Chapter 2
Linear and Nonlinear Systems

2.1 Linear Systems Theory

A magnificent tool for studying the behavior of practical systems is the linear system theory. Many devices and components routinely used to implement signal processing tools may be modelled (or at least approximated) as linear systems, allowing one to design and analyze complicated systems using frequency-domain approach. However, caution must be exercised when dealing with some elements of the problem under investigation that do not fit the profile. Many devices in the biomedical field, which presumed to be linear, operate as a linear device over only a limited range of parameters, imposing a restrict limitations on the range of validity of analyses that are conducted. We postpone a discussion on the subject of nonlinear systems to the ensuing section and proceed with a discussion of linear systems in this section.

Although we have discussed the concept of filtering in the previous chapter, a formal discussion of the concept of systems has not been provided so far. To that end, we begin by considering the impulse-response of a linear system, which fully characterizes the response of a linear system to any input signal using an integral equation. As the name implies, IR describes the response of a system to an impulse function.

Let $h(t,\tau)$ denote the response of a linear system at time t to a dirac delta function (*i.e.*, $\delta(t-\tau) = \begin{cases} \infty & t=\tau \\ 0 & \text{otherwise} \end{cases}$) applied to the system at $t-\tau$. Equivalently, in the discrete-time domain, we have $h[n,m]$ as the system output at time n to a delta function (*i.e.*, $\delta[n-m] = \begin{matrix} 1 & n=m \\ 0 & \text{otherwise} \end{matrix}$) at time $n-m$. The output of the linear system may then be described in terms of the input $x(t)$ (or $x[n]$) and $h(t,\tau)$ (or $h[n,m]$) as follows:

$$y[n] = \sum_{m=-\infty}^{\infty} h[n,m]\,x[n-m]\,. \tag{2.1}$$

K. Kiasaleh, *Biological Signals Classification and Analysis*,
Lecture Notes in Bioengineering, DOI 10.1007/978-3-642-54879-6_2

and in the continuous-time domain

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau)\, x(t - \tau)\, d\tau. \tag{2.2}$$

To arrive at the above results, the following identities are used:

$$x(t) = \int_{-\infty}^{\infty} \delta(t - \tau)\, x(\tau)\, d\tau \tag{2.3}$$

and

$$x[n] = \sum_{m=-\infty}^{\infty} \delta[n - m]\, x[m]. \tag{2.4}$$

The reader may recognize these types of systems as linear and time-variant as the impulse response seems to be a function of "when" the impulse has been applied to the system. Note that $h(t, \tau)$ is a function of both t and τ. Similarly, $h[n, m]$ is also a function of m and n.

A large class of systems may be modelled (or at least approximated) as time-invariant, resulting in a noticeable reduction in the complexity of the above computations. For linear time-invariant (LTI) systems, $h(t, \tau) = h(\tau)$ and $h[n, m] = h[m]$. This simplification simply implies that the response of the system is a function of time separation between when the impulse function has been applied to the system and when the observation has been made. Many systems in use today may fit this model, although all systems experience minor fluctuations in their behavior over time.

The reader may verify that this simplification reduces the computation of the output of a linear system to that of a convolutional integration (or summation). Namely,

$$y[n] = h[n] \copyright x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n - m] \tag{2.5}$$

and

$$y(t) = h(t) \copyright x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau \tag{2.6}$$

where © denotes convolution. A further simplification, which is of practical interest, can be made to the above by realizing that a realizable linear system is causal, and hence $h(\tau) = 0$ for $\tau < 0$ or $h[m] = 0$ for $m < 0$. Hence, for a linear, time-invariant, and causal system, we have

$$y[n] = \sum_{m=0}^{\infty} h[m]\, x[n - m] \tag{2.7}$$

and

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau. \tag{2.8}$$

For time-varying systems that are causal, we have

$$y[n] = \sum_{m=0}^{\infty} h[n,m]\, x[n-m]. \tag{2.9}$$

and in the continuous-time domain

$$y(t) = \int_{0}^{\infty} h(t,\tau)\, x(t-\tau)\, d\tau. \tag{2.10}$$

Linear systems must also provide bounded outputs for bounded inputs (BIBO) in order to be useful devices in signal processing. BIBO condition is satisfied if

$$\sum_{m=-\infty}^{\infty} |h[m]|^2 < \infty$$

or

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

A question that perhaps lingers in the mind of reader is whether there is a simple relationship between $h(\tau)$ and $h[m]$, and if so, what is the relationship? One is tempted to use $h[m] \stackrel{?}{=} h(mT_s)$ as the relationship between continuous and discrete time systems. However, as we will see shortly, that may not be the case. In order to explore this further, we need to resort to Fourier analysis.

2.1.1 System Function

It is widely accepted that the Fourier transform provides the most effective tool in linear system study. Although Laplace transform offers the benefit of taking initial conditions into account, the steady-state nature of most systems in practice makes the use of this transform less useful. Hence, we concern ourselves with FT analysis here. To that end, we define system function as

$$H[\omega] = \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m} \tag{2.11}$$

or, in the continuous-time domain,

$$H(\Omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\Omega t} dt. \tag{2.12}$$

An immediate and profound consequence of this transformation is the well-known and well-used relationship between input and output of a linear system, which is given by

$$Y[\omega] = H[\omega]\, X[\omega] \tag{2.13}$$

or

$$Y(\Omega) = H(\Omega)\, X(\Omega) \tag{2.14}$$

where

$$X[\omega] = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega m} \tag{2.15}$$

and

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt. \tag{2.16}$$

Note that FT has reduced the convolution operation for finding the output of the system to that of multiplication in the frequency domain. Such a simplification is not without cost; one has to obtain the FT of signal and impulse response and then convert the FT of the output back into the time domain. However, such a cost becomes trivial when we are dealing with cascaded systems. That is, when a series of q linear systems with IR of $h_1, h_2, \ldots, h_q$ are cascaded, the output can be found using

$$y[n] = h_1[n] \copyright h_2[n] \copyright \cdots \copyright h_n\left[q\right] \copyright x[n] \tag{2.17}$$

or

$$y(t) = h_1(t) \copyright h_2(t) \copyright \cdots \copyright h_n(t) \copyright x(t)\,. \tag{2.18}$$

This involves involving the cumbersome convolutional summation or integration q times! This is by any standard a tedious task. With the use of system function, the output can be obtained using

$$Y[\omega] = H_1[\omega]\, H_2[\omega] \ldots H_q[\omega]\, X[\omega] \tag{2.19}$$

or

$$Y(\Omega) = H_1(\Omega)\, H_2(\Omega) \ldots H_q(\Omega)\, X(\Omega)\,, \tag{2.20}$$

which is simply the product of the individual system functions by the FT of the input. Now, the cost of taking FT of the input and computing the inverse FT of the output becomes less burdensome for a large q.

2.1.2 Response of Linear Systems to Random Signals

A pertinent question to ask at this stage is whether a simple Fourier transform of a signal is sufficient to characterize a random process or sequence in frequency domain. The answer is clearly "no" as $X[\omega]$ and $X(\Omega)$ are both random variables. As noted in earlier sections, PSD is a more appropriate tool in spectrum analysis of random signals. We, then, are concerned with the PSD of the output of a linear system as a function of input PSD. To address this question, one has to consider the correlation of the output of the linear system as a function of the input correlation. We first consider the case of WSS input and LTI system. That is, let us consider

$$R_y[n,m] = E\left\{Y^*[n]\,Y[m]\right\} = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} E\left\{x^*[n-l_1]\,x[m-l_2]\right\} h^*[l_1]\,h[l_2] \tag{2.21}$$

Realizing that $E\{x^*[n-l_1]\,x[m-l_2]\} = R_x[n-l_1, m-l_2] = R_x[m-n+l_1-l_2]$,

$$\begin{aligned} R_y[n,m] &= \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} R_x[m-n+l_1-l_2]\,h^*[l_1]\,h[l_2] \\ &= R_y[m-n] \end{aligned} \tag{2.22}$$

Furthermore,

$$\begin{aligned} E\{y[n]\} = m_y[n] &= \sum_{m=-\infty}^{\infty} h[n-m]\,E\{x[m]\} \\ &= m_x \sum_{m=-\infty}^{\infty} h[n-m] \\ &= c m_x \end{aligned} \tag{2.23}$$

where $c = \sum_{m=-\infty}^{\infty} h[m]$ is a constant and the mean of $x[n]$ is assumed to be constant since $x[n]$ is WSS. Since $m_y[n] = m_y$ and $R_y[n,m] = R_y[m-n]$, *the output of a LTI linear system with WSS input is another WSS signal.* Further, the correlation of the output of a LTI system in terms of the correlation of the WSS input signal is give by

$$\boxed{R_y[m] = E\left\{Y_n^* Y_m\right\} = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} R_x[m+l_1-l_2]\,h^*[l_1]\,h[l_2].} \tag{2.24}$$

Taking the FT of $R_y[m]$,

$$S_y[\omega] = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_x[m + l_1 - l_2]\, h^*[l_1]\, h[l_2]\, e^{-j\omega m} \tag{2.25}$$

Letting $m' = m + l_1 - l_2$

$$\begin{aligned} S_y[\omega] = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} & R_X\left[m'\right] h^*[l_1]\, h[l_2] \\ & \times e^{-j\omega m'} e^{j\omega l_1} e^{-j\omega l_2} \end{aligned} \tag{2.26}$$

$$\begin{aligned} S_y[\omega] &= \left(\sum_{l_1=-\infty}^{\infty} h^*[l_1]\, e^{j\omega l_1}\right)\left(\sum_{m'=-\infty}^{\infty} R_X\left[m'\right] e^{-j\omega m'}\right) \\ &\quad \times \left(\sum_{l_2=-\infty}^{\infty} h[l_2]\, e^{-j\omega l_2}\right) \\ &= S_x[\omega]\, H[\omega]\, H^*[\omega] \end{aligned} \tag{2.27}$$

So,

$$\boxed{S_y[\omega] = |H[\omega]|^2\, S_x[\omega]} \tag{2.28}$$

Similarly,

$$\begin{aligned} R_y(t_1, t_2) &= E\left\{y^*(t_1)\, y(t_2)\right\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left\{x^*(t_1 - \tau_1)\, x(t_2 - \tau_2)\right\} h^*(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2 \end{aligned} \tag{2.29}$$

which when $x(t)$ is WSS, reduces to

$$\begin{aligned} R_y(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t_2 - t_1 - \tau_2 + \tau_1)\, h^*(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2 \\ &= R_y(t_2 - t_1) \end{aligned} \tag{2.30}$$

Also, for a WSS input,

$$\begin{aligned} E\{y(t)\} = m_y(t) &= \int_{-\infty}^{\infty} E\{x(t-\tau)\}\, h(\tau)\, d\tau \\ &= m_x \int_{-\infty}^{\infty} h(\tau)\, d\tau \\ &= c' m_x \end{aligned} \tag{2.31}$$

where $c' = \int_{-\infty}^{\infty} h(\tau)\, d\tau$.

Hence, $y(t)$ is also WSS, and the PSD of $y(t)$ can be found as

$$\begin{aligned} S_y(\Omega) &= \int_{-\infty}^{\infty} R_y(\tau)\, e^{-j\Omega\tau} d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau - \tau_2 + \tau_1)\, h^*(\tau_1)\, h(\tau_2) \\ &\quad \times e^{-j\Omega\tau} d\tau d\tau_1 d\tau_2. \end{aligned} \tag{2.32}$$

Letting $\tau - \tau_2 + \tau_1 = \tau'$

$$\begin{aligned} S_y(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x\left(\tau'\right) h^*(\tau_1)\, h(\tau_2) \\ \times e^{-j\Omega\tau'} e^{j\Omega\tau_1} e^{-j\Omega\tau_2} d\tau' d\tau_1 d\tau_2 \end{aligned} \tag{2.33}$$

$$\begin{aligned} S_y(\Omega) = \left(\int_{-\infty}^{\infty} R_x\left(\tau'\right) e^{-j\Omega\tau'} d\tau'\right)\left(\int_{-\infty}^{\infty} h^*(\tau_1)\, e^{j\Omega\tau_1} d\tau_1\right) \\ \times \left(\int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\Omega\tau_2} d\tau_2\right) \end{aligned} \tag{2.34}$$

which reduces to

$$\boxed{S_y(\Omega) = |H(\Omega)|^2 S_x(\Omega)} \tag{2.35}$$

Equations (2.28) and (2.35) are critical to system study as they shed light on how power of a signal is distributed in the frequency domain as a result of filtering. This is a very powerful tool for system engineering, where power distribution of a signal in the frequency domain is required to establish spectrum utilization.

Furthermore, or perhaps more importantly, these equations allow one to filter out undesired portion of the spectrum of a signal which has been corrupted by additive or other undesirable effects. Since almost all biomedical signals are non-stationary, one can only use the above results when the input signal is confined to a duration in time where its characteristics parallel those of a WSS signal. This implies that, although one cannot in general assume WSS condition for the input signal, the above results can be applied to signals over limited periods of time over which the signal characteristics are not subject to change. For this to be useful, however, one has to have a priori knowledge of the signal characteristics.

A critical question to ask at this stage is whether the above equations hold true for cyclostationary processes. This is of particular interest as most linearly modulated signals can be classified as cyclostationary. We, then, proceed to reconsider this problem when the input signal is cyclostationary. First, we establish that the output of a LTI system is cyclostationary when the input is the same. To that end, we consider the correlation function of the output, given by

$$\begin{aligned}
R_y[n,m] &= E\left\{y^*[n]\,y[m]\right\} \\
&= \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} R_x[l_1,l_2]\,h^*[n-l_1]\,h[m-l_2] \qquad (2.36)\\
R_y\left[n+lN_p, m+lN_p\right] &= \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} R_x[l_1,l_2]\,h^*\left[n+lN_p-l_1\right] \\
&\quad \times h\left[m+lN_p-l_2\right] \qquad (2.37)\\
&= \sum_{l_1'=-\infty}^{\infty}\sum_{l_2'=-\infty}^{\infty} R_x\left[n-l_1'+lN_p, m+lN_p-l_2'\right] \\
&\quad \times h^*\left[l_1'\right] h\left[l_2'\right] \\
&= R_y[n,m] \qquad (2.38)
\end{aligned}$$

Furthermore, since

$$E\{y[n]\} = m_y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,E\{x[m]\}\,, \qquad (2.39)$$

we have

$$\begin{aligned} E\left\{y\left[n+lN_p\right]\right\} &= m_y\left[n+lN_p\right] \\ &= \sum_{m=-\infty}^{\infty} h\left[n-m\right] E\left\{x\left[m+lN_p\right]\right\} \\ &= \sum_{m=-\infty}^{\infty} h\left[n-m\right] m_x\left[m+lN_p\right] \\ &= \sum_{m=-\infty}^{\infty} h\left[n-m\right] m_x\left[m\right] = m_y\left[n\right] \end{aligned} \tag{2.40}$$

Hence, the output is cyclostationary. The PSD of the output, hence, will be the FT of the average autocorrelation function. That is,

$$\begin{aligned} S_y\left[\omega\right] &= DTFT\left\{\langle R_y\left[m,m+n\right]\rangle_m\right\} \\ &= \sum_{n=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} \langle R_x\left[m-l_1,m+n-l_2\right]\rangle_m \\ &\quad \times h^*\left[l_1\right] h\left[l_2\right] e^{-j\omega n} \end{aligned} \tag{2.41}$$

$$\begin{aligned} S_y\left[\omega\right] = \frac{1}{N_p}\sum_{n=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\sum_{m=0}^{N_p-1} R_x\left[m-l_1,m+n-l_2\right] \\ \times h^*\left[l_1\right] h\left[l_2\right] e^{-j\omega n} \end{aligned} \tag{2.42}$$

Adding $e^{\pm j\omega l_2}$ and $e^{\pm j\omega l_1}$,

$$\begin{aligned} S_y\left[\omega\right] = \frac{1}{N_p}\sum_{n=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\sum_{m=0}^{N_p-1} R_x\left[m-l_1,m+n-l_2\right] \\ \times e^{-j\omega(n+l_1-l_2)} h^*\left[l_1\right] e^{j\omega l_1} h\left[l_2\right] e^{-j\omega l_2}. \end{aligned} \tag{2.43}$$

Now, making the change of variable $m-l_1=m'$

$$\begin{aligned} S_y\left[\omega\right] = \frac{1}{N_p}\sum_{n=-\infty}^{\infty}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\sum_{m'=-l_1}^{N_p-1-l_1} R_x\left[m',m'+n-l_2+l_1\right] \\ \times e^{-j\omega(n+l_1-l_2)} h^*\left[l_1\right] e^{j\omega l_1} h\left[l_2\right] e^{-j\omega l_2} \end{aligned} \tag{2.44}$$

Finally, making the change of variable $n' = n - l_2 + l_1$ and summing over n' first,

$$\begin{aligned} S_y[\omega] &= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \frac{1}{N_p} \sum_{m'=-l_1}^{N_p-1-l_1} R_x\left[m', m'+n'\right] \\ &\quad \times e^{-j\omega n'} h^*[l_1] e^{j\omega l_1} h[l_2] e^{-j\omega l_2} \\ &= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \langle R_x\left[m', m'+n'\right]\rangle_{m'} \\ &\quad \times e^{-j\omega n'} h^*[l_1] e^{j\omega l_1} h[l_2] e^{-j\omega l_2} \end{aligned} \tag{2.45}$$

Realizing that

$$\sum_{n'=-\infty}^{\infty} \langle R_x\left[m', m'+n'\right]\rangle_{m'} e^{-j\omega n'} = S_x[\omega] \tag{2.46}$$

for cyclostationary signals,

$$\begin{aligned} S_y[\omega] &= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} S_x[\omega] h^*[l_1] e^{j\omega l_1} h[l_2] e^{-j\omega l_2} \\ &= S_x[\omega] H^*(\omega) H[\omega] \\ &= S_x[\omega] |H(\omega)|^2 \end{aligned} \tag{2.47}$$

which leads to the same result as that for WSS signals! This is a very encouraging result in that it allows one to shape/filter the spectrum of cyclostationary signals using filtering in a manner that is identical to that used for WSS signals.

Similarly, in the continuous domain, we have

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t_1 - \tau_1, t_2 - \tau_2) h^*(\tau_1) h(\tau_2) d\tau_1 d\tau_2 \tag{2.48}$$

$$\begin{aligned} R_y\left(t_1 + lT_p, t_2 + lT_p\right) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x\left(t_1 + lT_p - \tau_1, t_2 + lT_p - \tau_2\right) \\ &\quad \times h^*(\tau_1) h(\tau_2) d\tau_1 d\tau_2 \end{aligned} \tag{2.49}$$

Since $x(t)$ is cyclostationary,

$$R_x\left(t_1 + lT_p - \tau_1, t_2 + lT_p - \tau_2\right) = R_x(t_1 - \tau_1, t_2 - \tau_2), \tag{2.50}$$

and hence $R_y\left(t_1 + lT_p, t_2 + lT_p\right) = R_y\left(t_1, t_2\right)$. Also,

$$\begin{aligned}
E\left\{y\left(t\right)\right\} = m_y\left(t\right) &= \int_{-\infty}^{\infty} E\left\{x\left(t-\tau\right)\right\} h\left(\tau\right) d\tau \\
E\left\{y\left(t + lT_p\right)\right\} &= m_y\left(t + lT_p\right) \\
&= \int_{-\infty}^{\infty} E\left\{x\left(t - \tau + lT_p\right)\right\} h\left(\tau\right) d\tau \\
&= \int_{-\infty}^{\infty} m_x\left(t + lT_p\right) h\left(\tau\right) d\tau \\
&= \int_{-\infty}^{\infty} m_x\left(t\right) h\left(\tau\right) d\tau \\
&= m_y\left(t\right)
\end{aligned} \tag{2.51}$$

Therefore, $y\left(t\right)$ is also cyclostationary and $S_y\left(\Omega\right)$ is given by

$$\begin{aligned}
S_y\left(\Omega\right) &= FT\left\{\langle R_y\left(t, t+\tau\right)\rangle_t\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle R_x\left(t-\tau_1, t+\tau-\tau_2\right)\rangle_t e^{-j\Omega\tau} \\
&\quad \times h^*\left(\tau_1\right) h\left(\tau_2\right) d\tau_1 d\tau_2 d\tau.
\end{aligned} \tag{2.52}$$

Making a change of variables $t - \tau_1 = t'$

$$\begin{aligned}
S_y\left(\Omega\right) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle R_x\left(t', t' + \tau + \tau_1 - \tau_2\right)\rangle_{t'} e^{-j\Omega\tau} \\
&\times h^*\left(\tau_1\right) h\left(\tau_2\right) d\tau_1 d\tau_2 d\tau
\end{aligned} \tag{2.53}$$

Now, adding $e^{\pm j\omega\tau_1}$ and $e^{\pm j\omega\tau_2}$

$$\begin{aligned}
S_y\left(\Omega\right) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle R_x\left(t', t' + \tau + \tau_1 - \tau_2\right)\rangle_{t'} e^{-j\Omega\left(\tau+\tau_1-\tau_2\right)} \\
&\times h^*\left(\tau_1\right) e^{j\Omega\tau_1} h\left(\tau_2\right) e^{-j\Omega\tau_2} d\tau_1 d\tau_2 d\tau
\end{aligned} \tag{2.54}$$

Finally, making the change of variable $\tau' = \tau + \tau_1 - \tau_2$,

$$S_y(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle R_x\left(t', t' + \tau'\right)\rangle_{t'} e^{-j\Omega\tau'} h^*(\tau_1)\, e^{j\Omega\tau_1} \times h(\tau_2)\, e^{-j\Omega\tau_2} d\tau' d\tau_1 d\tau_2 \tag{2.55}$$

and realizing that

$$\int_{-\infty}^{\infty} \langle R_x\left(t', t' + \tau'\right)\rangle_{t'} e^{-j\Omega\tau'} d\tau' = S_x(\Omega), \tag{2.56}$$

we have

$$\begin{aligned} S_y(\Omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\Omega)\, h^*(\tau_1)\, e^{j\Omega\tau_1} h(\tau_2)\, e^{-j\Omega\tau_2} d\tau' d\tau_1 d\tau_2 \\ &= S_x(\Omega)\, H(\Omega)\, H^*(\Omega) \\ &= S_x(\Omega)\, |H(\Omega)|^2 \end{aligned} \tag{2.57}$$

With this useful tool at hand, we can examine the output of linear systems when the input is a random process. In particular, this tool allows a systematic means of shaping the spectrum of signals. To illustrate this point, let us consider the following scenario: Let us assume that we are interested in computing the PSD of the signal

$$x(t) = \sum_{n=-\infty}^{\infty} I_n p(t - nT). \tag{2.58}$$

Based upon the above discussions, we are better off computing the PSD of

$$w(t) = \sum_{n=-\infty}^{\infty} I_n \delta(t - nT) \tag{2.59}$$

and then filter the above signal with a filter with an impulse response of $p(t)$. Finally, applying $S_y(\Omega) = S_x(\Omega)\, |H(\Omega)|^2$, we can obtain the PSD of the desired signal. To elaborate, one can view the random signal as the input to a linear system with impulse response $h(t) = p(t)$. It is rather easy to verify that

$$x(t) = w(t) \copyright h(t) \tag{2.60}$$

where © denotes convolution. Now, in the previous examples, we have shown that $w(t)$ is cyclostationary with period T. That is, $R_w(t, t+\tau)$ is a periodic function of t with period T. Hence, PSD of $w(t)$ can be found as

$$\begin{aligned}
FT\{\langle R_w(t, t+\tau)\rangle_t\} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} R_I(n_2-n_1) \\
&\quad\times \delta(t-n_1T)\,\delta(t+\tau-n_2T)\,e^{-i\Omega\tau}d\tau dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{n_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} R_I(n_2-n_1) \\
&\quad\times \delta(t-n_1T)\,e^{j\Omega(t-n_2T)}dt \\
&= \frac{1}{T}\sum_{n_1=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} R_I(n_2-n_1)\,e^{-i\Omega T(n_2-n_1)} \\
&= \frac{1}{T}S_I[\Omega T]
\end{aligned} \tag{2.61}$$

Now, using (2.35), we arrive at (1.176).

Example 1 Signal

$$x(t) = \sum_{n=-\infty}^{\infty} I_n p(t-nT)$$

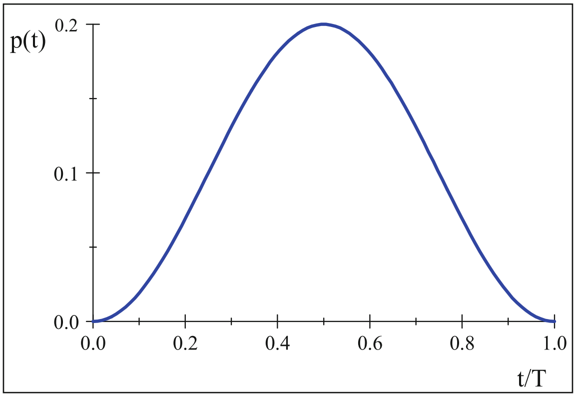
with

$$p(t) = \begin{cases} 0.1\left(1-\cos\left(\frac{2\pi t}{T}\right)\right) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(shown below)
is sampled at the rate of $\frac{10}{T}$ and the resulting signal is passed through a filter with the system function

$$H[\omega] = \begin{cases} 1; & |\omega| \le \frac{\pi}{8} \\ \frac{40}{3\pi}\left(-|\omega| + \frac{\pi}{5}\right); & \frac{\pi}{8} \le |\omega| \le \frac{\pi}{5} \\ 0; & \text{otherwise} \end{cases}. \tag{2.62}$$

$H[\omega] = \begin{cases} 1; & \text{if } |\omega| \le \frac{\pi}{8} \\ \frac{40}{3\pi}\left(-|\omega| + \frac{\pi}{5}\right); & \text{if } \frac{\pi}{8} \le |\omega| \le \frac{\pi}{5} \\ 0; & \text{if} \end{cases}$. Furthermore, I_n is a zero-mean random sequence with $R_I[q] = \delta[q] + 0.5\delta[q-1] + 0.5\delta[q+1]$. Plot the PSD of the input and output of this filter.



A plot of $p(t)$ for Example 1

Solution: *Using* (1.185),

$$S_x[\omega] = \frac{1}{10} S_I[10\omega] |P[\omega]|^2 ; |\omega| \le \pi$$

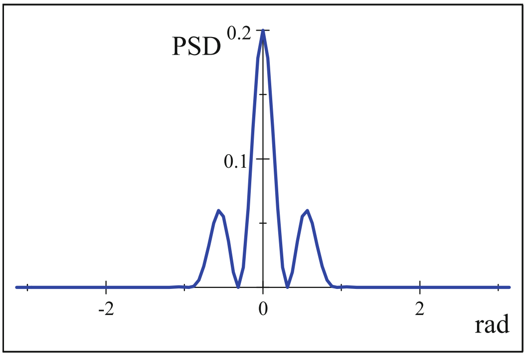
where

$$\begin{aligned} P[\omega] &= \sum_{n=0}^{10} P\left(\frac{nT}{10}\right) \exp(-j\omega n) \\ &= 0.1 \sum_{n=0}^{10} \left(1 - \frac{1}{2} e^{j\frac{\pi n}{5}} - \frac{1}{2}^{-j\frac{\pi n}{5}}\right) e^{-j\omega n} \\ &= 0.1 \frac{e^{-j11\omega} - 1}{e^{-j\omega} - 1} - \frac{0.1}{2} \frac{e^{-j\left(11\omega - \frac{11\pi}{5}\right)} - 1}{e^{-j\left(\omega - \frac{\pi}{5}\right)} - 1} - \frac{0.1}{2} \frac{e^{-j\left(11\omega + \frac{11\pi}{5}\right)} - 1}{e^{-j\left(\omega + \frac{\pi}{5}\right)} - 1}. \end{aligned} \tag{2.63}$$

Next, we examine the PSD of I_n, which is given by

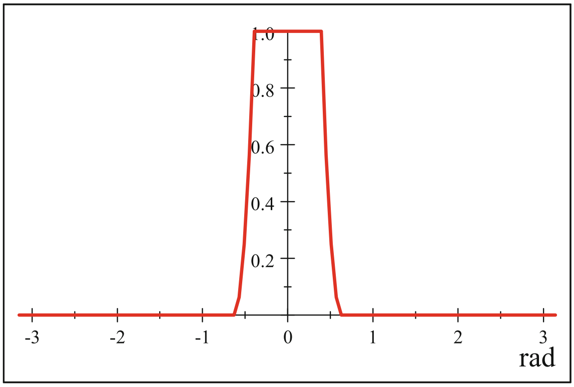
$$S_I[\omega] = 1 + 0.5e^{-j\omega} + 0.5e^{+j\omega} = 1 + \cos(\omega).$$

The PSD of input is plotted below.



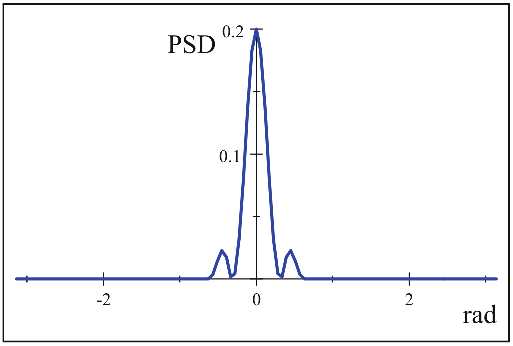
Input power spectrum as a function of ω *(in rad).*

The filter system function is depicted in the figure below.



$H[\omega]$ as a function of $|\omega| \leq \pi$.

The output spectrum is given by $S_y[\omega] = S_x[\omega]\,|H(\omega)|^2$ *and is shown in the following figure.*



Output power spectrum as a function of ω *(in rad).*

Note that filtering has a profound impact on the shape of the signal spectrum, and hence it is a powerful tool in reshaping the frequency content of a random signal.

Example 2 Repeat the previous example when the pulse shape is change to a raised-cosine (RC) pulse, given by $p\left(t\right)=\sin\mathrm{c}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\beta t}{T}\right)}{1-\frac{4\beta^2t^2}{T^2}}$ where $\sin\mathrm{c}\left(x\right)=\frac{\sin(\pi x)}{\pi x}$ with β denoting the 'roll-off' factor of the pulse. Plot the output spectrum for $\beta=0.25$ *and* $\beta=0.5$.

Solution: *This pulse is used very frequently in digital communications (as we observed in the preceding sections) due to its desirable properties. A plot of the pulse shape for different values of* β *(known as roll-off factor) is depicted in Fig. 2.1. Note that this pulse assumes a value of zero for integer multiples of a symbol period. That is, for the signal* $x\left(t\right)=\sum_{n=-\infty}^{\infty}I_n p\left(t-nT\right)$, *we have* $x\left[n\right]=I_n$ *for all n. Hence, a sample at the correct "time" will yield a desired result. Also, for* $\beta=0$, $p\left(t\right)$ *reduces to a* $\sin\mathrm{c}(x)$ *function, which implies that the RC filter with* $p\left(t\right)$ *impulse response is an ideal lowpass filter.*

Given this pulse shape, $P\left[\omega\right]$ *is given by*

$$P\left[\omega\right]=\sum_{n-\infty}^{\infty}p\left(nT_s\right)\exp\left(-j\omega n\right)=\sum_{n-\infty}^{\infty}\sin\mathrm{c}\left(\frac{nT_s}{T}\right)\frac{\cos\left(\frac{\pi n\beta T_s}{T}\right)}{1-\frac{4\beta^2n^2T_s^2}{T^2}}\rightarrow$$

$$P\left[\omega\right]=\begin{cases}T/T_s & \left|\omega\right|\le\frac{2\pi(1-\beta)T_s}{2T}\\ \frac{T}{2T_s}\left[1+\cos\left(\frac{\pi T}{\beta}\left[\frac{|\omega|}{2\pi T_s}-\frac{1-\beta}{2T}\right]\right)\right] & \frac{2\pi(1-\beta)T_s}{2T}\le\left|\omega\right|\le\frac{2\pi(1+\beta)T_s}{2T}\\ 0 & \left|\omega\right|>\frac{2\pi(1+\beta)T_s}{2T}\end{cases} \tag{2.64}$$

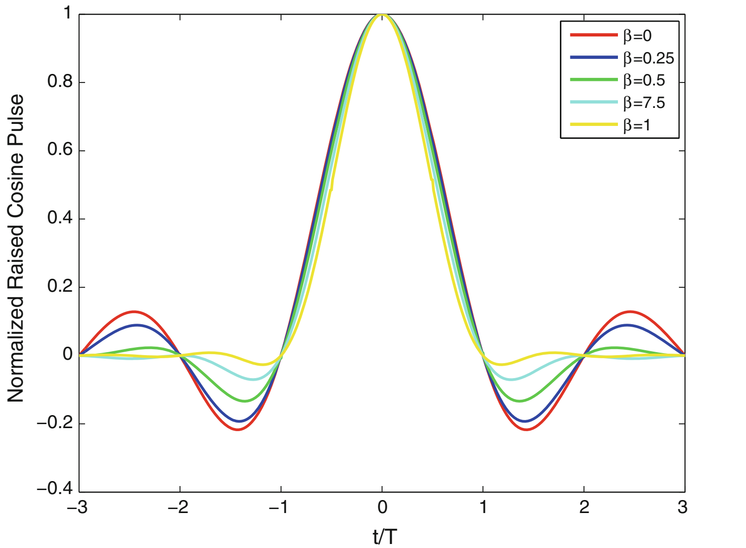
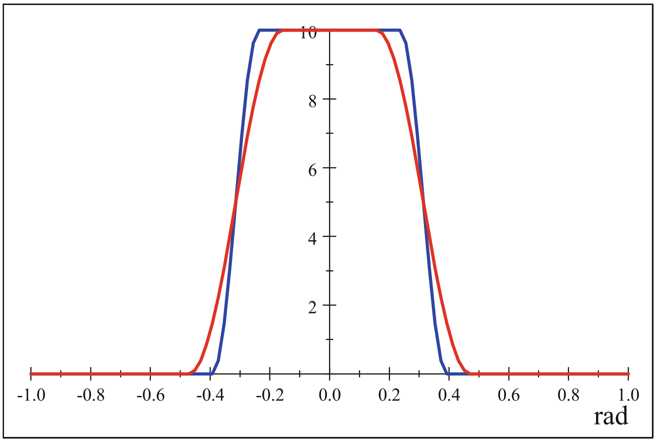


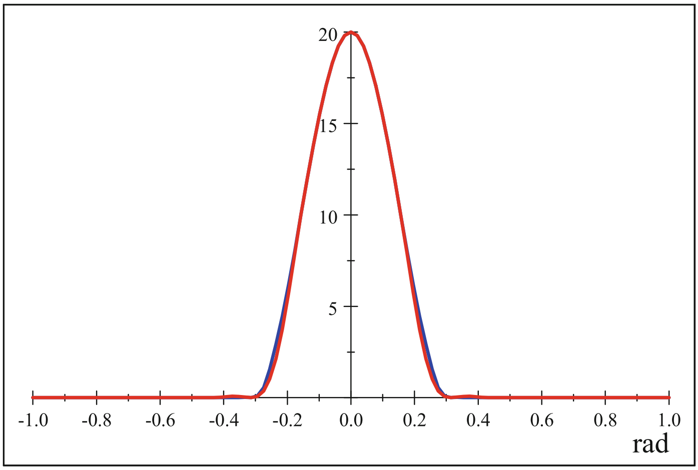
Fig. 2.1 Raised-cosine pulse for different roll-off factors

and is shown below for $T = 10T_s$ *and* $\beta = 0.5$ *and* 0.25*. Note that* β *has an impact on the 'bandwidth' of the pulse shape.*



$P[\omega]$ *for RC pulse as a function of* ω *for* $\beta = 0.25$ *(blue) and* $\beta = 0.5$ *(red).*

The output PSD is shown below. The filter has almost entirely suppressed the impact of β *as the response of the filter is pronounced around* $\omega = 0$ *and the two PSDs are almost identical around* $\omega = 0$ *(see above).*



$S_y[\omega]$ *as a function of* ω *for RC pulse with* $\beta = 0.25$ *(blue) and* $\beta = 0.5$ *(red).*

Since linear system theory can be applied here, one can consider this problem as a problem of filtering a signal (such as $w(t)$ shown in this section), where the filtering consists of two stages. One is $H[\omega]$ and the other $P[\omega]$. From linear system theory, we realize that the ordering of the filtering operation is immaterial here. Note that the filtering operation in the above example behaves quite similar to the upsampling/interpolation scenario. To elaborate, let us consider the signal

$$\zeta_q = \sum_{n=-\infty}^{\infty} I_n U\left[q-n\right] \tag{2.65}$$

where $U\left[q\right]$ is the unit step response given by $U\left[q\right] = \begin{cases} 1 & q \geq 0 \\ 0 & \text{otherwise} \end{cases}$. If ζ_q is upsampled by a factor of μ (i.e., sampling interval is $T_s = \frac{T}{\mu}$), the resulting signal is given by

$$\eta_q = \zeta_{\frac{q}{\mu}} = \sum_{n=-\infty}^{\infty} I_n \delta\left[q - n\mu\right] \tag{2.66}$$

where $\delta\left[n\right] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$. Obviously, filtering η_q with a filter with an impulse response $p\left[n\right]$ results in the sampled version of $x\left(t\right)$, given by (1.77). Now, if we choose to filter (2.66) using $H\left[\omega\right]$ first and then pass the resulting signal through a filter with the impulse response $p\left[n\right]$, an identical result to that of the above example is obtained. Hence, $x\left(t\right)$ may be viewed as the result of oversampling/interpolation/filtering of the signal ζ_q.

2.2 Response of Nonlinear Systems to Random Signals

In this section, we tackle a problem which does not allow the use of linear system theory, as discussed in previous section, to analyze the response of the system to random signals. For reasons which will become obvious shortly, these types of problems have received little attention in textbooks on the subject. Although initially one may conclude that nonlinear systems are of limited interest, a careful examination of some routine operations in biomedical or electronics applications in general contradicts this conclusion. To elaborate, virtually all linear devices are comprised of components which are designed to appear linear over a prescribed range of input parameters. For instance, power amplifiers, low-noise amplifiers, or mixers function as linear devices over a limited range of the input signal power level. Hence, one can be satisfied with the linear system theory approach toward characterizing the response of such systems to random inputs as long as one is confined to an appropriate range of system parameters. A more serious problem emerges when the intended operation is a nonlinear one. In biomedical signal processing, the signal of interest, which are used for diagnosis purposes, may experience a number of nonlinear effects before being detected by a receiving device. The nonlinearity may or may not be known to us. Either way, the recovered signal, which may be used to identify an underlying problem, has now gone through a nonlinear effect which will make the analysis of the signal rather difficult. It is also well known that biological systems behave in a 'chaotic' manner in a variety of scenarios. Epileptic brain response is a good example of such a scenario. The chaotic responses of systems may be attributed to their nonlinear characteristics. Hence, the response of nonlinear systems to random signals is of particular interest to those interested in studying chaotic systems. A complete study of nonlinear systems response to random signals is beyond the scope of this text. Therefore, we focus on some basic nonlinearity which can be used to model some common effects in electronic devices, which are often used to collect and process biomedical signals.

For instance, power-of-ν operation may be used to model a vast number of nonlinear devices. In this case, the input-output relationship of the system is given by

$$y[n] = |x[n]|^{\nu} \tag{2.67}$$

or

$$y(t) = |x(t)|^{\nu}. \tag{2.68}$$

The other model, which may allows delay propagation effect, is the delay-and-multiply (D&M) operation, given by

$$y[n] = x^*[n]\, x[n-m] \tag{2.69}$$

or

$$y(t) = x^*(t)\, x(t-T_d) \tag{2.70}$$

for some T_d and m. Note that $T_d = 0$ (or $m = 0$) leads to a special case of power-of-ν device, known as squaring device for $\nu = 2$. The results discussed in the previous sections will fail to model the responses of the system to random signals in these cases. One immediate observation which can be made is that the statistics of the output of such devices involve higher statistics of the input signal, which may or may not be readily available. In particular, when a non-Gaussian input signal is considered, these operations often lead to intractable results. A large class of signals in the engineering field can be modeled as Gaussian. In that case, the statistics of nonlinear signals, as noted earlier, can be obtained from its second order statistics. We, therefore, consider Gaussian signals in the remainder of this section. Although a full treatment of general signals is not presented here, the ensuing analysis presents a roadmap for further analysis of nonlinear devices with non-Gaussian inputs.

2.2.1 Nonlinear Processing of Gaussian Signals

We assume that $x[n]$ (or $x(t)$) is a "proper" zero-mean Gaussian random sequence (or process). That is,

$$E\{x[n]\, x[n-m]\} = 0 \tag{2.71}$$

for all n and m. Furthermore, the correlation function is defined as

$$R_x[n, n-m] = E\left\{x^*[n]\, x[n-m]\right\}. \tag{2.72}$$

The above implies that if

$$x[n] = x_r[n] + jx_i[n],$$

then

$$R_{x_i x_r}[n, n-m] = -R_{x_r x_i}[n, n-m] \tag{2.73}$$

and

$$R_{x_i}[n, n-m] = R_{x_r}[n, n-m]. \tag{2.74}$$

Similarly,

$$E\{x(t)\, x(t-\tau)\} = 0 \tag{2.75}$$

where

$$R_x(t, t-\tau) = E\left\{x^*(t)\, x(t-\tau)\right\} \tag{2.76}$$

This implies that if

$$x(t) = x_r(t) + jx_i(t), \tag{2.77}$$

then

$$R_{x_i x_r}(t, t-\tau) = -R_{x_r x_i}(t, t-\tau) \tag{2.78}$$

and

$$R_{x_i}(t, t-\tau) = R_{x_r}(t, t-\tau) \tag{2.79}$$

for all t and τ.

Since power is related to the second order statistics of a signal, we first consider the first two moments of a signal which has gone through a simple squaring or delay-and-multiply operation. Note that squaring operation is a special case of delay-and-multiply, and hence, we begin with this operation. To that end,

$$\begin{aligned} E\{y[n]\} &= E\left\{x^*[n]\, x[n-m]\right\} \\ &= R_x[n, n-m] \end{aligned} \tag{2.80}$$

or

$$E\{y(t)\} = R_x(t, t - T_d), \tag{2.81}$$

which implies that the first order statistics of the signal depends on the second order statistics of the input. To address the second order statistics of the output, we first re-write $y[n]$ (or $y(t)$) in terms of its the real and imaginary parts. That is,

$$\begin{aligned} y[n] &= y_r[n] + jy_i[n] \\ &= E\left\{x^*[n]\, x[n-m]\right\} \\ &= x_r[n]\, x_r[n-m] + x_i[n]\, x_i[n-m] \\ &\quad + j\left(x_r[n]\, x_i[n-m] - x_i[n]\, x_r[n-m]\right) \end{aligned} \tag{2.82}$$

or

$$\begin{aligned} y(t) &= y_r(t) + jy_i(t) \\ &= E\left\{y^*(t)\, y(t-T_d)\right\} \\ &= x_r(t)\, x_r(t-T_d) + x_i(t)\, x_i(t-T_d) \\ &\quad + j\left(x_r(t)\, x_i(t-T_d) - x_i(t)\, x_r(t-T_d)\right). \end{aligned} \tag{2.83}$$

Hence,

$$y_r[n] = x_r[n]\, x_r[n-m] + x_i[n]\, x_i[n-m] \tag{2.84}$$

$$y_i[n] = x_r[n]\, x_i[n-m] - x_i[n]\, x_r[n-m] \tag{2.85}$$

Similarly,

$$y_r(t) = x_r(t)\, x_r(t-T_d) + x_i(t)\, x_i(t-T_d) \tag{2.86}$$

$$y_i(t) = x_r(t)\, x_i(t-T_d) - x_i(t)\, x_r(t-T_d). \tag{2.87}$$

Hence, in general,

$$\begin{aligned} E\left\{y_r^2[n]\right\} &= E\left\{x_r^2[n]\, x_r^2[n-m]\right\} + E\left\{x_i^2[n]\, x_i^2[n-m]\right\} \\ &\quad + 2E\left\{x_r[n]\, x_r[n-m]\, x_i[n]\, x_i[n-m]\right\}, \end{aligned} \tag{2.88}$$

$$\begin{aligned} E\left\{y_i^2[n]\right\} &= E\left\{x_r^2[n]\, x_i^2[n-m]\right\} + E\left\{x_i^2[n]\, x_r^2[n-m]\right\} \\ &\quad - 2E\left\{x_r[n]\, x_i[n-m]\, x_i[n]\, x_r[n-m]\right\}, \end{aligned} \tag{2.89}$$

and

$$\begin{aligned} E\left\{y_i[n]\,y_r[n]\right\} &= E\left\{x_r^2[n]\,x_i[n-m]\,x_r[n-m]\right\} \\ &\quad - E\left\{x_i^2[n]\,x_i[n-m]\,x_r[n-m]\right\} \\ &\quad + E\left\{x_r[n]\,x_i[n]\,x_i^2[n-m]\right\} \\ &\quad - E\left\{x_r[n]\,x_i[n]\,x_r^2[n-m]\right\}. \end{aligned} \tag{2.90}$$

Similarly, for continuous time,

$$\begin{aligned} E\left\{y_r^2(t)\right\} &= E\left\{x_r^2(t)\,x_r^2(t-T_d)\right\} + E\left\{x_i^2(t)\,x_i^2(t-T_d)\right\} \\ &\quad + 2E\left\{x_r(t)\,x_r(t-T_d)\,x_i(t)\,x_i(t-T_d)\right\}, \end{aligned} \tag{2.91}$$

$$\begin{aligned} E\left\{y_i^2(t)\right\} &= E\left\{x_r^2(t)\,x_i^2(t-T_d)\right\} + E\left\{x_i^2(t)\,x_r^2(t-T_d)\right\} \\ &\quad - 2E\left\{x_r(t)\,x_i(t-T_d)\,x_i(t)\,x_r(t-T)\right\}, \end{aligned} \tag{2.92}$$

and

$$\begin{aligned} E\left\{y_r(t)\,y_i(t)\right\} &= E\left\{x_r^2(t)\,x_i(t-T_d)\,x_r(t-T_d)\right\} \\ &\quad - E\left\{x_i^2(t)\,x_i(t-T_d)\,x_r(t-T_d)\right\} \\ &\quad + E\left\{x_r(t)\,x_i(t)\,x_i^2(t-T_d)\right\} \\ &\quad - E\left\{x_r(t)\,x_i(t)\,x_r^2(t-T_d)\right\}. \end{aligned} \tag{2.93}$$

In general, the above equations can be applied to all signals. One glaring conclusion of the above is that the 4th order statistics of the signals are typically needed in order to proceed. Such statistical information may not be available to us. For Gaussian signals, however, significant simplification is possible. That is, for zero-mean Gaussian random processes, we have (use the characteristic function of a random vector to show this)

$$\begin{aligned} E\left\{x(t_1)\,x(t_2)\,x(t_3)\,x(t_4)\right\} &= E\left\{x(t_1)\,x(t_2)\right\}E\left\{x(t_3)\,x(t_4)\right\} \\ &\quad + E\left\{x(t_1)\,x(t_3)\right\}E\left\{x(t_2)\,x(t_4)\right\} \\ &\quad + E\left\{x(t_1)\,x(t_4)\right\}E\left\{x(t_2)\,x(t_3)\right\}. \end{aligned} \tag{2.94}$$

For the case the signal is non-zero mean, we can define

$$x(t) = m(t) + x'(t) \tag{2.95}$$

where $m(t) = E\{x(t)\}$ and $x'(t)$ is a zero-mean Gaussian random process which is statistically identical to $x(t)$. In that event,

$$\begin{aligned}
E\{x(t_1)\,x(t_2)\,x(t_3)\,x(t_4)\} = {} & E\left\{x'(t_1)\,x'(t_2)\right\}E\left\{x'(t_3)\,x'(t_4)\right\} \\
& + m(t_1)\,m(t_2)\,m(t_3)\,m(t_4) \qquad (2.96) \\
& + m(t_1)\,m(t_2)\,R_{x'}(t_3, t_4) \\
& + m(t_1)\,m(t_3)\,R_{x'}(t_2, t_4) \\
& + m(t_1)\,m(t_4)\,R_{x'}(t_2, t_3) \\
& + m(t_2)\,m(t_3)\,R_{x'}(t_1, t_4) \\
& + m(t_2)\,m(t_4)\,R_{x'}(t_1, t_3) \\
& + m(t_3)\,m(t_4)\,R_{x'}(t_1, t_2)\,. \qquad (2.97)
\end{aligned}$$

where we have used the following property of zero-mean Gaussian random processes:

$$E\left\{x'(t_1)\,x'(t_2)\,x'(t_3)\right\} = 0 \qquad (2.98)$$

for all t_1, t_2, and t_3. Hence, having access to the correlation of the process brings us back to computing $E\left\{x'(t_1)\,x'(t_2)\right\}E\left\{x'(t_3)\,x'(t_4)\right\}$ where $x'(t)$ is a zero-mean process.

For the following discussion, however, we assume that the Gaussian process is zero-mean (as would be the case for any electronic noise or additive noise observed in biomedical signals). Hence, for Gaussian signals, we have

$$\begin{aligned}
E\left\{y_r^2[n]\right\} = {} & R_{x_r}[n, n]\,R_{x_r}[n-m, n-m] + 2R_{x_r}^2[n, n-m] \\
& + R_{x_i}[n, n]\,R_{x_i}[n-m, n-m] + 2R_{x_i}^2[n, n-m] \\
& + 2R_{x_r}[n, n-m]\,R_{x_i}[n, n-m] + 2R_{x_r x_i}[n, n-m]\,R_{x_r x_i}[n-m, n] \\
& + 2R_{x_r x_i}[n, n]\,R_{x_r x_i}[n-m, n-m]\,, \qquad (2.99)
\end{aligned}$$

$$\begin{aligned}
E\left\{y_i^2[n]\right\} = {} & 2R_{x_r x_i}^2[n, n-m] + R_{x_r}[n, n]\,R_{x_i}[n-m, n-m] \\
& + 2R_{x_i x_r}^2[n, n-m] + R_{x_i}[n, n]\,R_{x_r}[n-m, n-m] \\
& - 2R_{x_r}[n, n-m]\,R_{x_i}[n, n-m] - 2R_{x_r x_i}[n, n]\,R_{x_r x_i}[n-m, n-m] \\
& - 2R_{x_r x_i}[n, n-m]\,R_{x_r x_i}[n-m, n]\,, \qquad (2.100)
\end{aligned}$$

and

$$\begin{aligned}
& E\{y_r[n]\,y_i[n]\} = R_{x_r}[n, n-m]\,R_{x_r x_i}[n, n-m] + R_{x_r}[n, n]\,R_{x_r x_i}[n-m, n-m] \\
& + R_{x_r}[n-m, n]\,R_{x_r x_i}[n, n-m] - R_{x_i}[n, n-m]\,R_{x_i x_r}[n, n-m] \\
& - R_{x_i}[n, n]\,R_{x_i x_r}[n-m, n-m] - R_{x_i}[n-m, n]\,R_{x_i x_r}[n, n-m] \\
& + R_{x_i}[n, n-m]\,R_{x_r x_i}[n, n-m] + R_{x_i}[n-m, n-m]\,R_{x_r x_i}[n, n] \\
& + R_{x_i}[n, n-m]\,R_{x_i x_r}[n-m, n] - R_{x_r}[n, n-m]\,R_{x_i x_r}[n, n-m] \\
& - R_{x_r}[n-m, n-m]\,R_{x_r x_i}[n, n] - R_{x_r}[n, n-m]\,R_{x_r x_i}[n-m, n] \qquad (2.101)
\end{aligned}$$

Realizing (2.73) and (2.74), these equations reduce to

$$E\left\{y_r^2[n]\right\} = 6R_{x_r}^2[n, n-m] + 2R_{x_r}[n, n]\,R_{x_r}[n-m, n-m] \\ - 2R_{x_r x_i}^2[n, n-m] + 2R_{x_r x_i}[n, n]\,R_{x_r x_i}[n-m, n-m], \quad (2.102)$$

$$E\left\{y_i^2[n]\right\} = 6R_{x_r x_i}^2[n, n-m] + 2R_{x_r}[n, n]\,R_{x_r}[n-m, n-m] \\ - 2R_{x_r}^2[n, n-m] - 2R_{x_r x_i}[n, n]\,R_{x_r x_i}[n-m, n-m] \quad (2.103)$$

and

$$E\{y_r[n]\,y_i[n]\} = 8R_{x_r}[n, n-m]\,R_{x_r x_i}[n, n-m] \quad (2.104)$$

Similarly,

$$E\left\{y_r^2(t)\right\} = R_{x_r}(t, t)\,R_{x_r}(t-T_d, t-T_d) + 2R_{x_r}^2(t, t-T_d) \\ + R_{x_i}(t, t)\,R_{x_i}(t-T_d, t-T_d) + 2R_{x_i}^2(t, t-T_d) \\ + 2R_{x_r}(t, t-T_d)\,R_{x_i}(t, t-T_d) + 2R_{x_r x_i}(t, t)\,R_{x_r x_i}(t-T_d, t-T_d) \\ + 2R_{x_r x_i}(t, t-T_d)\,R_{x_i x_r}(t, t-T_d), \quad (2.105)$$

$$E\left\{y_i^2(t)\right\} = R_{x_r}(t, t)\,R_{x_i}(t-T_d, t-T_d) + 2R_{x_r x_i}^2(t, t-T_d) \\ + R_{x_i}(t, t)\,R_{x_r}(t-T_d, t-T_d) + 2R_{x_i x_r}^2(t, t-T_d) \\ - 2R_{x_r}(t, t-T_d)\,R_{x_i}(t, t-T_d) - 2R_{x_r x_i}(t, t)\,R_{x_i x_r}(t-T_d, t-T_d) \\ - 2R_{x_r x_i}(t, t-T_d)\,R_{x_i x_r}(t, t-T_d), \quad (2.106)$$

and

$$E\{y_r(t)\,y_i(t)\} = E\left\{x_r^2(t)\,x_r(t-T_d)\,x_i(t-T_d)\right\} \\ - E\left\{x_i^2(t)\,x_r(t-T_d)\,x_i(t-T_d)\right\} \\ + E\left\{x_r(t)\,x_i(t)\,x_i^2(t-T_d)\right\} \\ - E\left\{x_r(t)\,x_i(t)\,x_r^2(t-T_d)\right\} \quad (2.107)$$

Substituting for the expectations in terms of the correlation function of $x_r(t)$ and $x_i(t)$,

$$E\{y_r(t)\,y_i(t)\} = R_{x_r}(t, t)\,R_{x_r x_i}(t-T_d, t-T_d) \\ + 2R_{x_r}(t, t-T_d)\,R_{x_r x_i}(t, t-T_d) \\ - R_{x_i}(t, t)\,R_{x_r x_i}(t-T_d, t-T_d)$$

$$
\begin{aligned}
&- 2R_{x_i}(t, t-T_d)\, R_{x_i x_r}(t, t-T_d) \\
&+ R_{x_i}(t-T_d, t-T_d)\, R_{x_r x_i}(t, t) \\
&+ 2R_{x_i}(t, t-T_d)\, R_{x_r x_i}(t, t-T_d) \\
&- R_{x_r}(t-T_d, t-T_d)\, R_{x_r x_i}(t, t) \\
&- 2R_{x_r}(t, t-T_d)\, R_{x_i x_r}(t, t-T_d)\,, \qquad (2.108)
\end{aligned}
$$

which reduce to

$$
\begin{aligned}
E\left\{y_r^2(t)\right\} &= R_{x_r}(t,t)\, R_{x_r}(t-T_d, t-T_d) + 2R_{x_r}^2(t, t-T_d) \\
&+ R_{x_i}(t,t)\, R_{x_i}(t-T_d, t-T_d) + 2R_{x_i}^2(t, t-T_d) \\
&+ 2R_{x_r}(t, t-T_d)\, R_{x_i}(t, t-T_d) + 2R_{x_r x_i}(t,t)\, R_{x_r x_i}(t-T_d, t-T_d) \\
&+ 2R_{x_r x_i}(t, t-T_d)\, R_{x_i x_r}(t, t-T_d)\,, \qquad (2.109)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left\{y_r^2(t)\right\} &= 6R_{x_r}^2(t, t-T_d) + 2R_{x_r}(t,t)\, R_{x_r}(t-T_d, t-T_d) \\
&- 2R_{x_r x_i}^2(t, t-T_d) + 2R_{x_r x_i}(t,t)\, R_{x_r x_i}(t-T_d, t-T_d)\,, \qquad (2.110)
\end{aligned}
$$

$$
\begin{aligned}
E\left\{y_i^2(t)\right\} &= 6R_{x_r x_i}^2(t, t-T_d) + 2R_{x_r}(t,t)\, R_{x_r}(t-T_d, t-T_d) \\
&- 2R_{x_r}^2(t, t-T_d) - 2R_{x_r x_i}(t,t)\, R_{x_r x_i}(t-T_d, t-T_d)\,, \qquad (2.111)
\end{aligned}
$$

and

$$
E\{y_r(t)\, y_i(t)\} = 8R_{x_r}(t, t-T_d)\, R_{x_r x_i}(t, t-T_d)\,.
$$

It becomes immediately obvious that the output of this device is a complex Gaussian random sequence with correlated real and imaginary parts. In most practical problems, the real and imaginary parts of the input complex Gaussian random sequence (or process) are uncorrelated. Hence, $R_{X_i X_r}[m, n] = 0$ for all n and m. We then have

$$
E\left\{y_r^2[n]\right\} = 6R_{x_r}^2[n, n-m] + 2R_{x_r}[n,n]\, R_{x_r}[n-m, n-m]\,, \qquad (2.112)
$$

$$
E\left\{y_i^2[n]\right\} = 2R_{x_r}[n,n]\, R_{x_r}[n-m, n-m] - 2R_{x_r}^2[n, n-m] \qquad (2.113)
$$

and

$$
E\{y_r[n]\, y_i[n]\} = 0 \qquad (2.114)
$$

which, interestingly enough, implies that the real and imaginary parts of $y[n]$ also become uncorrelated. In the continuous time domain,

$$E\left\{y_r^2(t)\right\} = 6R_{x_r}^2(t, t - T_d) + 2R_{x_r}(t, t)\, R_{x_r}(t - T_d, t - T_d) \tag{2.115}$$

$$E\left\{y_i^2(t)\right\} = 2R_{x_r}(t, t)\, R_{x_r}(t - T_d, t, t - T_d) - 2R_{x_r}^2(t, t - T_d) \tag{2.116}$$

$$E\{y_r(t)\, y_i(t)\} = 0. \tag{2.117}$$

2.2.1.1 WSS Gaussian Input Signal

In this case, which is often encountered when one uses electronic circuits and devices to observe a biomedical signal, the additive signal once processed by the nonlinearity will have the following properties:

$$\begin{aligned} E\left\{y_r^2[n]\right\} &= 6R_{x_r}^2[m] + 2R_{x_r}^2[0] \\ &\quad - 2R_{x_r x_i}^2[m] + 2R_{x_r x_i}^2[0], \end{aligned} \tag{2.118}$$

$$\begin{aligned} E\left\{y_i^2[n]\right\} &= 6R_{x_r x_i}^2[m] + 2R_{x_r}^2[0] \\ &\quad -2R_{x_r}^2[m] - 2R_{x_r x_i}^2[0] \end{aligned} \tag{2.119}$$

and

$$E\{y_r[n]\, y_i[n]\} = 8R_{x_r}[m]\, R_{x_r x_i}[m] \tag{2.120}$$

Furthermore, for the continuous-time domain,

$$\begin{aligned} E\left\{y_r^2(t)\right\} &= 6R_{x_r}^2(T_d) + 2R_{x_r}^2(0) \\ &\quad - 2R_{x_r x_i}^2(T_d) + 2R_{x_r x_i}^2(0), \end{aligned} \tag{2.121}$$

$$\begin{aligned} E\left\{y_i^2(t)\right\} &= 6R_{x_r x_i}^2(T_d) + 2R_{x_r}^2(0) \\ &\quad - 2R_{x_r}^2(T_d) - 2R_{x_r x_i}^2(0), \end{aligned} \tag{2.122}$$

and

$$E\{y_r(t)\,y_i(t)\} = 8R_{x_r}(T_d)\,R_{x_r x_i}(T_d)\,.$$

A further simplification occurs when the input signal is a white random process. In that case, the real and imaginary part of $x[n]$ (or $x(t)$) are uncorrelated and $R_x[n] = 0$ for $n \neq 0$ (or $R_x(\tau) = 0$ for $\tau \neq 0$). Hence,

$$E\left\{y_r^2[n]\right\} = E\left\{y_i^2[n]\right\} = 2R_{x_r}^2[0] \tag{2.123}$$

and

$$E\{y_r[n]\,y_i[n]\} = 0. \tag{2.124}$$

In the continuous domain, we arrive at similar results. That is,

$$E\left\{y_i^2(t)\right\} = E\left\{y_r^2(t)\right\} = 2R_{x_r}^2(0) \tag{2.125}$$

$$E\{y_r(t)\,y_i(t)\} = 0. \tag{2.126}$$

We will examine this case further in the next section.

2.2.2 Nonlinear Processing of WSS Gaussian Processes

As noted earlier, the Gaussian signal case is of particular interest as this noise is often present in all electronic measurements. We briefly tackled this problem earlier. In this section, we consider this case in some detail. Let us consider $y[n]$, given by

$$\begin{aligned} y[n] = {} & \{x_r[n]\,x_r[n-m] + x_i[n]\,x_i[n-m]\} \\ & + j\{x_r[n]\,x_i[n-m] - x_i[n]\,x_r[n-m]\} \end{aligned} \tag{2.127}$$

If one assumes that the input sequence is a zero mean, WSS (a condition which is often satisfied in practice) Gaussian process with the real and imaginary parts of the process being uncorrelated with identical correlation functions (i.e., $R_{x_r}[m] = R_{x_i}[m]$), we have the following simplifications:

$$E\left\{y_r^2[n]\right\} = 6R_{x_r}^2[m] + 2R_{x_r}^2[0] \tag{2.128}$$

$$E\left\{y_i^2[n]\right\} = 2R_{x_r}^2[0] - 2R_{x_r}^2[m] \tag{2.129}$$

and

$$E\left\{y_r[n]\,y_i[n]\right\} = 0. \tag{2.130}$$

Similarly, for the continuous-time case,

$$E\left\{y_r^2(t)\right\} = 6R_{x_r}^2(T_d) + 2R_{x_r}^2(0) \tag{2.131}$$

$$E\left\{y_i^2(t)\right\} = 2R_{x_r}^2(0) - 2R_{x_r}^2(T_d)$$

and

$$E\left\{y_r(t)\,y_i(t)\right\} = 0.$$

This situation is perhaps the most commonly encountered scenario observed in practice, making this analysis of significant importance. As we observed in the previous chapter, a time-domain random process is often projected onto two or more orthogonal eigenfunctions of the process, resulting in uncorrelated random variables (or sequence). For a Gaussian signal, this further implies that the projection results in an independent, identically-distributed (i.i.d) Gaussian random sequence. We then view the discrete case as not only the discrete "time" case, but also the case where the random sequence is the by-product of the projection of the process onto its eigenfunctions, resulting in an uncorrelated random sequence.

Before we discuss the continuous-time case, we need to obtain the correlation of $y[n]$ in order to find the power spectrum density of the output. For this case, we have

$$R_y[n_2, n_1] = E\left\{y^*[n_2]\,y[n_1]\right\} \tag{2.132}$$

$$\begin{aligned} R_y[n_2, n_1] = E\{&(x_r[n_1]\,x_r[n_1 - m] + x_i[n_1]\,x_i[n_1 - m] \\ &+ j\,(x_r[n_1]\,x_i[n_1 - m] - x_i[n_1]\,x_r[n_1 - m])) \\ &\times (x_r[n_2]\,x_r[n_2 - m] + x_i[n_2]\,x_i[n_2 - m] \\ &- j\,(x_r[n_2]\,x_i[n_2 - m] - x_i[n_2]\,x_r[n_2 - m]))\} \end{aligned} \tag{2.133}$$

This expression is rather involved. Although significant simplification will result if one assumes a WSS signal, for the sake of completeness, we will carry out the analysis for non-WSS case and then provide an expression for the case of WSS signal as a special case. We note that although WSS case is quite common, in some cases, the interest may be to obtain long term correlation function of the signal. In such cases, the underlying process that dictates the behavior of the random process may not lend itself to stationarity. In such cases, one has to assumed that the process is no longer WSS. In that event,

$$
\begin{aligned}
R_y[n_2,n_1] = {} & E\{x_r[n_1]x_r[n_1-m,]x_r[n_2]x_r[n_2-m]\} \\
& + E\{x_i[n_1]x_i[n_1-m]x_i[n_2]x_i[n_2-m]\} \\
& + E\{x_r[n_1]x_r[n_1-m]x_i[n_2]x_i[n_2-m]\} \\
& + E\{x_r[n_2]x_r[n_2-m]x_i[n_1]x_i[n_1-m]\} \\
& + E\{x_r[n_1]x_i[n_1-m]x_r[n_2]x_i[n_2-m]\} \\
& + E\{x_i[n_1]x_r[n_1-m]x_i[n_2]x_r[n_2-m]\} \\
& - E\{x_r[n_1]x_i[n_1-m]x_i[n_2]x_r[n_2-m]\} \\
& - E\{x_i[n_1]x_r[n_1-m]x_r[n_2]x_i[n_2-m]\} \\
& + jE\{x_r[n_1]x_i[n_1-m]x_r[n_2]x_r[n_2-m]\} \\
& - jE\{x_i[n_1]x_r[n_1-m]x_r[n_2]x_r[n_2-m]\} \\
& + jE\{x_r[n_1]x_i[n_1-m]x_i[n_2]x_i[n_2-m]\} \\
& - jE\{x_i[n_1]x_r[n_1-m]x_i[n_2]x_i[n_2-m]\} \\
& - jE\{x_r[n_2]x_i[n_2-m]x_r[n_1]x_r[n_1-m]\} \\
& + jE\{x_i[n_2]x_r[n_2-m]x_i[n_1]x_i[n_1-m]\} \\
& - jE\{x_r[n_2]x_i[n_2-m]x_i[n_1]x_i[n_1-m]\} \\
& + jE\{x_i[n_2]x_r[n_2-m]x_i[n_1]x_i[n_1-m]\}
\end{aligned} \tag{2.134}
$$

Noting that $x_r[n]$ and $x_i[n]$ are independent (since uncorrelated), zero-mean Gaussian random sequences and that

$$
E\{x_r[n_1]x_r[n_2]x_r[n_3]\} = E\{x_i[n_1]x_i[n_2]x_i[n_3]\} = 0 \tag{2.135}
$$

for all n_1, n_2, and n_3, we have

$$
\begin{aligned}
R_y[n_2,n_1] = {} & 2E\{x_r[n_1]x_r[n_1-m,]x_r[n_2]x_r[n_2-m]\} \\
& + 2E\{x_r[n_1]x_r[n_1-m]\}E\{x_i[n_2]x_i[n_2-m]\} \\
& + 2E\{x_i[n_1-m]x_i[n_2-m]\}E\{x_r[n_2]x_r[n_1]\} \\
& - 2E\{x_r[n_1]x_r[n_2-m]\}E\{x_i[n_2]x_i[n_1-m]\}
\end{aligned} \tag{2.136}
$$

For WSS Gaussian signal case (note that $R_{x_r}[n]$ is an even function since $x_r[n]$ is real. Note that, by definition, $R_{x_r}[n] = R^*_{x_r}[-n]$, and since $x_r[n]$ is real, $R_{x_r}[n] = R^*_{x_r}[n]$. Hence, $R_{x_r}[n] = R_{x_r}[-n]$ for real random sequence).

$$
R_y[n_2,n_1] = 4R^2_{x_r}[m] + 4R^2_{x_r}[n_1-n_2] \tag{2.137}
$$

Furthermore,

$$
\begin{aligned}
m_y[n] &= E\{y[n]\} = E\{x_r[n]x_r[n-m] + x_i[n]x_i[n-m]\} \\
&= 2R_{x_r}[m]
\end{aligned} \tag{2.138}
$$

which implies that the output sequence remains WSS! Interestingly enough, the output sequence now possesses a non-zero mean even though the input signal is a zero-mean signal. This is not a surprising result since the average of the output of a squaring device, for instance, is proportional to the power of the input signal, which is non-zero for all power signals. Hence, such a device may be viewed as an energy or power detector. There are other (and perhaps more interesting) aspects of the output of nonlinear devices which will be explored in the ensuing chapters.

We also note that

$$\begin{aligned} E\left\{|y[n]|^2\right\} &= E\left\{y_r^2[n]\right\} + E\left\{y_i^2[n]\right\} \\ &= 2R_{x_r}^2[0] + 6R_{x_r}^2[m] + 2R_{x_r}^2[0] - 2R_{x_r}^2[m] \\ &= 4R_{x_r}^2[m] + 4R_{x_r}^2[0], \end{aligned} \tag{2.139}$$

which is consistent with the result given by $R_Y[n_1, n_1]$ in (2.137). Now, if one is interested in the covariance function of the process

$$\begin{aligned} K_y[n_1, n_2] &= R_y[n_1, n_2] - m_y[n_1]\, m_y[n_2] \\ &= 4R_{x_r}^2[n_1 - n_2] \end{aligned} \tag{2.140}$$

For the continuous-time domain, using (2.115), (2.115), and (2.117), we have

$$E\left\{y_r^2(t)\right\} = 2R_{x_r}^2(0) + 6R_{x_r}^2(T_d) \tag{2.141}$$

$$E\left\{y_i^2(t)\right\} = 2R_{x_r}^2(0) - 2R_{x_r}^2(T_d) \tag{2.142}$$

$$R_{y_r y_i}(t_1, t_2) = 0. \tag{2.143}$$

Similarly, using the previous results shown above, we have the following autocorrelation and covariance functions for the output of the delay-and-multiply device for WSS Gaussian input, respectively:

$$R_y(\tau) = 4R_{x_r}^2(T_d) + 4R_{x_r}^2(\tau) \tag{2.144}$$

$$m_y = 2R_{x_r}(T_d) \tag{2.145}$$

$$K_y(\tau) = 4R_{x_r}^2(\tau) \tag{2.146}$$

2.2.3 Output PSD of DM Devices with WSS Gaussian Input

Since PSD is a key tool in studying the behavior of random signals in the frequency domain, we proceed to investigate the output of the nonlinear device discussed above from the PSD point of view. For the discrete domain, using (2.137), we have

$$\begin{aligned} S_y[\omega] &= \sum_{n=-\infty}^{\infty} R_y[n]\, e^{-j\omega n} \\ &= 8\pi R_{x_r}^2[m]\,\delta(\omega) + 4S_{x_r}[\omega] \otimes S_{x_r}[\omega] \end{aligned} \tag{2.147}$$

where $\otimes$ is convolution in the frequency domain. As noted earlier, the power spectrum will have to contain a dc term corresponding to the power of the input signal. Furthermore, due to the convolution operation, the spectrum in the output suffers from a "smearing" or dispersion effect. This is a common trademark of nonlinear devices where the extend of the spectrum content is expanded due to the nonlinear operation.

For continuous-time domain, we have a similar set of results. That is,

$$\begin{aligned} S_y(\Omega) &= \int_{-\infty}^{\infty} R_y(\tau)\, e^{-j\omega\tau} d\tau \\ &= 8\pi R_{x_r}^2(T_d)\,\delta(\Omega) + 4S_{x_r}(\Omega) \otimes S_{X_r}(\Omega) \end{aligned} \tag{2.148}$$

Example 3 Find the PSD of the output of a delay-and-multiple device for the signal

$$x(t) = \sum_{n=-\infty}^{\infty} I[n]\, p(t - nT) \tag{2.149}$$

where $p(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(t-T/2)^2}{2\sigma^2}\right\}$ with $\sigma = \frac{T}{7}$ and I_n is a zero-mean, white sequence which takes on $\{-1, 1\}$ with equal probability. Plot the PSD for $T_d = \frac{T}{7}$, T.

Solution: *We recognize $x(t)$ as a linearly modulated signal, and hence of some considerable importance to us. For this signal, we have obtained the PSD in previous examples. That is, from* (1.176),

$$S_x(\Omega) = \frac{1}{T} S_I[\Omega T]\, |P(\Omega)|^2 \tag{2.150}$$

where, in this case,

$$S_x(\Omega) = \frac{1}{T} |P(\Omega)|^2 \tag{2.151}$$

since the sequence I_n is a white sequence. Furthermore, $|P(\Omega)|^2 = \exp\left(-\sigma^2\Omega^2\right)$. Figure 2.2 depicts a sample of the signal for $I[n] = [+1, +1, -1, -1, -1, -1, +1, +1, +1, -1]$. The value of σ is small enough (as compared with T) so that the adjacent symbol pulses do not overlap significantly, resulting in negligible inter-symbol interference. In Fig. 2.3, the PSD of this signal is depicted. As can be seen, the spectrum stays significant over a bandwidth that stretches from $\frac{-15}{T}$ to $\frac{15}{T}$, resulting in a bandwidth that is almost 30 times the symbol rate. Hence, this pulse shape seems inefficient for a bandwidth-limited application. For biomedical signal processing, however, bandwidth is not a significant consideration. Hence, such pulses can be used to study the behavior of dispersive medium, such as tissue, where the medium will cause the broadening of the pulse. In tissue optics, the use of optical pulses with Gaussian temporal characteristics is common. It is noteworthy that this means of signaling (or signals similar to this) are used to generate ultra wide-band signals (UWB) in signal processing. Now, considering the DM device, the output for such a device is depicted in Fig. 2.4 for a delay of $T_d = T$. Note that the output in this case resembles the original signal, and hence no significant information can be extracted from the signal. Note that $R_{x_r}(T_d = T) = 0$, which implies that the output of DM is a zero-mean signal. A different result emerges when $T_d = \frac{T}{5}$. From Fig. 2.5, we can see that a delay substantially smaller than the symbol time result in a signal whose average value is not zero. In fact, for $T_d = 0$, the mean in the output is $m_Y = 2R_{x_r}(0) = 2P_x$, and hence the DM device may be used to measure the power of the input signal. More importantly, the signal Fig. 2.5 seems to be independent of the modulating sequence and exhibits a periodic behavior at the symbol rate.

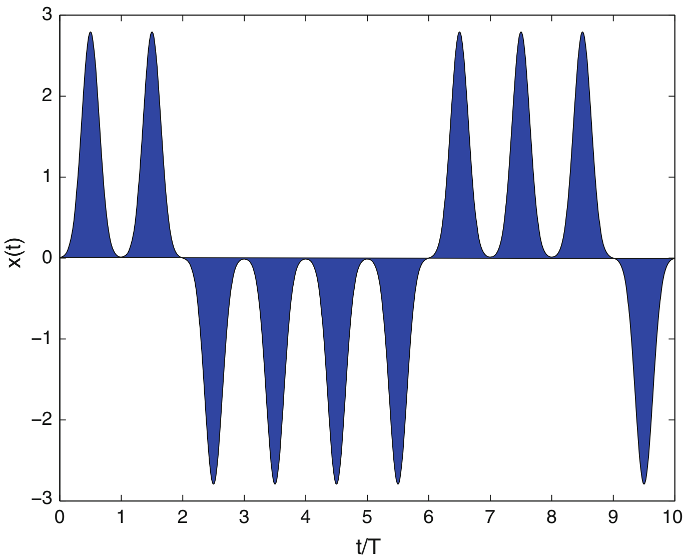


Fig. 2.2 Linearly modulate signal with Gaussian pulse shape for $I_n = [+1, +1, -1, +1, +1, -1, +1, -1, -1]$

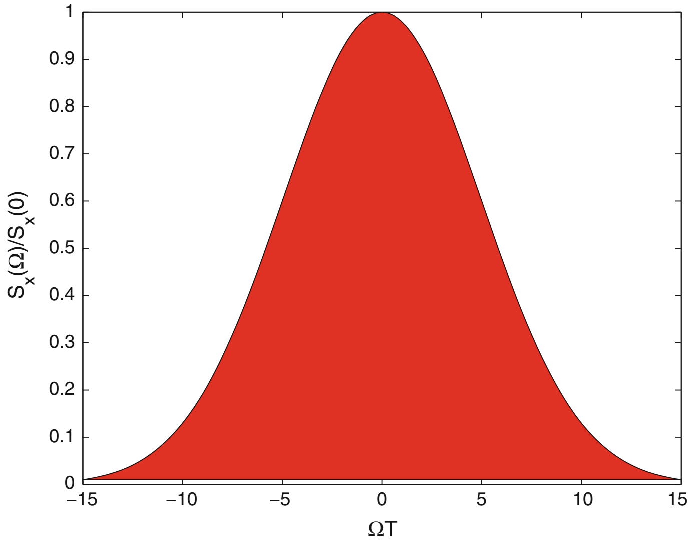


Fig. 2.3 PSD of the linearly modulated signal with Gaussian pulse

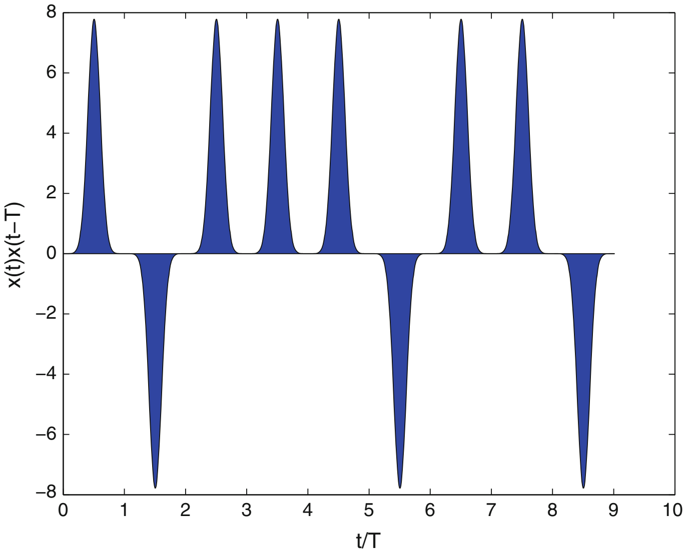


Fig. 2.4 Response of DM device to a linearly modulated Gaussian signal. Delay is assumed to be one symbol duration

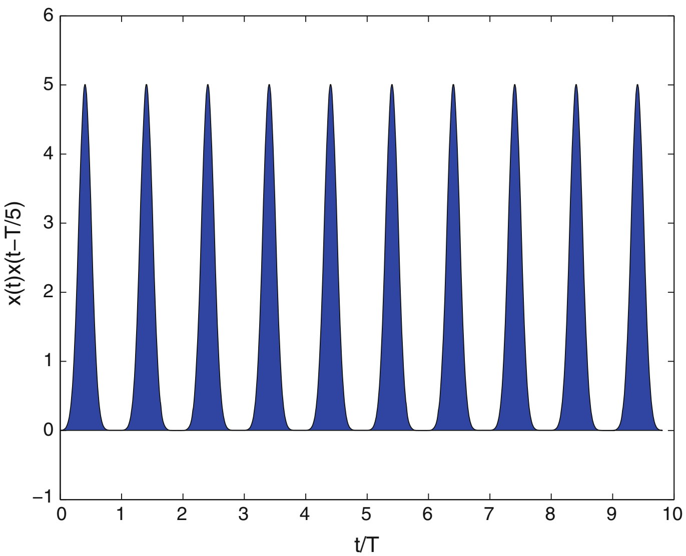


Fig. 2.5 Response of DM device to a linearly modulated Gaussian signal. Delay is assumed to be 20 % of the symbol duration

2.3 Systems with Signal + Noise

In the previous chapters, we have treated signals as random, with deterministic signals classified as a special class of random signals. In practice, electronic and other additive processes often present themselves at the output of an electronic measurement device as an "additive" noise. In biomedical signal processing domain, the additive noise is primarily due to electronic measurement processes, although some underlying biological processes may contribute to this noise. Additive noise is, in general, assumed to be "wideband" as compared with the signal of interest (or the signal that is being measured). That is, the PSD of the additive noise typically spreads over a bandwidth that is larger (if not substantially larger) than that of the "signal" portion of the observed signal.

For this reason, it is instructive to examine this scenario in some depth. Without loss of generality, we consider the discrete-time scenario as most modern systems dealing with biomedical signals take advantage of DSP systems, and hence the very first step of the signal processing chain is a sampling device. Hence, for most signals consider in this text, we arrive at the following signal at the output of the sampling device:

$$x[n] = y[n] + w[n] \tag{2.152}$$

where $y[n]$ denotes the "desired" signal to be measured while $w[n]$ denotes an additive, zero-mean noise (or unwanted) signal that is due to thermal noise in electronic devices or other unknown wideband noise processes in the observation chain. In some scenarios, where the measured signal is transmitted over a wireless channel (it is possible that the biomedical signal is sent over a wireless channel), $w[n]$ is due to background noise (or white noise) that is present in wireless communication links. It is important to note that $y[n]$ may very well be a random signal with known statistical characteristics, whereas $w[n]$ is typically assume to be an additive, zero-mean Gaussian noise signal with known (or measurable) statistics. In the remainder of this section, we assume that all measured signals adhere to this model and that the additive Gaussian noise is an ever present fixture of biomedical signal processing. In almost all cases of interests, it is assumed a statistical independent between $y[n]$ and $w[n]$. We follow the same approach hereafter. This does not limit us in any way as one can account for nonlinearity and other measurement impairments via modeling $y[n]$ in a proper fashion. Namely, $y[n]$ may be viewed as a nonlinear mapping of a desired signal.

It is often of interest to study means of reducing the impact of $w[n]$ on the observation signal. We will address this problem from a spectral point of view followed by a time-domain analysis. To that end, we need to study the PSD of $x[n]$. This is, in general, a difficult problem since one may not have any form of stationarity for $y[n]$. There is, however, one assumption that can be made; that is, $w[n]$ is an additive Gaussian noise, which is independent of $y[n]$. For the scenario that $y[n]$ and $w[n]$ are WSS,

$$R_x[n] = R_y[n] + R_w[n] \tag{2.153}$$

which leads to

$$S_x[\omega] = S_y[\omega] + S_w[\omega] \tag{2.154}$$

where $S_y[\omega]$ and $S_w[\omega]$ are the PSDs' of $y[n]$ and $w[n]$, respectively. This result confirms an intuitive assessment of this problem; that is, when two signals are independent, the power of the sum of the two signals is the sum of the individual power levels. This observation will certainly become invalid when there is nonzero correlation between the two signals. That is, when $y[n]$ and $w[n]$ are not only WSS, but also are jointly WSS

$$\begin{aligned} R_x[n] &= E\left\{x^*[m]\,x[m+n]\right\} \\ &= R_y[n] + R_w[n] + R_{yw}[n] + R_{wy}[n] \\ &= R_y[n] + R_w[n] + R_{yw}[n] + R^*_{yw}[-n] \end{aligned} \tag{2.155}$$

which leads to

$$S_x[\omega] = S_y[\omega] + S_w[\omega] + 2\,\mathrm{Re}\left\{S_{yw}[\omega]\right\} \tag{2.156}$$

with $S_{yw}[\omega]$ denoting the FT of $R_{yw}[n]$ (this is known as the cross PSD).

Now, if we relax the WSS condition imposed on $y[n]$, but keep $w[n]$ as an independent, zero-mean Gaussian noise that is WSS (this is always satisfied in practice), one can use the general definition of PSD to arrive at

$$S_x[\omega] = S_y[\omega] + S_w[\omega] \tag{2.157}$$

where now

$$S_y[\omega] = \lim_{N\to\infty} \frac{1}{2N+1} E\left\{ |y_N[\omega]|^2 \right\} \tag{2.158}$$

with $y_N[\omega] = \sum_{n=-N}^{N} y[n] e^{-j\omega n}$. Hence, the concept of "additive" noise in this case literally implies a scenario where PSD of noise is added to that of the signal.

Example 4 A signal $x[n]$ consists of the sum of a zero-mean, Gaussian noise and a sinusoidal signal with a random phase (uniformly distributed over the range $[-\pi, \pi]$), amplitude A, and frequency f_c Hz, which has been sampled at 4 times the Nyquist rate. The additive noise is independent of the sinusoidal signal and has the following property: $E\{w[n] w^*[m]\} = \sigma^2 \delta[n-m]$. Find the PSD of $x[n]$.

Solution: *From the definition of the problem,*

$$S_x[\omega] = S_y[\omega] + S_w[\omega] \tag{2.159}$$

and

$$S_w[\omega] = \sigma^2 \text{ for all } \omega \tag{2.160}$$

and

$$\begin{aligned} R_y[n,m] &= E\left\{ A^2 \sin(2\pi n f_c T_s + \theta) \sin(2\pi m f_c T_s + \theta) \right\} \\ &= \frac{A^2}{2} \cos(2\pi (n-m) f_c T_s) \end{aligned} \tag{2.161}$$

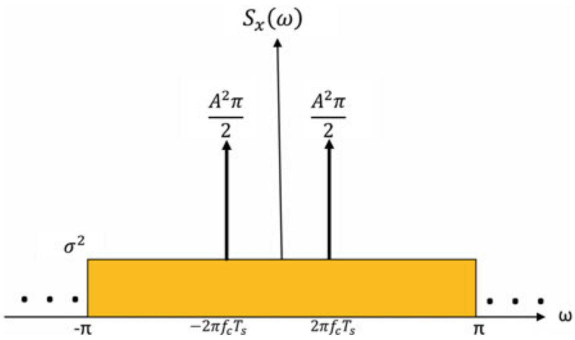
where $T_s = \frac{1}{8f_c}$. *In arriving at this, it is assumed that* $y[n] = A\sin(2\pi n f_c T_s + \theta)$. *Hence,*

$$\begin{aligned} S_y[\omega] &= \frac{A^2\pi}{2} \delta(\omega - 2\pi f_c T_s) + \frac{A^2\pi}{2} \delta(\omega + 2\pi f_c T_s) \\ &= \frac{A^2\pi}{2} \delta\left(\omega - \frac{2\pi}{8}\right) + \frac{A^2\pi}{2} \delta\left(\omega + \frac{2\pi}{8}\right) \end{aligned} \tag{2.162}$$

Furthermore,

$$S_x[\omega] = \sigma^2 + \frac{A^2\pi}{2} \delta\left(\omega - \frac{\pi}{4}\right) + \frac{A^2\pi}{2} \delta\left(\omega + \frac{\pi}{4}\right),$$

Fig. 2.6 The PSD of signal that has been corrupted by AWGN



which is depicted in Fig. 2.6. Note that the additive noise appears as a "white" noise, occupying the entire spectrum, while signal is narrow-band, confined to a very small (in this case, 0) portion of the spectrum. This figure offers a unique perspective as far as the concept of signal detection in noise is concerned; although in time domain one cannot distinguish between the signal and the additive noise, in the frequency domain, the presence of signal can be established in the fact of wideband noise.

Before we leave this section, it is helpful to consider the above example from the "detection of signal in noise" point of view. That is, how can we detect (or extract) signal from the signal+noise scenario described above? To that end, let us consider the above example in time domain. Figures 2.7 and 2.8 depict the signal+noise scenario in time domain where $A = 1$, $\sigma^2 = 0.25$, $\theta = 0$, and $f_c T_s = 1/8$ for 1000 and 300

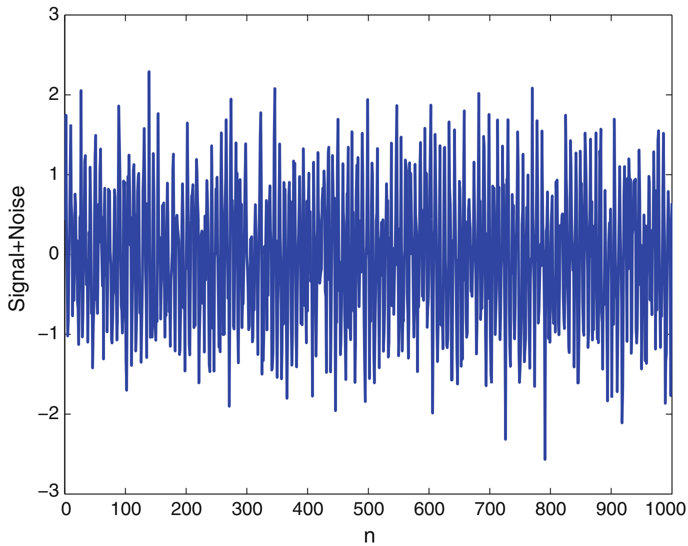


Fig. 2.7 1000-sample view of the signal plus noise waveform for Example 4

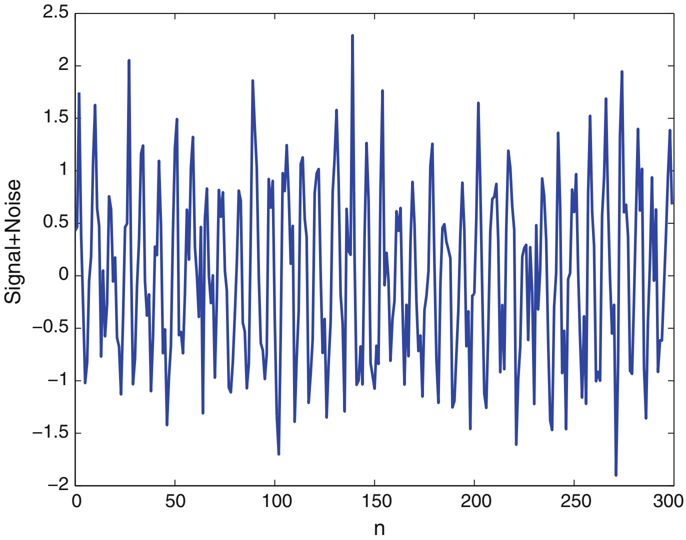


Fig. 2.8 300-sample view of the signal plus noise waveform for Example 4

samples, respectively. A quick glance at these figures reveal a sobering fact; even for a situation where the signal magnitude (A) is 2 times the standard deviation of noise (σ), the periodic signal is entirely buried in noise (which makes the detection of signal rather hard, if not impossible). By comparison, we observed in the previous example that the spectral analysis, such as PSD, allow one to discover a signal which may be hidden in a wideband noise with relative ease. That is, one can clearly identify the signal in Fig. 2.6 from noise, while this task will be cumbersome, if not impossible, using the time series of the signal. In fact, if one is asked the question "*How do you recover the signal here?*," the response of "*I will use a bandpass filter with a bandwidth that is as small as possible with its center frequency fixed at* $2\pi f_c T_s = \frac{\pi}{4}$ *rad*" will be the most likely answer one would receive in view of Fig. 2.6. As we will see in later sections that the above answer is indeed not too far from an "optimum" solution for the detection of signals buried in a wideband noise.

2.3.1 Signal-to-Noise Ratio (SNR)

One reasonable question to ask at this stage is whether one can devise a method to extract $y[n]$ as "accurately" as possible by observing $x[n]$? The term "accurately" is ambiguous at this point as we have not established a metric for accuracy. One well-known and widely used metric is the concept of signal-to-noise ratio (SNR), which as to do with the ration of the "power" of signal to that of the noise. In general, SNR may be defined as follows:

$$SNR_x = \frac{P_s}{P_n} \tag{2.163}$$

where P_s and P_n are the power levels of signal and "noise" portions of $x[n]$, respectively. For the additive scenario considered above (i.e., $x[n] = y[n] + w[n]$),

$$SNR_x = \frac{P_y}{P_w} = \frac{\int_{-\pi}^{\pi} S_y[\omega]\, d\omega}{\int_{-\pi}^{\pi} S_w[\omega]\, d\omega} \tag{2.164}$$

This formulation assumes that the PSD of $y[n]$ and $w[n]$ are available. In time domain, we have

$$SNR_x = \frac{\langle R_y[n,n]\rangle_n}{\langle R_w[n,n]\rangle_n} = \frac{\langle E\left\{|y[n]|^2\right\}\rangle_n}{\langle E\left\{|w[n]|^2\right\}\rangle_n} \tag{2.165}$$

where $\langle z[n]\rangle_n = \lim_{N\to\infty} \frac{1}{2N\_+1} \sum_{n=-N}^{N} z[n]$ may be viewed as the time-average of the enclosed. Note that, one can define an instantaneous SNR as

$$SNR_x[n] = \frac{R_y[n,n]}{R_w[n,n]}, \tag{2.166}$$

and when the additive noise is WSS (i.e., $R_w[n,n] = R_w[0]$), as is the case for most practical scenarios,

$$SNR_x = \langle SNR_x[n]\rangle_n. \tag{2.167}$$

That is,

$$\begin{aligned} SNR_x &= \langle SNR_x[n]\rangle_n \\ &= \langle \frac{R_y[n,n]}{R_w[0]}\rangle_n = \frac{\langle R_y[n,n]\rangle_n}{R_w[0]} \\ &= \frac{\int_{-\pi}^{\pi} S_y[\omega]\, d\omega}{\int_{-\pi}^{\pi} S_w[\omega]\, d\omega} \end{aligned} \tag{2.168}$$

and SNR_x may be viewed as the average SNR. Now, if we perform any filtering operation on $x[n]$, we have

$$SNR_{x_o} = \frac{P_{y_o}}{P_{w_o}} = \frac{\int_{-\pi}^{\pi} S_y[\omega]\, |H_o(\omega)|^2\, d\omega}{\int_{-\pi}^{\pi} S_w[\omega]\, |H_o(\omega)|^2\, d\omega}, \tag{2.169}$$

where we have assumed that ($\circ$ denotes convolution in time)

$$\begin{aligned} x_o[n] &= x[n] \circ h[n] \\ &= y[n] \circ h[n] + w[n] \circ h[n] \\ &= y_o[n] + w_o[n], \end{aligned} \tag{2.170}$$

2.3.2 Matched and Optimum Filtering

A question that comes up at this stage is whether there is an operation that can improve the SNR of $x[n]$. That is, is there an operation that can maximize SNR_x? In general, if such operation exits, then it may very well be a nonlinear operation. For the sake of implementability, we only consider linear operations instead. This is a significant assumption with non-trivial consequence of sacrificing performance for the sake of implementability.

It is noteworthy that *when $w[n]$ is a zero-mean additive Gaussian noise, the operation that will maximize the SNR is indeed a linear operation.* Before we address that problem, we will consider another approach for achieving the best "performance" in the face of additive noise system. As noted in the previous chapter, in addition to SNR, one can establish another criterion for extracting (or estimating) signal in the face of noise: that is, the concept of a system that minimizes mean-square error was shown to yield practical architectures. For the scenario at hand, this implies finding a filter that minimizes the mean-square error in estimating $y[n]$ by observing $x[n]$, given by

$$\begin{aligned} \text{MSE} &= E\left\{|e[n]|^2\right\} \\ &= E\left\{|y[n] - x[n] \circ h_o[n]|^2\right\}. \end{aligned} \tag{2.171}$$

Before we proceed any further, we need to address the stationarity of the processes involved. In particular, we assume that $y[n]$ and $w[n]$ are both WSS (or at least behave as WSS over the range of observation). This assumption is rather restrictive as signals observed in biomedical field do not tend to lend themselves to stationarity However, as was noted, for short observation intervals where the underlying elements that control the statistical behavior of the signal are somewhat stationary, the results presented in the following can be applied to such problems.

As shown previously, an estimate that minimizes the MSE is obtained by using the orthogonality principle, which states that such an estimate renders the error "orthogonal" to the observation. That is,

$$E\left\{e[n]\,x^*[n]\right\} = 0. \tag{2.172}$$

Carrying out the expectation, we arrive at $E\{w[n]\,y^*[m] = 0 \text{ for all } n, m\}$

$$\begin{aligned} E\left\{e[n]\,x^*[n]\right\} &= E\left\{(y[n] - x[n] \circ h_o[n])\,x^*[n]\right\} \\ &= R_y[0] - \sum_{m=-\infty}^{\infty} R_y[m]\,h_o[m] \\ &\quad - \sum_{m=-\infty}^{\infty} R_w[m]\,h_o[m] \end{aligned} \tag{2.173}$$

Taking this into the FT domain,

$$E\left\{e[n]\,x^*[n]\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_y[\omega] - H_o[\omega]\left\{S_y[\omega] + S_w[\omega]\right\} d\omega \tag{2.174}$$

In order for this to be zero,

$$H_o^{(\mathrm{WH})}[\omega] = \frac{S_y[\omega]}{S_y[\omega] + S_w[\omega]} \tag{2.175}$$

This filter is due to Wiener-Hof. The main issue here is whether this filter is causal (therefore, implementable). This all depends on the PSD of signal and noise. For this filter, the MSE (which is the minimum MSE) is given by

$$\begin{aligned} \mathrm{MSE}_{\min} &= E\left\{\left|y[n] - x[n] \circ h_o^{(\mathrm{WH})}[n]\right|^2\right\} \\ &= E\left\{\left(y[n] - x[n] \circ h_o^{(\mathrm{WH})}[n]\right) y^*[n]\right\} \\ &= R_y[0] - \sum_{m=-\infty}^{\infty} R_y[m]\,h_o^{(\mathrm{WH})}[m] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_y[\omega]\,d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} S_y[\omega]\,H_o^{(\mathrm{WH})}[\omega]\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_y[\omega]\left[1 - H_o^{(\mathrm{WH})}[\omega]\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S_y[\omega]\,S_w[\omega]}{S_y[\omega] + S_w[\omega]} d\omega \end{aligned} \tag{2.176}$$

The above results allow one to find an optimum filtering operation. Furthermore, one can obtain the error in estimating the signal using the suggested filtering operation.

Example 5 For the previous example, give an expression for system function of the optimum filter (a filter that maximizes SNR) and comment on its implementability. If the filter is not implementable, suggest a realizable system and compare its output SNR to that of the optimum filter.

Solution: *Based on the above analysis (and Eq. (2.175)), we have*

$$H_o^{(\mathrm{WH})}[\omega] = \frac{A^2\pi\delta\left(\omega - \frac{2\pi}{8}\right) + A^2\pi\delta\left(\omega + \frac{2\pi}{8}\right)}{A^2\pi\delta\left(\omega - \frac{2\pi}{8}\right) + A^2\pi\delta\left(\omega + \frac{2\pi}{8}\right) + 2\sigma^2} \tag{2.177}$$

This is clearly not an easily implementable filter, but an examination of the filter response provides a valuable insight; given the behavior of a delta function, we arrive at the following realization of the filter:

$$H_o^{(\mathrm{WH})}[\omega] = \begin{cases} 1 & |\omega| = \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases} \tag{2.178}$$

This implies that the optimum is a "perfect," zero bandwidth bandpass filter centered at the input frequency! This makes an intuitive sense as the filter should not consider any portion of the spectrum over which the signal is not present. However, such a filter is not realizable. Instead, one can come up with a sub-optimum strategy. That is a BPF with finite, but small bandwidth. Furthermore, linear phase shift must be introduced to insure causality. Specifically, we suggest (the phase does not impact the SNR)

$$\left|H_o^{(\mathrm{AF})}(\omega)\right| = \begin{cases} 1 & \left||\omega| - \frac{\pi}{4}\right| \le \Delta\omega \\ 0 & \text{otherwise} \end{cases} \tag{2.179}$$

This results in the output SNR

$$\begin{aligned} SNR_o^{(\mathrm{AF})} &= \frac{\frac{A^2}{2}}{2\sigma^2\frac{\Delta\omega}{2\pi}} = \frac{A^2/2}{\sigma^2 2\Delta\omega/2\pi} \\ &= \frac{A^2/2}{N_0} = \frac{A^2/2}{\sigma^2/2\pi}\frac{1}{2\Delta\omega} \end{aligned} \tag{2.180}$$

where $N_0 = \sigma^2 2\Delta\omega/2\pi$ may be viewed as the noise power. It is interesting to note that the noise power increases with $\Delta\omega$ (reducing SNR) without improving the signal power. Hence, smaller this parameter, better would be the output SNR. The SNR of the optimum filter is given by

$$\begin{aligned} SNR_o^{(\mathrm{WH})} &= \frac{\int_{-\pi}^{\pi} S_y[\omega]\left|H_o^{(\mathrm{WH})}[\omega]\right|^2 d\omega}{\int_{-\pi}^{\pi} S_w[\omega]\left|H_o^{(\mathrm{WH})}[\omega]\right|^2 d\omega} \\ &= \frac{A^2/2}{0} = \infty \end{aligned}$$

This implies that the optimum filter in this case extracts the signal "perfectly" without any loss (captures all of the signal energy while rejecting all of the noise contribution). This is apparent from the description of the filter. The approximate filter still has a problem; namely, it has an infinite number of taps and is not realizable. A remedy is to introduce delay (linear phase shift) to achieve causality for a truncated response case. That is, in order for the filter to be causal, i.e., $h'[n] = 0$ *for* $n < 0$, *we can construct the filter using*

$$h'[n] = h_0^{(\mathrm{AF})}[n-m] \tag{2.181}$$

for some m such that

$$h_0^{(\mathrm{AF})}[n-m] \approx 0 \text{ for } n < 0. \tag{2.182}$$

Although the above technique seems to provide an optimum solution, its implementability (as shown by the above simple example) is in question. Furthermore, even if one arrives at a realizable solution, it may very well be an infinite impulse response (IIR) solution, which may not be of interest in some practical scenarios. An alternate approach is to seek an finite impulse response (FIR) solution; that is

$$\hat{y}[n] = \sum_{k=0}^{L-1} x[n-k]\, h_n[k]; \tag{2.183}$$

where L is the number of taps (delays) for the FIR filter, which is set by hardware restrictions, processing delay, etc. This operation uses the past history of the observed signal to render a decision on the value of the signal portion of the signal. Note that $h_n[k]$ will have to be computed each time (n) a new data sample has been observed (that is the reason for using the subscript of n in the definition of impulse response, denoted by $h_n[k]$). One can then benefit from the orthogonality principle to arrive at a practical solution. That is, the coefficients of the filter must satisfy

$$E\left\{\left(y[n] - \hat{y}[n]\right) x^*\left[n-k'\right]\right\} = 0;\ \text{ for } k' \in [0, L-1] \tag{2.184}$$

This is a set of L equations with L unknowns, and hence it can readily be solved. To do so, we need to simplify the above as follows:

$$\begin{aligned} R_y\left[n-k',n\right] &= \sum_{k=0}^{L-1} R_x\left[n-k',n-k\right]h_n\left[k\right] \\ &= \sum_{k=0}^{L-1} R_y\left[n-k',n-k\right]h_n\left[k\right] \\ &\quad+\sum_{k=0}^{L-1} R_w\left[n-k',n-k\right]h_n\left[k\right], \end{aligned} \tag{2.185}$$

for $k' \in [0, L-1]$, which reduces to

$$\mathbf{A}\left[n\right]\mathbf{h}\left[n\right] = \mathbf{b}\left[n\right] \tag{2.186}$$

where $\mathbf{A}\left[n\right] = \left\{a_{ij}\left[n\right]\right\}$ and $L \times L$ matrix, $\mathbf{b}\left[n\right] = \left\{b_i\left[n\right]\right\}$ is an $L \times 1$ vector, and $\mathbf{h}\left[n\right] = \left\{h_n\left[i\right]\right\}$ is an $L \times 1$ vector with $a_{ij}\left[n\right] = R_y\left[n-i,n-j\right] + R_w\left[n-i,n-j\right]$ and $b_i\left[n\right] = R_y\left[n-i,n\right]$; $i, j \in [0, L-1]$. Hence, assuming that $\mathbf{A}^{-1}\left[n\right]$ exists, we have

$$\mathbf{h}\left[n\right] = \mathbf{A}^{-1}\left[n\right]\mathbf{b}\left[n\right] \tag{2.187}$$

Note that this solution allows for time variation (it must be computed for all n). Namely, one can utilize this technique over an observation interval and update the solution for the ensuing data frames. Furthermore, WSS is not required! This is particularly of interest in biomedical signal processing as the condition of WSS cannot be always assumed.

Assuming, however, that we have WSS at least over the observation interval, we have the following simplification:

$$\mathbf{h} = \mathbf{A}^{-1}\mathbf{b} \tag{2.188}$$

where $\mathbf{A} = \left\{a_{ij}\right\}$, $\mathbf{b} = \left\{b_i\right\}$, and $\mathbf{h} = \left\{h\left[i\right]\right\}$ with $a_{ij} = R_y\left[i-j\right] + R_w\left[i-j\right]$ and $b_i = R_y\left[i\right]$. Note that this estimation process requires a knowledge of the 2nd order statistics of the signal, which may be obtained from the PSD of the signal. In some applications, one can use spectrum analyzer to estimate the spectrum of a signal. In that event, one can invoke the above process for the estimation of $y\left[n\right]$, assuming that the additive noise $w\left[n\right]$ is independent of the signal.

This solution, although not optimum for all scenarios, provides an implementable strategy which can achieve a quasi-optimum performance for additive Gaussian noise channels.

There is yet another mechanism that one can use to compute $\mathbf{h}[n]$; that is, we can find a solution that maximizes SNR when $w[n]$ is WSS and $y[n]$ is not random and is limited to a length of L samples (this may be an observation interval). This situation may be encountered when the statistical variations to the signal over an observation interval are minute and the signal is corrupted by an additive, WSS noise. Using the definition of SNR provided above, we have

$$\begin{aligned} P_s &= |y_o[n]|^2 \\ &= \left| \sum_{k=0}^{L-1} y[k]\, h_o[n-k] \right|^2 \qquad (2.189) \\ &= \left| \int_{-\pi}^{\pi} Y[\omega]\, H_o[\omega]\, d\omega \right|^2 \qquad (2.190) \end{aligned}$$

$$\begin{aligned} P_n &= E\left\{ |w_o[n]|^2 \right\} \\ &= \sum_{k=0}^{L-1} \sum_{k=0}^{L-1} R_w[k_1 - k_2]\, h_o[k_1]\, h_o^*[k_2] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_w[\omega]\, |H_o[\omega]|^2 \, d\omega \qquad (2.191) \end{aligned}$$

where we have assumed that the filter is also L tap long. Hence,

$$SNR_{x_o} = \frac{\left| \int_{-\pi}^{\pi} Y[\omega]\, H_o[\omega]\, d\omega \right|^2}{\int_{-\pi}^{\pi} S_w[\omega]\, |H_o[\omega]|^2 \, d\omega}$$

A case of interest is when the additive noise is "white", implying that $S_w[\omega] = \sigma^2$; $|\omega| \le \pi$. In that scenario, we have

$$SNR_{x_o} = \frac{\left| \int_{-\pi}^{\pi} Y[\omega]\, H_o[\omega]\, d\omega \right|^2}{\sigma^2 \int_{-\pi}^{\pi} |H_o(\omega)|^2 \, d\omega} \qquad (2.192)$$

Using the Cauchy-Schwartz (CS) inequality, we have

$$\begin{aligned} SNR_{x_o} &\le \frac{\int_{-\pi}^{\pi} |Y[\omega]|^2 \, d\omega \int_{-\pi}^{\pi} |H_o(\omega)|^2 \, d\omega}{\sigma^2 \int_{-\pi}^{\pi} |H_o(\omega)|^2 \, d\omega} \\ &= \frac{\int_{-\pi}^{\pi} |Y[\omega]|^2 \, d\omega}{\sigma^2} = \frac{\sum_{n=0}^{L-1} |y[n]|^2}{\sigma^2} \qquad (2.193) \end{aligned}$$

Interestingly, the maximum possible SNR is the ratio of the power of $y[n]$ to the PSD of the additive noise. More important question is "*How does one achieve the maximum SNR?*" From the CS inequality, we know that the equality is achieved when

$$H_o^{(\mathrm{MF})}(\omega) = Y^*(\omega)\, e^{j\omega(L-1)} \tag{2.194}$$

(the linear phase is added for implementability sake). This simply implies that the filter system function is "matched" to the "shape" of the spectrum of the signal (hence the name matched filter). Using the inverse FT,

$$h_0[n] = y^*[L-1-n] \tag{2.195}$$

If the filter is to be implementable, $h_0[n] = 0$ for $n < 0$, which is insured by the assumption that $y[n] = 0$ for $n \notin [0, L-1]$. If the signal (or observation interval) is finite, this will be an easy condition to satisfy. For infinite observation interval, then one requires an infinite delay (impractical). Hence, a design parameter that needs to be selected carefully is the observation interval, which in turn will define the key parameters of the optimum filtering.

Example 6 For the previous example, find the matched filter and the SNR that it will achieve. Assume that the signal is confined to L taps. That is, $y[n] = A\sin(2\pi n f_c T_s + \theta)$ for $n \in [0, L-1]$, and 0, otherwise.

Solution: *The matched filter is given by* $h_0[n] = y^*[L-1-n]$, *which leads to the following:*

$$h_0^{(\mathrm{MF})}[n] = y[n] = A\sin(2\pi((L-1)-n)f_c T_s + \theta);\ 0 \le n \le L-1$$

and zero, otherwise. This assumes that one "knows" the phase θ *(this is one difficulty in realizing matched filter). The maximum SNR is*

$$SNR_o^{(\mathrm{MF})} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}|Y[\omega]|^2\, d\omega}{\sigma^2/2\pi} = \frac{\sum_{n=0}^{L-1}|y[n]|^2}{\sigma^2}$$

which for a relatively large L leads to

$$SNR_o^{(\mathrm{MF})} = \frac{A^2L/2}{\sigma^2/2\pi}$$

It is interesting to note that the ratio of the matched filter SNR and the SNR of the approximate filter is

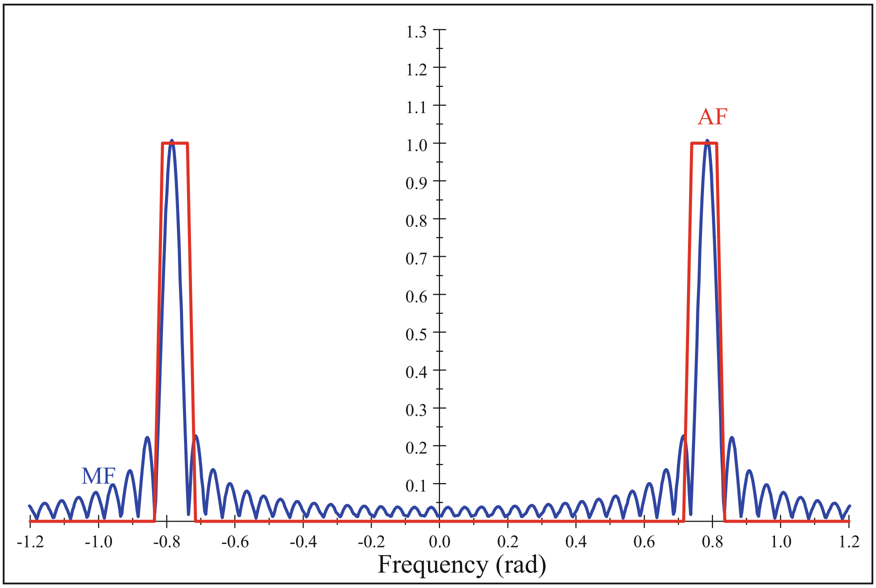
$$\frac{SNR_o^{(\mathrm{AF})}}{SNR_o^{(\mathrm{MF})}} = \frac{1}{2\Delta\omega L}$$

Furthermore, if one lets $\Delta\omega \rightarrow 0$, $SNR_o^{(\mathrm{AF})}\infty$, *which is* $SNR_o^{(\mathrm{WH})}$ *for this signal. It is instructive to note that the WH filter provides the minimum MSE for the noise that may not be white. In that sense, it is the most ideal form of filtering, although its implementability is in question. In this problem, however, we deal with white noise. Hence, we are able to find implementable solutions that achieve near ideal performance.*

For the situations where one is comparing the approximate filter (AF), motived by WH, to the MF, one has to consider $\frac{1}{2\Delta\omega L}$ *as the degradation factor. However, for* $\frac{1}{2\Delta\omega L} > 1$, *the use of AF is advantageous over the MF! This is due to the fact that AF for* $\frac{1}{2\Delta\omega L} > 1$ *is approximating the WH filter closely. Furthermore, an ideal filter with a bandwidth of* $2\Delta\omega$ *has infinite taps, whereas MF used here has only L taps. We can, then, use* $\frac{1}{2\Delta\omega L}$ *to assess the number of taps needed to achieve a performance that is better than that of the AF. It is instructive to consider the system function of the matched filter, which is given by*

$$H_o^{(\mathrm{MF})}(\omega) = \frac{A(L-1)\,e^{j\phi}}{2}\sin\mathrm{c}\left(\frac{(\omega-\omega_0)(L-1)}{2}\right)\exp\left\{-j\frac{\omega(L-1)}{2}\right\} - \frac{A(L-1)\,e^{-j\phi}}{2}\sin\mathrm{c}\left(\frac{(\omega+\omega_0)(L-1)}{2}\right)\exp\left\{-j\frac{\omega(L-1)}{2}\right\}$$

where $\omega_0 = 2\pi f_c T_s$ and $\phi = \theta + \frac{3}{2}(L-1)\,\omega_0$. *In this case,* $\omega_0 = \frac{\pi}{4}$ *(note that* $T_s = \frac{1}{8f_c}$*). As can be see, this filter can be implemented in time domain with relative ease, while a frequency domain implementation is cumbersome. The magnitude of this filter is shown in Figure below. The system function reflects a bandpass filter with theoptimum filter point of view, this filter focuses only on a portion of the spectrum that signal is present. Note that this system function, although similar to the approximate optimum filter, in principle, matches the spectrum of the signal and is not constant over the bandwidth of interest. However, one can argue that the most significant portion of the filter response resides in a null-to-null bandwidth of the spectrum. That is, one can use* $\frac{4\pi}{(L-1)} = 0.0982$ *for* $L = 128$ *as the bandwidth of the matched filter. If we set* $\Delta\omega = \frac{4\pi}{(L-1)} \approx \frac{4\pi}{L}$ *for the AF solution, then* $\frac{SNR_o^{(\mathrm{AF})}}{SNR_o^{(\mathrm{MF})}} = \frac{1}{8\pi}$, *which implies that the MF still outperforms the AF solution by a significant margin.*



$\frac{\left|H_o^{(\mathrm{MF})}(\omega)\right|}{\max\left(\left|H_o^{(\mathrm{MF})}(\omega)\right|\right)}$ and $\frac{\left|H_o^{(\mathrm{AF})}(\omega)\right|}{\max\left(\left|H_o^{(\mathrm{AF})}(\omega)\right|\right)}$ for the sinusoidal signal with $L = 128$ taps, $\omega_0 = \frac{\pi}{4}$, $A = 1$, and $\theta = 0$.

In the above example, of course, one can reduce $\Delta\omega$ for the AF solution to be competitive with MF solution, with the nontrivial cost of an increased impulse response length (hence, delay). However, caution must be exercised when using this comparison. To elaborate, the AF solution assumes that the power of sinusoid is entirely centered around ω_0, which further assumes $L = \infty$. In reality (and for the sake of fairness), one must compare $SNR_o^{(\mathrm{MF})}$ with the SNR of the AF solution when the AF system is excited by an $L-$tap sinusoid (note that as $\Delta\omega \to 0, SNR^{(AF)} \to \infty$ only if the spectrum of the signal is a pair of delta functions centered at $\pm\omega_0$ $(L = \infty)$, whereas the signal considered here will render $SNR^{(\mathrm{AF})} \to 0$ as $\Delta\omega \to 0$). In that event, the SNR of the AF solution will be even smaller than that computed above. For these reasons, the use of MF is highly recommended as it offers an implementable architecture (an FIR implementation with filter coefficients (i.e., $h[n]$) which can be stored and used in a DSP platform) and offers an SNR that is superior to that of the other linear filters.

Finally, note that the MF solution resembles a bandpass filter centered around $\pm\omega_0$. This was our "intuitive" answer to the question "*How do you recover the signal here?*" which was posed in the System with Signal + Noise section of this book. We caution the reader about using intuition for all cases of signal detection indiscriminately. One key reason that our intuition leads us to an optimum structure is due to the fact that noise presents itself as a "white" noise with identical power levels at all frequencies in this scenario. If one deviates from this assumption, the resulting optimum device will surely not follow our intuition, and in fact, the optimum solution may very well be counter-intuitive. In that event, one has to rely entirely on the optimum receiver principle presented earlier to arrive at a realizable implementation (or realizable approximation to the possibly unrealizable optimum solution).

http://www.springer.com/978-3-642-54878-9

Biological Signals Classification and Analysis
Kiasaleh, K.
2015, XI, 380 p. 231 illus., 222 illus. in color., Hardcover
ISBN: 978-3-642-54878-9