag{2.36}$$

$$\text{GHe-PMin}(A, C) \subseteq \text{Be-PMin}(A, C). \tag{2.37}$$

Moreover, if Y has the property that for any closed convex cone $K \subseteq Y$ there exists $y^* \in Y^*$ such that $\langle y, y^* \rangle > 0$ for every $y \in K \setminus (-K)$ (for example if Y is a separable normed vector space) then S-PMin$(A, C) =$ Hu-PMin(A, C).

(ii) *Assume that* Θ *is a base of* C *and* Y *is a locally convex space. Then* Sup-PMin$(A, C) \subseteq$ He-PMin(A, Θ). *Moreover, if* Θ *is bounded, then* Sup-PMin$(A, C) =$ He-PMin(A, Θ).

(iii) *Assume that* Θ *is a compact base of* C *and* Y *is a locally convex space. Then*

$$\text{S-PMin}(A, C) = \text{Hu-PMin}(A, C) \quad \textit{and} \quad \text{He-PMin}(A, \Theta) = \text{GHe-PMin}(A, C).$$

Moreover, if A *is closely* C*-convex, then* S-PMin$(A, C) =$ Be-PMin(A, C), *while if* A *is closely convex then* S-PMin$(A, C) =$ Bo-PMin(A, C).

(iv) *Assume that* StrMin$(A, C) \neq \emptyset$. *Then* StrMin$(A, C) =$ Min(A, C). *If* C *is closed (or, more generally,* $\operatorname{cl} C \cap (-C) = \{0\}$*), then*

$$\text{StrMin}(A, C) = \text{Hu-PMin}(A, C) = \text{Hu-PMin}(A + C, C); \tag{2.38}$$

if $C^{\#} \neq \emptyset$ *then*

$$\text{StrMin}(A, C) = \text{S-PMin}(A, C) = \text{GHe-PMin}(A, C), \tag{2.39}$$

and StrMin$(A, C) =$ He-PMin(A, Θ) *for every base* Θ *of* C *if, furthermore,* Y *is a locally convex space.*

Proof. (i) Take $\overline{y} \in$ S-PMin(A, C). Then $\overline{y} \in A$ and there exists $\overline{y}^* \in C^{\#}$ such that $\langle \overline{y}, \overline{y}^* \rangle \leq \langle y, \overline{y}^* \rangle$ for all $y \in A$. It follows that $0 \leq \langle y, \overline{y}^* \rangle$ for all $y \in E_1 := \operatorname{cl}\operatorname{conv}\operatorname{cone}(A + C - \overline{y})$, and so $E_1 \cap (-C) = \{0\}$, because $\overline{y}^* \in C^{\#}$. Hence the first inclusion in (2.35) holds.

Take $\overline{y} \in$ Hu-PMin$(A+C, C)$ $(\subseteq A+C)$. Hence $\overline{y} = y+v$ with $y \in A$, $v \in C$. Then $-v \in (A + C - \overline{y}) \cap (-C) \subseteq (\operatorname{cl}\operatorname{conv}\operatorname{cone}(A + C - \overline{y})) \cap (-C) = \{0\}$. Therefore, $v = 0$, and so $\overline{y} \in A$. It follows that $\overline{y} \in$ Hu-PMin(A, C).

Take $\overline{y} \in$ Hu-PMin(A, C). Then $(A - \overline{y}) \cup C \subseteq E_2 := \operatorname{cl}\operatorname{conv}\operatorname{cone}[(A - \overline{y}) \cup C]$, whence $(A - \overline{y}) + C \subseteq E_2$. It follows that $F := \operatorname{cl}\operatorname{cone}(A + C - \overline{y}) \subseteq E_2$, and so $\overline{y} \in$ Be-PMin(A, C). Assuming that A is closely C-convex and $\overline{y} \in$ Be-PMin(A, C), we have that F is a convex cone and $F \cap (-C) = \{0\}$. Since $(A - \overline{y}) \cup C \subseteq A + C - \overline{y}$, it follows that $E_2 \subseteq F$, and so $\overline{y} \in$ Hu-PMin(A, C).

The equalities in (2.35) and (2.36) follow directly from the definitions of the corresponding sets.

Take $\overline{y} \in$ Bo-PMin$(A + C, C)$. Then $\overline{y} = y + v$ for some $y \in A$, $v \in C$. Then $-v \in (A + C - \overline{y}) \cap (-C) \subseteq (-C) \cap \operatorname{cl}\operatorname{cone}(A + C - \overline{y}) = \{0\}$. Hence $\overline{y} \in A$, and so $\overline{y} \in$ Bo-PMin(A, C).

Take $\overline{y} \in$ Bo-PMin(A, C). Since $\operatorname{cl}\operatorname{cone}(A - \overline{y}) \supseteq A - \overline{y}$, it follows that $\overline{y} \in$ Min(A, C).

Take $\overline{y} \in$ GHe-PMin$(A, C) =$ GHe-PMin$(A + C, C)$. Then there exists $D \in \mathscr{D}_C$ such that $(A + C - \overline{y}) \cap (-\operatorname{int} D) = \emptyset$, and so $[\mathbb{R}_+(A + C - \overline{y})] \cap (-\operatorname{int} D) = \emptyset$, whence $[\operatorname{cl}\operatorname{cone}(A + C - \overline{y})] \cap (-\operatorname{int} D) = \emptyset$. Hence $\overline{y} \in$ Be-PMin(A, C). Therefore, (2.37) holds.

Assume now that for any closed convex cone $K \subset Y$ there exists $y^* \in Y^*$ such that $\langle y, y^* \rangle > 0$ for every $y \in K \setminus (-K)$, and take $\overline{y} \in \text{Hu-PMin}(A, C)$. Then $E_3 \cap (-C) = \{0\}$, or, equivalently, $(C \setminus \{0\}) \cap (-E_3) = \emptyset$, where $E_3 := \operatorname{cl}\operatorname{conv}\operatorname{cone}[(A - \overline{y}) \cup C]$. Then there exists $y^* \in Y^*$ such that $\langle y, y^* \rangle > 0$ for every $y \in E_3 \setminus (-E_3)$. Because $C \subseteq E_3$, if $y \in C \setminus \{0\}$ then $y \in E_3 \setminus (-E_3)$, and so $\langle y, y^* \rangle > 0$. Hence $y^* \in C^\#$. Therefore, $\overline{y} \in \text{S-PMin}(A, C)$.

(ii) Assume that Θ is a base of C and Y is a locally convex space. Because Θ is a base of C, there exists $V_0 \in \mathcal{N}_Y^c$ such that $(2V_0) \cap \Theta = \emptyset$, or, equivalently, $V_0 \cap (V_0 - \Theta) = \emptyset$. Take $\overline{y} \in \text{Sup-PMin}(A, C)$, that is $\overline{y} \in A$ and for every $V \in \mathcal{N}_Y$ there exists $U \in \mathcal{N}_Y$ such that $K \cap (U - C) \subseteq V$. Therefore, there exists $U_0 \in \mathcal{N}_Y^c$ such that $U_0 \subseteq V_0$ and $[K \cap (U_0 - \Theta)] \subseteq K \cap (U_0 - C) \subseteq V_0$. It follows that $K \cap (U_0 - \Theta) \subseteq V_0 \cap (U_0 - \Theta) \subseteq V_0 \cap (V_0 - \Theta) = \emptyset$. Hence $\overline{y} \in \text{He-PMin}(A, \Theta)$.

Assume, moreover, that Θ is bounded and take $\overline{y} \in \text{He-PMin}(A, \Theta)$. Then there exists $V_0 \in \mathcal{N}_Y$ such that $K \cap (V_0 - \Theta) = \emptyset$. Suppose that $\overline{y} \notin \text{Sup-PMin}(A, C)$. Then there exists $V_1 \in \mathcal{N}_Y^c$ such that for every $U \in \mathcal{N}_Y^c$ one has $K \cap (U - C) \not\subseteq V_0$. Hence, for every $U \in \mathcal{N}_Y^c$ there exists $y_U \in U$, $t_U \in \mathbb{R}_+$, $z_U \in \Theta$ such that $y_U - t_U z_U \in K \setminus V_0$. Clearly, $t_U > 0$ for every $U \in \mathcal{N}_Y^c$ with $U \subseteq V_0$; otherwise we get the contradiction $y_U \in (K \setminus V_0) \cap V_0 = \emptyset$. Taking $p := p_{V_0}$ the Minkowski functional of V_0, p is a continuous seminorm with $\operatorname{int} V_0 = \{y \in Y \mid p(y) < 1\}$ (see Proposition 6.2.1). Clearly, $t_U^{-1} y_U - z_U \in K$. Because $K \cap (V_0 - \Theta) = \emptyset$ we get $t_U^{-1} y_U \notin V_0$ for $U \subseteq V_0$, and so $t_U^{-1} p(y_U) = p(t_U^{-1} y_U) \geq 1$ for such U. It follows that $t_U \leq p(y_U)$ for $U \subseteq V_0$. Hence $(t_U)_{U \in \mathcal{N}_Y^c} \to 0$. Since $(y_U)_{U \in \mathcal{N}_Y^c} \to 0$ and $(z_U)_{U \in \mathcal{N}_Y^c}$ is bounded, it follows that $(y_U - t_U z_U)_{U \in \mathcal{N}_Y^c} \to 0$, contradicting the fact that $y_U - t_U z_U \in K \setminus V_0$ for every U. This contradiction shows that $\overline{y} \in \text{Sup-PMin}(A, C)$.

(iii) Assume that Θ is a compact base of C and Y is a locally convex space.

Take $\overline{y} \in \text{Hu-PMin}(A, C)$, that is $\overline{y} \in A$ and $E_4 \cap (-C) = \{0\}$, where $E_4 := \operatorname{cl}\operatorname{conv}\operatorname{cone}\big[(A - \overline{y}) \cup C\big]$. It follows that $E_4 \cap (-\Theta) = \emptyset$. Using a separation theorem, there exist $y^* \in Y^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\forall y \in (A - \overline{y}) \cup C,\ v' \in \Theta,\ t \in \mathbb{R}_+ : t \langle y, y^* \rangle \geq \alpha > \beta \geq \langle -v', y^* \rangle. \tag{2.40}$$

It follows that $\alpha \leq 0$, $2\gamma := \inf_\Theta y^* > 0$, whence $y^* \in C^\#$, and $\langle y - \overline{y}, y^* \rangle \geq 0$ for every $y \in A$. Hence $\overline{y} \in \text{S-PMin}(A, C)$. Taking into account (2.35) we obtain that $\text{S-PMin}(A, C) = \text{Hu-PMin}(A, C)$.

Take $\overline{y} \in \text{GHe-PMin}(A, C)$. There exists $D \in \mathcal{D}_C$ such that $(A - \overline{y}) \cap (-\operatorname{int} D) = \emptyset$. Clearly, $\Theta \subseteq \operatorname{int} D$. Since Θ is compact and Y is a l.c.s., there exists $V \in \mathcal{N}_Y^\Theta$ such that $\Theta + V \subseteq \operatorname{int} D$. Taking $D' := P_V^\Theta$ we have that $D' \in \mathcal{D}_\Theta$ and $\overline{y} \in \text{WMin}(A, D')$, and so $\overline{y} \in \text{He-PMin}(A, \Theta)$. Hence $\text{He-PMin}(A, \Theta) = \text{GHe-PMin}(A, C)$.

Assume now that A is closely C-convex and take $\overline{y} \in$ Be-PMin(A, C), that is $\overline{y} \in A$ and $\operatorname{cl}\operatorname{cone}(A + C - \overline{y}) \cap (-C) = \{0\}$. Since A is closely C-convex, we obtain that $\operatorname{cl}\operatorname{cone}(A + C - \overline{y})$ is a closed convex cone. From our hypothesis we have that $\operatorname{cl}\operatorname{cone}(A + C - \overline{y}) \cap (-\Theta) = \emptyset$. Using a separation theorem, there exist $y^* \in Y^*$ and $\alpha, \beta \in \mathbb{R}$ such that (2.40) holds with $y \in A + C - \overline{y}$ instead of $y \in (A - \overline{y}) \cup C$. It follows that $\alpha \leq 0$, $2\gamma := \inf_\Theta y^* > 0$, whence $y^* \in C^\#$, and $\langle y - \overline{y}, y^* \rangle \geq 0$ for every $y \in A$. Hence $\overline{y} \in$ S-PMin(A, C). Hence Be-PMin$(A, C) =$ S-PMin(A, C).

Assume now that A is closely convex and take $\overline{y} \in$ Bo-PMin(A, C), that is $\overline{y} \in A$ and

$$\operatorname{cl}\operatorname{cone}(A - \overline{y}) \cap (-C) = \{0\},$$

whence $\operatorname{cl}\operatorname{cone}(A - \overline{y}) \cap (-\Theta) = \emptyset$. Since A is closely convex, $\operatorname{cl}\operatorname{cone}(A - \overline{y})$ is convex. Using a separation theorem, as above, we get $\overline{y} \in$ S-PMin(A, C). Hence S-PMin$(A, C) =$ Bo-PMin(A, C).

(iv) Assume now that StrMin$(A, C) \neq \emptyset$. Because C is pointed we have that StrMin$(A, C) = \{\overline{y}\}$ for some $\overline{y} \in A$; hence $A \subseteq \overline{y} + C$. Clearly, $(A - \overline{y}) \cap (-C) \subseteq C \cap (-C) = \{0\}$, and so $\overline{y} \in$ Min(A, C). Conversely, take $y \in$ Min(A, C); then $\{0\} = (A - y) \cap (-C) \ni \overline{y} - y$, and so $y = \overline{y} \in$ StrMin(A, C). Therefore, StrMin$(A, C) =$ Min(A, C).

Since $A - \overline{y} \subseteq C$, we have that $(A - \overline{y}) \cup C = (A + C - \overline{y}) \cup C = C$.

Assume, moreover, that C is closed; then

$$\operatorname{cl}\operatorname{conv}\operatorname{cone}[(A + C - \overline{y}) \cup C] = C,$$

and so $\overline{y} \in$ Hu-PMin(A, C). Then (2.38) follows from Min$(A, C) = \{\overline{y}\}$ and relations (2.35), (2.36).

Assume that $C^\# \neq \emptyset$. Because $\overline{y} \in A \subseteq \overline{y} + C$, clearly, $\arg\min_A y^* = \{\overline{y}\}$ for every $y^* \in C^\#$. Hence S-PMin$(A, C) = \{\overline{y}\}$. The second equality in (2.39) follows using Proposition 2.4.5 (iv).

Assume that Y is a l.c.s. and Θ is a base of C. Since $0 \notin \operatorname{cl}\Theta$, there exists $y^* \in Y^*$ with $\inf_\Theta y^* > 0$. Hence $y^* \in C^\#$, and so $\arg\min_A y^* = \{\overline{y}\}$. Using Proposition 2.4.5 (v) we have that $\overline{y} \in$ S-PMin(A, C). Using (2.39) and Proposition 2.4.5 (vi) we get He-PMin$(A, \Theta) = \{\overline{y}\}$. □

It is worth observing that for $y^* \in Y^* \setminus \{0\}$ and $C := K_{y^*} := \{0\} \cup \{y \in Y \mid \langle y, y^* \rangle > 0\}$ we have that $C^\# = \mathbb{P}y^*$ and S-PMin$(A, C) =$ Min$(A, C) = \arg\min_A y^*$, and so all the efficiency sets used in Propositions 2.4.5, 2.4.6, excepting StrMin, reduce to $\arg\min_A y^*$. Moreover, StrMin$(A, C) \neq \emptyset$ if and only if y^* has a unique minimum point on A.

The inclusions in assertions (i), and (iii), (iv) of Proposition 2.4.5, as well as the corresponding equalities for C closed or A convex, respectively, are established in [175, Theorem 3.1]. The most part of the inclusions in assertions (i) and (iii) of Proposition 2.4.6, as well as the mentioned equalities, can be found [392]; note that the hypothesis that Θ is compact can be replaced by the fact that Θ is weakly compact (for this just apply the corresponding result for the weak topology on Y). Assertion (ii) of Proposition 2.4.6 can be found in [618], while the equality $\text{S-PMin}(A, C) = \text{Hu-PMin}(A, C)$ in Proposition 2.4.6 (i) is obtained in [271, Theorem V.2.4]; the case of separable normed vector spaces in (i) is based on the following extension of the Krein–Rutman theorem obtained by Hurwicz [271, Lemma V.2.2].

Theorem 2.4.7. *Let $(Y, \|\cdot\|)$ be a separable normed vector space and $C \subseteq Y$ be a closed convex cone. Then there exists $y^* \in Y^*$ such that $\langle y, y^* \rangle > 0$ for every $y \in C \setminus (-C)$.*

Proof. If $C \setminus (-C) = \emptyset$ (that is C is a linear space) we can take $y^* = 0$. So, assume that $C \setminus (-C) \neq \emptyset$. Because $C^{++} = C$, C^+ is not trivial, too. The set $C_1 := C^+ \cap U_{Y^*}$ is a weakly* closed subset of U_{Y^*} (hence C_1 is w^*-compact). Because Y is separable, the weak* topology on U_{Y^*} is metrizable (see [108, Theorem V.5.1]), and so C_1 (being w^*-compact) is w^*-separable. Let $A = \{y_1^*, y_2^*, \ldots\} \subseteq C_1$ be w^*-dense in C_1. Take

$$\overline{y}^* := \sum_{k=1}^{\infty} \frac{1}{2^k} y_k^*;$$

the series is strongly convergent because it is absolutely convergent and Y^* is a Banach space. Clearly, $\overline{y}^* \in C_1 \subseteq C^+$. Assume that there exists $\overline{y} \in C \setminus (-C)$ such that $\langle \overline{y}, \overline{y}^* \rangle = 0$. Because $\langle \overline{y}, y_k^* \rangle \geq 0$ for every $k \geq 1$, we obtain that $\langle \overline{y}, y^* \rangle = 0$ for every $y^* \in A$. With the set A being w^*-dense in C_1, we obtain that $\langle \overline{y}, y^* \rangle = 0$ for every $y^* \in C_1$, and so $\langle -\overline{y}, y^* \rangle = 0 \geq 0$ for every $y^* \in C^+ = \mathbb{R}_+ C_1$. Therefore, we get the contradiction $-\overline{y} \in C^{++} = C$. □

In order to describe weak and proper minimality in a unified way, we use the notation of Q-minimal points (compare Ha [228]).

Definition 2.4.8. Assume that $D \subset Y$ is a proper cone with nonempty interior and put $Q := \operatorname{int} D$. We say that $\overline{y}$ is a **Q-minimal point** of A ($\overline{y} \in \text{QMin}(A, C)$) if

$$A \cap (\overline{y} - Q) = \emptyset$$

or, equivalently,

$$(A - \overline{y}) \cap (-Q) = \emptyset.$$

In the paper by Gerstewitz and Iwanow [197] properly minimal elements are defined using a set $Q \subset Y$ with $0 \in \operatorname{bd} Q$ and $\operatorname{cl} Q + (C \setminus \{0\}) \subset \operatorname{int} Q$. This approach is related to the well-known concept of *dilating cone* (or a *dilation*) of C:

Definition 2.4.9. Suppose that $D \subset Y$ is a proper cone with nonempty interior and put $Q := \operatorname{int} D$. Q is said to be a *dilation* of C, or *dilating* C if it contains $C \setminus \{0\}$.

Remark 2.4.10. Makarov and Rachkovski [409] studied more detailed some concepts of proper efficiency and introduced the notion of $\mathscr{B}$-efficiency, i.e., efficiency w.r.t. a family of dilations of C. Namely, given $\mathscr{B} \in \mathscr{F}(C)$, where $\mathscr{F}(C)$ is the class of families of dilations of C, $\overline{y}$ is said to be a *$\mathscr{B}$-minimal point* of A ($\overline{y} \in \mathscr{B}\,\mathrm{Min}(A, C)$) if there exists $B \in \mathscr{B}$ such that

$$(A - \overline{y}) \cap (-B) = \emptyset.$$

It has been established that Borwein proper efficiency, Henig global proper efficiency, Henig proper efficiency, super efficiency and Hartley proper efficiency are $\mathscr{B}$-efficiency with $\mathscr{B}$ being appropriately chosen family of dilating cones. The reader will see that in contrast with $\mathscr{B}$-efficiency, the concept introduced in Definition 2.4.8 includes not only some concepts of proper efficiency among which are these ones considered in [409] but also the concepts of strong efficiency and weak efficiency.

In order to study the relationships between weakly / properly minimal points and Q-minimal points let Y be a n.v.s. and Θ as before a base of C. Setting

$$\delta := \delta_\Theta := d(0, \Theta) = \inf\{\|\theta\| \mid \theta \in \Theta\} > 0,$$

for each $0 < \eta < \delta$, we can associate to C a convex, pointed and open set V_η, defined by

$$V_\eta := \operatorname{cone}(\Theta + \eta \overset{\circ}{B}_Y). \tag{2.41}$$

For each scalar $\varepsilon > 0$, we also associate to C an open set $C(\varepsilon)$

$$C(\varepsilon) := \{y \in Y \mid d(y, C) < \varepsilon d(y, -C)\}.$$

We are going to show that the weakly / properly minimal points introduced in Definitions 2.4.2 and 2.4.4 are in fact Q-minimal points (Definition 2.4.8) with Q being appropriately chosen sets. The following result is shown in [228].

Theorem 2.4.11. (a) $\overline{y} \in \mathrm{WMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$ *with* $Q = \mathrm{int} C$.
(b) $\overline{y} \in \text{S-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$ *with* $Q = \{y \in Y \mid y^*(y) > 0\}$ *and* $y^* \in C^\#$.
(c) $\overline{y} \in \text{Hu-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$, *with* $Q = Y \setminus -\mathrm{cl\,conv\,cone}[(A - \overline{y}) \cup C]$.
(d) $\overline{y} \in \text{Be-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$, *with* $Q = Y \setminus -\mathrm{cl\,cone}[(A - \overline{y}) + C]$.
(e) $\overline{y} \in \text{Ha-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$ *with* $Q = C(\varepsilon)$ *for some* $\varepsilon > 0$.
(f) $\overline{y} \in \text{Bo-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$, *with Q being some dilation of C*.
(g) $\overline{y} \in \text{GHe-PMin}(A, C)$ *iff* $\overline{y} \in \mathrm{QMin}(A, C)$, *with* $Q = \mathrm{int}\, D$, *being some dilation of C, where D is a proper pointed convex cone in Y*.

(h) *(supposing that Y is a n.v.s.)* $\overline{y} \in$ He-PMin(A, C) *iff* $\overline{y} \in$ QMin(A, C) *with* $Q = V_\eta$ *and* η *is some scalar satisfying* $0 < \eta < \delta = d(0, \Theta)$.

(i) *(supposing that Y is a n.v.s. and C has a bounded base Θ)* $\overline{y} \in$ Sup-PMin(A) *iff* $\overline{y} \in$ QMin(A, C) *with* $Q = V_\eta$ *and* η *is some scalar satisfying* $0 < \eta < \delta = d(0, \Theta)$.

Proof. Using Definitions 2.4.4 and 2.4.8 one can easily prove the assertions (a)–(d) and (g). The assertions (e)–(f) are formulated in a slightly different form as established by Makarov and Rachkovski [409].

We prove now the assertion (h), namely, we show that $\overline{y} \in$ He-PMin(A, C) iff there is a scalar η with $0 < \eta < \delta$ such that

$$(A - \overline{y}) \cap (-V_\eta) = \emptyset. \tag{2.42}$$

Recall that by definition, $\overline{y} \in$ He-PMin(A, C) iff

$$\text{cl cone}(A - \overline{y}) \cap (\Theta + \varepsilon B_Y) = \emptyset. \tag{2.43}$$

It is also known [619] that $\overline{y} \in$ He-PMin(A, C) iff

$$(A - \overline{y}) \cap (-\overline{S}_n) = \{0\} \tag{2.44}$$

for some integer $n \in \mathbb{N}$, where $\overline{S}_n = \text{cl cone}(\Theta + \delta/(2n)B_Y)$. Now, suppose that $\overline{y} \in$ He-PMin(A, C). Then (2.43) holds. Without loss of generality we can assume that $0 < \varepsilon < \delta$. We show that (2.43) holds with $\eta = \varepsilon$. Suppose to the contrary that there is $y' \in A - \overline{y}$ such that $y' \in -V_\varepsilon$. Clearly, $y' \in \text{cl cone}(A - \overline{y}) \cap (-\text{cl cone}(\Theta + \varepsilon B_Y))$. On the other hand, as $0 < \eta = \varepsilon < \delta$ and by the definition of δ, $0 \notin V_\varepsilon$. Hence $y' \neq 0$. This is a contradiction to (2.43). Next, suppose that (2.42) holds for some η. Let n be an integer satisfying $n - 1 > \delta/(2\eta)$ or $\delta/(2n-2) < \eta$. By (2.42) we have

$$(A - \overline{y}) \cap (-V_{\delta/(2n-2)}) \subseteq (A - \overline{y}) \cap (-V_\eta) = \emptyset.$$

Then $(A - \overline{y}) \cap (-V_{\delta/(2n-2)} \cup \{0\}) = \{0\}$. On the other hand, [619, Lemma 2.1] states that if $(A - \overline{y}) \cap (-V_{\delta/(2n-2)} \cup \{0\}) = \{0\}$, then $(A - \overline{y}) \cap (-\overline{S}_n) = \{0\}$. Thus, (2.44) holds and therefore, $\overline{y} \in$ He-PMin(A, C), as it was to be shown.

To complete the proof note that the last assertion (i) of this theorem follows from (h) and the assertion (ii) in Proposition 2.4.6. □

Remark 2.4.12. The assertion (h) in the above theorem is inspired by the definition of Henig properly minimal points for sets in locally convex spaces given by Gong in [213]. One can deduce that any Henig properly minimal point is a global Henig properly minimal point.

Furthermore, we mention solution concepts for vector optimization problems introduced by ElMaghri–Laghdir[175] where it is not supposed that the ordering cone $C \subset Y$ is pointed and closed. This concept is based on a generalization of the concept of dilations of a cone $C \subset Y$ (see Definition 2.4.9). In the text below we follow the presentation in [175]. We assume that X, Y, Z are topological vector spaces and S is a subset of X. For the following notations and results the **lineality** of C, defined by

$$l(C) := C \cap (-C),$$

is very important. Of course, C is pointed if $l(C) = \{0\}$.

Unlike the assumptions made before we assume in the sequel in this section that Y is ordered by the proper convex cone C (C is proper if $C \neq l(C)$, or equivalently, C is not a linear subspace of Y).

Furthermore, using $l(C)$ we introduce

$$C^{\&} := \{y^* \in Y^* \mid \langle y, y^* \rangle > 0 \; \forall y \in C \setminus l(C)\} = (C \setminus l(C))^{\#}.$$

Hence, if C is pointed then $C^{\&} = C^{\#}$. Note that Theorem 2.4.7 gives sufficient conditions for $C^{\&} \neq \emptyset$.

We use the notations $y \leqq y'$ if $y' - y \in C$; $y < y'$ if $y' - y \in \operatorname{int} C$ and furthermore, $y \leq y'$ if $y' - y \in C \setminus l(C)$.

In the following we consider a proper vector-valued objective function $f : X \to Y^{\bullet}$, $S \subseteq X$ and use the notation

$$f(S) := \{f(x) \mid x \in S \cap \operatorname{dom} f\} \subset Y.$$

Consider now the vector optimization problem

$$\text{minimze } f(x) \text{ subject to } x \in S. \tag{VP}$$

Using the lineality of C and without assuming the pointedness and closedness of C we introduce the following solution concepts for (VP). These concepts are extensions of the solution concepts introduced in Definitions 2.4.1, 2.4.2 and 2.4.4.

Definition 2.4.13. $\overline{x} \in S \cap \operatorname{dom} f$ is

- **strongly l(C)-minimal** if $f(\overline{x}) \leqq f(x)$ for all $x \in S$, or equivalently

$$f(S) \subseteq f(\overline{x}) + C,$$

- **Pareto l(C)-minimal** if $f(x) \leqq f(\overline{x}) \Rightarrow f(\overline{x}) \leqq f(x)$ for all $x \in S$, or equivalently

$$f(S) \cap (f(\overline{x}) - (C \setminus l(C))) = \emptyset,$$

- **weakly l(C)-minimal** if $f(x) \not< f(\overline{x})$ for all $x \in S$, or equivalently

$$f(S) \cap (f(\overline{x}) - \operatorname{int} C) = \emptyset,$$

- **l(C)-properly minimal** if there exists $D \subset Y$ a proper convex cone with $C \setminus l(C) \subseteq \operatorname{int} D$ such that $\overline{x}$ is efficient with respect to D, $f(S) \cap (f(\overline{x}) - D \setminus l(D)) = \emptyset$.

Definition 2.4.14. Suppose that $D \subset Y$ is a proper convex cone with nonempty interior. Put $Q := \operatorname{int} D$. Q is said to be a **generalized dilation** of C or **generalized dilating** C if it contains $C \setminus l(C)$.

Set

$$\mathscr{Q}_C := \{D \subset Y \mid D \text{ is a proper convex cone with } C \setminus l(C) \subset \operatorname{int} D\}.$$

Lemma 2.4.15. *Let $C \subset Y$ be a proper convex cone. Then $\mathscr{Q}_C \neq \emptyset$ if and only if $C^{\&} \neq \emptyset$. Moreover, if $C^{\&} \neq \emptyset$ then*

$$C + \operatorname{int} D = \operatorname{int} D \quad \forall D \in \mathscr{Q}_C \tag{2.45}$$

and

$$C^{\&} = \cup_{D \in \mathscr{Q}_C} (D^+ \setminus \{0\}). \tag{2.46}$$

Proof. Assume that $\mathscr{Q}_C \neq \emptyset$ and take $D \in \mathscr{Q}_C$. Since D is proper, $\operatorname{int} D \neq Y$. Take $q_0 \in Y \setminus \operatorname{int} D$. By a separation theorem there exists $y^* \in Y^*$ such that $\langle q_0, y^* \rangle \leq 0 < \langle y, y^* \rangle$ for all $y \in \operatorname{int} D$. It follows that $y^* \in C^{\&}$.

Conversely, assume that $C^{\&} \neq \emptyset$ and take $y^* \in C^{\&}$. Consider $D := \{y \in Y \mid \langle y, y^* \rangle \geq 0\}$. From the very definition of $C^{\&}$ we have that $C \setminus l(C) \subset \{y \in Y \mid \langle y, y^* \rangle > 0\} = \operatorname{int} D$.

Let us prove (2.45). First note that $C \subseteq \operatorname{cl}(C \setminus \operatorname{cl} C)$. Indeed, there exists $k_0 \in C \setminus l(C)$. Let $k \in C$. Then $k + \lambda k_0 \in C + (C \setminus l(C)) = C \setminus l(C)$ for every $\lambda > 0$. The claim follows taking the limit for $\lambda \to 0$. Then, taking $D \in \mathscr{Q}_C$ and using repeatedly [79, Lemma 2.5] we get

$$\operatorname{int} D \subseteq C + \operatorname{int} D = \operatorname{cl}(C \setminus l(C)) + \operatorname{int} D = (C \setminus l(C)) + \operatorname{int} D \subseteq \operatorname{int} D,$$

and so (2.45) holds.

Take now $D \in \mathscr{Q}_C$; then $C \setminus l(C) \subseteq \operatorname{int} D$, and so $C^{\&} = (C \setminus l(C))^{\#} \supseteq (\operatorname{int} D)^{\#} = D^+ \setminus \{0\}$. Hence the inclusion $\supseteq$ holds in (2.46). Let now $y^* \in C^{\&}$. Then $D := \{y \in Y \mid \langle y, y^* \rangle \geq 0\} \in \mathscr{Q}_C$ and $y^* \in D^+ \setminus \{0\}$. It follows that (2.46) holds. $\square$

For $A \subset Y$ we set

$$E_s(A) := E_s^C(A) := \{\overline{a} \in A \mid A \subseteq \overline{a} + C\},$$

$$E_e(A) := E_e^C(A) := \{\overline{a} \in A \mid A \cap (\overline{a} - (C \setminus l(C))) = \emptyset\},$$

$$E_w(A) := E_w^C(A) := \{\overline{a} \in A \mid A \cap (\overline{a} - \operatorname{int} C) = \emptyset\},$$

$$E_p(A) := E_p^C(A) := \cup \left\{E_e^D(A) \mid D \in \mathcal{Q}_C\right\}.$$

Since $\operatorname{int} C \subseteq C \setminus l(C) \subseteq \operatorname{int} D$, we have that

$$E_p(A) \subseteq E_e(A) \subseteq E_w(A), \tag{2.47}$$

the last inclusion makes sense for $\operatorname{int} C \neq \emptyset$. It is worth to observe that

$$E_p(A) = \cup \left\{E_w^D(A) \mid D \in \mathcal{Q}_C\right\}. \tag{2.48}$$

Indeed, the inclusion $\subseteq$ in (2.48) follows from the last inclusion in (2.47). Take $\overline{a} \in E_w^D(A)$ for some $D \in \mathcal{Q}_C$ and consider $D' := \{0\} \cup \operatorname{int} D$. Then $D' \in \mathcal{Q}_C$ and $\overline{a} \in E_w^{D'}(A)$ because $D' \subseteq D$. Since $E_w^{D'}(A) = E_e^{D'}(A)$, we obtain that $\overline{a} \in E_p(A)$.

Lemma 2.4.16. *$E_\sigma(A) = A \cap E_\sigma(A+C)$ for $\sigma \in \{s, e, w, p\}$. Moreover, if $l(C) = \{0\}$, that is C is pointed, then $E_\sigma(A) = E_\sigma(A + C)$ for $\sigma \in \{s, e, p\}$. Generally, $E_w(A) = A \cap \operatorname{bd}(A + C)$ (for $\operatorname{int} C \neq \emptyset$), and so $E_w(A)$ and $E_w(A + C)$ ($= \operatorname{bd}(A + C)$) might be different (take $A = \{0\}$).*

Proof. For $\sigma = s$ use the fact that $A \subseteq a + C \Rightarrow A \subseteq A + C \subseteq a + C$. For $\sigma = e$ use the fact that $C \setminus l(C) = C + (C \setminus l(C))$, while for $\sigma = w$ use the fact that $C + \operatorname{int} C = \operatorname{int} C$.

For $\sigma = p$ one uses (2.45). If $a \in E_p(A)$ then there exists $D \in \mathcal{Q}_C$ such that $A \cap (a - \operatorname{int} D) = \emptyset$. Hence $0 \notin A - a + \operatorname{int} D = (A + C) - a + \operatorname{int} D$ by (2.45), and so $a \in E_p(A + C)$. Conversely, let $a \in A \cap E_p(A + C)$. Then $0 \notin (A + C) - a + \operatorname{int} D = A - a + \operatorname{int} D$, and so $a \in E_p(A)$.

Assume that $l(C) = \{0\}$. We have to prove that $E_\sigma(A + C) \subseteq A$ for $\sigma \in \{s, e, p\}$. Assume that $a + k \in E_\sigma(A + C)$ for some $a \in A$ and $k \in C \setminus \{0\}$. For $\sigma = s$ we get $a + k \leqq a$, whence the contradiction $k \in -C$. For $\sigma = e$ we get the contradiction $a \in (A + C) \cap (a + k - (C \setminus \{0\})$. For $\sigma = p$, we have that $E_p(A) \subseteq E_e(A) \subseteq A$. □

Generally, $E_s(A) = \emptyset$.

Lemma 2.4.17. *If $E_s(A) \neq \emptyset$ then $E_s(A) = E_e(A)$. Moreover, if $C^\& \neq \emptyset$ then $E_s(A) = E_p(A) = E_e(A)$.*

Proof. Because $0 \notin C \setminus l(C) = C + (C \setminus l(C))$ we have that

$$C \cap (-(C \setminus l(C))) = \emptyset.$$

Fix $\overline{a} \in E_s(A)$. Let $a \in E_s(A)$. Then $A \subseteq a + C$, and so

$$A \cap (a - (C \setminus l(C))) \subseteq (a + C) \cap (a - (C \setminus l(C))) = a + [C \cap (-(C \setminus l(C)))] = \emptyset.$$

Hence $a \in E_e(A)$, and so $E_s(A) \subseteq E_e(A)$.

Take now $a \in E_e(A) \subseteq A \subseteq \overline{a} + C$. Hence $a = \overline{a} + k$ for some $k \in C$. If $k \notin l(C)$ then $\overline{a} \in A \cap ((a - (C \setminus l(C)))$, contradicting the fact that $a \in E_e(A)$. Hence $k \in l(C)$, and so $A \subseteq \overline{a} + C = a - k + C \subseteq a + C$, which shows that $a \in E_s(A)$.

Assume, moreover, that $C^{\&} \neq \emptyset$, and take $y^* \in C^{\&}$. It follows that $D := \{y \in Y \mid \langle y, y^* \rangle \geq 0\} \in \mathcal{Q}_C$. Using (2.45) we have that $C \cap (-\operatorname{int} D) = \emptyset$, and so

$$A \cap (\overline{a} - \operatorname{int} D) \subseteq (\overline{a} + C) \cap (\overline{a} - \operatorname{int} D) = \overline{a} + (C \cap (-\operatorname{int} D)) = \emptyset;$$

therefore, $\overline{a} \in E_p A$. Hence $E_s(A) \subseteq E_p(A)$. Since always $E_p(A) \subseteq E_e(A)$ we get the conclusion. □

The corresponding sets of solutions for (VP) are denoted by $E_s(f, S)$, $E_e(f, S)$, $E_w(f, S)$, $E_p(f, S)$, respectively. More precisely,

$$E_\sigma(f, S) := \{\overline{x} \in S \cap \operatorname{dom} f \mid f(\overline{x}) \in E_\sigma(f(S \cap \operatorname{dom} f))\}, \quad \sigma \in \{s, e, w, p\}.$$

2.5 Vector Optimization Problems with Variable Ordering Structure

Yu introduced in [606] nondominated solutions of vector optimization problems with variable ordering structure based on general domination set mappings, compare also Chen, Huang, Yang [91]. Vector optimization problems with variable ordering structure are studied intensively by Eichfelder in [162, 164–166], Eichfelder, Ha [168] where corresponding solution concepts, characterizations by scalarization methods, optimality conditions and numerical procedures are presented. Eichfelder [163, 166] gives a very detailed overview on solution concepts for vector optimization problems with variable ordering structure and presents a complete characterization of these solution concepts (see also Eichfelder, Kasimbeyli [170] and Eichfelder, Gerlach [167]).

Let X and Y be Banach spaces, $\emptyset \neq S \subset X$, $f : X \to Y$ and let $C : X \rightrightarrows Y$ be a set-valued map such that for each $x \in X$, $C(x)$ is a nonempty convex set with $0 \in \operatorname{bd} C(x)$.

We consider the following vector optimization problems with variable ordering structure

$$\text{v-minimize} \quad f(x) \quad \text{subject to } x \in S.$$

The solution concept for this problem is given in the following definition (compare [91, Definition 1.15]):

Definition 2.5.1 (v-Minimal Points, Weakly v-Minimal Points). Let $C : X \rightrightarrows Y$ be a set-valued map with $0 \in \operatorname{bd} C(x)$, $C(x)$ a convex set for all $x \in X$, $S \subset X$ and $f : S \to Y$. An element $\overline{x} \in S$ is said to be a **v-minimal point** of f w.r.t. $C(\cdot)$ if

$$(f(S) - f(\overline{x})) \cap (-(C(\overline{x}) \setminus \{0\})) = \emptyset.$$

The set of all $f(x)$ with x a v-minimal point of f w.r.t. $C(\cdot)$ is denoted by $\operatorname{Min}(f(S), C(\cdot))$.

Suppose that for all $x \in X$, $\operatorname{int} C(x) \neq \emptyset$. An element $\overline{x} \in S$ is said to be a **weakly v-minimal point** of f w.r.t. $C(\cdot)$ if

$$(f(S) - f(\overline{x})) \cap (-\operatorname{int} C(\overline{x})) = \emptyset.$$

The set of all $f(x)$ with x a weakly v-minimal point of f w.r.t. $C(\cdot)$ is denoted by $\operatorname{WMin}(f(S), C(\cdot))$.

Remark 2.5.2. For further solution concepts, especially nondominated elements, of vector optimization problems with variable ordering structure see Eichfelder [166].

In the case that we ask for v-minimal points of f w.r.t. $C(\cdot)$ the vector optimization problem with variable ordering structure is given by

$$\operatorname{Min}(f(S), C(\cdot)). \tag{VP_v}$$

When we are looking for weakly v-minimal points of f w.r.t. $C(\cdot)$ we study the problem

$$\operatorname{WMin}(f(S), C(\cdot)).$$

The following relationships between v-minimal solutions of the vector optimization problem (VP_v) and solutions of suitable scalarized problems are shown by Chen, Huang and Yang [91, Theorem 2.18].

Theorem 2.5.3. *Consider the vector optimization problem with variable ordering structure (VP_v), where $C : X \rightrightarrows Y$ is a set-valued map such that for each $x \in X$, $C(x)$ is a convex subset of Y with $0 \in \operatorname{bd} C(x)$ and $\operatorname{int} C(x) \neq \emptyset$. Then:*

(a) *Let $\overline{x} \in S$. Suppose that there exists $y^* \in Y^*$ with $y^*(y) > 0$ for all $y \in C(\overline{x}) \setminus \{0\}$ such that $\overline{x} \in S$ is a minimal solution of the scalar optimization problem*

$$\min_{x \in S} y^*(f(x)). \tag{P_{y^*}}$$

Then $\overline{x}$ is a v-minimal point of f w.r.t. $C(\cdot)$ concerning the vector optimization problem (VP_v).

(b) *Let $f(S)$ be a convex subset of Y and $\overline{x}$ a v-minimal point of the problem (VP_v). Then there exists $y^* \in Y^*$ satisfying $y^*(y) > 0$ for all $y \in \operatorname{int} C(\overline{x})$, such that $\overline{x}$ is a minimal solution of the scalar optimization problem (P_{y^*}).*

Remark 2.5.4. Characterizations of solutions of general vector optimization problems with variable ordering structure by means of nonlinear scalarizing functionals are given by Eichfelder in [163, 164, 166] and by Eichfelder, Ha [168].

2.6 Solution Concepts in Set-Valued Optimization

Unless otherwise mentioned, let Y be a linear topological space, partially ordered by a proper pointed convex closed cone C. Let $\mathscr{P}(Y) = 2^Y$ be the **power set** of Y.

We consider a set-valued optimization problem with a general geometric constraint:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in S, \tag{SP}$$

where S is a subset of X, X is a linear space and the cost mapping $F : S \rightrightarrows Y$ is a set-valued mapping. As already introduced, we use the notations $F(S) = \cup_{x \in S} F(x)$ and $\operatorname{dom} F = \{x \in S \mid F(x) \neq \emptyset\}$.

In Sects. 2.6.1, 2.6.2 and 2.6.3 we introduce different solution concepts for the problem (*SP*). Furthermore, in Sect. 2.6.4 we present the embedding approach by Kuroiwa [353, 354, 357], in Sect. 2.6.5 we discuss solution concepts with respect to abstract preference relations by Bao and Mordukhovich [28], in Sect. 2.6.6 we introduce solution concepts for set-valued optimization problems with variable ordering structure, in Sect. 2.6.7 we study approximate solutions of set-valued optimization problems and finally, in Sect. 2.7 we discuss relations between the solution concepts.

2.6.1 Solution Concepts Based on Vector Approach

First, we introduce a solution concept where "minimization" in (*SP*) is to be understood with respect to the partial order $\leq_C$ defined in (2.9). In contrast to single-valued functions, for every $\overline{x} \in \operatorname{dom} F$ there are many distinct values $\overline{y} \in Y$ such that $\overline{y} \in F(\overline{x})$. Hence, in the first approach, when studying minimizers of a set-valued mapping, we fix one element $\overline{y} \in F(\overline{x})$, and formulate the following solution concept based on the concept of Pareto minimality introduced in Definition 2.4.1.

Definition 2.6.1 (Minimizer of (SP)). Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \operatorname{graph} F$. The pair $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is called a **minimizer** of the problem (*SP*) if $\bar{y} \in \operatorname{Min}\big(F(S), C\big)$.

Furthermore, also the other notions of weakly / properly minimal points for sets (see Definitions 2.4.2 and 2.4.4) naturally induce corresponding notions of *weak / proper minimizers* to the corresponding set optimization problems (see Ha [228]).

Definition 2.6.2 (Weak Minimizer of (SP)). Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \operatorname{graph} F$. The pair $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is called a **weak minimizer** of the problem (*SP*) if $\bar{y} \in \operatorname{WMin}\big(F(S), C\big)$.

Let $D \subset Y$ be as before (see Definition 2.4.8) a proper cone with nonempty interior and $Q := \operatorname{int} D$.

Definition 2.6.3 (Q-Minimizer of (SP)). Consider the set-valued optimization problem (SP). Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \operatorname{graph} F$. We say that $(\bar{x}, \bar{y})$ is an **S-proper minimizer** (**Hurwicz proper minimizer**, **Hartley proper minimizer**, **Benson proper minimizer**, **Borwein proper minimizer**, **Henig global proper minimizer**, **Henig proper minimizer**, **super minimizer** and **Q-minimizer**, respectively) of (SP) if $\bar{y}$ is an S-properly minimal (Hurwicz properly minimal, Hartley properly minimal, Benson properly minimal, Borwein properly minimal, Henig global properly minimal, Henig properly minimal, super efficient and Q-minimal, respectively) point of $F(S)$, i.e., $\bar{y} \in$ S-PMin$(F(S), C)$ ($\bar{y} \in$ Hu-PMin$(F(S), C)$, $\bar{y} \in$ Ha-PMin$(F(S), C)$, $\bar{y} \in$ Be-PMin$(F(S), C)$, $\bar{y} \in$ Bo-PMin$(F(S), C)$, $\bar{y} \in$ GHe-PMin$(F(S), C)$, $\bar{y} \in$ He-PMin$(F(S), C)$, $\bar{y} \in$ Sup-PMin$(F(S), C)$, $\bar{y} \in$ Q-Min$(F(S), C)$, respectively).

Moreover, especially in Chaps. 8 and 15 we study set-valued optimization problems, where the set-valued objective map $F : X \rightrightarrows Y$ is to be maximized over the feasible set $S \subseteq X$ (X is a linear space)

$$\text{maximize} \quad F(x) \quad \text{subject to} \quad x \in S. \qquad (SP_{max})$$

Analogously to Definitions 2.6.1 and 2.6.2 we now introduce maximizers and weak maximizer of (SP_{max}).

Definition 2.6.4 (Maximizer of (SP_{max})). Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \operatorname{graph} F$. The pair $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is called a **maximizer** of the problem (SP_{max}) if $\bar{y} \in \operatorname{Max}\big(F(S), C\big)$.

Definition 2.6.5 (Weak Maximizer of (SP_{max})). Consider the set-valued optimization problem (SP_{max}). Let $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \operatorname{graph} F$. We say that $(\bar{x}, \bar{y})$ is a **weak maximizer** of (SP_{max}) if $\bar{y}$ is a weakly maximal point of $F(S)$, i.e., $\bar{y} \in \operatorname{WMax}(F(S), C)$.

Furthermore, we consider set-valued optimization problems with a special structure concerning the restrictions, namely inequality restrictions: Let X, Y, Z be real locally convex Hausdorff spaces, Y, Z be ordered by proper pointed closed convex cones C, K, respectively, $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$. Under these assumptions we study a set-valued optimization problem of the following form (see Tasset [570]):

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in S, \tag{SP_T}$$

where $M \subset X$ is a set satisfying $M \subseteq \operatorname{dom} F \cap \operatorname{dom} G$ and

$$S := \{x \in M \mid G(x) \cap (-K) \neq \emptyset\}. \tag{2.49}$$

For set-valued problems (SP_T) with a feasible set S given by (2.49) we derive duality assertions in Sect. 8.1 using the following solution concept with respect to the quasi(-relative) interior of a cone $C \subset Y$.

Let $B \subset Y$ be a nonempty convex set; the **quasi interior of** B is

$$\operatorname{qi} B := \{y \in B \mid \operatorname{cl}(\mathbb{R}_+(B - y)) = Y\}$$

and the **quasi-relative interior of** B is

$$\operatorname{qri} B := \{y \in B \mid \operatorname{cl}(\mathbb{R}_+(B - y)) \text{ is a linear space}\}.$$

Because $\operatorname{cl}(\mathbb{R}_+(B - y)) \subset \operatorname{cl}\operatorname{aff} B - y$, we have that $\operatorname{cl}\operatorname{aff} B = Y$ whenever $\operatorname{qi} B \neq \emptyset$. In fact we have

$$0 \in \operatorname{qi}(B - B) \iff \operatorname{cl}\operatorname{aff} B = Y \Longrightarrow \operatorname{qi} B = \operatorname{qri} B. \tag{2.50}$$

It is worth to observe that for $y_0 \in B$ we have that

$$y_0 \notin \operatorname{qri} B \iff \exists y^* \in Y^* : \inf y^*(B) \geq \langle y_0, y^* \rangle < \sup y^*(B); \tag{2.51}$$

in particular,

$$y_0 \in B \setminus \operatorname{qri} B \Longrightarrow \exists y^* \in Y^* \setminus \{0\} : \inf y^*(B) = \langle y_0, y^* \rangle. \tag{2.52}$$

Note that in the above implications we do not assume that $\operatorname{qri} B \neq \emptyset$. Note that (2.51) covers [72, Theorem 2.7].

Observe also that for $B = C$ a convex cone,

$$y \in \operatorname{qi} C,\ y^* \in C^+ \setminus \{0\} \Longrightarrow \langle y, y^* \rangle > 0. \tag{2.53}$$

Indeed, if $\langle y, y^* \rangle = 0$ then $\langle y' - y, y^* \rangle \geq 0$ for every $y' \in C$, and so $\langle y'', z^* \rangle \geq 0$ for every $y'' \in \operatorname{cl}(\mathbb{R}_+(C - y)) = Y$. We get so the contradiction $y^* = 0$.

Using these notations Tasset [570] introduced the following solution concept for the set-valued problem (SP_T) with restrictions given by (2.49).

Definition 2.6.6 (Quasi-Weak Minimizer of (SP_T)**).** Assume $\operatorname{qi} C \neq \emptyset$ and consider the set-valued optimization problem (SP_T) with restrictions given by (2.49). Let $\overline{x} \in S$ and $(\overline{x}, \overline{y}) \in \operatorname{graph} F$. The pair $(\overline{x}, \overline{y})$ is called a **quasi-weak minimizer** of the problem (SP_T) with restrictions given by (2.49) if $F(S) \cap (\overline{y} - \operatorname{qi} C) = \emptyset$, and we denote this by $\overline{y} \in \operatorname{Min}(F(S), \operatorname{qi} C)$.

2.6.2 Solution Concepts Based on Set Approach

Although the concept of a minimizer of the set-valued problem (*SP*) given in Definitions 2.6.1 and 2.6.3 is of mathematical interest, it cannot be often used in practice. It is important to mention that a minimizer $(\overline{x}, \overline{y})$ depends on only certain special element $\overline{y}$ of $F(\overline{x})$ and other elements of $F(\overline{x})$ are ignored. In other words, an element $\overline{x} \in S$ for that there exists at least one element $\overline{y} \in F(\overline{x})$ which is a Pareto minimal point (Definition 2.4.1) of the image set of F even if there exist many bad elements in $F(\overline{x})$, is a solution of the set-valued optimization problem (*SP*). For this reason, the solution concepts introduced in Sect. 2.6.1 are sometimes improper.

In order to avoid this drawback it is necessary to work with practically relevant order relations for sets. This leads to solution concepts for set-valued optimization problems based on comparisons among values of the set-valued objective map F.

First, we will introduce several order relations that are used in order to formulate corresponding solution concepts for the set-valued problem (*SP*). The **set less order relation** $\preceq^s_C$ is introduced independently by Young [605] and Nishnianidze [443] (cf. Eichfelder, Jahn [169]) for the comparison of sets:

Definition 2.6.7 (Set Less Order Relation).

Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \in \mathscr{P}(Y)$ be arbitrarily chosen nonempty sets. Then the set less order relation is defined by

$$A \preceq^s_C B :\Longleftrightarrow A \subseteq B - C \text{ and } A + C \supseteq B.$$

Remark 2.6.8. Of course, we have

$$A \subseteq B - C \iff \forall a \in A\ \exists b \in B : a \leq_C b$$

and

$$A + C \supseteq B \iff \forall b \in B\ \exists a \in A : a \leq_C b.$$

Kuroiwa [347, 349, 351] introduced the following order relations:

Definition 2.6.9 (Lower (Upper) Set Less Order Relation). Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \in \mathscr{P}(Y)$ be arbitrary nonempty sets. Then the lower set less order relation $\preceq^l_C$ is defined by

$$A \preceq^l_C B :\Longleftrightarrow A + C \supseteq B$$

and the upper set less order relation $\preceq^u_C$ is defined by

$$A \preceq^u_C B :\Longleftrightarrow A \subseteq B - C.$$

The lower set less order relation $\preceq^l_C$ is illustrated in Fig. 2.2 and the upper set less order relation $\preceq^u_C$ in Fig. 2.3.

Remark 2.6.10. There is the following relationship between the lower set less order relation $\preceq^l_C$ and the upper set less order relation $\preceq^u_C$:

$$A \preceq^l_C B :\Longleftrightarrow A+C \supseteq B \Longleftrightarrow B \subseteq A-(-C) \Longleftrightarrow: B \preceq^u_{-C} A \Longleftrightarrow (-B) \preceq^u_C (-A).$$

Remark 2.6.11. It is easy to see that $A \preceq^l_C B$ is equivalent to

$$A + C \supseteq B + C.$$

Furthermore, $A \preceq^u_C B$ is equivalent to

$$A - C \subseteq B - C.$$

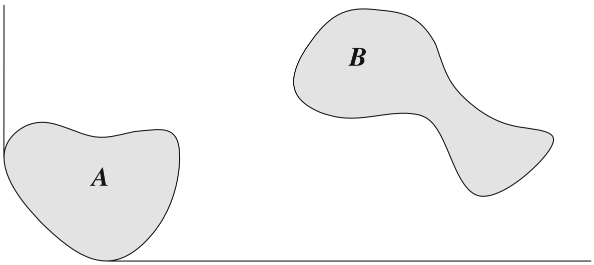

Fig. 2.2 Lower set less order relation $\preccurlyeq^l_C$

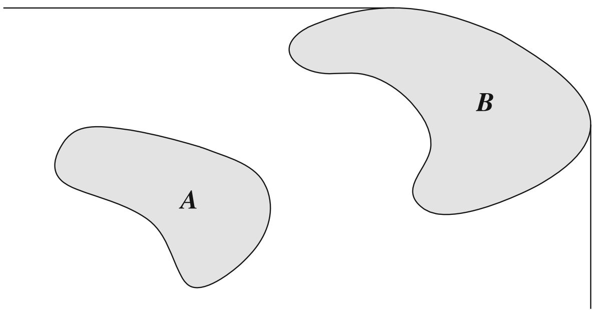

Fig. 2.3 Upper set less order relation $\preccurlyeq^u_C$

It is important to mention that

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \iff A + C = B + C.$$

Under our assumption that C is a pointed closed convex cone it holds $\mathrm{Min}(A + C, C) = \mathrm{Min}(A, C)$ and $\mathrm{Min}(B + C, C) = \mathrm{Min}(B, C)$ such that we get

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \Longrightarrow \mathrm{Min}(A, C) = \mathrm{Min}(B, C).$$

Under the additional assumptions $A \subseteq \mathrm{Min}(A, C) + C$ and $B \subseteq \mathrm{Min}(B, C) + C$ (domination property, see [47, 400]) we have

$$\mathrm{Min}(A, C) = \mathrm{Min}(B, C) \iff A + C = B + C$$

and so

$$A \preceq^l_C B \text{ and } B \preceq^l_C A \iff \mathrm{Min}(A, C) = \mathrm{Min}(B, C).$$

Similarly,

$$A \preceq^u_C B \text{ and } B \preceq^u_C A \iff A - C = B - C$$

and because of $\mathrm{Max}(A - C, C) = \mathrm{Max}(A, C)$ and $\mathrm{Max}(B - C, C) = \mathrm{Max}(B, C)$ it holds

$$A \preceq^u_C B \text{ and } B \preceq^u_C A \Longrightarrow \mathrm{Max}(A, C) = \mathrm{Max}(B, C).$$

Under the additional assumption $A \subseteq \mathrm{Max}(A, C) - C$ and $B \subseteq \mathrm{Max}(B, C) - C$ it holds

$$\mathrm{Max}(A, C) = \mathrm{Max}(B, C) \iff A - C = B - C$$

and so

$$A \preceq^u_C B \text{ and } B \preceq^u_C A \iff \mathrm{Max}(A, C) = \mathrm{Max}(B, C).$$

In interval analysis there are even more order relations in use, like the certainly less order relation $\preceq^c_C$ (Kuroiwa [347–349, 351], compare Eichfelder, Jahn [169]):

Definition 2.6.12 (Certainly Less Order Relation $\preceq^c_C$). For arbitrary nonempty sets $A, B \in \mathscr{P}(Y)$ the certainly less order relation $\preceq^c_C$ is defined by

$$A \preceq^c_C B :\iff (A = B) \text{ or } (A \neq B, \forall a \in A\ \forall b \in B : a \leq_C b).$$

An illustration of the certainly less order relation $\preceq^c_C$ is given in Fig. 2.4.

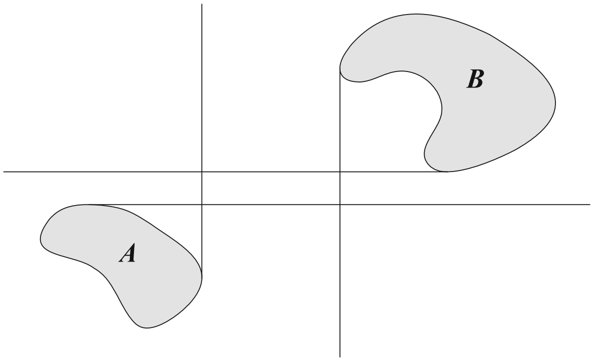

Fig. 2.4 Certainly less order relation $\preccurlyeq^c_C$

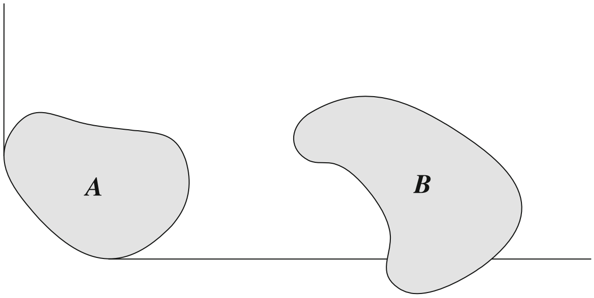

Fig. 2.5 Possibly less order relation $\preccurlyeq^p_C$

Moreover, the possibly less order relation $\preceq^p_C$ (Kuroiwa [348, 349, 351]) is given in the following definition:

Definition 2.6.13 (Possibly Less Order Relation $\preceq^p_C$). For arbitrary nonempty sets $A, B \in \mathscr{P}(Y)$ the possibly less order relation $\preceq^p_C$ is defined by

$$A \preceq^p_C B :\Longleftrightarrow (\exists a \in A, \exists b \in B : a \leq_C b).$$

The possibly less order relation $\preceq^p_C$ is illustrated in Fig. 2.5.

Remark 2.6.14. It is clear that $A \preceq^c_C B$ implies

$$\exists a \in A \quad \text{such that} \quad \forall b \in B : a \leq_C b. \tag{2.54}$$

Moreover, (2.54) implies $A \preceq^l_C B$ (see Definition 2.6.9) such that

$$A \preceq^c_C B \Longrightarrow A \preceq^l_C B.$$

Furthermore, $A \preceq^l_C B$ implies

$$\exists a \in A, \ \exists b \in B \text{ such that } a \leq_C b. \tag{2.55}$$

Taking into account Definition 2.6.13, we have

$$A \preceq_C^c B \quad \Longrightarrow \quad A \preceq_C^l B \quad \Longrightarrow \quad A \preceq_C^p B. \tag{2.56}$$

Remark 2.6.15. The relation $A \preceq_C^c B$ implies

$$\exists b \in B \quad \text{such that} \quad \forall a \in A : a \leq_C b. \tag{2.57}$$

Moreover, (2.57) implies $A \preceq_C^u B$ (see Definition 2.6.9) such that

$$A \preceq_C^c B \Longrightarrow A \preceq_C^u B.$$

Furthermore, $A \preceq_C^u B$ implies

$$\exists a \in A, \ \exists b \in B \text{ such that } a \leq_C b, \tag{2.58}$$

such that we get

$$A \preceq_C^c B \quad \Longrightarrow \quad A \preceq_C^u B \quad \Longrightarrow \quad A \preceq_C^p B$$

taking into account Definition 2.6.13.

Furthermore, the minmax less order relation $\preceq_C^m$ is introduced for sets A, B belonging to

$$\mathscr{F} := \{A \in \mathscr{P}(Y) \mid \operatorname{Min}(A, C) \neq \emptyset \text{ and } \operatorname{Max}(A, C) \neq \emptyset\}.$$

Note that for instance in a topological real linear space Y for every compact set in $\mathscr{P}(Y)$ minimal and maximal elements exist.

Definition 2.6.16 (Minmax Less Order Relation). Let A, B be sets belonging to $\mathscr{F}$. Then the minmax less order relation $\preceq_C^m$ is defined by

$$A \preceq_C^m B :\Longleftrightarrow \operatorname{Min}(A, C) \preceq_C^s \operatorname{Min}(B, C) \text{ and } \operatorname{Max}(A, C) \preceq_C^s \operatorname{Max}(B, C).$$

The minmax certainly less order relation $\preceq_C^{mc}$ is introduced in the next definition:

Definition 2.6.17 (Minmax Certainly Less Order Relation). For arbitrary $A, B \in \mathscr{F}$ the *minmax* certainly less order relation $\preceq_C^{mc}$ is given by

$$\begin{aligned} A \preceq_C^{mc} B :\Longleftrightarrow (A = B) \; or \; (A \neq B, &\operatorname{Min}(A, C) \preceq_C^c \operatorname{Min}(B, C) \\ &\text{and } \operatorname{Max}(A, C) \preceq_C^c \operatorname{Max}(B, C)). \end{aligned}$$

Finally, we introduce the minmax certainly nondominated order relation $\prec_C^{mn}$ (see Jahn, Ha [295]).

Definition 2.6.18 (Minmax Certainly Nondominated Order Relation). For arbitrary nonempty $A, B \in \mathscr{P}(Y)$ the *minmax certainly nondominated order relation* $\preceq_C^{mn}$ is defined by

$$A \preceq_C^{mn} B :\Longleftrightarrow (A = B) \; or \; (A \neq B, \mathrm{Max}(A, C) \preceq_C^s \mathrm{Min}(B, C)).$$

The set less order relation $\preceq_C^s$ and the order relations $\preceq_C^l$, $\preceq_C^u$, $\preceq_C^m$, $\preceq_C^{mc}$ and $\preceq_C^{mn}$ are preorders. If $\preceq_C$ denotes one of these order relations, then we can define optimal solutions with respect to the preorder $\preceq_C$ and the corresponding set-valued optimization problem is given by

$$\preceq_C - \text{minimize} \quad F(x), \quad \text{subject to} \quad x \in S, \qquad (SP - \preceq_C)$$

where we assume again (compare (SP)) that Y is a linear topological space, partially ordered by a proper pointed convex closed cone C, S is a subset of X, X is a linear space, $F : X \rightrightarrows Y$.

Definition 2.6.19 (Minimal Solutions of ($SP - \preceq_C$) w.r.t. the Preorder $\preceq_C$). An element $\overline{x} \in S$ is called a minimal solution of problem ($SP - \preceq_C$) w.r.t. the preorder $\preceq_C$ if

$$F(x) \preceq_C F(\overline{x}) \quad \text{for some} \quad x \in S \Longrightarrow F(\overline{x}) \preceq_C F(x).$$

Remark 2.6.20. When we use the set relation $\preceq_C^l$ introduced in Definition 2.6.9 in the formulation of the solution concept, i.e., when we study the set-valued optimization problem ($SP - \preceq_C^l$), we observe that this solution concept is based on comparisons among sets of minimal points of values of F (see Definition 2.4.1). Furthermore, considering the upper set less order relation $\preceq_C^u$ (Definition 2.6.9), i.e., considering the problem ($SP - \preceq_C^u$) we recognize that this solution concept is based on comparisons of maximal points of values of F (see Definition 2.4.1).

When $\overline{x} \in S$ is a minimal solution of problem ($SP - \preceq_C^l$) there does not exist $x \in S$ such that $F(x)$ is strictly smaller than $F(\overline{x})$ with respect to the set order $\preceq_C^l$.

In the following we give three examples (see Kuroiwa [347]) of set-valued optimization problems in order to illustrate the different solution concepts introduced in Definitions 2.6.1 and 2.6.19.

Example 2.6.21. Consider the set-valued optimization problem

$$\text{minimize} \quad F_1(x), \quad \text{subject to} \quad x \in S,$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $S = [0, 1]$ and $F_1 : S \rightrightarrows Y$ is given by

$$F_1(x) := \begin{cases} [(1, 0), (0, 1)] & \text{if } x = 0 \\ [(1 - x, x), (1, 1)] & \text{if } x \in (0, 1], \end{cases}$$

where $[(a, b), (c, d)]$ is the line segment between (a, b) and (c, d).

Only the element $\overline{x} = 0$ is a minimal solution in the sense of Definition 2.6.19 w.r.t. $\preceq^l_C$. However, all elements $(\overline{x}, \overline{y}) \in \operatorname{graph} F_1$ with $\overline{x} \in [0,1]$, $\overline{y} = (1 - \overline{x}, \overline{x})$ for $\overline{x} \in (0,1]$ and $\overline{y} = (1,0)$ for $\overline{x} = 0$ are minimizers of the set-valued optimization problem in the sense of Definition 2.6.1. This example shows that the solution concept with respect to the set relation $\preceq^l_C$ (see Definition 2.6.19) is more natural and useful than the concept of minimizers introduced in Definition 2.6.1.

Example 2.6.22. Now we discuss the set-valued optimization problem

$$\text{minimize} \quad F_2(x), \quad \text{subject to} \quad x \in S,$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $S = [0,1]$ and $F_2 : S \rightrightarrows Y$ is given by

$$F_2(x) := \begin{cases} \left[(1, \frac{1}{3}), (\frac{1}{3}, 1)\right] & \text{if } x = 0 \\ [(1-x, x), (1,1)] & \text{if } x \in (0,1]. \end{cases}$$

The set of minimal solutions in the sense of Definition 2.6.19 w.r.t. $\preceq^l_C$ is the interval $[0,1]$, but the set of minimizers in the sense of Definition 2.6.1 is given by

$$\{(\overline{x}, \overline{y}) \in \operatorname{graph} F_2 \mid \overline{x} \in (0,1],\ \overline{y} = (1 - \overline{x}, \overline{x})\}.$$

Here we observe that $\overline{x} = 0$ is a $\preceq^l_C$-minimal solution but the set $F_2(\overline{x})$ ($\overline{x} = 0$) has no Pareto minimal points.

Example 2.6.23. In this example we are looking for minimal solutions of a set-valued optimization problem with respect to the set relation $\preceq^u_C$ introduced in Definition 2.6.9.

$$\preceq^u_C -\text{minimize} \quad F_3(x), \quad \text{subject to} \quad x \in S, \qquad (SP - \preceq^u_C)$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $S = [0,1]$ and $F_3 : S \rightrightarrows Y$ is given by

$$F_3(x) := \begin{cases} [[(1,1),(2,2)]] & \text{if } x = 0 \\ [[(0,0),(3,3)]] & \text{if } x \in (0,1], \end{cases}$$

where $[[(a,b),(c,d)]] := \{(y_1, y_2) \mid a \leq y_1 \leq c,\ b \leq y_2 \leq d\}$.

Then a minimal solutions of $(SP - \preceq^u_C)$ in the sense of Definition 2.6.19 is only $\overline{x} = 0$. On the other hand, $x \in (0,1]$ are not minimal solutions of $(SP - \preceq^u_C)$ in the sense of Definition 2.6.19, but for all $\overline{x} \in (0,1]$ there are $\overline{y} \in F_3(\overline{x})$ such that $(\overline{x}, \overline{y})$ are minimizers in the sense of Definition 2.6.1.

Further relationships between different solution concepts in set-valued optimization are discussed in Sect. 2.7.

Applications of solution concepts based on set approach introduced in this section are given in Sect. 15.4 concerning robustness, in Sect. 1.1 concerning game theory. Furthermore, in Sect. 8.2 we present duality assertions for the primal problem $(SP - \preceq^l_C)$.

2.6.3 Solution Concepts Based on Lattice Structure

We recall in this section the concept of an **infimal set** (resp. **supremal set**), which is due to Nieuwenhuis [442], was extended by Tanino [563], and slightly modified with respect to the elements $\pm\infty$ by Löhne and Tammer [397]. We will shortly discuss the role of the space of self-infimal sets, which was shown in [397] to be a complete lattice. As we will see in Sect. 15.1, this complete lattice is useful for applications of set-valued approaches in the theory of vector optimization, especially in duality theory.

First, we recall the definitions of lower and upper bounds as well as the infimum and supremum of a subset of a partially ordered set. Consider a partially ordered set $(Y, \leq)$ and $A \subseteq Y$. As already introduced in Definition 2.1.6, an element $l \in Y$ is called a **lower bound** of A if $l \leq y$ for all $y \in A$. Furthermore, an element $u \in Y$ is called an **upper bound** of A if $u \geq y$ for all $y \in A$. Using lower and upper bounds, the infimum and supremum for a subset A of a partially ordered set $(Y, \leq)$ is defined in Definition 2.1.6. An element $\bar{l} \in Y$ is called greatest lower bound or **infimum** of $A \subseteq Y$ if $\bar{l}$ is a lower bound of A and for every other lower bound l of A it holds $l \leq \bar{l}$. If the infimum of A exists we use the notation $\bar{l} = \inf A$ for it. Analogously, we define the least upper bound or **supremum** of $A \subseteq Y$ and denote it by $\sup A$.

Based on the definition of the infimum and supremum we introduce the notion of a complete lattice that is important for the approach in this section.

Definition 2.6.24. A partially ordered set $(Y, \leq)$ is called a **complete lattice** if the infimum and supremum exist for every subset $A \subseteq Y$.

A characterization of a complete lattice based on the existence of the infimum of subsets $A \subseteq Y$ is given by Löhne [395, Proposition 1.6].

Proposition 2.6.25. *A partially ordered set* $(Y, \leq)$ *is a complete lattice if and only if the infimum exists for every subset* $A \subseteq Y$.

Let us give some examples for complete lattices (compare [395]).

Example 2.6.26. It is well known that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ equipped with the natural order relation $\leq$ provide a complete lattice.

Example 2.6.27. Consider a nonempty set Y and let $\mathscr{P}(Y) = 2^Y$ be the power set of Y. $(\mathscr{P}(Y), \supseteq)$ is a complete lattice. The infimum and supremum of a nonempty subset $\mathscr{A} \subseteq \mathscr{P}(Y)$ are described by

$$\inf \mathscr{A} = \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A. \tag{2.59}$$

If $\mathscr{A}$ is empty, we put $\sup \mathscr{A} = Y$ and $\inf \mathscr{A} = \emptyset$. $Y \in \mathscr{P}(Y)$ is the least element and $\emptyset \in \mathscr{P}(Y)$ is the greatest element in $(\mathscr{P}(Y), \supseteq)$.

Example 2.6.28. Consider a linear space Y and let $\mathscr{C}(Y)$ be the family of all convex subsets of Y. $(\mathscr{C}(Y), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathscr{A} \subseteq \mathscr{C}(Y)$ is described by

$$\inf \mathscr{A} = \operatorname{conv} \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A. \tag{2.60}$$

If $\mathscr{A}$ is empty, we put again $\sup \mathscr{A} = Y$ and $\inf \mathscr{A} = \emptyset$.

Example 2.6.29. Consider a topological space Y and let $\mathscr{F}(Y)$ be the family of all closed subsets of Y. $(\mathscr{F}(Y), \supseteq)$ provides a complete lattice. By

$$\inf \mathscr{A} = \operatorname{cl} \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A \tag{2.61}$$

the infimum and supremum of a nonempty subset $\mathscr{A} \subseteq \mathscr{F}(Y)$ are given.

If $\mathscr{A}$ is empty, we put again $\sup \mathscr{A} = Y$ and $\inf \mathscr{A} = \emptyset$.

Results concerning the infimum and supremum in the space of upper closed sets are given in Proposition 2.6.40.

In the sequel, in this section we assume that $(Y, \leq)$ is a partially ordered linear topological space, where the order is induced by a proper pointed convex cone C satisfying $\emptyset \neq \operatorname{int} C \neq Y$. Here we do not assume that C is closed. However, in Sect. 15.1 we will give a reformulation of a vector optimization problem as $\mathscr{I}$-valued problem, where the closedness of C is important (compare Proposition 2.17 in [395] and Sect. 2.7). We write $y \leq y'$ iff $y' - y \in C$ and $y < y'$ iff $y' - y \in \operatorname{int} C$. We denote by $Y^\bullet := Y \cup \{-\infty\} \cup \{+\infty\}$ the **extended space**, where the ordering is extended by the convention

$$\forall y \in Y: \ -\infty \leq y \leq +\infty.$$

The linear operations on $Y^\bullet$ are extended by the following calculus rules in analogy to that ones stated for the extended real space $\overline{\mathbb{R}}$:

$$\begin{array}{ll}
0 \cdot (+\infty) = 0, & 0 \cdot (-\infty) = 0, \\
\forall \alpha > 0 & : \alpha \cdot (+\infty) = +\infty, \\
\forall \alpha > 0 & : \alpha \cdot (-\infty) = -\infty, \\
\forall y \in Y^\bullet & : y + (+\infty) = +\infty + y = +\infty, \\
\forall y \in Y \cup \{-\infty\} & : y + (-\infty) = -\infty + y = -\infty.
\end{array}$$

The extended space $Y^\bullet$ is not a linear space.

In the following definition we introduce the upper closure of $A \subseteq Y^\bullet$ (see [395, 397]) that is important for the formulation of the solution concept.

Definition 2.6.30. The **upper closure** (with respect to C) of $A \subseteq Y^\bullet$ is defined to be the set

$$\mathrm{Cl}_+ A := \begin{cases} Y & \text{if} \quad -\infty \in A \\ \emptyset & \text{if} \quad A = \{+\infty\} \\ \{y \in Y \mid \{y\} + \mathrm{int}\, C \subseteq A \setminus \{+\infty\} + \mathrm{int}\, C\} & \text{otherwise.} \end{cases}$$

We have [395, Proposition 1.40]

$$\mathrm{Cl}_+ A := \begin{cases} Y & \text{if} \quad -\infty \in A \\ \emptyset & \text{if} \quad A = \{+\infty\} \\ \mathrm{cl}\,((A \setminus \{+\infty\}) + C) & \text{otherwise.} \end{cases} \tag{2.62}$$

As introduced in Definition 2.4.2, the set of *weakly minimal points* of a subset $A \subseteq Y$ (with respect to C) is defined by

$$\mathrm{WMin}(A, C) = \{y \in A \mid A \cap (\{y\} - \mathrm{int}\, C) = \emptyset\},$$

and the set of *weakly maximal points* of A is defined by

$$\mathrm{WMax}(A, C) = \{y \in A \mid A \cap (\{y\} + \mathrm{int}\, C) = \emptyset\}.$$

If $A \neq \emptyset$, $A \subseteq Y$ we have [442, Theorem I-18]

$$\mathrm{WMin}(\mathrm{Cl}_+ A, C) = \emptyset \iff A + \mathrm{int}\, C = Y \iff \mathrm{Cl}_+ A = Y.$$

In order to formulate set-valued optimization problems where the solution concept is based on the lattice structure we introduce the notion of an infimal set for a subset of $Y^\bullet$ (see Nieuwenhuis [442], Tanino [563, 566] and Löhne, Tammer [397]).

Definition 2.6.31 (Infimal Set). The **infimal set** of $A \subseteq Y^\bullet$ (with respect to C) is defined by

$$\mathrm{Inf}\, A := \begin{cases} \mathrm{WMin}(\mathrm{Cl}_+ A, C) & \text{if } \emptyset \neq \mathrm{Cl}_+ A \neq Y \\ \{-\infty\} & \text{if } \mathrm{Cl}_+ A = Y \\ \{+\infty\} & \text{if } \mathrm{Cl}_+ A = \emptyset. \end{cases}$$

Remark 2.6.32. We see that the infimal set of a nonempty set $A \subset Y$ (with respect to C) coincides with the set of weakly minimal elements of the set $\mathrm{cl}\,(A + C)$ with respect to C ($\mathrm{WMin}(\mathrm{cl}\,(A + C), C)$), if $\mathrm{cl}\,(A + C) \neq Y$. Note that if $A \subset Y$ then $\mathrm{WMin}(A, C) = A \cap \mathrm{Inf}\, A$.

By our conventions, $\mathrm{Inf}\, A$ is always a nonempty set. Clearly, if $-\infty$ belongs to A, we have $\mathrm{Inf}\, A = \{-\infty\}$, in particular, $\mathrm{Inf}\{-\infty\} = \{-\infty\}$. Furthermore, it holds $\mathrm{Inf}\, \emptyset = \mathrm{Inf}\{+\infty\} = \{+\infty\}$ and $\mathrm{Cl}_+ A = \mathrm{Cl}_+(A \cup \{+\infty\})$ and hence $\mathrm{Inf}\, A = \mathrm{Inf}(A \cup \{+\infty\})$ for all $A \subseteq Y^\bullet$.

Considering the set of weakly maximal points $\mathrm{WMax}(A, C)$ of a set $A \subset Y$ with respect to C (see Definition 2.4.2) we define analogously the **lower closure** $\mathrm{Cl}_- A$ and the **supremal set** $\operatorname{Sup} A$ of a set $A \subseteq Y^\bullet$. It holds

$$\operatorname{Sup} A = -\operatorname{Inf}(-A).$$

The following assertions were proved by Nieuwenhuis [442] and, in an extended form, by Tanino [563].

Proposition 2.6.33. *For $A, B \subseteq Y$ with $\emptyset \neq \mathrm{Cl}_+ A \neq Y$ and $\emptyset \neq \mathrm{Cl}_+ B \neq Y$ it holds*

(i) $\operatorname{Inf} A = \{y \in Y \mid y \notin A + \operatorname{int} C,\ \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C\}$,
(ii) $A + \operatorname{int} C = B + \operatorname{int} C \iff \operatorname{Inf} A = \operatorname{Inf} B$,
(iii) $A + \operatorname{int} C = \operatorname{Inf} A + \operatorname{int} C$,
(iv) $\mathrm{Cl}_+ A = \operatorname{Inf} A \cup (\operatorname{Inf} A + \operatorname{int} C)$,
(v) $\operatorname{Inf} A$, $(\operatorname{Inf} A - \operatorname{int} C)$ *and* $(\operatorname{Inf} A + \operatorname{int} C)$ *are disjoint,*
(vi) $\operatorname{Inf} A \cup (\operatorname{Inf} A - \operatorname{int} C) \cup (\operatorname{Inf} A + \operatorname{int} C) = Y$.

Proposition 2.6.34. *For $A \subseteq Y^\bullet$ it holds*

(i) $\operatorname{Inf}\operatorname{Inf} A = \operatorname{Inf} A$, $\mathrm{Cl}_+\mathrm{Cl}_+ A = \mathrm{Cl}_+ A$, $\operatorname{Inf}\mathrm{Cl}_+ A = \operatorname{Inf} A$, $\mathrm{Cl}_+ \operatorname{Inf} A = \mathrm{Cl}_+ A$,
(ii) $\operatorname{Inf}(\operatorname{Inf} A + \operatorname{Inf} B) = \operatorname{Inf}(A + B)$,
(iii) $\alpha \operatorname{Inf} A = \operatorname{Inf}(\alpha A)$ *for* $\alpha > 0$.

Proposition 2.6.35. *Let $A_i \subset Y^\bullet$ for $i \in I$, where I is an arbitrary index set. Then it holds*

(i) $\mathrm{Cl}_+ \bigcup_{i\in I} A_i = \mathrm{Cl}_+ \bigcup_{i\in I} \mathrm{Cl}_+ A_i$,
(ii) $\operatorname{Inf} \bigcup_{i\in I} A_i = \operatorname{Inf} \bigcup_{i\in I} \operatorname{Inf} A_i$.

Proof. (i) As $\mathrm{Cl}_+ \{+\infty\} = \emptyset$ and $\mathrm{Cl}_+ A = \mathrm{Cl}_+(A \setminus \{+\infty\})$ we can assume that $+\infty \notin \bigcup_{i\in I} A_i$. We also assume $\{-\infty\} \notin \bigcup_{i\in I} A_i$, because the statement is otherwise obvious.

So we have

$$\mathrm{Cl}_+ \bigcup_{i\in I} A_i = \mathrm{cl}\left(\bigcup_{i\in I} A_i + C\right) = \mathrm{cl}\left(\bigcup_{i\in I} \mathrm{cl}(A_i + C)\right) + C = \mathrm{Cl}_+ \bigcup_{i\in I} \mathrm{Cl}_+ A_i.$$

(ii) Follows from (i) and Proposition 2.6.33 (iv). $\square$

In order to formulate set-valued optimization problems in an appropriate form for deriving duality assertions (see Sects. 8.3 and 15.1) we introduce in the following definitions the hyperspaces of upper closed sets and self-infimal sets.

Using the upper closure of $A \subseteq Y$ (Definition 2.6.30) we introduce the hyperspace of upper closed sets (compare Löhne, Tammer [397]).

Definition 2.6.36 (Hyperspace of Upper Closed Sets).
The family $\mathscr{F} := \mathscr{F}_C(Y)$ of all sets $A \subseteq Y$ with

$$\mathrm{Cl}_+ A = A$$

is called the **hyperspace of upper closed sets**.

In $\mathscr{F}$ we introduce an addition $\oplus_{\mathscr{F}} : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$, a multiplication by non-negative real numbers $\odot_{\mathscr{F}} : \mathbb{R}_+ \times \mathscr{F} \to \mathscr{F}$ and an order relation $\preccurlyeq_{\mathscr{F}}$ as follows:

$$A \oplus_{\mathscr{F}} B := \mathrm{cl}\,(A + B),$$

$$\alpha \odot_{\mathscr{F}} A := \mathrm{Cl}_+(\alpha \cdot A)$$

$$A \preccurlyeq_{\mathscr{F}} B :\iff A \supseteq B.$$

We use the rule $0 \cdot \emptyset = \{0\}$. This implies $0 \odot_{\mathscr{F}} \emptyset = \mathrm{Cl}_+\{0\} = \mathrm{cl}\,C$.

Furthermore, using the infimal set of $A \subseteq Y^\bullet$ (Definition 2.6.31) we introduce the hyperspace of self-infimal sets (compare [397]).

Definition 2.6.37 (Hyperspace of Self-Infimal Sets). The family $\mathscr{I} := \mathscr{I}_C(Y^\bullet)$ of all self-infimal subsets of $Y^\bullet$, i.e., all sets $A \subseteq Y^\bullet$ satisfying

$$\mathrm{Inf}\, A = A$$

is called **hyperspace of self-infimal sets**.

In $\mathscr{I}$ we introduce an addition $\oplus_{\mathscr{I}} : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$, a multiplication by non-negative real numbers $\odot_{\mathscr{I}} : \mathbb{R}_+ \times \mathscr{I} \to \mathscr{I}$ and an order relation $\preccurlyeq_{\mathscr{I}}$ as follows:

$$A \oplus_{\mathscr{I}} B := \mathrm{Inf}(A + B),$$

$$\alpha \odot_{\mathscr{I}} A := \mathrm{Inf}(\alpha \cdot A)$$

$$A \preccurlyeq_{\mathscr{I}} B :\iff \mathrm{Cl}_+ A \supseteq \mathrm{Cl}_+ B.$$

Note that the definition of $\oplus_{\mathscr{I}}$ is based on the inf-addition in $Y^\bullet$. As a consequence we obtain $\{-\infty\} \oplus_{\mathscr{I}} \{+\infty\} = \{+\infty\}$. Of course, for all $A \in \mathscr{I}$ we get $0 \odot_{\mathscr{I}} A = \mathrm{Inf}\{0\} = \mathrm{bd}\,C$. In the space of self-supremal sets the sup-addition in $Y^\bullet$ is the underlying operation (Fig. 2.6).

Lemma 2.6.38. *For $A, B \in \mathscr{I}$ with $\emptyset \neq \mathrm{Cl}_+ A \neq Y$ and $\emptyset \neq \mathrm{Cl}_+ B \neq Y$ we have*

$$A \preccurlyeq_{\mathscr{I}} B \quad \iff \quad A \cap (B + \mathrm{int}\, C) = \emptyset.$$

Proof. If $\mathrm{Cl}_+ A = \emptyset$ or $\mathrm{Cl}_+ B = \emptyset$ the proof is immediate, such that we assume that $\mathrm{Cl}_+ A \neq \emptyset$ and $\mathrm{Cl}_+ B \neq \emptyset$.

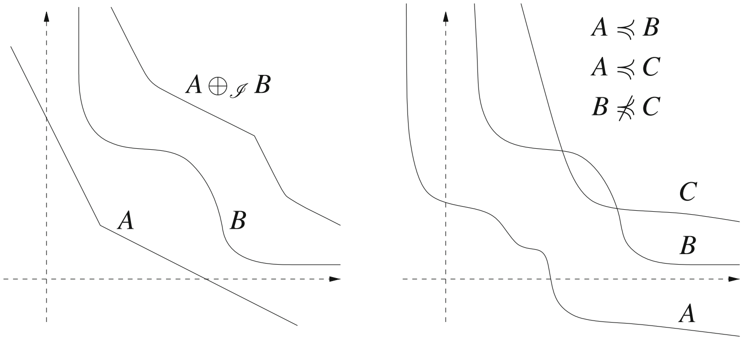

Fig. 2.6 The addition and the ordering in $\mathscr{I}$ for $C = \mathbb{R}^2_+$

Suppose that $A \preccurlyeq_{\mathscr{I}} B$. Taking into account the definition of $\preccurlyeq_{\mathscr{I}}$ it holds $\mathrm{Cl}_+ B \subseteq \mathrm{Cl}_+ A$. By Proposition 2.6.33 (iv) we have $\mathrm{Cl}_+ B = B \cup (B + \operatorname{int} C)$. This yields $(B \cup (B + \operatorname{int} C)) \cap (A - \operatorname{int} C) = \emptyset$ because of $\mathrm{Cl}_+ B \subseteq \mathrm{Cl}_+ A$. Therefore $A \cap (B + \operatorname{int} C) = \emptyset$.

Conversely, if $A \cap (B + \operatorname{int} C) = \emptyset$ then $B + \operatorname{int} C \subset A + \operatorname{int} C$. Hence $\mathrm{Cl}_+ B \subset \mathrm{Cl}_+ A$ and so $A \preccurlyeq_{\mathscr{I}} B$. □

Proposition 2.6.39. *The spaces* $(\mathscr{F}, \oplus_{\mathscr{F}}, \odot_{\mathscr{F}}, \supset)$ *and* $(\mathscr{I}, \oplus_{\mathscr{I}}, \odot_{\mathscr{I}}, \preccurlyeq_{\mathscr{I}})$ *are isomorphic and isotone. The corresponding bijection is given by*

$$j : \mathscr{F} \to \mathscr{I}, \quad j(\cdot) = \mathrm{Inf}(\cdot), \quad j^{-1}(\cdot) = \mathrm{Cl}_+(\cdot).$$

Proof. By Proposition 2.6.34 (i), j is a bijection between $\mathscr{F}$ and $\mathscr{I}$.

For $A_1, A_2 \in \mathscr{F}$, we have $j(A_1) \oplus_{\mathscr{I}} j(A_2) = j(A_1 \oplus A_2)$. This follows from Proposition 2.6.34 (ii).

Similarly, we can easily verify that for $\alpha \geq 0$ and $A, B \in \mathscr{F}$ we have

$$\alpha \odot_{\mathscr{I}} j(A) = j(\alpha \odot_{\mathscr{F}} A) \quad \text{and} \quad A \supseteq B \iff j(A) \preccurlyeq j(B).$$ □

Proposition 2.6.40. $(\mathscr{F}, \supseteq)$ *and* $(\mathscr{I}, \preccurlyeq)$ *are complete lattices. For nonempty subsets* $\mathscr{A} \subseteq \mathscr{F}$ *and* $\mathscr{B} \subseteq \mathscr{I}$ *the infimum and supremum can be expressed by*

$$\inf \mathscr{A} = \mathrm{cl} \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A,$$

$$\inf \mathscr{B} = \mathrm{Inf} \bigcup_{B \in \mathscr{B}} \mathrm{Cl}_+ B, \qquad \sup \mathscr{B} = \mathrm{Inf} \bigcap_{B \in \mathscr{B}} \mathrm{Cl}_+ B.$$

Proof. For the space $(\mathscr{F}, \supseteq)$ the statements are obvious and for $(\mathscr{I}, \preccurlyeq)$ they follow from Proposition 2.6.39. □

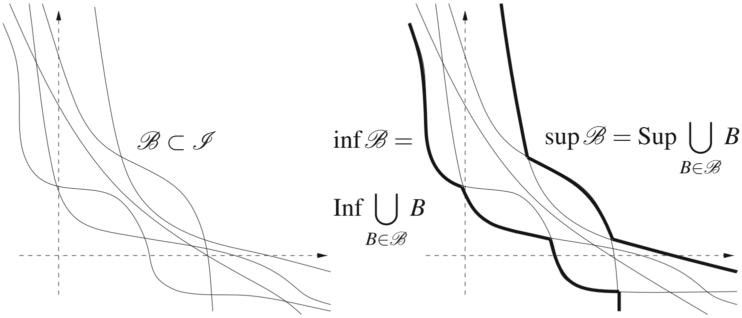

Fig. 2.7 The infimum and supremum in $\mathscr{I}$ for $C = \mathbb{R}^2_+$

As usual, if $\mathscr{A} \subset \mathscr{F}$ and $\mathscr{B} \subset \mathscr{I}$ are empty we define the infimum (supremum) to be the largest (smallest) element in the corresponding complete lattice, i.e., $\inf \mathscr{A} = \emptyset$, $\sup \mathscr{A} = Y$, $\inf \mathscr{B} = \{+\infty\}$ and $\sup \mathscr{B} = \{-\infty\}$.

It follows the main result of this section, which shows that the infimum as well as the supremum in $\mathscr{I}$ can be expressed in terms that frequently are used in vector optimization (compare [442], [145, 563, 566]), but up to now not in the framework of complete lattices (see Fig. 2.7).

Theorem 2.6.41. *For nonempty sets $\mathscr{B} \subset \mathscr{I}$ it holds*

$$\inf \mathscr{B} = \operatorname{Inf} \bigcup_{B\in\mathscr{B}} B, \qquad \sup \mathscr{B} = \operatorname{Sup} \bigcup_{B\in\mathscr{B}} B.$$

Proof. (i) It holds $\inf \mathscr{B} = \operatorname{Inf} \bigcup_{B\in\mathscr{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B\in\mathscr{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B\in\mathscr{B}} B = \operatorname{Inf} \bigcup_{B\in\mathscr{B}} B$.

(ii) We have to show that

$$\operatorname{Sup} \bigcup_{B\in\mathscr{B}} B = \operatorname{Inf} \bigcap_{B\in\mathscr{B}} \operatorname{Cl}_+ B.$$

Then the assertion follows with Proposition 2.6.40.

a) If $\{+\infty\} \in \mathscr{B}$ we have $+\infty \in \bigcup_{B\in\mathscr{B}} B$ and hence $\operatorname{Sup} \bigcup_{B\in\mathscr{B}} B = \{+\infty\}$. On the other hand, since $\operatorname{Cl}_+ \{+\infty\} = \emptyset$, we have $\operatorname{Inf} \bigcap_{B\in\mathscr{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \emptyset = \{+\infty\}$.
b) Let $\{+\infty\} \notin \mathscr{B}$ but $\{-\infty\} \in \mathscr{B}$. If $\{-\infty\}$ is the only element in $\mathscr{B}$ the assertion is obvious, otherwise we can omit this element without changing the expressions.
c) Let $\{+\infty\} \notin \mathscr{B}$ and $\{-\infty\} \notin \mathscr{B}$. Then, $B \subseteq Y$ and $\emptyset \neq \operatorname{Cl}_+ B \neq Y$ for all $B \in \mathscr{B}$, i.e., we can use the statements of Proposition 2.6.33. Define the sets

$$V := \bigcup_{B\in\mathscr{B}} (B - \operatorname{int} C) = \left(\bigcup_{B\in\mathscr{B}} B \right) - \operatorname{int} C$$

and

$$W := \bigcap_{B\in\mathscr{B}} \mathrm{Cl}_+ B.$$

We show that $V \cap W = \emptyset$ and $V \cup W = Y$. Assume there exists some $y \in V \cap W$. Hence there is some $\bar{B} \in \mathscr{B}$ such that $y \in \big(\bar{B} - \operatorname{int} C\big) \cap \mathrm{Cl}_+ \bar{B} = \emptyset$, a contradiction. Let $y \in Y \setminus W$ (we have $W \neq Y$, because otherwise it holds $\mathrm{Cl}_+ B = Y$ for all $B \in \mathscr{B}$ and hence $\{-\infty\} \in \mathscr{B}$). Then there exists some $\bar{B} \in \mathscr{B}$ such that $y \notin \mathrm{Cl}_+ \bar{B}$. By Proposition 2.6.33 (iv), (vi) we obtain $y \in \bar{B} - \operatorname{int} C \subset V$.

If $\mathrm{Cl}_- V = Y$ we have $W = \emptyset$, hence $\operatorname{Sup} \bigcup_{B\in\mathscr{B}} B = \operatorname{Sup} V = \{+\infty\} = \operatorname{Inf} \emptyset = \operatorname{Inf} W$. Otherwise, we have $\emptyset \neq \mathrm{Cl}_- V \neq Y$ and $\emptyset \neq \mathrm{Cl}_+ W \neq Y$. By Proposition 2.6.33, we obtain

$$\begin{aligned}
\operatorname{Sup} \bigcup_{B\in\mathscr{B}} B &= \{y \in Y \mid y \notin V,\ \{y\} - \operatorname{int} C \subset V\} \\
&= \{y \in Y \mid y \in W, (\{y\} - \operatorname{int} C) \cap W = \emptyset\} \\
&= \operatorname{WMin}(W, C) = \operatorname{WMin}(\mathrm{Cl}_+ W, C) = \operatorname{Inf} W
\end{aligned}$$

and so the proof is completed. □

We will see in Sect. 15.1 that the infimum/supremum in $\mathscr{I}$ is closely related to the solution concepts of vector optimization because the infimal/supremal set is closely related to the set of weakly minimal/maximal elements (see also Sect. 2.7).

We next show some calculus rules in the hyperspace $\mathscr{I}$ of self-infimal sets given by Löhne, Tammer [398] and Löhne [395, Proposition 1.56].

Proposition 2.6.42. *For subsets $\mathscr{A}, \mathscr{B} \subseteq \mathscr{I}$ we have*

(i) $\inf \mathscr{A} \oplus_{\mathscr{I}} \mathscr{B} = \inf \mathscr{A} \oplus_{\mathscr{I}} \inf \mathscr{B}$
(ii) $\sup \mathscr{A} \oplus_{\mathscr{I}} \mathscr{B} \preccurlyeq \sup \mathscr{A} \oplus_{\mathscr{I}} \sup \mathscr{B}$.

Proof. (i) If $\mathscr{A} = \emptyset$, it holds $\inf \mathscr{A} \oplus_{\mathscr{I}} \mathscr{B} = \inf \mathscr{A} = \{+\infty\}$ and so

$$\inf \mathscr{A} \oplus_{\mathscr{I}} \mathscr{B} = \inf \mathscr{A} \oplus_{\mathscr{I}} \inf \mathscr{B} = \{+\infty\}.$$

Otherwise, we have

$$\begin{aligned}
\inf \mathscr{A} \oplus_{\mathscr{I}} \mathscr{B} &= \operatorname{Inf} \bigcup_{A\in\mathscr{A}, B\in\mathscr{B}} A \oplus_{\mathscr{I}} B = \operatorname{Inf} \bigcup_{A\in\mathscr{A}, B\in\mathscr{B}} A + B \\
&= \operatorname{Inf}\Big(\bigcup_{A\in\mathscr{A}} A + \bigcup_{B\in\mathscr{B}} B\Big) = \operatorname{Inf} \bigcup_{A\in\mathscr{A}} A \oplus_{\mathscr{I}} \operatorname{Inf} \bigcup_{B\in\mathscr{B}} B \\
&= \inf \mathscr{A} \oplus_{\mathscr{I}} \inf \mathscr{B}.
\end{aligned} \tag{2.63}$$

(ii) For all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have $A \oplus_{\mathscr{I}} B \preccurlyeq \sup \mathscr{A} \oplus_{\mathscr{I}} \sup \mathscr{B}$. As $\mathscr{I}$ is a complete lattice we can take the supremum on the left-hand side of the inequality. This yields (ii).

□

In the same manner like $\mathscr{F}$ and $\mathscr{I}$ we define the space $\mathscr{F}^{\diamond}$ of lower closed subsets of Y and the space $\mathscr{S}$ of self-supremal subsets of $Y^{\bullet}$, where we underlie the sup-addition in $Y^{\bullet}$ in the latter case.

Using the *infimal set* given by Definition 2.6.31 we formulate set-valued optimization problems based on lattice structure:

For an arbitrary set S, consider the following set-valued optimization problem

$$\operatorname{Inf} \bigcup_{x \in S} F(x) \qquad (SP_Y)$$

with a set-valued objective map $F : S \rightrightarrows Y^{\bullet}$. We denote the **domain** of $F : S \rightrightarrows Y^{\bullet}$ by $\operatorname{Dom} F := \{x \in S \mid F(x) \neq \emptyset \text{ and } F(x) \neq \{+\infty\}\}$.

Next, we introduce a solution concept for set-valued optimization problems in the hyperspace of self-infimal sets $(\mathscr{I}, \oplus_{\mathscr{I}}, \odot_{\mathscr{I}, \preccurlyeq_{\mathscr{I}}})$ (see Definition 2.6.37 and Definition 2.6.31 for infimal sets).

Definition 2.6.43 (Infimal Set of (SP_Y)**).** The set $\bar{P} := \operatorname{Inf} \bigcup_{x \in S} F(x)$ is the solution set of the set-valued optimization problem (SP_Y).

Since $\operatorname{Inf} \bigcup_{x \in S} F(x) = \operatorname{Inf} \bigcup_{x \in S} \operatorname{Inf} F(x)$, (SP_Y) can be expressed as an $\mathscr{I}$-valued problem; without loss of generality we can assume that the sets $F(x)$ are self-infimal, i.e., $F : S \to \mathscr{I}$. Thus we consider the following problem.

$$\bar{P} = \operatorname{Inf} \bigcup_{x \in S} F(x) = \inf_{x \in S} F(x). \qquad (SP- \preceq_{\mathscr{I}})$$

Furthermore, in Sect. 8.3.1 we derive duality assertions for $\mathscr{F}$-valued optimization problems. We consider the space $(\mathscr{F}, \supseteq)$ (see Definition 2.6.36), where the order relation is given by

$$A \preccurlyeq_{\mathscr{F}} B : \Longleftrightarrow \; A \supseteq B$$

for subsets A, B of $\mathscr{F}$.

In order to formulate the $\mathscr{F}$-valued optimization problem we study an objective map $F : X \to \mathscr{F}$, i.e., the objective function values of F are subsets of the hyperspace of upper closed sets introduced in Definition 2.6.36. Using these notations we study the $\mathscr{F}$-valued problem

$$\bar{P} := \inf_{x \in X} P(x) = \operatorname{cl} \bigcup_{x \in X} F(x). \qquad (SP- \preceq_{\mathscr{F}})$$

Remark 2.6.44. In Sect. 8.3 we will show duality assertions for $\mathscr{F}$-valued ($\mathscr{I}$-valued, respectively) primal problems and corresponding dual problems based on conjugation as well as Lagrangian technique. Furthermore, in Sect. 15.1 we will use the lattice approach for deriving duality assertions for vector optimization problems.

Remark 2.6.45. In the special case of single-valued functions $F = f : S \longrightarrow Y$ the problems ($SP\!-\preceq_{\mathscr{I}}$) and ($SP\!-\preceq_{\mathscr{F}}$) coincide.

In Sect. 14.2 we will present an algorithm for solving set-valued optimization problems where the objective map has a polyhedral convex graph. There we will need the following notions and assertions concerning the infimum and supremum in a subspace of the hyperspace of upper closed sets $(\mathscr{F}, \oplus_{\mathscr{F}}, \odot_{\mathscr{F}}, \supseteq)$ (see Definition 2.6.36), see Löhne [395].

Definition 2.6.46. The **subspace of all closed convex subsets** of an extended partially ordered linear topological space $Y^\bullet$ is given by

$$\mathscr{F}_{\text{conv}} := \{A \subset \mathscr{F} \mid \forall \lambda \in (0,1)\ :\ \lambda \odot A\ \oplus\ (1-\lambda) \odot A = A\}.$$

The space $\mathscr{F}_{\text{conv}}$ can be characterized using the convex hull (compare Löhne [395, Proposition 1.59]).

Proposition 2.6.47. *Assume that Y is a linear topological space ordered by a proper pointed convex cone $C \subset Y$ with* $\operatorname{int} C \neq \emptyset$ *and let $\mathscr{F} = \mathscr{F}_C(Y)$. Then*

$$\mathscr{F}_{\text{conv}} = \{A \subseteq Y \mid \operatorname{Cl}_+ \operatorname{conv} A = A\}.$$

Proof. Taking into account

$$A = \operatorname{Cl}_+ \operatorname{conv} A \iff A = \operatorname{conv} A \wedge A = \operatorname{Cl}_+ A \iff$$

$$\forall \lambda \in [0,1] : A = \operatorname{Cl}_+(\lambda A + (1-\lambda)A) \iff$$
$$\forall \lambda \in [0,1] : A = \lambda \odot A\ \oplus\ (1-\lambda) \odot A$$
□

Proposition 2.6.48. $(\mathscr{F}_{\text{conv}}, \supseteq)$ *is a complete lattice. For nonempty subsets $\mathscr{A} \subseteq \mathscr{F}_{\text{conv}}$ the infimum and supremum can be expressed by*

$$\inf \mathscr{A} = \operatorname{cl} \operatorname{conv} \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A.$$

Proof. For all $A \in \mathscr{A}$, where $\mathscr{A}$ is a nonempty subset of $\mathscr{F}_{conv}$ we get with Proposition 2.6.47 $A = \operatorname{Cl}_+ \operatorname{conv} A$. This yields

$$\operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathscr{A}} A = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathscr{A}} \operatorname{Cl}_+ \operatorname{conv} A = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathscr{A}} \operatorname{cl}\operatorname{conv}(A + C)$$

$$= \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathscr{A}} (A + C) = \operatorname{cl}\big(\big(\operatorname{conv} \bigcup_{A \in \mathscr{A}} A\big) + C\big)$$
$$= \operatorname{Cl}_+ \operatorname{conv} \bigcup_{A \in \mathscr{A}} A.$$

So we get $\operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathscr{A}} A \in \mathscr{F}_{conv}$. Since $\bigcap_{A \in \mathscr{A}} A$ is convex and upper closed, we can conclude $\bigcap_{A \in \mathscr{A}} A \in \mathscr{F}_{conv}$. So we get the assertions of the proposition. □

2.6.4 The Embedding Approach by Kuroiwa

An important approach for deriving optimality conditions and algorithms for solving set-valued optimization problems is based on the introduction of an **embedding space** into which the set-valued optimization problem is embedded (see Kuroiwa [353, 354]). With this approach Kuroiwa [354] defines notions of directional derivatives for set-valued maps and derives corresponding necessary and sufficient optimality conditions (compare Sect. 12.10). In this section we present results given by Kuroiwa in [353, 354, 357].

Let Y be a n.v.s., let C be a proper closed convex pointed cone in Y with $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} C^+ \neq \emptyset$.

The set relation $\preceq_C^{cl}$ discussed in this section is defined as follows (cf. Definition 2.6.9): For $A, B \subset Y$,

$$A \preceq_C^{cl} B :\Longleftrightarrow \operatorname{cl}(A + C) \supset B. \tag{2.64}$$

We consider minimal solutions with respect to the quasi-order $\preceq_C^{cl}$ given in (2.64) in the sense of Definition 2.6.19, and the corresponding set-valued optimization problem is given by

$$\preceq_C^{cl} -\text{minimize} \quad F(x), \quad \text{subject to} \quad x \in S, \qquad (SP_A - \preceq_C^{cl})$$

where $F : S \rightrightarrows Y$ is a set-valued objective mapping and S is a set.

We are looking for minimal solutions of $(SP_A - \preceq_C^{cl})$ in the sense of Definition 2.6.19, i.e., for elements $\overline{x} \in S$ with

$$F(x) \preceq_C^{cl} F(\overline{x}) \quad \text{for some} \quad x \in S \Longrightarrow F(\overline{x}) \preceq_C^{cl} F(x).$$

As already mentioned, a subset A of Y is said to be ***C*-convex** if $A + C$ is convex. Furthermore, $A \subseteq Y$ is said to be **C^+-bounded** if $\langle y^*, A\rangle$ is bounded from below for any $y^* \in C^+$, where C^+ is the positive dual cone of C.

Let $\mathscr{G}$ be the **family of all nonempty *C*-convex and C^+-bounded subsets of** Y. In the following, we introduce a process of construction of a normed space $\mathscr{V}$ into which $\mathscr{G}$ is embedded. This approach goes back to Kuroiwa and Nuriya [357].

At first, we introduce an **equivalence relation** $\equiv$ on $\mathcal{G}^2$: For all $(A, B),\ (D, E) \in \mathcal{G}^2$,

$$(A, B) \equiv (D, E) :\Longleftrightarrow \mathrm{cl}(A + E + C) = \mathrm{cl}(B + D + C).$$

The **quotient space** $\mathcal{G}^2/\equiv$ is denoted by $\mathcal{V}$, where

$$\mathcal{V} := \{[A, B] | (A, B) \in \mathcal{G}^2\},$$

where $[A, B] := \{(D, E) \in \mathcal{G}^2 | (A, B) \equiv (D, E)\}$.

Furthermore, we define addition and scalar multiplication on the quotient space $\mathcal{V}$ as follows:

$$[A, B] + [D, E] := [A + D, B + E]\,,$$

$$\lambda \cdot [A, B] := \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{V}, +, \cdot)$ is a vector space over $\mathbb{R}$.

In order to introduce an order relation on $\mathcal{V}$ we define the following subset of $\mathcal{V}$ by

$$\mu(C) := \{[A, B] \in \mathcal{V} | B \preceq_C^{cl} A\}. \tag{2.65}$$

It is easy to see that $\mu(C)$ is a pointed convex cone in $\mathcal{V}$.

Using the pointed convex cone $\mu(C)$, we define an **order relation** $\preceq_{\mu(C)}$ on $\mathcal{V}$ as follows:

$$[A, B] \preceq_{\mu(C)} [D, E] :\Longleftrightarrow [D, E] - [A, B] \in \mu(C).$$

Then, $(\mathcal{V}, +, \cdot, \preceq_{\mu(C)})$ is an ordered vector space over $\mathbb{R}$.

Let a function φ from $\mathcal{G}$ to $\mathcal{V}$ be given by

$$\varphi(A) := [A, \{0\}] \quad \text{for all} \quad A \in \mathcal{G},$$

then

$$A \preceq_C^{cl} B \Leftrightarrow \varphi(A) \preceq_{\mu(C)} \varphi(B),$$

for any $A,\ B \in \mathcal{G}$.

By using this function φ, the set optimization problem $(SP_A - \preceq_C^{cl})$ can be transformed into a **vector optimization problem** in the following sense: If F is a map from S to $\mathscr{G}$, then $\overline{x} \in S$ is a minimal solution of $(SP_A - \preceq_C^{cl})$ if and only if

$$\varphi \circ F(S) \cap (\varphi \circ F(\overline{x}) - \mu(C)) = \{\varphi \circ F(\overline{x})\}. \tag{2.66}$$

Formula (2.66) means that $\varphi \circ F(\overline{x})$ is a Pareto minimal point of $\varphi \circ F(S)$ with respect to $\mu(C)$, i.e., $\varphi \circ F(\overline{x}) \in \mathrm{Min}(\varphi \circ F(S), \mu(C))$ (see Definition 2.4.1).

Finally, we introduce a norm $|\cdot|$ in $\mathscr{G}^2/\equiv$. Consider $c \in \operatorname{int} C$ and a weak* compact base $W := \{y^* \in C^+ \mid \langle y^*, c\rangle = 1\}$ of C^+, then for each $[A, B] \in \mathscr{V}$,

$$|[A, B]| := \sup_{y^* \in W} |\inf\langle y^*, A\rangle - \inf\langle y^*, B\rangle|,$$

is well-defined. Furthermore, let

$$\mathscr{V}(W) := \{[A, B] \in \mathscr{V} \mid |[A, B]| < +\infty\},$$

then $(\mathscr{V}(W), |\cdot|)$ is a normed vector space, and $\mu(C)$ is closed in $(\mathscr{V}(W), |\cdot|)$.

In Sect. 12.10 we derive necessary and sufficient conditions for solutions of $(SP_A - \preceq_C^{cl})$ using this embedding approach.

2.6.5 Solution Concepts with Respect to Abstract Preference Relations

In this section we present a solution concept for set-valued optimization problems with geometric constraints useful in welfare economics (introduced by Bao and Mordukhovich [28]).

$$\text{minimize} \qquad F(x) \quad \text{subject to} \quad x \in S, \tag{SP_A}$$

where the cost mapping $F : X \rightrightarrows Y$ is a set-valued mapping, X and Y are Banach spaces and S is a subset of X.

The "minimization" in (SP_A) is understood with respect to a certain preference relation on Y. This general (abstract) preference relation on Y is defined as follows (see Sect. 2.1 or [431, Subsection 5.3.1]): For a given nonempty subset $\mathscr{R} \subset Y \times Y$, one says that y^1 is preferred to y^2 (we write $y^1 \prec y^2$) if $(y^1, y^2) \in \mathscr{R}$.

Following Bao and Mordukhovich [28] we will study a preference on Y directly in terms of a given preference mapping $L : Y \rightrightarrows Y$ instead of a preference $\prec$ on Y described via a subset $\mathscr{R} \subset Y \times Y$. The **level-set mapping** $L : Y \rightrightarrows Y$ associated to the preference relation $\prec$ is defined by

$$L(y) := \{u \in Y \mid u \prec y\}. \tag{2.67}$$

This means that $u \in Y$ is preferred to y if $u \in L(y)$.

An abstract preference $\prec$ has to satisfy some requirements in order to be useful in optimization and applications, especially in economics and engineering. In [431, Definition 5.55] three properties imposed in order to postulate the notion of **closed preference relations** as follows:

- The preference relation $\prec$ is **nonreflexive**, this means that $(y, y) \notin \mathscr{R}$ for all $y \in Y$;
- Given some $\overline{y} \in Y$ (a local minimizer in the sequel), the preference $\prec$ is **locally satiated** around $\overline{y}$ in the sense that $y \in \operatorname{cl} L(y)$ for all y in some neighborhood of $\overline{y}$.
- The preference $\prec$ is **almost transitive** meaning that

$$[u \in L(y),\ v \in \operatorname{cl} L(u)] \Longrightarrow v \in L(y). \tag{2.68}$$

Especially, in the study of vector optimization problems the almost transitivity property is widely used (under different names) ; see, e.g., [41, 431, 433, 624, 625] and the references therein. However, the almost transitivity property turns out to be rather restrictive, in contrast to the first two properties of $\prec$ formulated above. In particular, for the Pareto minimality (compare Definition 2.4.1) defined by

$$y^1 \prec y^2 :\Longleftrightarrow y^2 - y^1 \in C \setminus \{0\} \tag{2.69}$$

via an ordering cone $C \subset Y$, the preference $\prec$ is almost transitive if and only if C is convex and pointed; see [431, Proposition 5.56]. For the lexicographical order on $\mathbb{R}^q$ and other natural preference relations important in vector optimization and its applications including those to welfare economics (see Sect. 15.3) this property of the ordering cone is not fulfilled.

Bao and Mordukhovich [28] developed an approach to preference relations in set-valued optimization that is motivated by applications to models in welfare economics and allows to avoid the almost transitivity condition (2.68), which does not usually hold for many economies. They required the most natural local satiation property of the preference mapping L formulated at the reference optimal solution. The following solution concept is introduced by Bao and Mordukhovich [28, Definition 3.1]:

Definition 2.6.49 (Fully Localized Minimizers for Constrained Set-Valued Optimization Problems). Consider the problem (SP_A). Let $(\overline{x}, \overline{y}) \in \operatorname{graph} F$ and $\overline{x} \in S$. Then

- $(\overline{x}, \overline{y})$ is called a **fully localized weak minimizer for** (SP_A) if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that there is no $y \in F(S \cap U) \cap V$ preferred to $\overline{y}$, i.e.,

$$F(S \cap U) \cap L(\overline{y}) \cap V = \emptyset. \tag{2.70}$$

- $(\overline{x}, \overline{y})$ is called a **fully localized minimizer for** (SP_A) if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that there is no $y \in F(S \cap U) \cap V$ with $y \neq \overline{y}$ and $y \in \operatorname{cl} L(\overline{y})$, i.e.,

$$F(S \cap U) \cap \operatorname{cl} L(\overline{y}) \cap V = \{\overline{y}\}. \tag{2.71}$$

- $(\overline{x}, \overline{y})$ is called a **fully localized strong minimizer for** (SP_A) if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that there is no $(x, y) \in \operatorname{graph} F \cap (U \times V)$ with $(x, y) \neq (\overline{x}, \overline{y})$ satisfying $x \in S$ and $y \in \operatorname{cl} L(\overline{y})$, i.e.,

$$\operatorname{graph} F \cap (S \times \operatorname{cl} L(\overline{y})) \cap (U \times V) = \{(\overline{x}, \overline{y})\}. \tag{2.72}$$

In Sect. 12.11, Theorem 12.11.3 we derive first order necessary conditions for fully localized minimizers. Furthermore, we study applications in welfare economics in Sect. 15.3.

Remark 2.6.50. It is easy to see that each fully localized strong minimizer for (SP_A) is also a fully localized minimizer for (SP_A). Furthermore, each fully localized minimizer for (SP_A) is also a fully localized weak minimizer for (SP_A). In the case $S = X$ in Definition 2.6.49, we speak about the corresponding fully localized minimizers for the mapping F.

Remark 2.6.51. For all of the concepts in Definition 2.6.49 (see Bao and Mordukhovich [28, Definition 3.1]) the underlying feature is that one introduces the image localization of minimizers in constructions (2.70), (2.71), (2.72). These concepts introduced in [28, Definition 3.1] are different from the concepts discussed before even for minimal points and weakly minimal points of single-valued objectives $F := f : X \longrightarrow Y$ and allow to study local Pareto-type optimal allocations of welfare economics (see Sect. 15.3). The concept of (global or local) strong minimizers was first time introduced in [28, Definition 3.1] for set-valued optimization problems; it is related to Khan's notion of strong Pareto optimal allocations for models of welfare economics (compare Khan [324]) and the corresponding relationships established in Sect. 15.3.

Example 2.6.52. In this example we will see that a fully localized strong minimizer may not provide a partially localized (i.e., with $V = Y$) minimum or weak minimum in (2.70) and (2.71). To see this, we consider a set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := \begin{cases} \{-x\} & \text{if } x < 0 \\ \{0, 1\} & \text{if } x = 0 \\ \{x + 1\} & \text{if } x > 0, \end{cases}$$

with respect to the usual order on $\mathbb{R}$ generating the level sets $L(y) = (-\infty, y)$, it is easy to see that condition (2.72) is fulfilled at $(0, 1)$ with $U = (-\frac{1}{2}, \frac{1}{2})$ and $V = (\frac{1}{2}, \frac{3}{2})$ but conditions (2.70) and (2.71) do not hold with $V = \mathbb{R}$.

Example 2.6.53. Now, we will see that the localized minimizers and weak minimizers in Definition 2.6.49 are identical in the case of scalar set-valued optimization with $Y = \mathbb{R}$ and $L(y) = (-\infty, y)$, but they may be quite different in the vector-valued case. For example, if we consider an objective map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ with $F(x) \equiv \mathbb{R}^2 \setminus \operatorname{int} \mathbb{R}^2_-$ and the usual weak preference on $\mathbb{R}^2$ with the level sets $L(y) = y - \operatorname{int} \mathbb{R}^2_+$ (see Definition 2.6.2), we have that $(0, 0) \in \mathbb{R} \times \mathbb{R}^2$ is a localized weak minimizer for F, but it is not a localized minimizer for this mapping. Note finally that localized strong minimizers reduce to standard isolate minimizers for scalar single-valued optimization problems.

2.6.6 *Set-Valued Optimization Problems with Variable Ordering Structure*

In the book by Chen, Huang and Yang [91] set-valued optimization problems with variable ordering structure are introduced, where the solution concept is related to the solution concept for vector optimization problems with variable ordering structure given in Definition 2.5.1.

Let X and Y be Banach spaces , $S \subset X$ be nonempty. Furthermore, let $C : X \rightrightarrows Y$ be a cone-valued mapping. We assume that for every $x \in X$, the set $C(x)$ is a proper closed convex cone with nonempty interior $\operatorname{int} C(x)$.

We consider a set-valued objective mapping $F : X \rightrightarrows Y$ and a set-valued optimization problem with variable ordering structure

$$\text{v-minimize } F(x) \qquad \text{subject to} \quad x \in S, \tag{SP_v}$$

where "v-minimize" stands for problems with variable ordering structure with respect to a cone-valued mapping $C : X \rightrightarrows Y$ in the following sense:

Definition 2.6.54 ((Weak) v-Minimizer of (SP_v)). Let $\overline{x} \in S$ and $\overline{y} \in F(\overline{x})$.

(a) The pair $(\overline{x}, \overline{y})$ is called a **v-minimizer** of (SP_v) if

$$(F(S) - \overline{y}) \cap (-C(\overline{x})) = \{0\}.$$

(b) The pair $(\overline{x}, \overline{y})$ is called a **weak v-minimizer** of (SP_v) if

$$(F(S) - \overline{y}) \cap (-\operatorname{int} C(\overline{x})) = \emptyset.$$

In the following we will show that a set-valued optimization problem (SP_v) can be transformed into an equivalent vector-optimization problem in the sense that their v-minimal solution pairs are identical.

Definition 2.6.55. A cone-valued map $C : X \rightrightarrows Y$ is **pointed** on $S \subset X$ if the cone $\cup_{x \in S} C(x)$ is pointed, i.e.,

$$(\cup_{x \in S} C(x)) \cap (- \cup_{x \in S} C(x)) = \{0\}.$$

Remark 2.6.56. A cone-valued map $C : X \rightrightarrows Y$ is pointed on S if and only if

$$\forall x_1, x_2 \in S : \qquad C(x_1) \cap (-C(x_2)) = \{0\}.$$

For deriving the relationships between v-minimizers of a set-valued problem with variable ordering structure (SP_v) in the sense of Definition 2.6.54 and v-minimal points of a corresponding vector optimization problem with an objective function $f : X \to Y$ (see Definition 2.5.1) we need a certain monotonicity property concerning the set-valued map C with respect to f.

Definition 2.6.57. Let $f : X \to Y$ be a vector-valued function and $C : X \rightrightarrows Y$ be a cone-valued mapping. The cone-valued mapping C is called **weakly f-monotone**, if for all $x_1, x_2 \in X$, $c_1 \in C(x_1)$,

$$f(x_1) - f(x_2) \in c_1 + C(x_2) \quad \Longrightarrow C(x_2) \subset C(x_1).$$

The following relationships between v-minimal points of a vector optimization problem (VP_v) and v-minimizers of a set-valued optimization problem (SP_v) are shown by Chen, Huang and Yang [91, Proposition 2.63].

Proposition 2.6.58. *Let $S \subset X$ and $C : X \rightrightarrows Y$ be a pointed cone-valued mapping on S. Furthermore, let $f : X \to Y$ be a vector-valued function and $F : X \rightrightarrows Y$ be given by*

$$F(x) = f(x) + C(x) \quad (x \in X). \tag{2.73}$$

(a) *Suppose that C is weakly f-monotone. If $\overline{x} \in S$ is a v-minimal point of the vector optimization problem (VP_v):*

$$\mathrm{Min}(f(S), C(\cdot)),$$

then $(\overline{x}, f(\overline{x}))$ is a v-minimizer of the set-valued optimization problem (SP_v):

$$\textit{v-minimize } F(x) \quad \textit{subject to} \quad x \in S.$$

(b) *If $(\overline{x}, \overline{y})$ is a v-minimizer of the set-valued optimization problem (SP_v), then $\overline{x}$ is a v-minimal point of the vector optimization problem (VP_v) and $\overline{y} = f(\overline{x})$.*

Proof. First, we show that (a) holds. Consider a v-minimal point $\overline{x} \in S$ of the problem (VP_v). Then

$$\forall x \in S : \quad f(x) - f(\overline{x}) \notin -C(\overline{x}) \setminus \{0\}.$$

This yields

$$\forall x \in S: \quad (f(x) - f(\overline{x}) + C(x)) \cap (-C(\overline{x}) \setminus \{0\}) = \emptyset. \tag{2.74}$$

Indeed, if there exists $\tilde{x} \in S$ with

$$(f(\tilde{x}) - f(\overline{x}) + C(\tilde{x})) \cap (-C(\overline{x}) \setminus \{0\}) \neq \emptyset,$$

then there exists $\overline{c} \in C(\overline{x})$ and $\overline{c} \neq 0$ such that

$$-\overline{c} \in f(\tilde{x}) - f(\overline{x}) + C(\tilde{x}).$$

This means

$$f(\overline{x}) - f(\tilde{x}) \in \overline{c} + C(\tilde{x}).$$

Taking into account the weak f-monotonicity of C, we get

$$f(\tilde{x}) - f(\overline{x}) \in -\overline{c} - C(\overline{x}).$$

Furthermore, since C is pointed and $\overline{c} \neq 0$, it follows that

$$f(\tilde{x}) - f(\overline{x}) \in -C(\overline{x}) \setminus \{0\},$$

in contradiction to the assumption that $\overline{x}$ is a v-minimal point of the problem (VP_v).

Taking into account (2.73) and (2.74) we get

$$\forall y \in F(x),\ x \in S: \quad y - f(\overline{x}) \notin -C(\overline{x}) \setminus \{0\}.$$

Hence, $(\overline{x}, f(\overline{x}))$ is a v-minimizer of the set-valued optimization problem (SP_v).

Now, we will prove (b). Let us assume that $(\overline{x}, \overline{y})$ is a v-minimizer of the set-valued optimization problem (SP_v). Then,

$$\overline{y} \in F(\overline{x}) = f(\overline{x}) + C(\overline{x})$$

and

$$\forall y \in F(S): \qquad y - \overline{y} \notin -C(\overline{x}) \setminus \{0\}. \tag{2.75}$$

Of course, it holds $\overline{y} = f(\overline{x})$. We have to show that $\overline{x}$ is a v-minimal point of the vector optimization problem (VP_v). Contrarily, suppose that $\overline{x}$ is not a v-minimal point of the vector optimization problem (VP_v). Then, for some element $\tilde{x} \in S \setminus \{\overline{x}\}$,

$$f(\tilde{x}) - f(\overline{x}) \in -C(\overline{x}) \setminus \{0\}.$$

So we get

$$f(\tilde{x}) - \overline{y} \in -C(\overline{x}) \setminus \{0\},$$

because of $f(\overline{x}) = \overline{y}$, in contradiction to (2.75). Hence, $\overline{x}$ is a v-minimal point of the vector optimization problem (VP_v). □

Remark 2.6.59. In the book by Chen, Huang, Yang [91, Theorem 2.64] necessary conditions for **weak v-minimizers** of (SP_v) are shown using the contingent derivative.

Remark 2.6.60. Necessary and sufficient optimality conditions in form of the Fermat rule for **nondominated** solutions of unconstrained set-valued optimization problems with variable ordering structure and the Lagrange multiplier rule for the constrained set-valued problems with variable ordering structure are given by Eichfelder and Ha in [168].

2.6.7 Approximate Solutions of Set-Valued Optimization Problems

In this section we introduce a concept of approximate solutions in set-valued optimization. Approximate solutions are of interest from the theoretical as well as computational point of view. Especially, in order to formulate set-valued versions of Ekeland's variational principle (compare Chap. 10) and a subdifferential variational principle for set-valued mappings (see Sect. 12.9) one is dealing with approximate solutions.

We consider a set-valued optimization problem:

$$\text{minimize} \qquad F(x) \quad \text{subject to} \quad x \in X, \tag{SP}$$

where X is a linear space, Y is a linear topological space, $C \subset Y$ is a proper closed convex cone and the cost mapping $F : X \rightrightarrows Y$ is a set-valued mapping.

The following concepts for approximate solutions of the set-valued problem (SP) was given by Bao and Mordukhovich [27, Definition 3.4] and is related to minimizers introduced in Definition 2.6.1.

Definition 2.6.61 (Approximate Minimizers of Set-Valued Optimization Problems). Let $\overline{x} \in X$ and $(\overline{x}, \overline{y}) \in \operatorname{graph} F$. Then:

(i) Consider $\varepsilon > 0$ and $k^0 \in C \setminus \{0\}$. The pair $(\overline{x}, \overline{y}) \in \operatorname{graph} F$ is called an **approximate** εk^0**-minimizer** for F if

$$y + \varepsilon k^0 \notin \overline{y} - C \qquad \text{for all} \;\; y \in F(x) \text{ with } x \neq \overline{x}.$$

(ii) Consider $\varepsilon > 0$ and $k^0 \in C \setminus \{0\}$. The pair $(\overline{x}, \overline{y}) \in \operatorname{graph} F$ is called a **strict approximate εk^0-minimizer** for F if there is a number $0 < \tilde{\varepsilon} < \varepsilon$ such that $(\overline{x}, \overline{y})$ is an approximate $\tilde{\varepsilon} k^0$-minimizer of this mapping.

In Sect. 12.9, Theorem 12.9.1, we will show necessary conditions for strict approximate εk^0-minimizers of F.

2.7 Relationships Between Solution Concepts

In this section we study the relationships between different solution concepts in set-valued optimization. Furthermore, we discuss the special case that the objective map is single-valued.

Let Y be a linear topological space, partially ordered by a proper pointed convex closed cone C, X a linear space, S a subset of X and $F : X \rightrightarrows Y$. We consider the set-valued optimization problem (SP):

$$\text{minimize} \qquad F(x) \quad \text{subject to} \quad x \in S. \tag{2.76}$$

In the formulation of the solution concepts based on set approach the underlying space is a linear topological space Y whereas the extended space $Y^\bullet := Y \cup \{-\infty\} \cup \{+\infty\}$ is considered in the formulation of the solution concepts based on lattice approach in order to work with infimum and supremum.

Remark 2.7.1. The differences between the solution concepts based on set-approach in Definition 2.6.19 and the solution concepts based on vector approach in Definition 2.6.1 are already discussed in Examples 2.6.21, 2.6.22 and 2.6.23.

Remark 2.7.2. In the special case of single-valued functions $F = f : X \longrightarrow Y$ the concept of minimizers of the set-valued problem (SP) (see Definition 2.6.1) coincides with the solution concept for Pareto minimal points of $f(S)$ with respect to C introduced in Definition 2.4.1: $(\overline{x}, f(\overline{x})) \in \operatorname{graph} f$ is a minimizer in the sense of Definition 2.6.1 if and only if $f(\overline{x})$ is a Pareto minimal point of $f(S)$ with respect to C, i.e., $f(\overline{x}) \in \operatorname{Min}(f(S), C)$.

Remark 2.7.3. In the special case of single-valued functions $F = f : X \longrightarrow Y$ the concept of minimal solutions of the problem $(SP - \preceq)$ (see Definition 2.6.19) w.r.t. the order relations introduced in Definition 2.6.9 coincides with the solution concept for Pareto minimal points given in Definition 2.4.1.

In the following we consider a linear topological space Y, a linear space X, $S \subseteq X$, a set-valued map $F : X \rightrightarrows Y$, $F(S) = \bigcup_{x \in S} F(x) \neq \emptyset$ and a proper pointed closed convex cone $C \subset Y$ with $\operatorname{int} C \neq \emptyset$. The relationship between the infimal set of $F(S)$ (Definition 2.6.31) and weak minimizers of $F(S)$ in the sense of Definition 2.6.2 is given in the next proposition.

Proposition 2.7.4. *Under the assumption that* $F(S) = \mathrm{cl}(F(S) + C)$ *we get*

$$\mathrm{WMin}(F(S), C) = \mathrm{Inf}\, F(S).$$

Proof. Taking into account the assumption $F(S) = \mathrm{cl}(F(S)+C)$ and the definition of the infimal set we get

$$\mathrm{WMin}(F(S), C) = \mathrm{WMin}(\mathrm{cl}(F(S)+C), C) = \mathrm{WMin}(\mathrm{Cl}_+ F(S), C) = \mathrm{Inf}\, F(S).$$

The proof is completed. □

The assertion of Proposition 2.7.4 says that the solution concept for $\mathscr{I}$-valued problems coincides with the set $\mathrm{WMin}(F(S), C)$ in Definition 2.6.3 for weak minimizers.

Furthermore, if we assume that $F(S) + C$ is closed we get the following assertion.

Proposition 2.7.5. *Under the assumption that* $F(S) + C$ *is closed we get*

$$\mathrm{Inf}\, F(S) = \mathrm{WMin}(F(S) + C, C).$$

Proof. Because of the closedness of $F(S) + C$ we get

$$\mathrm{WMin}(F(S) + C, C) = \mathrm{WMin}(\mathrm{cl}(F(S) + C), C) = \mathrm{Inf}\, F(S),$$

taking into account Definition 2.6.31. □

Corollary 2.7.6. *Assuming that* $F(S) + C$ *is closed and* $\mathrm{WMin}(F(S) + C, C) = \mathrm{WMin}(F(S), C)$ *we get*

$$\mathrm{Inf}\, F(S) = \mathrm{WMin}(F(S), C).$$

In Sect. 15.1 we will use methods of set-valued optimization for deriving duality assertions for vector optimization problems. The relationships between vector optimization problems and $\mathscr{I}$-valued problems are discussed by Löhne and Tammer [397] and in a comprehensive and detailed way by Löhne [395].

Let Y be a linear topological space, partially ordered by a proper pointed convex closed cone C, X a linear space, S a subset of X and $f : X \longrightarrow Y^\bullet$ a vector-valued function. We consider the vector optimization problem

$$\mathrm{Min}(f(S), C). \tag{VOP}$$

In Sect. 15.1 we will see that it is very useful to assign to (VOP) a corresponding $\mathscr{I}$-valued problem such that one can use the complete lattice structure of $(\mathscr{I}, \preceq := \preceq_{\mathscr{I}})$, where $(\mathscr{I}, \preceq)$ is defined with respect to the ordering cone C of the vector optimization problem.

For a given vector-valued function $f : X \longrightarrow Y^\bullet$ we put

$$\overline{f} : X \longrightarrow \mathscr{I}, \qquad \overline{f}(x) := \operatorname{Inf}\{f(x)\}$$

and assign to (VOP) the $\mathscr{I}$-valued problem

$$\preceq -\text{minimize} \quad \overline{f} \quad \text{subject to} \quad x \in S. \tag{$VOP_{\mathscr{I}}$}$$

Problem ($VOP_{\mathscr{I}}$) is said to be the $\mathscr{I}$-extension of the vector optimization problem (VOP) (see Löhne [395]). The lattice extension of the vector optimization problem (VOP) allows us to handle the problem in the framework of complete lattices. For this extension it is important that the ordering cone C is closed as we will see in the proof of the following proposition.

The following assertion is shown by Löhne [395, Proposition 2.17].

Proposition 2.7.7. *For all $x, u \in X$ it holds*

$$f(x) \leq_C f(u) \quad \Longleftrightarrow \quad \overline{f}(x) \preceq \overline{f}(u).$$

Proof. Consider $y = f(x)$ and $z = f(u)$. Let $\operatorname{Inf}\{y\} \preceq \operatorname{Inf}\{z\}$, then $\mathrm{Cl}_+\{y\} \supseteq \mathrm{Cl}_+\{z\}$. With (2.62) we can conclude $z \in \mathrm{cl}(\{z\} + C) \subseteq \mathrm{cl}(\{y\} + C)$. Because of the assumption that C is closed, we get $z \in \{y\} + C$. This means $y \leq_C z$. The opposite inclusion is obvious. □

As a direct consequence of Proposition 2.7.7 we get corresponding assertions concerning the solutions of (VOP) and ($VOP_{\mathscr{I}}$) (see Löhne [395, Proposition 2.18]).

Finally, it is important to mention the following references. In the paper by Hernández, Jiménez, Novo [244], Benson proper efficiency in set-valued optimization is discussed. Hernández, Jiménez, Novo study in [245] weak and proper efficiency in set-valued optimization. Flores-Bazán, Hernández characterize efficiency without linear structure in [189]. Moreover, Hernández, Rodríguez-Marín, Sama describe solutions of set-valued optimization problems [253]. Furthermore, in Hernández, Rodríguez-Marín [250] certain existence results for solutions of set optimization problems are derived.

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