2 Theoretical Background

This chapter presents the theoretical principles for this thesis. On the one hand, the Markowitz portfolio theory is outlined, which is the basis for the portfolio models. To obtain the investor's individual portfolio, the principles of the utility theory are needed, which are also presented. Moreover, the consistency of the Expected Utility Theory with the Markowitz portfolio theory is displayed. On the other hand, the asset pricing theory is presented, which is the theoretical explanation for different asset returns in the capital market and the fundament for testing return anomalies.

2.1 Modern Portfolio Theory

The foundation of the portfolio models consists of the modern portfolio theory of Markowitz and its extension through Tobin. Therefore the basis is outlined at first, before the new 'risk portfolio models' are presented in the next chapter.

2.1.1 Mean Variance Framework

Markowitz (1959) mean-variance optimization is the classical technique to allocate capital among a set of assets (Michaud, 1998, p. 1). Since the return is measured by the expected value of the random portfolio return, while the risk is quantified by the variance of the portfolio return, it is called mean-variance framework (Recchia, 2010, p.14). The portfolio allocation process implies the conflicting goals, return maximizing and risk minimizing. Markowitz was the first to show theoretically the observed diversification effect, that is, the reduction of the risk through splitting the capital to different

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assets. Given the returns, variances and correlations of the assets, the mean-variance approach allows to determine efficient portfolios through maximizing the return while constraining risk or minimizing the risk subject to a desired target return.

To outline the theory, at first, it is important to operationalize the relevant characteristics, that is returns and variances of the assets as well as of the portfolio. The returns can be calculated as discrete or logarithmic returns (Poddig et al., 2005, p. 31, p. 35):

$$r_t^D = \frac{p_t - p_{t-1}}{p_{t-1}} \tag{2.1}$$

with

$$\begin{aligned} r_t^D &: \text{discrete return for the period } t-1 \text{ until } t \\ p_t &: \text{asset price at time } t \\ p_{t-1} &: \text{asset price at time } t-1 \end{aligned}$$

$$r_t^S = ln\left(\frac{p_t}{p_{t-1}}\right) = ln(p_t) - ln(p_{t-1}) \tag{2.2}$$

with

$$\begin{aligned} r_t^S &: \text{logarithmic return for the period } t-1 \text{ until } t \\ p_t &: \text{asset price at time } t \\ p_{t-1} &: \text{asset price at time } t-1 \end{aligned}$$

Whereas in case of discrete returns, a discrete compounding of the capital is assumed (once at the end of the calculation period), in case of logarithmic returns a continuous compounding of the capital is assumed. The advantage of using the logarithmic returns lies in the transformation of the returns into different periods and the statistical properties. E.g., if there are daily returns, the monthly return can be simply calculated as the sum of the daily returns in this month. This is not so simple in case of discrete returns. Moreover, continuous returns display a symmetric density and are more in line with normal distribution, which is an assumption in many financial theories (Poddig et al., 2003, p. 105). However, discrete returns are easier to interpret. But both return calculations can simply be transformed into another and the differences are small if short periods are used.

The forecast of the mean return requires the knowledge of the returns in different scenarios and the probabilities of these scenarios (Poddig et al., 2005, p. 43):

$$\mu_i = \sum_{j=1}^{Z} p_j \cdot r_{ij} \tag{2.3}$$

with

μ_i : expected return of asset i
Z : amount of possible scenarios
p_j : probability of occurrence of scenario j
r_{ij} : return of asset i in scenario j

To measure the risk, Markowitz (1952) uses the variance. This is given by (Poddig et al., 2005, p. 44)

$$\sigma_i^2 = \sum_{j=1}^{Z} p_j (r_{ij} - \mu_i)^2 \tag{2.4}$$

$$\sigma_i = \sqrt{\sigma_i^2} \tag{2.5}$$

with

$$\sigma_i^2 : \text{variance of asset } i$$
$$\sigma_i : \text{standard deviation of asset } i$$

However, as the amount of possible scenarios and the probabilities of the scenarios are mostly unknown, the mean and the variance are estimated through historical returns (Poddig et al., 2005, p. 128f):

$$\mu_i = \frac{1}{T}\sum_{t=1}^{T} r_{it} \tag{2.6}$$

with

$$\mu_i : \text{empirical mean of asset } i$$
$$T : \text{amount of returns}$$

$$\sigma_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \mu_i)^2 \tag{2.7}$$

with

$$\sigma_i^2 : \text{empirical variance of asset } i$$
$$\sigma_i : \text{empirical standard deviation of asset } i$$

The expected portfolio return is calculated through the sum of the products of the asset returns with their asset weight in the portfolio (Poddig et al., 2005, p. 47):

$$\mu_p = \sum_{i=1}^{n} w_i \mu_i \tag{2.8}$$

with

$$\mu_p : \text{expected return of portfolio } p$$
$$w_i : \text{weight of asset } i$$
$$\mu_i : \text{mean return of asset } i$$

Or in vector form:

$$\mu_p = \boldsymbol{w}'\boldsymbol{r} \tag{2.9}$$

with

$$\boldsymbol{w} : \text{nx1 vector of asset weights}$$
$$\boldsymbol{r} : \text{nx1 vector of mean asset returns}$$

The portfolio variance is calculated as follows (Poddig et al., 2005, p. 51f):

$$\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} w_i w_j \sigma_{ij}. \tag{2.10}$$

with

$$\sigma_p^2 : \text{portfolio variance}$$
$$\sigma_i^2 : \text{variance of asset } i$$
$$w_i : \text{weight of asset } i$$
$$\sigma_{ij} : \text{covariance of asset } i \text{ and } j$$

where the empirical covariance between asset i and j is given by

$$\sigma_{ij} = \frac{1}{T-1} \sum_{i=1}^{T} (r_{it} - \mu_i)(r_{jt} - \mu_j). \tag{2.11}$$

with

$$\sigma_{ij} : \text{covariance of asset } i \text{ and } j$$

The portfolio variance is calculated in vector form as

$$\sigma_p^2 = \boldsymbol{w}'\boldsymbol{V}\boldsymbol{w} \tag{2.12}$$

with

$$\boldsymbol{V} : \text{variance-covariance matrix}$$

The observed diversification effect, which can reduce the portfolio risk through building a portfolio and has induced Markowitz (1959) portfolio theory, can be summarized as follows (Poddig et al., 2005, p. 53f). Assume a naive portfolio, where all n assets have the same weight $1/n$.

Then the portfolio variance is given by (Poddig et al., 2005, p. 54)

$$\sigma_p^2 = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma_i^2 + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{1}{n}\frac{1}{n}\sigma_{ij}. \tag{2.13}$$

with

$$\begin{aligned} \frac{1}{n} &: \text{asset weight} \\ \sigma_p^2 &: \text{portfolio variance} \\ \sigma_i^2 &: \text{variance of asset } i \\ \sigma_{ij} &: \text{covariance of asset } i \text{ and } j \end{aligned}$$

Rearranging the equation leads to

$$\begin{aligned} \sigma_p^2 &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sigma_i^2 + \frac{n-1}{n}\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i \neq j}}^{n}\frac{1}{n(n-1)}\sigma_{ij} \\ &= \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \frac{n-1}{n}\left(\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i \neq j}}^{n}\sigma_{ij}\right). \end{aligned} \tag{2.14}$$

Both bracket terms can be interpreted as the mean of the variances and the mean of the covariances in the portfolio:

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}_{Var}^2 + \frac{n-1}{n}\bar{\sigma}_{Cov} \tag{2.15}$$

with

$\bar{\sigma}_{Var}^2$: average variance of an asset in portfolio p

$\bar{\sigma}_{Cov}$: average covariance of the assets in portfolio p

If the amount of assets in the portfolio gets large, $n \to \infty$, then

$$\frac{1}{n}\bar{\sigma}_{Var}^2 \to 0, \tag{2.16}$$

$$\frac{n-1}{n}\bar{\sigma}_{Cov} \to \bar{\sigma}_{Cov} \quad \text{and thus} \tag{2.17}$$

$$\sigma_p^2 \to \bar{\sigma}_{Cov} \tag{2.18}$$

The risk of the portfolio can be divided in two parts: the unsystematic, asset specific risk and the systematic, covariance or market risk. Whereas the first component can be diversified through building a portfolio, the second component influences all assets and remains with the investor.

This effect explains why it is wiser to invest in an portfolio rather to hold individual assets. Based on this effect, Markowitz (1959) formulates the Modern Portfolio Theory or the Mean-Variance Framework.

The Mean-Variance Framework has the following assumptions:

- Investors care only about mean and standard deviation of asset returns
- Investors are risk averse (prefer same return for less risk or higher return with same risk, the aspect of risk aversion is also an important part of the utility theory in section 2.1.3)

More formally, let a and b be two different Portfolios.

- *Portfolio a dominates portfolio b, if it has a higher expected return with the same variance or a smaller variance with the same expected return or both (Poddig et al., 2005, p. 79):*

$$\mu(a) \geq \mu(b),\ if\ \sigma^2(a) = \sigma^2(b),$$

or

$$\sigma^2(a) \leq \sigma^2(b),\ if\ \mu(a) = \mu(b),$$

or

$$\mu(a) \geq \mu(b),\ and\ \sigma^2(a) \leq \sigma^2(b).$$

- *The portfolio is efficient, if there does not exist another portfolio, which dominates it.*

The efficient frontier represents all efficient portfolios. Markowitz assumes risk-averse investors, as the choice of the efficient portfolios depends on the two assumptions, that investors prefer a higher return vs. a lower return given the same level of risk, and a lower risk of a portfolio with the same return.

The objective of the portfolio theory is to obtain efficient portfolios. As it is obvious from the dominance- and efficiency-criteria, a trade-off between return and risk exists. To obtain a portfolio on the efficient frontier, the following optimization problem has to be solved (Poddig et al., 2005, p. 81):

$$min \quad \sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

or in matrix-form

$$min \quad \sigma_P^2 = \boldsymbol{w'Vw}$$

subject to

$$\sum_{i=1}^{N} w_i \mu_i = r^* \quad or \quad \boldsymbol{w'r = r^*}$$

and

$$\sum_{i=1}^{N} w_i = 1; \quad w_i \geq 0, \quad for\, all\, i = 1, ..., N$$

with

σ_P^2 : portfolio volatility
r^* : target return
$\boldsymbol{w}$: weights vector
$\boldsymbol{V}$: covariance matrix

Through varying the target return r^*, the efficient frontier can be obtained point by point. On the left and right end of the efficient frontier lies the Minimum-Variance (MV) portfolio and the Maximum-Return portfolio. For the MV portfolio, the risk is minimized without considering the return and for the Maximum-Return portfolio, the return is maximized without considering the risk (Poddig et al., 2005, p. 109f):

Minimum-Variance Portfolio

$$min \quad \sigma_P^2 = \boldsymbol{w'Vw} \tag{2.19}$$

subject to

$$\sum_{i=1}^{N} w_i = 1; \quad w_i \geq 0, \quad for\, all\, i = 1, ..., N$$

Maximum-Return Portfolio

$$\mu_P = \boldsymbol{w'r} \Rightarrow max!$$

subject to

$$\sum_{i=1}^{N} w_i = 1; \quad w_i \geq 0, \quad for\, all\, i = 1, ..., N$$

Alternative formulations of the optimization problem lead also to efficient portfolios and are common in practice. Instead of minimizing the risk, one can maximize the return subject to a risk constraint (Fabozzi, 2007, p. 34):

$$max \quad \boldsymbol{w'\mu} \tag{2.20}$$

subject to

$$\boldsymbol{w'Vw} = \sigma^{2*}$$
$$\sum_{i=1}^{N} w_i = 1; \quad w_i \geq 0, \quad for\, all\, i = 1, ..., N$$

Figure 2.1 illustrates the efficient frontier with 100 points (different target returns) for a dataset of nine assets (Swiss pension fund benchmark, dataset LPP2005.RET of the R package fPortfolio (Würtz and Rmetrics Core Team, 2011)). The points represent random generated (inefficient) portfolios, whereas the square represents the Minimum Variance Portfolio (2.19). (The other points represent the portfolio models which are explained in chapter 3.2: the circle is the tangency portfolio, the triangle the naive portfolio and the rhombus the equally risk contribution portfolio based on the covariance risk budgets.)

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