2 Chain Geometry over Clifford Algebras

In the following chapter we give an introduction to chain geometry. It is not the aim of this work to give a complete treatise of this topic, we just introduce the concepts we need for our purposes. For a more detailed introduction the reader is referred to [11]. The roots of chain geometry can be found in Benz [5]. Benz investigated projective lines over commutative two-dimensional algebras and the corresponding chain geometries. A more recent treatise is [33].

To provide the preliminaries we follow [11] in order to define the setting that is necessary for our purposes. After this introduction to chain geometry and chain geometric concepts we focus on chain geometry over Clifford algebras. We introduce chain geometries over Pin and Spin groups. Moreover, we show that the connected components of the Pin and the Spin group define sub chain geometries of the Clifford algebra. Furthermore, we show that subgroups of these groups also define sub chain geometries. For application to kinematics we study the chain geometry over the Clifford algebra $\mathcal{C}\ell_{(3,0,1)}$ and its Spin group in detail and classify the occurring chains.

2.1 Chain Geometry

2.1.1 Distance Spaces

In this section we recall the concept of a distance space and give some examples that we use later on. Projective lines over rings serve as examples for distance spaces.

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Definition 2.1. *A pair* $(\mathcal{P}, \Delta)$*, where* $\mathcal{P}$ *is a non empty set and* $\Delta \subseteq \mathcal{P} \times \mathcal{P}$ *is a relation on* $\mathcal{P}$ *is called a* distance space *if the following conditions are satisfied:*

(1) Δ *is a* symmetric *relation, i.e.,* $a\Delta b \Rightarrow b\Delta a$*,*

(2) Δ *is an* anti-reflexive *relation, i.e.,* $\neg(a\Delta a)$ *is valid for* $a \in \mathcal{P}$*.*

The elements of $\mathcal{P}$ *are called points and the relation* Δ *is the* distance relation*. Two points* $a, b \in \mathcal{P}$ *are called distant if* $a\Delta b$*.*

Definition 2.2. *Elements* a, b *that are not distant are called parallel* $(a \parallel b)$*. The parallel relation is symmetric and reflexive.*

Examples for distance spaces will be given in the following sections.

2.1.2 The projective Line over an $\mathcal{L}$-algebra

First, we give the definition of an $\mathcal{L}$-algebra.

Definition 2.3. *Let* $\mathcal{L}$ *be a subring of* $\mathcal{R}$ *contained in the* center $\mathcal{C}(\mathcal{R})$*, where*

$$\mathcal{C}(\mathcal{R}) := \{a \in \mathcal{R} \mid ax = xa \text{ for all } x \in \mathcal{R}\}\,.$$

We call $\mathcal{R}$ *an* $\mathcal{L}$-algebra*. An* $\mathcal{L}$*-algebra* $\mathcal{R}$ *is a module over the commutative ring* $\mathcal{L}$*.*

Remark 2.1. *In the most cases we deal with algebras over the real numbers. In this case we have* $\mathcal{L} = \mathbb{R}$ *and we call the algebra an* $\mathbb{R}$*-algebra.*

Definition 2.4. *Let* $\mathcal{R}$ *be a* $\mathcal{K}$*-algebra over the field* $\mathcal{K}$*.*

(1) For $x \in \mathcal{R}$ *we define* $e(x) := \{a \in \mathcal{K} \mid x + a \in \mathcal{R}^\times\}$*.*

(2) The $\mathcal{K}$*-algebra* $\mathcal{R}$ *is called* strong*, if for all* $x \in \mathcal{R}$ *the inequality* $\mathrm{card}(e(x)) > \mathrm{card}(\mathcal{K} \backslash e(x))$ *holds.*

Definition 2.5. *The* general linear group $\mathrm{GL}(\mathcal{R}, 2)$ *over a ring* $\mathcal{R}$ *is defined as the set of invertible* 2×2 *matrices with entries in* $\mathcal{R}$*.*

With the general linear group we can now define the projective line over a ring $\mathcal{R}$.

Definition 2.6. *The projective line over a ring $\mathcal{R}$ is defined as the set $\mathbb{P}^1(\mathcal{R})$ of all cyclic submodules $\mathcal{R}(a,b)$ of $\mathcal{R}^2$, where (a,b) is the first row of an invertible 2×2 matrix over $\mathcal{R}$:*

$$\mathbb{P}^1(\mathcal{R}) := \left\{ \mathcal{R}(a,b) \mid \exists c,d : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\mathcal{R},2) \right\}.$$

Such pairs are called admissible, *see [32]. Note, that we define points of the projective line as left-homogeneous equivalence classes.*

Furthermore, we define equivalence classes of admissible pairs, respectively points.

Definition 2.7. *Two points $(a,b),(c,d) \in \mathbb{P}^1(\mathcal{R})$ are equivalent if there is a unit $r \in \mathcal{R}^\times$ with $ra = c$ and $rb = d$. This relation is an equivalence relation and denoted by $\sim$. For an equivalence class we write $\mathcal{R}(a,b)$.*

The homogeneous component of the pair is the second coordinate. This means the algebra element 0 has the projective coordinates $U = \mathcal{R}(0,1)$, the neutral element 1 has the coordinates $V = \mathcal{R}(1,1)$, and the projective point $W = \mathcal{R}(1,0)$ corresponds to an ideal point. In the following we denote points of the projective line $\mathbb{P}^1(\mathcal{R})$ with capital latin letters.

Remark 2.2. *If $\mathcal{R}$ is a field, Def. 2.6 describes the classical projective line.*

From Def. 2.6 we see, $\mathcal{R}(1,0)$ is a point, since the 2×2 identity matrix is in $\mathrm{GL}(\mathcal{R},2)$. If we take a general matrix $\mathrm{M} \in \mathrm{GL}(\mathcal{R},2)$ all points $P \in \mathbb{P}^1(\mathcal{R})$ can be obtained by $\mathcal{R}(1,0) \cdot \mathrm{M}$. Hence, the point set $\mathbb{P}^1(\mathcal{R})$ can also be described as the orbit of $\mathcal{R}(1,0)$ under the action of $\mathrm{GL}(\mathcal{R},2)$. Two points $X = \mathcal{R}(x_1,x_0)$, $Y = \mathcal{R}(y_1,y_0)$ are called distant $(X \Delta Y)$ if they are complementary, this means if $\mathcal{R}^2 = X \oplus Y$. Hence, $(\mathbb{P}^1(\mathcal{R}),\Delta)$ is a distance space called the projective line over $\mathcal{R}$.

Remark 2.3. *We consider only algebras where the left-inverse elements are also right-inverse elements. Thus, every point may be represented by an admissible pair, see [32].*

2.1.3 The Projective Linear Group $\mathrm{PGL}(\mathcal{R},2)$

The group of all linear mappings that map the projective line $\mathbb{P}^1(\mathcal{R})$ onto itself can be described by the general linear group over $\mathcal{R}$. The kernel of $\mathrm{GL}(\mathcal{R},2)$ is given by

$$\ker \mathrm{GL}(\mathcal{R},2) = \left\{ \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \mid q \in \mathcal{C}(\mathcal{R}) \cap \mathcal{R}^{\times} \right\}.$$

Note, that the kernel is equal to the center of the general linear group

$$\ker \mathrm{GL}(\mathcal{R},2) = \mathcal{C}(\mathrm{GL}(\mathcal{R},2)).$$

The *projective linear group* $\mathrm{PGL}(\mathcal{R},2)$ acting on $\mathbb{P}^1(\mathcal{R})$ can be defined as the quotient group

$$\mathrm{PGL}(\mathcal{R},2) := {}^{\mathrm{GL}(\mathcal{R},2)}/_{\mathcal{C}(\mathrm{GL}(\mathcal{R},2))}.$$

Due to the fact, that the projective line is defined as left-module, a mapping of $\mathbb{P}^1(\mathcal{R})$ onto itself can be described with the matrix vector product $(a,b)\cdot \mathrm{M}$, where the admissible pair (a,b) is a row vector.

Remark 2.4. *In this chapter we denote the action of an element $\gamma \in \mathrm{PGL}(\mathcal{R},2)$ as superscript. For example the image of a point $P \in \mathbb{P}^1(\mathcal{R})$ under γ is denoted by P^γ.*

Theorem 2.1. *The group $\mathrm{PGL}(\mathcal{R},2)$ acts on $\mathbb{P}^1(\mathcal{R})$ and leaves the distance relation invariant. This means for $\gamma \in \mathrm{PGL}(\mathcal{R},2)$ and $P,Q \in \mathbb{P}^1(\mathcal{R})$ we have $P\Delta Q \Rightarrow P^\gamma \Delta Q^\gamma$, where P^γ and Q^γ denote the images of P and Q under γ.*

A proof of this theorem can be found in [11]. Now we give another theorem that describes the action of $\mathrm{PGL}(\mathcal{R},2)$ on $\mathbb{P}^1(\mathcal{R})$ more detailed. Therefore, we need:

Definition 2.8. *Let $(\mathcal{P},\Delta)$ be a distance space and Γ a subgroup of the group $\mathrm{Aut}(\mathcal{P},\Delta)$, i.e., the group of automorphisms of the distance space $(\mathcal{P},\Delta)$. The action of Γ on $(\mathcal{P},\Delta)$ is called*

(1) 2-Δ-*transitive, if Γ acts transitive on the set of all pairs of distant points of $\mathcal{P}$,*

(2) 3-Δ-*transitive, if* Γ *acts transitive on the set of all triples of distant points of* $\mathcal{P}$.

Remark 2.5. *A group acts transitive on a set if for each pair of elements* a, b *there is a group element that maps* a *to* b.

Theorem 2.2. *The group* $\mathrm{PGL}(\mathcal{R}, 2)$ *acts* 3-Δ-*transitive on* $\mathbb{P}^1(\mathcal{R})$.

For a proof we refer to [33].

Table 2.1: Elements of $\mathrm{PGL}(\mathcal{R}, 2)$ corresponding to addition, multiplication, and reciprocation.

addition	right-multiplic.	left-multiplic.	reciprocation
$x \mapsto x + t$	$x \mapsto xq$	$x \mapsto qx$	$x \mapsto x^{-1}$
$\mathcal{R}(x,1)\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$	$\mathcal{R}(x,1)\begin{pmatrix}q & 0\\ 0 & 1\end{pmatrix}$	$\mathcal{R}(x,1)\begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix}$	$\mathcal{R}(x,1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

Now we want to investigate a subgroup of the projective group that corresponds to special operations on the ring $\mathcal{R}$. Let $\mathcal{R}(x_1, x_0)$ be an arbitrary point on $\mathbb{P}^1(\mathcal{R})$ then addition, multiplication, and reciprocation in $\mathcal{R}$ can be described by the elements of $\mathrm{PGL}(\mathcal{R}, 2)$ listed in Table 2.1. These are all permutations of $\mathbb{P}^1(\mathcal{R})$ for that the inverse map is of the same type. The subgroup of $\mathrm{PGL}(\mathcal{R}, 2)$ generated by these four elements is denoted by $\Gamma(\mathcal{R})$.

2.1.4 The projective Line over a Subring

For a chain geometry we need one more ingredient. Therefore, we look at pairs of rings $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is a subring of $\mathcal{R}$. The projective line over $\mathcal{L}$ is a distance space $(\mathbb{P}^1(\mathcal{L}), \Delta_{\mathcal{L}})$. Hence, we are interested in the relation between $(\mathbb{P}^1(\mathcal{L}), \Delta_{\mathcal{L}})$ and $(\mathbb{P}^1(\mathcal{R}), \Delta_{\mathcal{R}})$. We interpret the projective line over $\mathcal{L}$ as a substructure of $(\mathbb{P}^1(\mathcal{R}), \Delta_{\mathcal{R}})$. It is clear that for a pair $(a, b) \in \mathcal{L}^2$ the equivalence classes have the form $\mathcal{L}(a, b)$ and if we interpret the point as element of $(\mathbb{P}^1(\mathcal{R}), \Delta_{\mathcal{R}})$, the equivalence classes have the form $\mathcal{R}(a, b)$. Hence, we can define the map

$$\iota : \mathbb{P}^1(\mathcal{L}) \to \mathbb{P}^1(\mathcal{R}), \quad \mathcal{L}(a, b) \mapsto \mathcal{R}(a, b).$$

This map is an injective morphism of distance spaces, see [11].

Remark 2.6. *In general we can not expect that non-distant points are mapped to non-distant points under* ι*. As example we can take a look at the ring pair* $(\mathbb{Z}, \mathbb{Q})$*, cf. [11, p. 26].*

If we equip $\mathbb{P}^1(\mathcal{L})^\iota$ with the distance relation $\Delta_\mathcal{R}$ the inverse map ι^{-1} does not need to be a morphism of distance spaces. A criteria for the inverse map ι^{-1} to be a morphism of distance spaces is given in:

Theorem 2.3. *Let* $\mathcal{L} \leq \mathcal{R}$ *and let* $\iota : (\mathbb{P}^1(\mathcal{L}), \Delta_\mathcal{L}) \to (\mathbb{P}^1(\mathcal{R}), \Delta_\mathcal{R})$ *be a morphism of distance spaces. The following statements are equivalent:*

(1) $\mathcal{L}^\times = \mathcal{R}^\times \cap \mathcal{L}$.

(2) ι *maps non-distant points to non-distant points.*

In this case, the map $\iota : \mathbb{P}^1(\mathcal{L}) \to \mathbb{P}^1(\mathcal{L})^\iota$ *is a isomorphism of distance spaces.*

For a proof we refer to [11].

2.2 Chain Geometry as Incidence Geometry

In this section we define the term chain geometry. Therefore, we need the projective line $\mathbb{P}^1(\mathcal{R})$ over an algebra as the point set $\mathcal{P}$ of our chain geometry. Furthermore, there are special subsets of $\mathcal{P}$ that we call chains.

Definition 2.9. *Let* $\mathcal{R}$ *be an* $\mathcal{L}$*-algebra and* $\gamma \in \mathrm{PGL}(\mathcal{R}, 2)$*. The subset* $\mathbb{P}^1(\mathcal{L})^\gamma$ *of* $\mathbb{P}^1(\mathcal{R})$ *is called a* chain *in* $\mathbb{P}^1(\mathcal{R})$*, if* γ *is induced by a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\mathcal{R}, 2)$*. The set of all chains in* $\mathbb{P}^1(\mathcal{R})$ *is denoted by* $\mathfrak{C}(\mathcal{L}, \mathcal{R})$ *and defined as*

$$\mathfrak{C}(\mathcal{L}, \mathcal{R}) := \left\{ \mathbb{P}^1(\mathcal{L})^\gamma \middle| \gamma \in \mathrm{PGL}(\mathcal{R}, 2) \right\}.$$

We call the incidence structure $\Sigma(\mathcal{L}, \mathcal{R}) := (\mathbb{P}^1(\mathcal{R}), \mathfrak{C}(\mathcal{L}, \mathcal{R}))$ chain geometry *over the* $\mathcal{L}$*-algebra* $\mathcal{R}$*.*

The set of points of $\Sigma(\mathcal{L},\mathcal{R})$ is additionally equipped with the distance relation $\Delta_{\mathcal{R}}$. We need that the distance relation $\Delta_{\mathcal{R}}$ on the chain $\mathbb{P}^1(\mathcal{L})$ coincides with $\Delta_{\mathcal{L}}$. With Th. 2.3 this is the case if $\mathcal{L}^\times = \mathcal{R}^\times \cap \mathcal{L}$. Hence, we assume this from now on and for $\Delta_{\mathcal{R}}$ we simply write Δ. Furthermore, we provide the more general definition of a chain space that also can be found in [11].

Definition 2.10. *Let $(\mathcal{P},\mathfrak{C})$ be an incidence structure and Δ a relation on $\mathcal{P}$ such that $(\mathcal{P},\Delta)$ is a distance space.*

(1) Then $(\mathcal{P},\mathfrak{C},\Delta)$ is called incidence structure with distance relation.

(2) Let $(\mathcal{P},\mathfrak{C},\Delta)$ be an incidence structure with distance relation. The elements of $\mathfrak{C}$ are called chains. $(\mathcal{P},\mathfrak{C},\Delta)$ is called a weak chain space*, if the following axioms hold:*

C1 *Each chain $c \in \mathfrak{C}$ contains at least three points and each point $p \in \mathcal{P}$ is contained by at least one chain.*

C2 *Three pairwise distant points $p, q, r \in \mathcal{P}$ are incident with exactly one chain $c \in \mathfrak{C}$. We denote this chain by $c =: (pqr)$.*

A weak chain space $(\mathcal{P},\mathfrak{C},\Delta)$ is called a weak chain space in the proper sense if the following additional axiom is satisfied.

C3 *Two points $p, q \in \mathcal{P}$ are distant if, and only if, they are different and connected by a chain.*

We need a further definition.

Definition 2.11. *Let $\mathbb{A} = (P,\mathfrak{L},\|)$ be an affine space and let $\mathfrak{L}' \subseteq \mathfrak{L}$ be a non-empty union of sets of parallel lines from $\mathbb{A}$. The incidence structure $(P,\mathfrak{L}')$ is called a partial affine space.*

With Def. 2.10 and Def. 2.11, we are able to define chain spaces.

Definition 2.12. *Let $\Sigma = (\mathcal{P},\mathfrak{C},\Delta)$ be a weak chain space in the proper sense.*

(1) For $P \in \mathcal{P}$ let $\mathfrak{C}_P := \{c\setminus\{P\} \mid P \in c \in \mathfrak{C}\}$. Let $\Delta(P)$ denote all points distant from $P \in \mathcal{P}$. Than $\Sigma_P := (\Delta(P),\mathfrak{C}_P)$ is called the residuum of Σ at the point P.

(2) If Σ satisfies the axiom

C4 *For each $P \in \mathcal{P}$, the residuum Σ_P is a partial affine space.*

Σ is called a chain space.

Remark 2.7. *All chain geometries over algebras are also chain spaces, see [11] or [33]. When we talk about chain spaces we mention chain geometries over algebras. A chain geometry over an $\mathcal{L}$-algebra $\mathcal{R}$ is denoted by $\Sigma(\mathcal{L}, \mathcal{R})$, where the set of points is given by $\mathcal{P} = \mathbb{P}^1(\mathcal{R})$ and the set of chains by*

$$\mathfrak{C}(\mathcal{L}, \mathcal{R}) := \left\{ (\mathbb{P}^1(\mathcal{L}))^\gamma \middle| \gamma \in \mathrm{PGL}(\mathcal{R}, 2) \right\}.$$

In the following we cite some theorems concerning chain geometries over algebras without proofs. For proofs we refer to [11] and [33].

Theorem 2.4. *Let $\mathcal{R}$ be an $\mathcal{L}$-algebra with $\mathcal{L}^\times = \mathcal{R}^\times \cap \mathcal{L}$. Then for $\Sigma(\mathcal{L}, \mathcal{R})$ it is true: Through three pairwise distant points there is exactly one chain.*

Theorem 2.5. *For $\Sigma(\mathcal{L}, \mathcal{R})$ it is equivalent:*

(1) Let $\mathcal{L}^\times = \mathcal{R}^\times \cap \mathcal{L}$. Two pairwise distant points of $\mathbb{P}^1(\mathcal{R})$ are distant if, and only if, they are incident with one chain.

(2) The ring $\mathcal{L}$ is a field.

Lemma 2.1. *Let $\mathcal{R}$ be an algebra over the ring $\mathcal{L}$. Then the group $\mathrm{PGL}(\mathcal{R}, 2)$ is a subgroup of the group of automorphisms $\mathrm{Aut}(\Sigma(\mathcal{L}, \mathcal{R}))$ of the incidence structure $\Sigma = \Sigma(\mathcal{L}, \mathcal{R})$. The group acts transitive on the set $\mathfrak{F} = \{(p, c) \in (\mathcal{P} \times \mathfrak{C}) \mid p \in c \in \mathfrak{C}(\mathcal{L}, \mathcal{R})\}$ of* flags *of Σ.*

Proposition 2.1. *The parallel relation in $\Sigma(\mathcal{L}, \mathcal{R})$ on $\mathbb{P}^1(\mathcal{R})$ is transitive, i.e., it is an equivalence relation, if $\mathcal{R}$ is a local ring.*

A proof can be found in [33].

Definition 2.13. *Let $\mathcal{K}$ be a field.*

(1) The chain geometry $\Sigma(\mathcal{K}, \mathcal{R})$ is called a Möbius geometry *if the parallel relation is the equality relation. This means that any two different points are incident with a single chain.*

(2) $\Sigma(\mathcal{K},\mathcal{R})$ *is called a* Laguerre geometry *provided that the parallel relation is an equivalence relation on* $\mathbb{P}^1(\mathcal{R})$ *and every chain meets every parallel class of points.*

(3) $\Sigma(\mathcal{K},\mathcal{R})$ *is called a* Minkowski geometry *of dimension* n *if*

$$\mathcal{R} = \mathcal{K} \times \mathcal{K} \times \ldots \times \mathcal{K} \ \textit{(n-times)},$$

where addition and multiplication are defined component-wise.

Remark 2.8. *The chain geometry* $\Sigma(\mathcal{K},\mathcal{R})$ *is a Laguerre geometry if* $\mathcal{R}$ *is a Laguerre algebra.*

2.2.1 Definition of a Cross Ratio

For a better description of chains and as natural invariant we define the cross ratio of four points.

Definition 2.14. *For* $A, B, C, D \in \mathbb{P}^1(\mathcal{R})$, *where* A, B, C *are mutually distant and* A, D *are distant, we define the* cross ratio $cr(A,B,C,D)$ *as a subset of* $\mathcal{R}$ *in the following way:*

$$d \in cr(A,B,C,D) \Leftrightarrow \exists \gamma \in \mathrm{PGL}(\mathcal{R},2)\colon (A^\gamma, B^\gamma, C^\gamma, D^\gamma) = (U, V, W, \mathcal{R}(d,1)),$$

with $U = \mathcal{R}(0,1)$, $V = \mathcal{R}(1,1)$, $W = \mathcal{R}(1,0)$.

Every cross ratio is a class of conjugates under $\mathcal{R}^\times$:

$$d \in cr(A,B,C,D) \Leftrightarrow cr(A,B,C,D) = \left\{ z^{-1}dz : \, z \in \mathcal{R}^\times \right\}.$$

Furthermore, the cross ratio is invariant under the action of $\mathrm{PGL}(\mathcal{R},2)$:

$$cr(A,B,C,D) = cr(A^\gamma, B^\gamma, C^\gamma, D^\gamma), \text{ for all } \gamma \in \mathrm{PGL}(\mathcal{R},2).$$

If $d \in \mathcal{C}(\mathcal{R})$ the conjugacy class $cr(A,B,C,D)$ becomes a single element and we write $cr(A,B,C,D) = d$. A theorem taken from [33] states:

Theorem 2.6. *(1) Four mutually distant points* $A, B, C, D \in \mathbb{P}^1(\mathcal{R})$ *are cocatenal, i.e., incident with one and the same chain, if* $cr(A,B,C,D) \in \mathcal{L}^\times$.

(2) Let A, B, C be mutually distant points. The chain containing A,B,C is the set

$$\{A\} \cup \big\{X \in \mathbb{P}^1(\mathcal{R}) \mid X \Delta A \ \textit{and}\ cr(A,B,C,X) \in \mathcal{L}\big\}\,.$$

Th. 2.6 allows a parametrisation of chains with the help of the cross ratio. Therefore, we take a closer look at the cross ratio. The map $\gamma \in \mathrm{PGL}(\mathcal{R},2)$ that takes three arbitrary mutually distant proper points $A, B, C \in \mathbb{P}^1(\mathcal{R})$ to the points U, V, W can be constructed in the following way:

$$\mathrm{M} = \mathrm{M}_1\mathrm{M}_2\mathrm{M}_3\mathrm{M}_4 = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} tv & 1 \\ (1-ct)v & -c \end{pmatrix},$$

with $t = (c-a)^{-1}$ and $v = (t + (b-c)^{-1})^{-1}$. In this product the first two matrices $\mathrm{M}_1\mathrm{M}_2$ put the point C to W, where it is fixed under the action of $\mathrm{M}_3\mathrm{M}_4$. The third factor M_3 maps the image of A under the first two factors to U. The last factor M_4 does not affect the image of A under the first three matrices. Finally, the last factor has traced B through the first three factors and maps the image of B to V. To check this we apply the mapping γ described by the matrix M to the three points:

$$\begin{aligned}
\mathcal{R}(a,1)\mathrm{M} &= \mathcal{R}(a,1)\begin{pmatrix} tv & 1 \\ (1-ct)v & -c \end{pmatrix} = \mathcal{R}(atv + (1-ct)v), a-c) \\
&= \mathcal{R}((a-c)t+1)v, a-c) = \mathcal{R}((-1+1)v, a-c) \sim \mathcal{R}(0,1) = U, \\
\mathcal{R}(b,1)\mathrm{M} &= \mathcal{R}(b,1)\begin{pmatrix} tv & 1 \\ (1-ct)v & -c \end{pmatrix} = \mathcal{R}(btv + (1-ct)v, b-c) \\
&= \mathcal{R}((b-c)t+1)v, b-c) \sim \mathcal{R}(b-c)t+1, (b-c)v^{-1}) \\
&= \mathcal{R}((b-c)t+1, (b-c)t+1) \sim \mathcal{R}(1,1) = V, \\
\mathcal{R}(c,1)\mathrm{M} &= \mathcal{R}(c,1)\begin{pmatrix} tv & 1 \\ (1-ct)v & -c \end{pmatrix} = \mathcal{R}(ctv + 1 - ctv, c-c) \\
&= \mathcal{R}(1,0) = W.
\end{aligned}$$

Let us now consider the action on an arbitrary proper point $\mathcal{R}(x,1) \in \mathbb{P}^1(\mathcal{R})$:

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