

Preface

Usually, dual unit quaternions are used to describe Euclidean displacements in three-dimensional kinematics. Engineers widely use this elegant calculus next to a real projective geometric model called Study's quadric. Moreover, there exists a "sphere-model" in a four-dimensional module space over the ring of dual numbers. However, there are no investigations connecting these three models up to now. After recalling Study's quadric, the "sphere-model", and other models, we present a survey-like introduction to Clifford algebras and their Spin and Pin groups. The advantage of Clifford algebras is that geometric objects and transformation may be represented as algebra elements. Using the so-called sandwich operator, it is possible to apply transformations that are represented by algebra elements directly to geometric objects, also described by algebra elements. We introduce the homogeneous and conformal Clifford algebra model. Furthermore, we show how to model special Cayley-Klein geometries and their isometry groups in a homogeneous Clifford algebra model. As an example, we focus on the homogeneous Clifford algebra model corresponding to line geometry and derive the correspondence between projective transformations of three-dimensional projective space and the Pin group $\text{Pin}_{(3,3,0)}$. Therefore, we introduce the Clifford algebra $\mathcal{Cl}_{(3,3,0)}$, constructed over the quadratic space $\mathbb{R}^{(3,3)}$, and describe how points on Klein's quadric are embedded as null vectors. We discuss how geometric entities that are known from Klein's model, *i.e.*, linear line manifolds can be transferred to this homogeneous Clifford algebra model. All entities known from line geometry occur naturally in this model and can be transformed projectively by the application of the sandwich operator. The action of grade-1 elements corresponds to the action of null polarities on $\mathbb{P}^3(\mathbb{R})$, *i.e.*, correlations that are involutions as the basic elements building up the group of regular projective transformations. It is proven that every regular projective transformation of $\mathbb{P}^3(\mathbb{R})$ can be expressed as the product

of six null polarities, *i.e.*, skew-symmetric 4×4 matrices at the most. The results achieved for Klein's quadric may be transferred to any quadric, hence, we present the homogeneous Clifford algebra model corresponding to Lie sphere geometry as an example. Additionally, a new geometric algebra allowing the description of inversions with respect to quadrics in principal position as Pin group is presented. This model serves as a generalization for the conformal geometric algebra and is constructed for dimension two and three in detail. Furthermore, the generalization to arbitrary dimension is shown.

A further focus of the thesis is applying chain geometry to Clifford algebras in order to examine the cross ratio in Clifford algebras. It is well known that the cross ratio of four complex numbers is real if, and only if, they all lie on a Möbius circle, *i.e.*, a circle or a line augmented by a point at infinity. A generalization of the so called Möbius geometry is obtained by using different algebras instead of complex numbers. This leads to a branch of geometry called chain geometry. Chains are subsets of the projective line over a ring which can be parametrised with the cross ratio. Therefore, it is natural to apply this theory to dual quaternions and to examine the kinematic and geometric interpretation. A more general point of view can be achieved by the use of Clifford algebras and Spin groups instead of dual quaternions and dual unit quaternions. After recalling the fundamental chain geometric background, we define the cross ratio for Clifford algebras and their Pin and Spin groups. We present a quadric model corresponding to the dual unit quaternions and homogeneous Clifford algebra models of Klein's, Study's and Lie's quadric where chains that are contained in the grade-1 subspace correspond to conic sections. Moreover, we derive an algebraic biarc construction with the help of contact spaces. Chains of the grade-1 subspace that are in contact at a certain point are parametrized with the cross ratio and correspond to conic sections. Moreover, it has been proven that the connected components of the Pin- and Spin groups define subspaces of chain spaces. Every element contained in a chain defined by three elements of the same connected component of the Pin- or the Spin group is contained in the same connected component of the Pin- or the Spin group. The question for the cross ratio of dual unit quaternions has been answered in detail.

In the last chapter we recall the concept of kinematic mappings. Different kinematic mappings have been found, for example the kinematic mapping of STUDY that maps the group of Euclidean displacements to Study's quadric, or the kinematic mapping of Blaschke and Grünwald. Furthermore, we use the Clifford algebra calculus to unify different kinematic mappings. With this construction it is possible to unify the concept of kinematic mappings for isometry groups of Cayley-Klein geometries and for orthogonal groups $SO(p, q)$. Collineations in any kinematic image and the corresponding Cayley-Klein space can be derived from the homogeneous Clifford algebra model. The kinematic images of Pin- and Spin groups are projective varieties. Due to the fact that projective varieties correspond to ideals, methods of Gröbner basis calculus can be applied to kinematic image spaces. We present kinematic mappings for Euclidean spaces of dimensions two, three and four in detail.

Dresden

Daniel Klawitter

Clifford Algebras

Geometric Modelling and Chain Geometries with
Application in Kinematics

Klawitter, D.

2015, XVIII, 216 p. 18 illus., 10 illus. in color., Softcover

ISBN: 978-3-658-07617-7