2. Putting in context Cardano's works and mathematics

This chapter is an introduction to Cardano's works from both a non-mathematical and a mathematical viewpoint. In the following, I give an overview of Cardano's mathematical treatises, including the few manuscripts still available, and discussing in particular the writings in which cubic equations are dealt with. At a second stage, I introduce the *Aliza* and provide a first estimate in time. It is nevertheless worthwhile to address from now a few issues concerning this very controversial book, such as the meaning of the title and its supposed aim. Its detailed analysis will be presented in Chapter 5. Finally, I account in this section for the secondary literature concerning the *Aliza*. Being such an unfamiliar book, it had a handful of readers during the four and a half centuries that have passed. Most of them only take episodically into account a very few topics. Pietro Cossali, instead, made a quite detailed analysis at the end of the 18^{th} century, which I discuss. Finally, in order to fully grasp Cardano's reasoning, I recall what our mathematics can nowadays say about quadratic, quartic, and especially cubic equations.

2.1. Cardano as a mathematical writer

Cardano has been a very prolific writer on a great variety of topics. Although on two occasions he destroyed almost one hundred thirty books, a consistent number of treatises and manuscripts has come to us. In the latest version of his autobiography, which was completed in 1576, a few months before his death, he lists one hundred thirty-one published treatises and one hundred eleven manuscripts. At Cardano's

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death, they were left[1] to Rodolfo Silvestri, one of his good friends in Rome and a former pupil of him. Subsequently, Cardano's writings were handed to another of his pupils, Fabrizio Cocanaro, and, in 1654, the majority of the manuscripts ended up at a Parisian bookseller. Eventually, they were bequeathed to Marc-Antoine Ravaud and Jean-Antoine Huguetan, who, with the help of the physician Charles Spon, published in 1663 in Lyon Cardano's *Opera Omnia*. It consists of ten volumes and one hundred twenty-seven treatises in all.

In the *Opera Omnia* there are thirteen mathematical treatises, some of which are published there for the first time, while others had previous editions. Chronologically listed according to the date of the first publication, then in the order in which they appear in the *Opera Omnia*, they are:[2]

- *Practica Arithmeticæ* (Milan 1539, Lyon 1663),
- *Ars Magna* (Nuremberg 1545, Basel 1570, Lyon 1663),
- *De Proportionibus* (Basel 1570, Lyon 1663),
- *De Regula Aliza* (Basel 1570, Lyon 1663),
- *De Ludo Aleæ* (Lyon 1663),
- *De Numerorum Proprietatibus* (Lyon 1663),
- *Libellus qui dicitur Computus Minor* (Lyon 1663),
- *Ars Magna Arithmeticæ* (Lyon 1663),
- *Sermo de Plus et Minus* (Lyon 1663),
- *Encomium Geometriæ* (Lyon 1663),
- *Exæreton Mathematicorum* (Lyon 1663),
- *De Artis Arithmeticæ Tractatus de Integris* (Lyon 1663),

[1] See [46].
[2] See [44].

- *De Mathematicis quæSitis* or *Paralipomena* (Lyon 1663).

They are all contained in the fourth volume of the *Opera Omnia*, which is devoted to mathematics, except for the *De Ludo Aleæ* in the first volume, and for the *De Artis Arithmeticæ Tractatus de Integris* together with the *Paralipomena* in the tenth volume. We remark that in the fourth volume of the *Opera Omnia* two spurious writings, the *Operatione della Linea* and the *Della Natura de Principij e Regole Musicali*,[3] have also been counted in.

We have moreover three mathematical manuscripts available. The Manuscript Plimpton 510/1700 at the Columbia University Library in New York is an Italian translation of the *Ars Magna* entitled *L'algebra*. The Manuscript N 187 at the Biblioteca Trivulziana in Milan, which was firstly entitled *Supplementum practicæ Hieronimi Castillionei Cardani medici mediolanensis in arithmetica* and then *Hieronimi Castillionei Cardani medici mediolanensis in libruum suum artis magnæ sive quadraginta capitulorum et quadraginta quæstionum*, partially attests to the *Ars Magna Arithmeticæ*. Finally, there is the Manuscript Lat. 7217, called *Commentaria in Euclidis Elementa*, at the Bibliothèque Nationale de France in Paris. It is a treatise in nine books on geometry, which has many common points with one of Cardano's unfinished projects, the *Nova Geometria*, which should have likely completed and enhanced the *Elements*.[4]

We must also consider another work that Cardano would have liked to write, but in the end he never wrote. It is the *Opus Arithmeticæ Perfectum*, probably conceived between the 1530s and the 1560s. It should have been an encyclopedic, arithmetical work in fourteen books, a sort of counterpart to the *Nova Geometria* (for more details, see below, Section 2.3). Finally, there are some non-mathematical

[3] Contrary to all the others, they are written in Vernacular. The *Operatione della Linea* is actually a work by Galileo Galilei contained in his *Le Operazioni del Compasso Geometrico e Militare* (1606). Concerning the *Della Natura de Principij e Regole Musicali*, see [91, page 97]. Also Ian MacLean includes them in a list of manuscripts written in Vernacular and not acknowledged by Cardano himself, see [48, page 60].

[4] For more details, see [63].

treatises, which anyway contain some mathematics as examples. They are the logical treatise *Dialectica* (Basel 1566, Lyon 1663) and the philosophical writing *De Subtilitate* (Lyon 1550, then fifteen editions more).

Among this huge amount of pages, we will only consider the works that deal with equations and, more precisely, principally with cubic equations. As said, they are the *Practica Arithmeticæ*, the *Ars Magna* (of which we have also available an English translation made by Richard Witmer in 1968), the *Ars Magna Arithmeticæ*, and – obviously – the *De Regula Aliza*. Except for the *Ars Magna Arithmeticæ*, they were all published during Cardano's lifespan. The *Ars Magna* also had a second edition, and all the mathematical treatises were posthumously republished in the *Opera Omnia*.

During the 14$^{\text{th}}$ and 15$^{\text{th}}$ centuries in the region of northern Italy (where Cardano grew up and subsequently lived for the first part of his life), a significant, even if modest, tradition of *scuole d'abaco* developed.[5] It is plausible[6] that Cardano had some contacts with this environment, since at that time these were the only places where one could learn algebra, or rather what was called so at that time. Anyway, there is no evidence that he had been himself an *abaco* master. It is moreover very likely[7] that Cardano pit himself against the more widespread Italian treatises, namely the inheritors of the *abaco* treatises like the *Nuovo Lume* by Giovanni Sfortunati and the *Libro de Abacho* by Pietro Borghi, and obviously the *Summa de Arithmetica, Geometria, Proportioni et Proportionalita* by Luca Pacioli. It indeed seems that the real link between Cardano and the *abaco* school was Gabriel Arator, one of Pacioli's students.

The general image of practical mathematics conveyed by these treatises is a double-sided one. On the one hand, there are miscellaneous problems solved case by case where the calculation skills are

[5] See [62, page 273].

[6] See [61]. Moreover, Cardano frequently mentions a certain Gabriel Arator, who assigns to himself the name of *magister*, see [2, Chapter LI, paragraph 17] and [62, page 275].

[7] See [62, page 276].

overwhelming. On the other hand, there is an encyclopedic work mainly focused on theory, that is Pacioli's treatise. The 1539 *Practica Arithmeticæ* – the first work in which Cardano deals with cubic equations – can be inserted in this tradition. Even if it is written in Latin and not in Vernacular, it contains all the major features of an *abaco* treatise: topics, choice of problems, and also terminology. It sometimes provides original reworkings of some techniques in the *abaco* tradition. Moreover, Cardano himself made the connection with Pacioli, even if in a conflicting way, which was a common publishing strategy after that the *Summa* appeared (given the turmoil that this work caused in the long term). In the *Practica Arithmeticæ*, cubic equations are only one among various problems and are seen as tightly depending on proportions.

According to the above chronological list, the *Ars Magna* should follow. Actually, thanks to some recent reappraisals of Cardano's mathematical writings,[8] we have come to the knowledge that the *Ars Magna Arithmeticæ* was preexistent to the *Ars Magna*. For a long time, the *Ars Magna Arithmeticæ* has been considered as a minor work and in any case a late work.[9] Cardano himself did not mention this treatise in his autobiographies and never had it printed. The (supposedly chronological) position of the published version in the framework of the 1663 *Opera Omnia* contributed to the misunderstanding, since it was wrongly placed after the *Ars Magna*. Nowadays it is commonly agreed that *Ars Magna Arithmeticæ* had been conceived before – or at least at the same time at – the *Ars Magna*. Massimo Tamborini suggests[10] that the *Ars Magna Arithmeticæ* and the *Ars Magna* were at the beginning planned together, and then separated and revised, maybe in order to publish the *Ars Magna* as soon as possible. In fact, in his 1544 autobiography, Cardano spoke of an "Ars Magna, which contains sixty-seven chapters",[11] whereas both the three editions of

[8] See the edition project of Cardano's works by Maria Luisa Baldi and Guido Canziani [44] and the conference proceedings [71], [45], [47], and [48].

[9] See [50], [74, page 298], and [22, Footnote 20, pages xvii-xviii].

[10] See [93, pages 177-9].

[11] "*Ars magna, continet sexaginta septem capitula*", see [24, page 131].

the *Ars Magna* and the *Ars Magna Arithmeticæ* published in 1663 both consist of only forty chapters. In this context, Cardano was talking about his project of an *Opus Arithmeticæ Perfectum*, crediting this "*Ars Magna*" to be its tenth volume. We take a more detailed look at the *Opus Arithmeticæ Perfectum* in Section 2.3.

Ian MacLean prefaces his edition of Cardano's autobiographies[12] by a commented chronology of Cardano's treatises. While speaking about the description of the *Opus Arithmeticæ Perfectum* given in Cardano's 1557 autobiography,[13] he identifies the tenth volume of the *Opus Arithmeticæ Perfectum* or the "*De regulis magnis, atque ideo ars magna vocatur*" with the *Ars Magna Arithmeticæ* itself. But MacLean's interpretation is confusing. In fact, the full 1557 quotation says: "De regulis magnis, and for that reason it is called ars magna. And only this one among the others is published",[14] whereas the *Ars Magna Arithmeticæ* has never been printed during Cardano's lifespan. Moreover, in the footnote to the full quotation, MacLean contradicts, and (fairly) identifies the same "De regulis magnis" with the *Ars Magna*.

Finally, Veronica Gavagna argues[15] that the *Ars Magna Arithmeticæ* had been composed between 1539 and 1542. She reasons as follows. Firstly, the *Ars Magna Arithmeticæ* is dedicated to "Philippum Archintum, bishop of Borgo Santo Sepolcro", who was in office between 1539 and 1546. Secondly, the Manuscript N 187 (1r-62v) at the Biblioteca Trivulzian in Milan, which partially attests to the *Ars Magna Arithmeticæ*, should have been at the beginning an addition to the *Practica Arithmeticæ*, as its first title '*Supplementum practicæ Hieronimi Castillionei Cardani medici mediolanensis in arithmetica*' suggests. Then, the title was modified into the final heading '*Hieronimi Castillionei Cardani medici mediolanensis in libruum suum artis magnæ sive quadraginta capitulorum et quadraginta quæstionum*'. When collated,

[12] See [24].
[13] See [24, page 65].
[14] "*De regulis magnis, atque ideo ars magna vocatur: atque hic solus ex omnibus editus est*", see [24, page 185].
[15] See [65].

the *Supplementum* and the *Ars Magna Arithmeticæ* show a word by word correspondence in entire chapters, especially the ones related to cubic equations. Then, it is quite likely that the latter originated from a reorganisation of the former. Moreover, in the *Practica Arithmeticæ* we read this:

> in the book titled Supplementum Practicae, where I solved algebraic equations of whatever degree, possible and impossible, general and not; so in that book there is everything people want to know in algebra. I have added new algebraic cases, but I can't publish them in the Practica because is too large [...] furthermore Supplementum is the crowning achievement of the whole of algebra and it is inspired to book X of the Elements.[16]

And this:

> there is a section lacking, that I can't add for the sake of brevity, because this volume is already too large: since this [new] book is a completion of algebra, its title will be Ars magna. Here you find the rules to solve all equations and all my outcomes obtained by the arithmetisation of book X. In particular, I have added some beautiful rules and two new cases.[17]

[16] This is a translation by Gavagna. "*[L]ibello qui dicit supplementum practice in quo ostendi omnia capitula algebræ possibilia et impossibilia usque in infinitum et quæ sint generalia et quæ non itaque non est aliquid desiderabile in tota arte quantacumque difficile quod non habeat radicem dantem cognitionem in illo libro et addidi plura capitula nova in ipso et non potui edere in ipsunt propter nimiam magnitudinem huius libelli eo quod est impressus in forma parva licet liber ille non transiendit tria aut quattuor folia et est consumatio totius artis et est exctractus ex decimo euclidis*", see [2, Chapter LXVIII, Paragraph 18, no page numbering].

[17] This is a translation by Gavagna. "*[D]eest in opusculum quod ob exiguam formulam cum in nimiam liber hic auctus sit magnitudinem adiici non potuit, ad artis totius complementum hoc artis magne titulo dicat: in quo universorumque capitulorum algebræ usque in infinitum inveniendi formula descripta est et quod super euclidis decimum ad normam numero rerum reducti inveneram congessi*", see [2, *Ad lectorem*, no page numbering].

We will see in Section 4.2 that this description – a book that deals with cubic and quartic equations, which is entitled "Ars Magna", and that contains the arithmetisation of *Elements*, Book X – fits the *Ars Magna Arithmeticæ*. Moreover, at the very beginning of the *Ars Magna Arithmeticæ*, we read this:

> [a]fter writing the Practica, it has seemed to me that it was necessary illustrating the cases, considered by the majority impossible, that I have found thanks to the proofs contained in our three books on Euclid, except for two rules,[18]

which is exactly the *incipit* of the Manuscript N 187, except that here the words 'the former has been revealed by *magister* Nicolò Tartaglia, the latter by Ludovico Ferrari' are struck through. Therefore, the *Supplementum* comes truly to be the link between the *Practica Arithmeticæ* and the *Ars Magna Arithmeticæ*, which follow one the other in this order. These two evidences would moreover place the minimal time limit of the composition of the *Ars Magna Arithmeticæ* on 1539. Gavagna's third remark is that Scipione del Ferro is never mentioned in the *Ars Magna Arithmeticæ*, whereas it is in the *Ars Magna*. Since Cardano and Ferrari discovered Scipione's contribution in 1542 (see below, Appendix A), this suggests to fix the maximum time limit on this year. But Gavagna goes even further. Her main thesis is that the *Ars Magna Arithmeticæ*, originally an addition to the *Practica Arithmeticæ*, was the first writing that should have released the cubic and quartic formulae. The time passing by, this work became so considerably filled with results that it conflicted with the project of the *Opus Arithmeticæ Perfectum*. Thus, it should have been reorganised

[18] This is a translation by Gavagna. "*Post compositionem Libri Practicæ visum est mihi necessarium ea ostendere, quæ a pluribus impossibilia iudicata sunt, quæ omnia nos invenimus ex demonstrationibus trium librorum a nobis supra Euclidem, exceptis duabus regulis harum*", see [13, page 303]. The mentioned "three books on Euclid" are the *Nova geometria*, a treatise on geometry started in 1535, which should have been composed at the beginning by three, then seven, nine, and finally fifteen books.

in two parts. The first part devoted to irrational numbers should have formed Book III of the *Opus Arithmeticæ Perfectum*, while the second one on equations became the core of the *Ars Magna*. This is indeed the structure of the *Ars Magna Arithmeticæ* as we know it nowadays. We consider therefore the *Ars Magna Arithmeticæ* as an intermediate step between the *Practica Arithmeticæ* and the *Ars Magna*. In the following, we come across several links between the two treatises that go in this direction, confirming Gavagna's hypothesis. Even though the *Ars Magna Arithmeticæ* is originated from the *Practica Arithmeticæ*, a lot of work had been done in the meanwhile concerning cubic equations. As Gavagna indicates and as we will see, the *Ars Magna Arithmeticæ* marks the rupture with the *abaco* schools and opens the way to equations as a proper subject of inquiry.

Finally, in 1545 came what has been considered Cardano's masterpiece, the *Ars Magna*. There, the equations, and in particular the cubic ones, play the main role. One can only grasp the importance of the *Ars Magna* in opposition to what came before. At last, all the scattered methods, rules, and special cases that had been discovered up to that time gave birth to a systematic treatment of cubic equations, including geometrical proofs of the solving methods.

2.2. A puzzling book

Nevertheless, in the *Ars Magna* the things are revealed to be not so plain as Cardano wished. As we will see, when a cubic equation has three real distinct solutions, it happens that square roots of negative numbers appear in the formula – which is the *casus irreducibilis*, as it has been lately called.

Twenty-five years after the publication of the first edition of the *Ars Magna* came the *De Regula Aliza*, in which Cardano conveyed his hope to overcome the above problem. In the 1570 *Ars Magna*, Chapter XII, Cardano himself provides the reader with this tip (for the full quotation see the next section on page 18). This is a key

point, since my whole interpretation of the book starts from the link that is established there between the *Aliza* and the *casus irreducibilis*.

Unfortunately, the *Aliza* is an extremely cryptic work, starting from the title. It immediately strikes any reader who is trying to get acquainted with the book. What is the meaning of the term '*aliza*' in the title? This is neither a common Latin word nor was one of the brand-new mathematical words loan-translated into Latin at that time. Moreover, this term does not seem to belong to Cardano's own mathematical terminology. As far as I know, only three other occurrences can be found in the whole of Cardano's writings, and only one of these refers to the book itself. In chronological order, the first reference is in the *Ars Magna*, Chapter XII,[19] while speaking of the *casus irreducibilis*. There, we read of an "*aliza* problem" (in 1545) or of a "book of *aliza*" (in 1570 and 1663), which are supposed to amend the cubic formula in the case $\Delta_3 < 0$. The second occurrence steps outside the topic of cubic equations and is a mere mention. It is in the 1554 (and following) edition(s) of the *De subtilitate*, while Cardano deals with the "reflexive ratio [*proportio reflexa*]".[20] Nevertheless, I could not retrieve the corresponding passage in the *Aliza*. The last occurrence is in the *Sermo de plus et minus*, which contains a specific mention of *Aliza*, Chapter XXII (see here, Section 5.6 on page 385). Thereafter, except in the title page of the *Aliza*, the term is nowhere else mentioned. We also remark that Cardano does not pay any attention to trying to explain such a rare term, and this is telling about his overall attitude towards his public.

In 1799 Pietro Cossali hinted that the term '*aliza*' means 'unsolvable'.[21] In 1892, Moritz Cantor related that Armin Wittstein suggests that this term comes from a wrong transcription of the Arabic word

[19] See [4, Chapter XII, page 251] or see here, page 114.

[20] More precisely, Cardano makes reference to the *Ars Magna* and to the *De Regula Aliza* ("*ex demonstratis in Arte magna et regula Aliza*") concerning the ratios of the side of a regular heptagon to any diagonal subtended to two or three sides, see [5, Book XVI, page 428].

[21] "De Regula Aliza*, cioè* De regula Irresolubili", see [57, volume II, page 441] or [60, Chapter I, paragraph 2, page 26].

'*a'izzâ*' and means 'difficult to do', 'laborious', 'ardous'.[22] In 1929 Gino Loria advised as a common opinion that the term comes from a certain Arabic word that means 'difficult'.[23] None of the above hypotheses were supported by a precise etymology. Very recently, Paolo D'Alessandro has confirmed Cossali's hypothesis.[24] The term '*aliza*', or '*aluza*', is likely a misspelling based on the Byzantine pronunciation 'áliʤja' of the Greek word 'ἀλυθεῖα', which is composed by the negative particle 'α' and by the feminine singular passive aorist participle of the verb 'λύω', 'to unbind', 'to unfasten', 'to loosen', 'to dissolve', 'to break up', 'to undo', 'to solve'.[25] Thus, '*aluza*' means 'non-solvable'. This etymology soundly agrees with Cardano's words in *Ars Magna*, Chapter XII, where the *Aliza* and the *casus irreducibilis* are linked. Note that, if we understand '*irreducibilis*' as the fact that the cubic formulae (which should convey three real roots) cannot reduce to some real radicals, then in Cardano's mathematics '*irreducibilis*' completely overlaps with 'non-solvable'.

If the enigma of the title can be relatively easily fixed, the same cannot be said about other difficulties. Basically, there is one fundamental justification for the high unfamiliarity of the *Aliza* among modern readers. Spoiling the whole surprise, Cardano did not reach his aim (or at least the aim that concerns the *casus irreducibilis*) in the *Aliza*. In very truth, he could not reach it, but this had only been showed three centuries later.[26] The book then tells of a failed

[22] "*[D]ie dritte führte den nie und nirgend erklärten Titel De regula Aliza, der durch unrichtige Trasscription aus dem arabischen Worte a'izzâ (schwierig anzustellen, mühselig, beschwerlich) entstanden sein kann [Diese Vermuthung rührt von H. Armin Wittstein her.], und alsdann Regel der schwierigen Fälle bedeuten würde*", see [52, page 532].

[23] "*[. . .] il titolo, sinora inesplicato* De Regula Aliza *(secondo alcuni* aliza *deriverebbe da una parola araba significante* difficile*)*", see [74, p. 298].

[24] By personal communication.

[25] See [73] and [85, page 367]. If it is so, then note that '*aluza*' and 'analysis' have a common origin.

[26] Indeed, it has been proved that imaginary numbers have necessarily to appear in the cubic formula when the equation has three real, different roots by Pierre Laurent Wantzel in 1843, Vincenzo Mollame in 1890, Otto Hölder in 1891, and Adolf Kneser in 1892. Paolo Ruffini also provided an incomplete proof in

attempt – rather, of an unattainable attempt – so that it is not surprising to find a good amount of looseness inside.

There are anyway other extrinsic reasons that contributed to worsen the situation and that could depend on the previous justification. However common the typos and mistakes in calculations were at the time, the book is particularly filled with them. Sometimes the misprints concern the correspondence of the lettering between the diagram and the text, and then we can only hope that the text is sufficiently enough devoid of incongruities to manage to correct the wrong letter(s). Sometimes we find some lettering in the text and no diagram at all. In other cases, there must have been full lines that were skipped at the page break. Moreover, Cardano's peculiar style in writing in Latin is not the plainest one. Speaking the mathematics as Cardano and his contemporaries did without a complete and stable formalism implied a certain amount of ambiguity, and Cardano was not used to simplify the task to his readers. The most favourable case is when a general statement is accompanied by numerical examples, so that we can untangle the text relying on calculations. Apart from that, Cardano's general observations are usually long, convoluted sentences with more than one possible interpretation. Unfortunately, the 1663 edition cannot help the reader: Spon, who revised Cardano's writings, was not a mathematician and the best that he could do was to proofread the grammar and to correct here and there some clear mistakes.

The *Aliza*'s obscurity is displayed by the lack of any global ordering in the flowing of the chapters' topics. At the beginning of Chapter 5, I provide a general overview of the kinds of issues that are dealt with in the *Aliza*. A same topic is usually scattered all along the book, with repetitions and logical gaps, so that in the end the reader feels highly disoriented. Under these conditions, the *Aliza* is more pragmatically provided as a miscellany rather than as a unitary book. To resume, using Pietro Cossali's words:

1799. See [42], [33] and [34], [30], [31], [27], and [36] and [37]. I plan to develop further this topic in a next work.

> [t]o the obscurity of that book caused by the difference of language, another [obscurity] is coupled, [which is the one] produced by an infinite, very pernicious mass of errors both in the numbers in calculations and in the letters in figures and related proofs. And another one, which is absolute and intrinsic, has to be added. It is the lack of order due to which it is very hard and uncomfortable to pull the strings, to see the result, and to evaluate the discovery. We spot at Cardano trying, opening new roads, retracing his steps, turning to one side or another according to his intellect's suggestions. In a word, the book is the action of attempting, not the ordering of the discoveries. And in the end it is one of Cardano's books, that his contemporaneous Bombelli called obscure in the telling.[27]

It is exactly because of these long-range incongruities that one needs to master as good as possible the contents of the other treatises by Cardano that deal with equations. It is an obviously recommended practice to contextualise a work in the frame of the other writings of the same author. But more than that, in this particular case where we miss the guide of a general structure, we need all the short, coherent problems, exercises, and mathematical notes that can be gathered from Cardano's other works (and in particular from the *Practica Arithmeticæ*, the *Ars Magna Arithmeticæ*, and the *Ars Magna*) to get oriented while reading the *Aliza*. This is why we devote two whole chapters to those treatises and other contextual matters.

[27] "*Alla tenebria di quel libro a noi cagionata dalla diversità del linguaggio se ne accoppia un'altra prodotta da un'infinita perversissima folla di errori, e di numeri ne' calcoli e di lettere si' nelle figure, si' nelle relative dimostrazioni; e bisogna eziandio aggiungersene altra assoluta, ed intrinseca proveniente da mancanza di ordine, per la quale riesce cosa faticosissima, e malegevolissima l'unire le fila, vedere il risultato, valutare il discoprimento. Si scorge Cardano, che tenta, che si apre nuove strade, che ritorna sulle battute, che si volge or da un lato, or dall'altro seguendo i suggerimenti varij dell'ingegno; in una parola il libro è l'atto del tentare, non un ordine delle scoperte; e finalmente è un libro di Cardano, che da Bombelli pur suo coetaneo fu denominato nel dire oscuro*", see [57, volume II, page 442] e [60, page 27].

2.3. From the "*aliza* problem" to the *De Regula Aliza*: changes in the editorial plan of the *Opus Arithmeticæ Perfectum* and an attempt to loosely assign a date

Before addressing the mathematical details, it is fully worthwhile to spend a few words to clarify as much as possible the chronological boundaries of the *Aliza*. Indeed, this work has appeared to be a miscellany of mathematical writings, notes, remarks, and observations that shine out neither for consistency nor for cohesion – at least, at first sight. Then, the question that one immediately asks is whether it is possible to date these writings and, if it is so, to when – even if approximately. If one takes a look at the (supposedly chronological) order in which the books are displayed in the fourth volume of the *Opera Omnia*, he will notice that the *Aliza* comes immediately after the *Ars Magna Arithmeticæ*, which in turn follows the *Ars Magna*. But this is completely ineffective, since, as we have seen in the preceding section, the *Ars Magna Arithmeticæ* has been very likely composed before the *Ars Magna*.

It is a very hard issue to give a reasonable estimate in time of the *Aliza* starting from the internal relations, namely from the mutual relationship of the contents of the chapters. At present, I content myself with checking the external references to this book. Let us firstly address the references to the *Aliza* that are present in the other mathematical writings by Cardano. As already said, the term '*aliza*' appears only four times in the whole of Cardano's mathematical writings, once in the title of the book itself, once in the *Ars Magna*, Chapter XII, once in the *De Subtilitate* (1550), and once in the *Sermo de Plus et Minus*. I leave aside the reference in the *De subtilitate* (see above, Footnote 20 on page 14), since it is a very brief mentioning of a certain result from the *Aliza*, which is not better specified and that I could not retrieve. I analyse the reference in the *Sermo de Plus et Minus* in Section 5.6 on page 385. Here follows the quotation from

the 1545 *Ars Magna*'s edition:[28]

> [w]hen the cube of one-third the coefficient of x is greater that the square of one-half the constant of the equation, which happens whenever the constant is less than three-fourths of this cube or when two-thirds the coefficient of x multiplied by the square root of one-third the same number is greater than the constant of the equation [that is, for short, when $\Delta_3 < 0$], then the solution of this [$x^3 = a_1x + a_0$] can be found by the aliza problem which is discussed in the book of the geometrical problems[29]

and from the 1570 and 1663 editions:

> [w]hen the cube of one-third the coefficient of x is greater that the square of one-half the constant of the equation, which happens whenever the constant is less than three-fourths of this cube or when two-thirds the coefficient of x multiplied by the square root of one-third the same number is greater than the constant of the equation [that is, for short, when $\Delta_3 < 0$], then consult the Aliza book appended to this work.[30]

These quotations not only allow to fix the *Aliza*'s overall subject topic, but they also couple (in 1545) the name '*aliza*' with a certain "book of

[28] See also below, Footnotes 49 and 51, page 114.

[29] "*At ubi cubus tertiæ partis numeri rerum excedat quadratum dimidii numeri æquationis, quod accidit quandocunque numerus æquationis est minor* $\frac{3}{4}$ *cubi illius, vel ubi ex* $\frac{2}{3}$ *numeri rerum producitur in R* $\frac{1}{3}$ *esiusedm numeri maior numerus numero æquationis, tunc hoc dissolvitur per quæstionem Alizam, de qua in libro de quæstionibus geometricis dictum est*", see [4, Chapter XII, page 62] or see [22, page 103].

[30] "*At ubi cubus tertiæ partis numeri rerum excedat quadratum dimidii numeri æquationis, quod accidit quandocunque numerus æquationis est minor* $\frac{3}{4}$ *cubi illius, vel ubi ex* $\frac{2}{3}$ *numeri rerum producitur in R* $\frac{1}{3}$ *esiusedm numeri maior numerus numero æquationis, tunc consules librum Alizæ hic adjectum*", see [10, Chapter XII, page 31v] and [12, Chapter XII, page 251] or see Footnote 5 in [22, page 103].

the geometrical problems". Let us store away for now this information. Note moreover that in 1545 the "*aliza*" is merely a "problem", while in 1570 it has reached the extent of a whole "book" (and it must be so, since in the latter quotation Cardano makes reference to one of the books appended to the new *Ars Magna*'s edition). Therefore, the *Aliza* likely arose as a problem, more precisely as a geometrical problem, and then grew bigger and bigger until it became a book. This could (at least partially) explain the lack of order in its structure. But, much more important is the fact that the *Aliza* is firstly published together with a book – the *Ars Magna* – that contains, even though in a few lines, the key to interpret it. This is fundamental in order to understand how Cardano possibly regarded the *Aliza*, namely as the treatise that (hopefully) contains the solution to the problem entailed by the *casus irreducibilis*.

With regard to the other way round, that is the references to other Cardano's mathematical writings that appear in the *Aliza*, it is implausible that they could help in locating the *Aliza* in time. It is possible – actually, likely – that Cardano subsequently added some references in a text that had already been written a while ago. We should remark, indeed, that he mainly made reference to the *De Proportionibus* and *Ars Magna*, which were published together with the *Aliza* in 1570.

So far, so good, but we did not progress much. Cardano already had in mind a certain "*aliza* problem" around 1545 – which makes sense, since he already most have come across the *casus irreducibilis* developing his cubic formulae in the *Ars Magna*. Afterwards, we can check for the references to the *Aliza* in Cardano's non-mathematical treatises, and there we are extremely lucky. We do not only have his autobiography *De Libris Propriis*, but, with Cardano being the true graphomaniac that he was, he left five versions of it, written over a period of more than thirty years. They are [3] (which is also in the *Opera Omnia* in [14]), [23] (which dates back to 1550), [6] (which is also in the *Opera Omnia* in [15]), [7] (which is also in the

Opera Omnia in [16]), and [17] (which dates back to 1576).[31] There, Cardano recalled not only his personal life, but also his career and his writings. Referring to his autobiography, Cardano says that

> [t]he book was often modified [...]. Thus I set forth for myself in it a certain image of everything that I had written, not only as an aide-memoire, and a mean of selecting those books which I would finish and correct first, but also to set down why, when and in what order I wrote what I did [...]. Indeed I did not only set down here the titles of my books, but also their size, incipits, contents, order, the utility of the division and order of the books, and what they contained of importance.[32]

Each of these versions consecrate to the *Aliza* only a few lines – and most of the time as a fringe topic – but we will see that we can anyway draw some conclusions.

In the lists of his works that Cardano makes in the different versions of his autobiography, the *Aliza* as a published book is of course only mentioned once, in 1576.[33] Most of the information concerning the *Aliza* comes collaterally to the description of the *Opus Arithmeticæ Perfectum* that can be found in the autobiography. This should have been a mathematical, encyclopedic work, composed by fourteen books, and probably conceived between the 1530s and the 1560s.[34] Unfortu-

[31] For more information on the chronology of Cardano's works, see Maclean's essay in the preface of [24].

[32] The English translation is by Maclean in [24, page 11], or "*[d]einde auxi, iterumque mutavi [...]. Iconem ergo quendam mihi in eo proposui omnium eorum quæ a me conscripta sunt, non solum ad memoriam confirmandam, eligendosque mihi libros quos prius absolverem et castigarem: sed ut doceam, quibus causis, temporibus, quoque ordine talia conscripserim, et ut vim numunis suo loco testarer [...]. Verum non hic solum docui librorum nomina, sed et magnitudinem, initium materiam pertractatam, ordinem divisionis, ordinisque librorum inter se utilitatem, quidque in se præcipuum contineret*" in [7, pages 23-24].

[33] See [17, page 40]. The other versions of Cardano's autobiography are all previous to 1570.

[34] See [93], [64], and [65].

nately, it has never been accomplished. It is quite difficult to gather some consistent information on the supposed table of contents of the *Opus Arithmeticæ Perfectum*.[35] It should have included calculations with integer, fractional, irrational, and denominated numbers, proportions, properties of numbers, commercial arithmetic, algebra, plane and solid geometry, and some arithmetical and geometrical problems. Veronica Gavagna[36] suggests that, though the general outfit follows the *Practica Arithmeticæ*'s structure, it also reveals an evident symmetry with Cardano's unpublished comment to the *Elements* (which will be later referred to as the *Nova Geometria*), suggesting that the *Opus Arithmeticæ Perfectum* should be considered as a sort of arithmetical counterpart. Moreover, it is likely that some of Cardano's printed mathematical treatises that are nowadays available should have filled up some of the books, or part of them, in the table of contents of the *Opus Arithmeticæ Perfectum*. Namely,[37] it is the case for the *Tractatus de Integris*, the *De Proportionibus*, and the *De Numerorum Proprietatibus*, which can be found in the fourth and tenth volumes of the *Opera Omnia*. The *Ars Magna Arithmeticæ*, the *Ars Magna*, and the *De Regula Aliza* also play a role in this game, but the matter is more ticklish, as we will see.

All the versions of Cardano's autobiography agree that Book X of the *Opus Arithmeticæ Perfectum* should have been on algebra and equations. In 1544, he says that "the tenth [book] is entitled Ars magna, it contains sixty-seven chapters",[38] whereas Books XIII and XIV "are assigned to arithmetical and geometrical problems".[39] This quotation raises the incidental question to know what was that "Ars magna" of which Cardano is speaking, since all the editions that we have with that title (handwritten or printed) contain only forty

[35] For a detailed discussion, see [24, pages 64-66].
[36] See [64, page 65].
[37] See below, Footnote 43, page 24.
[38] "*[D]ecimus inscribitur Ars magna, continet sexaginta septem capitula*", see [3, page 426].
[39] "*[T]ertiusdecimus ac quartusdecimus, quæstionibus Arithmeticis et Geometricis destinantur*", see [3, page 426].

chapters. Massimo Tamborini in [93, pages 178-179], Ian Maclean in [24, page 65], and Veronica Gavagna in [65] report on this incongruity, and I refer to them for an accurate discussion. In a nutshell, as we have seen and despite Maclean's interpretation, it seems more likely that both the *Ars Magna Arithmeticæ* and the *Ars Magna* (or, at least, their original common core on equations) should have concurred to Book X of the *Opus Arithmeticæ Perfectum*.[40] We moreover recall that, in Chapter XII of the 1545 *Ars Magna*, the "*aliza* problem" was coupled with a certain "book on geometrical problems" (see above, on page 19). Then, in 1544 the situation is the following. Book X of the *Opus Arithmeticæ Perfectum* is devoted to equations and possibly consists of some parts of the *Ars Magna Arithmeticæ* and *Ars Magna*. The *Aliza* is not explicitly mentioned, but is connected (at least, starting from 1545) to certain geometrical problems in the *Opus Arithmeticæ Perfectum*'s last book.

Later on, in 1550, Cardano devoted Book X to "all the chapters on the square together with the Aliza rule". Book XIV still concerned geometry, but was restricted to the "measure of figures".[41] Any reference to the "Ars magna" or to our *Ars Magna* had disappeared. Now, the *Aliza* was shifted from filling the last book to filling the tenth one and, accordingly, the last book of the *Opus Arithmeticæ Perfectum* was reduced in contents.

Afterwards, from 1557 on, the *Aliza* disappeared again, coming back to the same situation as in 1544.[42] This time Cardano gave the *incipit* and the lengths of the books in the *Opus Arithmeticæ Perfectum* that he has already written. In particular, he said that

[40] We also recall that, according to Veronica Gavagna, Book III of the *Opus Arithmeticæ Perfectum* (the *incipit* of which is "*Cum in radicibus quantitatum*") should have been covered by the first part of the *Ars Magna Arithmeticæ* on the arithmetisation of Book X of the *Elements*, see here page 13.

[41] "*In [decimo] omnia capitula supra quadratum cum regula Aliza*" and "*[i]n [quartodecimo] ad mensuram figurarum pertinentia, quæ geometrica vocantur*", see [23, page 9v].

[42] "*Decimus de regulis magnis, atque ideo ars magna vocatur: atque hic solus ex omnibus editus est*" and "*Tertiusdecimus quæstiones Arithmeticas, ut Quartusdecimus Geometricas*" in [6, pages 37-38] and [7, page 16].

Book X began with "*[h]æc ars olim a Mahomete*" and is made up of 83 *folia*. This enables[43] us to identify for sure Book X with the *Ars Magna*'s that we have nowadays available.

Finally, but almost twenty years later in 1576, Cardano told that in 1568 he joined the *Aliza* and the *De Proportionibus* to the *Ars Magna* and had them printed.[44]

We can now try to fix some temporal limits for the *Aliza*. A certain "*aliza*" was firstly mentioned in 1545 (but we can stretch this time limit of a year, up to 1544) and it sank into oblivion at least from 1557. It reappeared in 1568, but it was only a quick mention. We can therefore conjecture that the miscellany of writings that compose the *Aliza* (or a part of them) started to be conceived at worst between 1544 and 1545. Then, its proofreading could have lasted at best until 1557, when Cardano had already lost interest in it. Afterwards, it is likely that during the 1560s, due to his personal misfortune, Cardano desisted from the *Opus Arithmeticæ Perfectum* project and, in general, gave up all of his mathematical projects. Moreover, regarding at least the *casus irreducibilis*, the changes in the editorial project of the *Opus Arithmeticæ Perfectum* could also express Cardano's hope to find some new results. In particular, the lapse of time during which only the *Aliza* appears as a part of the *Opus Arithmeticæ Perfectum* could correspond to the lapse of time during which Cardano believed to manage to avoid the *casus irreducibilis*, namely from 1550 to (at best) 1557. But, when the *Aliza* was printed in 1570, its second title (just above 'Chapter I') recalled its origins as a problem, since it contains the word '*libellus*', which means 'pamphlet', 'booklet', 'small book'. No matter at that time to replace the *Ars Magna*.

We can finally address the sources that mention the *Aliza*, but that were not written by Cardano himself. I am only aware of one,[45] which

[43] By the way, this also enables us to stick the *Tractatus de integris* to the first book, the *De Proportionibus* to the fifth book, and the *De numerorum proprietatibus* to the sixth book of the *Opus Arithmeticæ Perfectum*, see [65].

[44] "*De proportionibus, et Aliza regula addidi anno MDLXVIII ad librum Artis magnæ et edidi*" in [17, page 41].

[45] I thank Veronica Gavagna for this reference.

is in [49, page 163]. Gian Luigi Betti relates about Ercole Bottrigari, printer, scholar, music theorist, and bibliophile, who lived in Bologna between 1531 and 1612. We do not know when he met Cardano for the first time, but he wrote in his *La Mascara*[46] that he questioned Cardano about the *Aliza* "two or three years" before 1570.[47] As Bottrigari testified, Cardano avoided to answer, but in doing so he dropped that the *Aliza* was going to be printed in Germany.[48] This is in accordance with Cardano's assertion in his 1576 autobiography.

Summing up, a certain "*aliza* problem", which, according to the *Ars Magna*, concerned the *casus irreducibilis* and which should have originally been linked to geometry, was already there in 1544 or 1545. We get the strong impression that, from that moment on, the problem at issue grew more and more in importance and attained its best around 1550. The reference to the "*aliza*" disappeared then in 1557. The *Aliza* (as we know it) was ready for press in 1568 and printed in 1570. Therefore, the core of the book on the *casus irreducibilis* had already been conceived a long time before its publication. At a certain point, a more or less sizeable number of pages on fringe topics could have been added and the name of the problem was handed on to the whole miscellany. In this way, what at the beginning was nothing but an "*aliza* problem" could have become an entire book – our *De*

[46] This is one of Bottigari's late works. Gian Luigi Betti found the autographic Manuscript MS B 45 in the library G. B. Martini at the Academy of music in Bologna.

[47] "*É invece certo che [Bottigari] gli [Cardano] abbia direttamente posto una questione 'sopra a quel suo capitolo Alizam algebratico [...] nella sua Arte magna, il qual da poi publicò due o tre anni, facendo ristampare essa Arte Magna insieme con il libro De propositionibus [sic in Betti] l'anno 1570'*" in [49, page 163].

[48] "*La risposta ricevuta al quesito proposto fu invero piuttosto singolare, poiché il Cardano avrebbe affermato : 'ch'egli non lo sapea, affermandomi che il suo genio era stato e non egli che lo aveva scritto, con soggiungermi che per ciò spesse volte egli stesso non sapea quello che avesse scritto, et che leggendolo non lo intendeva. Cosa che veramente mi scandalizzò molto a prima faccia, poi mi diè che molto a meravigliarmi. Finalmente mi spedì dicendomi ch'io aspettassi che tosto (come ho detto, che poi veramente fu) ei verrebbe in istampa da Alemagna'*" in [49, page 163] referring to the *Mascara*, page 84.

Regula Aliza. Obviously, giving a title to a miscellany of heterogeneous chapters does not automatically make the text consistent. Indeed, we see in Section 5.1 that the *Aliza* has not been proofread or proofread very quickly.

2.4. The readers of the *De Regula Aliza*

Due to the problems that I have detailed in the preceding section, the *Aliza* was, and still remains, a very unfamiliar book. During the centuries, it had nevertheless a handful of reader. In the following, I give an overview on the studies that have already been done on this book. Note that I do not mention the authors who simply quote the title.

According to the last sentence of the above quotation by Cossali,[49] one of the very first readers of the *Aliza* must have been Rafael Bombelli. Anyway, I did not manage to retrieve any comments in his *Algebra* (1572) about the *Aliza.* There is nevertheless an indirect link that deserves to be mentioned. It is Cardano himself that coupled in the first paragraph of the *Sermo de Plus et Minus* a reference to the *Aliza* and Bombelli's name.[50] What we know for sure is that Bombelli read the *Ars Magna*, since in the preface to the readers in his *Algebra* (1572), he criticised the obscure way in which the *Ars Magna* had been written.[51] This also shows that Cossali's last sentence in the above quotation is inaccurate.

Federico Commandino read for sure the *Aliza*, or at least some parts. In his edition of the *Elements* (1572), he briefly mentioned Cardano's alternative sign rule from *Aliza*, Chapter XXII (see below, Section 5.6 from page 378).[52]

[49] See above, on page 16.

[50] For more details, see below, Section 5.6.

[51] "*Hanno poi, e Barbari, e Italiani à nostri tempi scritto [...], oltre che il Cardano Mediolanese nella sua arte magna, ove di questa scientia assai disse, ma nel dire fu oscuro*", see [1, Agli Lettori, no page number].

[52] "*[C]oguntur dicere si minus per minus multiplicetur produci plus. Quod verum non esse primus animadvertit Hieronymus Cardanus non solum mathematicus,*

Belonging to the next generation of mathematicians, Simon Stevin made reference to the *Aliza* in his *Arithmétique* (1585). While solving the equation $x^3 = 6x + 40$, Stevin shortly stated that Cardano also put some examples in his *Aliza*.[53]

Also Thomas Harriot made reference to the *Aliza*, and twice. We know that he read (at least) one chapter of the *Aliza*. In the British Library,[54] Rosalind Tanner has found[55] the following note to one of Harriot's colleagues, Walter Warner.

> Although Cardane in the beginning the first Chapter (pag. 6) of his 10^{th} booke of Arithmeticke wold have $\sqrt{9}$ to be $+3$ or -3, yet in his Aliza being a latter worke he was of another opinion. I prey read his 32 Chapter being at the 42 page.[56]

The chapter of the *Aliza* to which Harriot was referring is actually Chapter XXII, which fairly starts at the *folio* 42 of the 1570 edition. In her two articles,[57] Tanner establishes a relation between Harriot's remarks on the fact that there is a choice behind the sign rule and the new sign rule proposed by Cardano in the *Aliza* (see Section 5.6).

As far as I know, there is only one scholar who studied the *Aliza* pretty in details. It is Pietro Cossali, born in Verona in 1748 and

sed et philosophus ac medicus prestantissimus, ut apparet in libro de regula aliza", see [26, Book X, Proposition 34, Theorem III, page 149].

[53] "*Cardane met aussi en son* Aliza *quelques exemples, servans à ceste matiere, mais pas generaux*", see [39, Book II, page 309].

[54] It is the Add. Ms. 6783, folio 121. The page has no date and Tanner remarks that no sufficient indication can be found to place it in time. Anyway, she remarks that all the dated pages in the manuscript are between 1589 and 1619. Note that Tanner says that the folio 121 is a stray sheet among a batch of unconnected papers, but in truth it is associated with other notes on the *Ars Magna*, see [38].

The second reference to the *Aliza* is in Add. Ms. 6785, folio 197. I thank Jackie Stedall for these connections.

[55] See [95].

[56] I have picked the quotation up from [95, page 144]. The '10^{th} booke of Arithmeticke' is in truth the *Ars Magna*, which at a certain point should have become Book X of the *Opus Arithmeticæ Perfectum*, see above, Section 2.3.

[57] See [95] and [94].

deceased in Padua in 1815. He was a priest of the Teatini's order in Milan, calculus, physics, and astronomy professor at the university in Padua, member of the Società italiana delle scienze, and pensionnaire of the Reale istituto italiano di scienze, lettere ed arti.[58] He also was a historian of mathematics. In 1799 his history of algebra in Renaissance Italy, the *Origine, trasporto in Italia, primi progressi in essa dell'Algebra*[59] was published in two volumes. Moreover, since 1966 we have available the commented transcription by Romano Gatto of Cossali's manuscript *Storia del caso irriducibile*,[60] which has a considerable number of pages in common with the history of algebra.

In the *Storia del caso irriducibile* and mostly in the second volume of the history of algebra, Cardano played a central role and the *Aliza* was handled on the same footing as the *Ars Magna*, as well as the *Ars Magna Arithmeticæ*. Nevertheless, it must be said that most of the time Cossali's accuracy as a historian is not up to standards. He seldom gave the exact references, had no hesitation in integrating or completely rewriting Cardano's proofs, and paid no attention to the interconnections between Cardano's treatises. In short, he was doing mathematics from a historical starting point rather than history of mathematics.

Actually, Cossali was a mathematician, and I believe that this could partially explain his attitude as a historian. Around this period, mathematicians started to study in depth the *casus irreducibilis* for cubic equations. In 1781 the Academy of Padua announced a competition to prove whether it is possible to free the cubic formula from imaginary numbers, but finally the prize was not assigned. Then, in 1799, Ruffini began to get involved in the study of equations and in 1813 he published an incomplete proof of the fact that, when a cubic equation undergoes the *casus irreducibilis*, none of its solutions can be expressed only by real radicals.[61] Cossali was very interested in the topic of the *casus irreducibilis*. His very first work, the *Particularis*

[58] For a more detailed biography, see [60, page 3]
[59] See [57].
[60] See [60].
[61] See [36] and [37].

methodi de cubicarum æquationum solutione a Cardano luci traditæ. Generalis posteriorum analystarum usus ex cap. I De Regula Aliza ipsius Cardani vitio luculentissime evictus. Atque mysterium casus irreducibilis post duo sœcula prorsus retecta causa sublatum specimen analyticum primum (1799) was devoted to it, and the *Disquisizione sui varj metodi di eliminazione con il componimento di uno nuovo* (1813) ranged around the same issue.[62] He wanted to take part in the 1781 competition, but he could not finish on time his *Sul quesito analitico proposto all'Accademia di Padova per il premio dell'anno 1781 di una assoluta dimostrazione della irriducibilità del binomio cubico*.[63] There, he also started a long lasting controversy with Anton Maria Lorgna on some mistakes that he made while dealing with the *casus irreducibilis*. Briefly, the mathematician Cossali worked pretty much on equations, and in particular on cubic equations. Cossali's strong interest toward the *casus irreducibilis* is the reason why – I suggest – he struggled with the *Aliza* under a mainly mathematical viewpoint.

In our chronological review of the *Aliza*'s readers, we stumble at this point upon some scholars who wrote an overall history of mathematics and limited themselves to choose and explain a bunch of tiny mathematical techniques from the *Aliza*. They are Charles Hutton (*Tracts on mathematical and philosophical subjects* (1812), on pages[64] 219-224), Moritz Cantor (*Vorlesungen über die Geschichte der Mathematik* (1892), on pages[65] 532-537), and Gino Loria (*Storia delle matematiche* (1931), on pages[66] 298-299). In particular, concerning the overall flowing of the *Aliza*, Loria said that

> [w]hile [...] all of Cardano's writings do not shine out for clarity, this last [referring to the *Aliza*] is despairingly unclear. Nobody or a few people (one of these is the

[62] See [58] and [59].
[63] See [56].
[64] See [70].
[65] See [52].
[66] See [74].

> historian Cossali) struggled to understand it and its influence was poor or none. The one who goes through it follows Cardano while he tries one thousand ways to solve that distressing enigma. He sees him stopping, moving backwards, then resuming his way ahead, pushed by the longing to discover the limits of applicability of the solving method conceived by Tartaglia.[67]

Finally, we have the contemporary readers, like the already mentioned Rosalind Tanner (*The alien realm of the minus: deviatory mathematics in Cardano's writings* (1980), on pages[68] 166-168), Silvio Maracchia (*Storia dell'algebra* (2005), on pages[69] 277 and 331-335), and Jacqueline Stedall (*From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra* (2011), on page[70] 10). In a greater or lesser measure, they conform to their predecessors and give an episodic account of the *Aliza*.

It is also worthwhile to mention the new impulse that has been given in Milan from the early 90s to the studies of Cardano's writings thanks to the project of the edition of his works (see the web site `http://www.cardano.unimi.it/`).

2.5. The mathematical context: how we deal with equations nowadays

As we will see in the following sections, a focus of Cardano's interest (especially in the *De Regula Aliza*) is, rather than numbers themselves,

[67] "*Mentre [...] tutti gli scritti del Cardano non brillano per chiarezza, quest'ultimo è di una disperante oscurità; nessuno o pochi (uno di questi è lo storico Cossali) si sforzarono di comprenderlo, ove scarsa per non dire nulla ne fu l'influenza. Chi lo percorre segue ll Cardano mentre tenta mille vie per scioglire quell'angoscioso enigma; lo vede arrestarsi, retrocedere per poi riprendere il cammino in avanti, sospinto dalla brama di scoprire i limiti di applicabilità del metodo di risoluzione ideato dal Tartaglia*", see [74, page 298].

[68] See [94].

[69] See [76].

[70] See [92].

how numbers can be expressed. This is what we could call the 'algebraic shapes' of numbers, that is expressions written in Cardano's stenography like $\sqrt{a}+b$ or $\sqrt[3]{a}+\sqrt[3]{b}$ where a, b are rational numbers such that respectively their square roots or cubic roots are irrational. When Cardano addressed the problem entailed by the *casus irreducibilis*, he turned to these algebraic shapes rather than to the numerical values of the solutions or to the structural relationships between them (as Lagrange, Ruffini, Abel, and Galois will subsequently do). In the end, he was trying for a general theory of the shapes under which certain numbers can be written. Indeed, a recurring strategy in Cardano's mathematical treatises is to establish whether these shapes can convey a solution of an equation or not (see below, Section 4.2.4 and Section 5.3.1). As such, Cardano's attempt is doomed to failure. Nevertheless, this study of the algebraic shapes is very up-to-date and marks a trend that will go on for about two centuries up to the 18$^{\text{th}}$ century.

It must be said, moreover, that Cardano mainly deals with procedures or algorithms rather than with formulae: in these terms, the most problematic of Cardano's algebraic treatises, that is the *Aliza*, should not be considered only as a failure, but also as a treasury of results.

Let us now try to understand why, from a nowadays mathematical viewpoint, Cardano had to fail. In the following sections, I recall our solving methods that lead to the formulae for quadratic, cubic, and quartic equations. I deal in particular with the cases in which the coefficients are real. Then, especially concerning cubic equations, I explain, using Galois theory, why imaginary numbers necessarily appear in the cubic formula when the equation falls into the *casus irreducibilis*. Finally, I provide some paradigmatic examples in the cubic case.

2.5.1. Solving quadratic equations

We consider the generic quadratic equation

$$\alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0, \tag{2.5.1}$$

with $\alpha_2, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\alpha_2 \neq 0$. We are looking for its solutions in $\mathbb{C}$.

Dividing by α_2, we get the monic equation

$$x^2 + \frac{\alpha_1}{\alpha_2}x + \frac{\alpha_0}{\alpha_2} = 0.$$

We rewrite the above equation in the following way

$$x^2 + \frac{\alpha_1}{\alpha_2}x + \left(\frac{(\alpha_1)^2}{4(\alpha_2)^2} - \frac{(\alpha_1)^2}{4(\alpha_2)^2}\right) = -\frac{\alpha_0}{\alpha_2}.$$

Thus, using the formula for the square of a binomial, we find a square on the left side of the equation

$$\left(x + \frac{\alpha_1}{2\alpha_2}\right)^2 = -\frac{\alpha_0}{\alpha_2} + \frac{(\alpha_1)^2}{4(\alpha_2)^2}.$$

Taking the square root and developing calculations, we obtain

$$x_1 = \frac{-\alpha_1 + \sqrt{(\alpha_1)^2 - 4\alpha_2\alpha_0}}{2\alpha_2} \quad \text{and} \quad x_2 = \frac{-\alpha_1 - \sqrt{(\alpha_1)^2 - 4\alpha_2\alpha_0}}{2\alpha_2}. \tag{2.5.2}$$

The relation (2.5.2) is called the 'quadratic formula' and $\Delta_2 = (\alpha_1)^2 - 4\alpha_2\alpha_0$ is called the 'discriminant' of a quadratic equation.

2.5.2. Quadratic equations with real coefficients

If α_2, α_1, α_0 in equation (2.5.1) are real, we can then discuss the nature of its solutions starting from the sign of its discriminant Δ_2. In the case $\Delta_2 > 0$, it is easily seen that we have two distinct real solutions. Analogously, in the case $\Delta_2 = 0$, we have two coincident real solutions. Finally, in the case $\Delta_2 < 0$, we have two complex conjugate solutions.

The main idea behind this method is the so-called completion of the square. It is exposed above as an algebraic trick, but whenever the coefficients are real we can interpret it in a geometrical way, according to the following diagram.

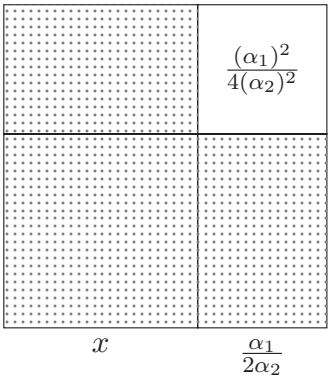


Figure 2.1. – Geometrical interpretation of the completion of the square for the equation $\alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0$ with real coefficients.

The dotted part is given. If we add $\frac{(\alpha_1)^2}{4(\alpha_2)^2}$ to it, we obtain a square the side of which gives x. In this way, a support for intuition is provided.

2.5.3. Solving cubic equations

We consider the general cubic equation

$$\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0, \tag{2.5.3}$$

with $\alpha_3, \alpha_2, \alpha_1, \alpha_0 \in \mathbb{C}$ and $\alpha_3 \neq 0$. We are looking for its solutions in $\mathbb{C}$.

The substitution $x = y - \frac{\alpha_2}{3\alpha_3}$ yields to the so-called depressed cubic equation, which is monic and lacks in the second degree term. It is

$$y^3 + py + q = 0 \tag{2.5.4}$$

with

$$p = \frac{3\alpha_3\alpha_1 - (\alpha_2)^2}{3(\alpha_3)^2}, \qquad (2.5.5)$$
$$q = \frac{2(\alpha_2)^3 - 9\alpha_3\alpha_2\alpha_1 + 27(\alpha_3)^3\alpha_0}{27(\alpha_3)^3}.$$

We introduce two variables u, v such that $y = u + v$. Substituting in equation (2.5.4), we get

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0. \qquad (2.5.6)$$

Let us notice that we can force on u, v (which have been arbitrarily chosen) another condition, for instance $3uv + p = 0$ suggested by the previous equation.[71] Therefore, we have $u^3v^3 = -\frac{p^3}{27}$ and equation (2.5.6)

$$u^3 + v^3 + q = 0.$$

becomes simpler. Thus, if we manage to solve the system coming from the two conditions on u, v

$$\begin{cases} u^3 + v^3 = -q \\ u^3v^3 = -\frac{p^3}{27} \end{cases}, \qquad (2.5.7)$$

we can also solve equation (2.5.3). Let us notice that this system has two solutions in u^3, v^3 and eighteen solutions in u, v. We are interested in the sum $u + v$, which will give the three solutions of cubic equation (2.5.4), then of equation (2.5.3).

[71] In fact, the linear system given by

$$\begin{cases} y = u + v \\ 3uv + p = 0 \end{cases}, \quad \text{that is} \quad \begin{cases} y = u + v \\ uv = -\frac{p}{3} \end{cases},$$

always has solutions in $\mathbb{C}$.

We consider u^3 and v^3 as unknowns. Since we know their sum and their product, we associate[72] the system (2.5.7) with the equation

$$t^2 + qt - \frac{p^3}{27} = 0. \tag{2.5.8}$$

This is called the 'Lagrange resolvent' for the cubic equation (2.5.4). Now we use the quadratic formula to draw backwards in chain all the solutions of the other equations: we firstly find the two solutions t_1, t_2 of equation (2.5.8), from which we obtain the two couples of solutions

$$\begin{cases} u^3 = t_1 \\ v^3 = t_2 \end{cases} \quad \text{and} \quad \begin{cases} u^3 = t_2 \\ v^3 = t_1 \end{cases}$$

of system (2.5.7).

We consider the first couple. We want to draw u, v. The three cubic roots of $t_1, t_2 \in \mathbb{C}$ are respectively

$$\sqrt[3]{t_1},\ \omega\sqrt[3]{t_1},\ \omega^2\sqrt[3]{t_1} \quad \text{and} \quad \sqrt[3]{t_2},\ \omega\sqrt[3]{t_2},\ \omega^2\sqrt[3]{t_2},$$

with ω a primitive third root of unity[73]. Thus, there are nine rearrangements for $u + v$. At first, we arbitrarily choose u. Since the condition $uv = -\frac{p}{3}$ must hold, it follows that only three rearrangements out of nine fit, which are:

$$y_1 = \sqrt[3]{t_1} + \sqrt[3]{t_2}, \quad \text{or} \quad y_2 = \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2}, \quad \text{or} \quad y_3 = \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2}.$$

These are the three solutions of equation (2.5.4).

Considering then the second couple of solutions of system (2.5.7), where the values for u^3, v^3 are exchanged, we find the same values for

[72] Since

$$(\alpha - t)(\beta - t) = t^2 - (\alpha + \beta)t + \alpha\beta,$$

if we search two numbers of which we know their sum S and product P, it is enough to solve the quadratic equation $t^2 - St + P = 0$.

[73] That is, $\omega^3 = 1$ e $\omega^k \neq 1$ per $k = 1, 2$. In particular, if ω is a primitive third root of unity, then $\omega = -\frac{1}{2} + \imath\frac{\sqrt{3}}{2}$ e $\omega^2 = -\frac{1}{2} - \imath\frac{\sqrt{3}}{2}$.

the symmetric expression $u+v$.

Hence, the solutions of equation (2.5.3) are

$$x_i = y_i - \frac{\alpha_2}{3\alpha_3} \quad \text{for} \quad i=1,2,3,$$

that is,

$$\begin{aligned} x_1 &= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \\ x_2 &= \omega \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \omega^2 \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \\ x_3 &= \omega^2 \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \omega \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} - \frac{\alpha_2}{3\alpha_3}, \end{aligned} \tag{2.5.9}$$

explicitly written as function of the coefficients p, q in (2.5.5). This is called the 'cubic formula' for equation (2.5.3) and

$$\Delta_3 = \frac{q^2}{4}+\frac{p^3}{27}, \tag{2.5.10}$$

$$\text{where } p = \frac{3\alpha_3\alpha_1 - (\alpha_2)^2}{3(\alpha_3)^2}, q = \frac{2(\alpha_2)^3 - 9\alpha_3\alpha_2\alpha_1 + 27(\alpha_3)^3\alpha_0}{27(\alpha_3)^3},$$

is called[74] 'discriminant' of the cubic equation (2.5.3).

The main idea behind this method is to reveal the structure of a cubic equation introducing two new variables u, v linked by the linear condition $u+v=y$, and then to exploit the degree of freedom

[74]Note that, usually, the discriminant is defined from a higher viewpoint by $(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2$, where x_1, x_2, x_3 are the roots of the cubic polynomial (see, for instance, [72, Chapter IV, Sections 6 and 8].). My definition differs from the usual one by the factor $-\frac{1}{108}$. More precisely, if Δ_3' is the usual discriminant, we have $\Delta_3 = -1/108\Delta_3'$. Nevertheless, I choose to use this definition, since in Cardano's text we explicitly find the value $\frac{q^2}{4}+\frac{p^3}{27}$ without the factor $-\frac{1}{108}$.

obtained in this way by imposing a further condition on the product uv. The method works because it takes advantage of symmetries in a fundamental way. In fact, the number of solutions of the system (2.5.7) decreases thanks to the symmetry in the expression $u + v$.

2.5.4. Cubic equations with real coefficients

We first remark that, if α_3, α_2, α_1, α_0 in equation (2.5.3) are real, then p, q in (2.5.5) are real too.

We can then discuss the nature of the solutions of equation (2.5.3) starting from the sign of its discriminant Δ_3. We know that each odd degree equation with real coefficients has at least one real solution, for instance as a simple consequence of the Intermediate values theorem.

It is convenient to neglect the term $\frac{\alpha_2}{3\alpha_3}$, which has no effect since it is real, and discuss the nature of the solutions of equation (2.5.4). In the case $\Delta_3 > 0$, we have one real solution and two complex conjugate solutions. In fact, we get that $\sqrt[3]{t_1}$, $\sqrt[3]{t_2}$ are real. Thus,

$$
\begin{aligned}
y_1 &= \sqrt[3]{t_1} + \sqrt[3]{t_2} \in \mathbb{R},\\
y_2 &= \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2} = -\frac{1}{2}(\sqrt[3]{t_1} + \sqrt[3]{t_2}) + \imath\frac{\sqrt{3}}{2}(\sqrt[3]{t_1} - \sqrt[3]{t_2}) \in \mathbb{C},\\
y_3 &= \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2} = -\frac{1}{2}(\sqrt[3]{t_1} + \sqrt[3]{t_2}) - \imath\frac{\sqrt{3}}{2}(\sqrt[3]{t_1} - \sqrt[3]{t_2}) \in \mathbb{C},
\end{aligned}
$$

and y_2, y_3 are complex conjugate.

In the case $\Delta_3 = 0$, we have three real solutions, two of which coincide. In fact, we get $\sqrt[3]{t_1} = \sqrt[3]{-\frac{q}{2}} = \sqrt[3]{t_2}$. Thus,

$$
\begin{aligned}
y_1 &= 2\sqrt[3]{t_1} = -2\sqrt[3]{\frac{q}{2}} \in \mathbb{R},\\
y_2 &= (\omega + \omega^2)\sqrt[3]{t_1} = -\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{\frac{q}{2}} = (\omega^2 + \omega)\sqrt[3]{t_1} = y_3 \in \mathbb{R}.
\end{aligned}
$$

Finally, let $\Delta_3 < 0$. Then, we have three distinct real solutions. In fact, we get $\sqrt[3]{t_1}, \sqrt[3]{t_2} \in \mathbb{C}$. But, since the equality $\sqrt[3]{t_1}\sqrt[3]{t_2} = -\frac{p}{3} \in \mathbb{R}$ holds, $\sqrt[3]{t_1}, \sqrt[3]{t_2}$ are complex conjugate. Thus,

$$\begin{aligned} y_1 &= \sqrt[3]{t_1} + \sqrt[3]{t_2} = \sqrt[3]{t_1} + \overline{\sqrt[3]{t_1}} \in \mathbb{R}, \\ y_2 &= \omega\sqrt[3]{t_1} + \omega^2\sqrt[3]{t_2} = \omega\sqrt[3]{t_1} + \overline{\omega\sqrt[3]{t_1}} \in \mathbb{R}, \\ y_3 &= \omega^2\sqrt[3]{t_1} + \omega\sqrt[3]{t_2} = \omega^2\sqrt[3]{t_1} + \overline{\omega^2\sqrt[3]{t_1}} \in \mathbb{R}. \end{aligned}$$

We observe that $y_2+y_3 = (\omega+\omega^2)\sqrt[3]{t_1}+(\overline{\omega}+\overline{\omega^2})\overline{\sqrt[3]{t_1}} = -(\sqrt[3]{t_1}+\sqrt[3]{t_2}) = -y_1$.

More precisely (but only for equation (2.5.4)), when we write t_1, t_2 in trigonometric form, we have

$$\begin{aligned} \sqrt[3]{t_1} &= \sqrt[3]{\rho}\sqrt[3]{\cos\theta + \imath\sin\theta} = \sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \imath\sin\frac{\theta}{3}\right), \\ \sqrt[3]{t_2} &= \overline{\sqrt[3]{t_1}} = \sqrt[3]{\rho}\left(\cos\frac{\theta}{3} - \imath\sin\frac{\theta}{3}\right), \end{aligned}$$

with $\rho \in [0, 2\pi)$ and $\theta \in [0, +\infty)$, and then

$$\begin{aligned} y_1 &= \sqrt[3]{t_1} + \overline{\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\frac{\theta}{3}, \\ y_2 &= \omega\sqrt[3]{t_1} + \overline{\omega\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\left(\frac{\theta}{3} + \frac{2}{3}\pi\right), \\ y_3 &= \omega^2\sqrt[3]{t_1} + \overline{\omega^2\sqrt[3]{t_1}} = 2\sqrt[3]{\rho}\cos\left(\frac{\theta}{3} + \frac{4}{3}\pi\right). \end{aligned}$$

We remark that the three angles in the cosine function argument are at the distance $\frac{2}{3}\pi$ one from the other, so that one of them necessarily lies in the first or fourth quadrant where the cosine function is positive. Another of them necessarily lies in the second or third quadrant where the cosine function is negative. On the contrary, the remaining one varies in such a way that one cannot establish *a priori* the sign of its cosine. Hence, we obtain a positive and a negative solution, whereas the sign of the third solution is changeable.

This last case – when a cubic equation has three distinct real solutions, or $\Delta_3 < 0$ – is called the '*casus irreducibilis*'.[75] Note that $\Delta_3 < 0$ means that imaginary numbers appear in the cubic formula (see below, Section 2.5.6).

Let us notice that, in the cubic case with real coefficients, no geometric interpretation provides a good support for intuition as the completion of the square was in the quadratic case. In other words, the completion of the cube does not have an analogous heuristic value. Actually, it does not work at all. In fact, if we write equation (2.5.3) as

$$x^3 + \frac{\alpha_2}{\alpha_3}x^2 + \frac{\alpha_1}{\alpha_3}x = -\frac{\alpha_0}{\alpha_3}$$

and we complete the cube, we find the condition $\alpha_1 = \frac{(\alpha_2)^2}{3}$.

Suppose then $\alpha_1 \neq \frac{(\alpha_2)^2}{3}$. We can try to write equation (2.5.3) as

$$x^3 + \frac{\alpha_2}{\alpha_3}x^2 = -\frac{\alpha_1}{\alpha_3}x - \frac{\alpha_0}{\alpha_3},$$

and add and subtract the quantity $\frac{\alpha_2^2}{\alpha_3^2}\frac{x}{3} + \frac{\alpha_2^3}{27\alpha_3^3}$ in order to obtain the cube of a binomial

$$\left(x + \frac{\alpha_2}{3}\right)^3 = -\frac{\alpha_1}{\alpha_3}x - \frac{\alpha_0}{\alpha_3} + \frac{\alpha_2^2}{\alpha_3^2}\frac{x}{3} + \frac{\alpha_2^3}{27\alpha_3^3}.$$

But then we can in no way write the right side of the equation as a cube of a polynomial one term of which is x, since x^3 does not appear on the right side. Then, one needs to follow another way and introduce the new variables u, v.

[75] This is a late terminology that we do not find in Cardano's works. Up to now, the earliest reference to the *casus irreducibilis* that I have found is in the 1781 contest by the Accademia in Padova, where they asked to prove that Cardano's formula necessarily entails imaginary numbers when the equation falls into the *casus irreducibilis*.

2.5.5. Cubic equations with real coefficients solved in a trigonometrical way

In the preceding section, we have remarked that, when a cubic equation has three distinct real solutions, imaginary numbers appear in the cubic formula, since $\Delta_3 < 0$ and the discriminant is under square roots. Nevertheless, there is a way to obtain a real expression for the solutions. One can achieve this by exploiting the triple-angle formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Consider the depressed equation 2.5.4 and set $y = u\cos\theta$. Then, $u^3\cos^3\theta + pu\cos\theta + q = 0$. We want to choose u in order to compare the equation with the triple-angle formula. We multiply both sides of the equation by $\frac{4}{u^3}$ and we have $4\cos^3\theta + \frac{4p}{u^2}\cos\theta + \frac{4q}{u^3} = 0$. There, we are allowed to choose $u = \sqrt{-\frac{4}{3}p}$, so that we get $4\cos^3\theta - 3\cos\theta + \frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}} = 0$. Note that u is well-defined, since $\Delta_3 < 0$ implies that $p < 0$. Then, by comparison, we have

$$-\cos 3\theta = \frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}$$

and, since $y = u\cos\theta$, we finally have that

$$\begin{aligned}
y_1 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right)\right),\\
y_2 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right) + \frac{2}{3}\pi\right),\\
y_3 &= \sqrt{-\frac{4}{3}p}\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\frac{q}{p}\sqrt{-\frac{3}{p}}\right) + \frac{4}{3}\pi\right).
\end{aligned}$$

The arccosine is defined only between -1 and 1, which gives the condition on the discriminant $27q^2 + 4p^3 \leq 0$. We remark that, as a matter of fact, the above formulae do not contain imaginary numbers.

Anyway, these formulae are not as interesting as the algebraic ones in Section 2.5.3, since they do not give as much information. Indeed, it is still out of reach to exactly calculate the cosine and arccosine functions. The above formulae look very easy, but in fact they hide the complex structure of the equation in an artificially introduced function (and its almost inverse function) that has a particular relationship with a certain cubic polynomial (the triple-angle formula). Moreover, when $\Delta_3 > 0$, one is compelled to use hyperbolic functions.

2.5.6. What Galois theory can say about cubic equations

There[76] is a deeper reason that explains why imaginary numbers necessarily appear in the cubic formula, and it surfaces using Galois theory. Galois' original definition of the group of an equation easily leads to impracticable calculations when the degree of the equation is greater than four. Nevertheless, his approach has the advantage of giving a very simple proof of the impossibility of overcoming the use of imaginary numbers to solve an irreducible cubic equation with real coefficients that falls into the *casus irreducibilis*.

Firstly, I will quickly sum up how to calculate the Galois group in the general case of an n^{th} degree equation. Let $f(x) = 0$ be the equation and denote its (simple) solutions by $x_1, x_2, \ldots, x_n$. Then, Galois states[77] – as obvious – that it is possible to find some integers $A_1, A_2, \ldots, A_n$ such that the function of the solutions (the so-called Galois resolvent)

$$V(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = A_1 x_{i_1} + A_2 x_{i_2} + \ldots + A_n x_{i_n}$$

takes $n!$ different values on the set of the permutations of the solutions.

Once a particular value of this function, for instance

$$V_1 = A_1 x_1 + A_2 x_2 + \ldots + A_n x_n$$

[76] I thank Massimo Galuzzi for this section.

[77] See [35, page 110]. I have slightly modified Galois' original notation.

obtained by the identical permutation, is given, we have, in modern terms, a primitive element. All the solutions of $f(x) = 0$ may be expressed as rational functions of V_1, with the coefficients that are in the same field as $f(x)$.

Due to the fundamental theorem of symmetric polynomials, the coefficients of the polynomial

$$\varphi(X) = \prod_{i_1,i_2,\ldots,i_n} (X - (A_1 x_{i_1} + A_2 x_{i_2} + \ldots + A_n x_{i_n}))$$

are in the same field as $f(x)$. Let $\Phi(X)$ be the irreducible factor of $\varphi(X)$ that contains the root[78] V_1. The Galois group is given by the permutations that correspond to the roots of $\Phi(X)$.

What does it mean, in this approach, to solve an equation? It amounts to "reduce its group successively to the point where it does not contain more than a single permutation".[79] This reduction is, of course, accomplished by adjoining quantities that can be supposed as known.

Let us now detail this approach in the particular case of cubic equations. We consider the irreducible equation

$$x^3 + px + q = 0, \tag{2.5.11}$$

with p, q real, and let x_1, x_2, x_3 be its solutions. In this case, a Galois resolvent may be given by[80]

$$V(x_i, x_j, x_k) = x_i - x_k.$$

[78] Of course, in the general case, we have $\Phi(X) = \varphi(X)$. The polynomial $\Phi(x)$ is the minimal polynomial of V_1.

[79] See [35, page 121].

[80] There are two kinds of tests to make. First, if $x_1 - x_2 = x_2 - x_1$, then $x_1 = x_2$. Second, if $x_1 - x_2 = x_2 - x_3$, then $2x_2 = x_1 + x_3$, that is $3x_2 = x_1 + x - 2 + x_3 = 0$. The other cases are similar.

By means of this resolvent, the polynomial $\varphi(X)$ becomes

$$\varphi(X) = \prod_{x_i \neq x_j} (X - (x_i - x_j)) = X^6 + 6pX^4 + 9p^2X^2 + 4p^3 + 27q^2,$$

or

$$\varphi(X) = (X^3 + 3pX)^2 + 4p^3 + 27q^2.$$

In this calculation, we remark that the quantity $4p^3 + 27q^2$ has a real structural value and is not a mere device to make the demonstration work.

The result, namely that imaginary numbers are indispensable tools to solve a cubic equation when it falls into the *casus irreducibilis*, has been proved, during the 19$^{\text{th}}$ century, by Paolo Ruffini in 1813 (see [36] for an incomplete proof, and [37]), Pierre Laurent Wantzel in 1843 (see [42]), Vincenzo Mollame in 1890 (see [33] and [34]), Otto Hölder in 1891 (see [30]), Adolf Kneser in 1892 (see [31]), and Leopold Gegenbauer in 1893 (see [27]). I plan to develop the study of this topic in a forthcoming work.

2.5.7. Paradigmatic examples for cubic equations

We have already touched on the fact that Cardano is interested in the algebraic shapes for the solutions of a cubic equation. In particular, let us have a look at the family of equations $x^3 = a_1x + a_0$, which is fundamental in Cardano's arguments. We will try to deduce some information on the shapes of its solutions.

Since for most of the time Cardano uses rational coefficients and since in his treatises there is the well-identifiable topic of equations with rational coefficients, let us assume that a_1, a_0 are rational. In this case, consider a, b positive, rational. Cardano states that the (positive) irrational solutions go either under the shape $\sqrt{a} + b$, with a such that $\sqrt{a}$ is irrational, or under the shape $\sqrt[3]{a} + \sqrt[3]{b}$, with a, b such that $\sqrt[3]{a}$, $\sqrt[3]{b}$ are irrational and $\sqrt[3]{ab}$ is rational. Indeed, it turns out that these two shapes can never be solution of the same cubic equation, since, if a rational equation has a solution of the first shape,

then it has $\Delta_3 < 0$ and, if it has a solution of the second shape, then it has $\Delta_3 > 0$.

In fact, we firstly consider a rational, irreducible polynomial of degree three and we suppose that it exists a, b positive, rational such that $x_1 = \sqrt{a} + b$ is irrational and is one of its roots. Then, by Galois theory, we have that the polynomial is irreducible over $\mathbb{Q}$, that is, if x_2 is another root of the polynomial, then it is rational. Moreover, $x_3 = -\sqrt{a} + b$. Since we get three real roots, the discriminant is negative.

We consider then a rational, irreducible polynomial of degree three and we suppose that it exists a, b positive, rational such that $x_1 = \sqrt[3]{a} + \sqrt[3]{b}$ is irrational and is one of its roots. Then, again by Galois theory, we have that, if x_2 is another root of the polynomial, then $x_2 = \omega^r \sqrt[3]{a} + \omega^s \sqrt[3]{b}$, with ω a primitive third root of unity and $r, s \in \mathbb{Z}/3\mathbb{Z}$ such that $r, s \not\equiv 0 \bmod 3$. If we moreover assume that $\sqrt[3]{ab}$ is rational, then $r + s \equiv 0 \bmod 3$. We can assume that $r = 1$ and $s = 2$ and $x_3 = \omega \sqrt[3]{a} + \omega^2 \sqrt[3]{b}$ (up to interchanging x_2 and x_3). It is easily seen that x_2, x_3 cannot be real, so that the discriminant of the equation is in the end positive.

For instance, consider the equation $x^3 = 6x + 4$, which is in *Aliza*, Chapters XXIV and LIX. It has $\Delta_3 < 0$ and its real solutions are $x_1 = 1 + \sqrt{3}$, $x_2 = -2$, and $x_3 = 1 - \sqrt{3}$. Instead, the equation $x^3 = 6x + 6$ (which is in *Ars Magna Arithmeticæ*, Chapters XXIII, XXVII, XXXII, in *Ars Magna*, Chapter XII, and in *Aliza*, Chapters XVIII, XIX, XXV, LVIII) has $\Delta_3 > 0$ and its real solution is $\sqrt[3]{4} + \sqrt[3]{2}$.

Moreover, there is the case in which an equation of the family $x^3 = a_1 x + a_0$ has $\Delta_3 > 0$ and the real solution is rational. For instance, 4 is the real solution of $x^3 = 6x + 40$, which is in *Ars Magna*, Chapter XII and *Aliza*, Chapters XLIX, LIII, LIX.

A consequence of the method in Section 2.5.3 is that solving cubic equations is often brought back to cubic root calculations, which is not always easy to carry out. For instance, when we want to draw a solution of $x^3 + 6x = 40$ using the cubic formula, $\sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}}$ is returned. In truth, this solution is 4, since the terms

under the cubic roots are two cubes, namely $20+\sqrt{392}=(2+\sqrt{2})^3$ and $20-\sqrt{392}=(2-\sqrt{2})^3$, but this is not evident at all.[81] Indeed, in *Ars Magna*, Chapter XII Cardano does not know that the terms under the cubic roots are two cubes. In *Aliza*, Chapters XLIX and LIII he knows that a solution is 4, but there is no mention of the cubic formula. Instead, in *Aliza*, Chapter LIX he knows it, and he explicitly states it in the third corollary.

Much more interesting – in the context of Cardano's inquiries – is the case in which we use the cubic formula to try to solve $x^3=6x+4$. In fact, the formula returns $\sqrt[3]{2+\sqrt{-4}}+\sqrt[3]{2-\sqrt{-4}}$, which in truth is real and is $1+\sqrt{3}$, but Cardano can in no way remark the equality.

2.5.8. Solving quartic equations

We consider the general quartic equation

$$\alpha_4x^4+\alpha_3x^3+\alpha_2x^2+\alpha_1x+\alpha_0=0, \tag{2.5.12}$$

with $\alpha_4,\alpha_3,\alpha_2,\alpha_1,\alpha_0\in\mathbb{C}$ and $\alpha_4\neq 0$. We are looking for its solutions in $\mathbb{C}$.

The substitution $x=y-\frac{\alpha_3}{4\alpha_4}$ yields to the so-called depressed quartic equation, which is monic and lacks in the third degree term

$$y^4+py^2+qy+r=0, \tag{2.5.13}$$

[81] In short, if one wants to know if $\sqrt{a}+b$, with a,b rational, is a cube in $\mathbb{Q}(\sqrt{a})$, one has to search for two rational x,y such that $\sqrt[3]{\sqrt{a}+b}=\sqrt{x}+y$ holds. For more details about the calculation, see for instance [57, Capo V.4, p. 291] or [76, pages 242-244].

with

$$p = \frac{-3(\alpha_3)^2 + 8\alpha_4\alpha_2}{8(\alpha_4)^2}, \tag{2.5.14}$$
$$q = \frac{(\alpha_3)^3 - 4\alpha_4\alpha_3\alpha_2 + 8(\alpha_4)^2\alpha_1}{8(\alpha_4)^3},$$
$$r = \frac{-3(\alpha_3)^4 + 16\alpha_4(\alpha_3)^2\alpha_2 - 64(\alpha_4)^2\alpha_3\alpha_1 + 256(\alpha_4)^3\alpha_0}{256(\alpha_4)^4}.$$

We rewrite equation (2.5.13) in the following way

$$y^4 + py^2 = -qy - r,$$

in order to obtain two perfect squares on the left and right side of the equation.

First, we exploit the formula to expand the square of a binomial. Adding the identity $(y^2 + p)^2 - y^4 - 2py^2 = p^2$ to equation (2.5.13) yields to

$$(y^2 + p)^2 = py^2 - qy + p^2 - r. \tag{2.5.15}$$

Then, in short, we introduce a variable u such that the perfect square on the left side of equation (2.5.15) is preserved (but it will be the square of a trinomial rather than of a binomial). We will force then a further condition on u in order to have a perfect square on the right side. We will finally exploit the formula to expand the square of a trinomial. Following this plan, we add the identity $(y^2 + p + u)^2 - (y^2 + p)^2 = 2uy^2 + 2pu + u^2$ to equation (2.5.15). This yields to

$$(y^2 + p + u)^2 = (p + 2u)y^2 - qy + (p^2 + 2pu + u^2 - r). \tag{2.5.16}$$

We choose a value for u such that the right side of equation (2.5.16) becomes a perfect square. For that, we consider the second degree polynomial in y on the right side of equation (2.5.16), and we force

its discriminant

$$\Delta_2 = q^2 - 4(p+2u)(p^2+2pu+u^2-r)$$

to be zero.

In this way, we want to solve the nested cubic equation in u

$$u^3 + \frac{5}{2}pu^2 + (2p^2 - r)u + (\frac{1}{2}p^3 - \frac{1}{2}pr - \frac{1}{8}q^2) = 0 \qquad (2.5.17)$$

in order to solve equation (2.5.16). We apply to it the method in Section 2.5.3 and we get the depressed nested cubic equation

$$v^3 + Pv + Q = 0, \qquad (2.5.18)$$

with

$$P = -\frac{p^2}{12} - r,$$
$$Q = -\frac{p^3}{108} + \frac{pr}{3} - \frac{q^2}{8},$$

that is $P, Q \in \mathbb{C}$ depend on the coefficients $\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0$ of equation (2.5.12). The solutions of equation (2.5.17) are

$$u_1 = \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} - \frac{5}{6}p,$$
$$u_2 = \omega \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} + \omega^2 \sqrt[3]{-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} - \frac{5}{6}p,$$
$$u_3 = \omega^2 \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} + \omega \sqrt[3]{-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} - \frac{5}{6}p.$$

Each one of the above solutions gives a polynomial with a double root on the right side of equation (2.5.16), allowing us to factorise it in the following way.

Using such a u, we have that

$$(p+2u)y^2 - qy + (p^2 + 2pu + u^2 - r)$$

is a perfect square of the form

$$(\alpha y + \beta)^2 = \alpha^2 y^2 + 2\alpha\beta y + \beta^2.$$

By comparison, we get $\alpha = \sqrt{p+2u}$ and $\beta = \frac{-q}{2\sqrt{p+2u}}$. Equation (2.5.16) can be then rewritten as

$$(y^2 + p + u)^2 = \left((\sqrt{p+2u})y + \frac{-q}{2\sqrt{p+2u}}\right)^2.$$

Taking the square root and collecting like powers of y, we obtain

$$y^2 + (\mp\sqrt{p+2u})y + \left(p + u \pm \frac{-q}{2\sqrt{p+2u}}\right) = 0. \qquad (2.5.19)$$

These are two quadratic equations. We apply the method in Section 2.5.1 to get the solutions

$$y_1, y_2 = \frac{\sqrt{p+2u} \pm \sqrt{(p+2u) - 4\left(p + u + \frac{-q}{2\sqrt{p+2u}}\right)}}{2},$$

$$y_1', y_2' = \frac{-\sqrt{p+2u} \pm \sqrt{(p+2u) - 4\left(p + u - \frac{-q}{2\sqrt{p+2u}}\right)}}{2}.$$

The solutions of equation (2.5.12) are then

$$x_i = y_i - \frac{\alpha_3}{4\alpha_4} \quad \text{with} \quad i = 1, 2, 3, 4,$$

that is

$$x_1 = \frac{\sqrt{p+2u} + \sqrt{(p+2u) - 4\left(p+u+\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4},$$
$$x_2 = \frac{\sqrt{p+2u} - \sqrt{(p+2u) - 4\left(p+u+\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4}, \tag{2.5.20}$$
$$x_3 = \frac{-\sqrt{p+2u} + \sqrt{(p+2u) - 4\left(p+u-\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4} = x_1',$$
$$x_4 = \frac{-\sqrt{p+2u} - \sqrt{(p+2u) - 4\left(p+u-\frac{-q}{2\sqrt{p+2u}}\right)}}{2} - \frac{\alpha_3}{4\alpha_4} = x_2',$$

explicitly written as function of the coefficients p, q, r in (2.5.14).

2.5.9. Quartic equations with real coefficients

We firstly remark that, if α_4, α_3, α_2, α_1, α_0 in equation (2.5.12) are real, then p, q, r in (2.5.14) are real too, that is the nested cubic equation (2.5.17) has real coefficients.

We know that equation (2.5.17) has at least one real solution. We take u to be such a real solution.

We can assume that $u > -\frac{p}{2}$, so that $p + 2u > 0$ and $\sqrt{p+2u} \in \mathbb{R}$. We evaluate the polynomial $R(v) = v^3 + Pv + Q$ on the right side of equation (2.5.18) at $v = \frac{p}{3}$ and we get

$$R\left(\frac{p}{3}\right) = -\frac{q^2}{8} < 0.$$

But

$$\lim_{v \to +\infty} R(v) = +\infty.$$

This means that equation (2.5.18) has a real solution $v > \frac{p}{3}$, so that equation (2.5.17) has a real solution $u > v - \frac{5}{6}p = -\frac{p}{2}$.

Using such a u in equation (2.5.19), we get $\sqrt{p+2u} \in \mathbb{R}$ and the two quadratic equations have real coefficients.

Therefore, if α_4, α_3, α_2, α_1, α_0 in equation (2.5.12) are real, we can choose u such that equation (2.5.19) has real coefficients.

Afterwards, let us consider equation (2.5.12). For the sake of ease, we take $\alpha_4 = 1$, since we can always easily reduce any equation to its monic form

$$x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0. \tag{2.5.21}$$

The coefficients of the depressed quartic equation (2.5.13) become then

$$\begin{aligned} p &= \frac{-3(\alpha_3)^2 + 8\alpha_2}{8}, \\ q &= \frac{(\alpha_3)^3 - 4\alpha_3\alpha_2 + 8\alpha_1}{8}, \\ r &= \frac{-3(\alpha_3)^4 + 16(\alpha_3)^2\alpha_2 - 64\alpha_3\alpha_1 + 256\alpha_0}{256}. \end{aligned}$$

Developing calculations in (2.5.20), we get four huge formulae that deliver the four solutions x_1, x_2, x_3, x_4 depending only on the coefficients of (2.5.12). There, a special role is played by

$$\begin{aligned} \Delta_4 = -4\Big(\alpha_2^2 - 3\alpha_3\alpha_1 + 12\alpha_0\Big)^3 + \\ \Big(2\alpha_2^3 - 9\alpha_3\alpha_2\alpha_1 + 27\alpha_1^2 + 27\alpha_3^2\alpha_0 - 72\alpha_2\alpha_0\Big)^2 \\ = -4B^3 + A^2, \end{aligned}$$

which is called the 'discriminant' of a quartic equation.

As in the case of quadratic and cubic equations, we would like to discuss the nature of the solutions of equation (2.5.12) starting from the sign of its discriminant Δ_4. Let us consider the quartic monic

polynomial on the left side of equation (2.5.21)

$$x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0. \tag{2.5.22}$$

The discriminant Δ_4 of a quartic polynomial is defined as the square of the product of the differences of the roots.[82] Let x_i with $i = 1, 2, 3, 4$ be the roots. Then

$$\Delta_4 = (x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2.$$

The discriminant is well defined, in the sense that it does not depend on the order of the roots. Indeed, rearranging the x_i boils down to a permutation of the factors of the product, since the differences are raised to the square.

Using this formula, it is easily seen that if (at least) two of the roots coincide, then $\Delta_4 = 0$.

Let us suppose now that all the roots are distinct. Since the polynomial factors as the product of two quadratic polynomials, there are three possibilities for its roots. We can have four real roots, or two real roots and two complex conjugate roots, or four complex roots conjugate in pairs.

If the four roots are real, it is easily seen that $\Delta_4 > 0$, since it is the product of squares of real numbers.

Let us suppose that two roots are distinct and real, say $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, and two roots are complex conjugate, say $x_3, x_4 \in \mathbb{C}$ with $x_4 = \overline{x_3}$. We write $x_3 = a + \imath b$ and $x_4 = a - \imath b$. Then, the factor $(x_1 - x_2)^2 > 0$, since it is the square of a real number. We remark that four differences out of six occur as pairs of complex conjugate numbers, so that their products are real, and thus positive. In fact, considering for example $(x_1 - x_3)^2(x_1 - x_4)^2$, we get $((x_1 - a - \imath b)(x_1 - a + \imath b))^2$. The remaining factor is $x_3 - x_4 = a + \imath b - (a - \imath b) = 2\imath b$, the square of which is negative. In this way, we get $\Delta_4 < 0$.

Finally, let us suppose that there are two pairs of complex conjugate roots, say x_1, x_2, x_3, $x_4 \in \mathbb{C}$ with $x_2 = \overline{x_1}$, and $x_4 = \overline{x_3}$. We write

[82] See [72, Chapter IV, Sections 6 and 8].

$x_1 = \alpha_1 + \imath b_1$, $x_2 = \alpha_1 - \imath b_1$, $x_3 = \alpha_3 + \imath b_3$, and $x_4 = \alpha_3 - \imath b_3$. As before, four of the six differences occur as pairs of complex conjugate numbers, so that their products are real, and then positive. The two remaining factors are $x_1 - x_2 = \alpha_1 + \imath b_1 - (\alpha_1 - \imath b_1) = 2\imath b_1$ and $x_3 - x_4 = \alpha_3 + \imath b_3 - (\alpha_3 - \imath b_3) = 2\imath b_3$, the square of which are negative. In this way, we get $\Delta_4 > 0$.

In particular, we see that $\Delta_4 < 0$ if and only if the polynomial (2.5.22) has exactly two real roots and two complex conjugate roots, and $\Delta_4 > 0$ if and only if the polynomial (2.5.22) has four real roots or four complex roots that are not real. This result can also be obtained by brute force calculations using the above formula for the roots x_1, x_2, x_3, x_4.

Therefore, we observe that in the quartic case the sign of the discriminant does not provide complete information on the nature of the roots. In particular, if $\Delta_4 \geq 0$, there is some ambiguity.

It is worth noting that if we have a polynomial of degree five or higher it is impossible to have an algebraic formula for the roots. This is a deep result of Galois theory.[83]

[83] See [68, Theorem 5.7.3].



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