Chapter 2

Stochastic effects in nonlinear systems

Oscillations are a frequently studied phenomenon in science. Examples of this widespread behaviour range from economy to natural sciences. Neurons also exhibit oscillatory properties. The Hodgkin-Huxley model [46] and the FitzHugh-Nagumo model [47, 48] belong to the most famous models to describe the dynamics of neurons. The FitzHugh-Nagumo model is used more widely, because the calculations for large networks of neurons are easier to handle with this system. It is also the prototype of an excitable system [49].
Excitable systems can be described by three different states: a rest state, an excited state and a refractory phase. If the system is perturbed sufficiently above a certain threshold, the trajectory is kicked out of a locally stable fixed point (rest state) and makes a long excursion through the phase space emitting a spike (excited state). After the excursion the trajectory moves back to the stable fixed point (refractory phase). Such a perturbation can be realised by noise. Noise-induced oscillations can be found in excitable system (in the excitable regime), as already mentioned, but also in system close to bifurcations (non-excitable systems). The latter will be considered in this work.
A famous example for a non-excitable system is the Van der Pol model [50], which is also the prototype a nonlinear oscillator. The behaviour of this system can be divided into two parameter regions: in one parameter region the oscillator shows damped oscillations, whereas the other regime is characterised by self-sustained oscillations. To change from one regime to the other, the system undergoes a Hopf bifurcation. A generic model to describe the behaviour of non-excitable systems close to a Hopf bifurcation is the Hopf normal form, which is also known under the name Stuart-Landau oscillator.

In this thesis, a Hopf normal form with a random force and time-delayed coupling term is studied. Before we investigate the full stochastic and time-delayed dynamics, we start with a single oscillator system with noise in this chapter. First, we will understand the deterministic behaviour of our system.

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2.1 Deterministic dynamics

The deterministic equation of motion reads

$$\dot{z}(t) = (\lambda + i\omega_0 - a|z(t)|^2 - b|z(t)|^4)z(t). \tag{2.1}$$

$z(t)$ is a complex variable, λ denotes the bifurcation parameter, and ω_0 is the intrinsic frequency of the system. The real parameters a and b are used to distinguish between the supercritical $(a = 1,\ b = 0)$ and the subcritical $(a = -1,\ b = 1)$ Hopf normal form.
The complex variable z can be expressed in polar coordinates, $z = re^{i\phi}$ $(r \geq 0)$, and decomposed into its real and imaginary part. We obtain the following equations.

$$\dot{r} = \lambda r - ar^3 - br^5, \tag{2.2}$$
$$\dot{\phi} = \omega_0. \tag{2.3}$$

For the bifurcation diagram of the supercritical Hopf normal form, we set $a = 1$ and $b = 0$. The stationary solution or fixed point $(\dot{r} = 0)$ of Eq. (2.2) is given by

$$r_1^* = 0. \tag{2.4}$$

For $\lambda > 0$ there exists a periodic solution (limit cycle)

$$z_2 = r_2^* e^{i\omega t} \text{ with } r_2^* = \sqrt{\lambda}. \tag{2.5}$$

For the subcritical Hopf normal form, we set $a = -1$ and $b = 1$. Then the fixed point is

$$r_1^* = 0, \tag{2.6}$$

and the periodic solutions are

$$r_2^* = \sqrt{\frac{1 + \sqrt{1 + 4\lambda}}{2}},\ r_3^* = \sqrt{\frac{1 - \sqrt{1 + 4\lambda}}{2}}. \tag{2.7}$$

Now we have to investigate the stability of the fixed point r^* and the limit cycles and, therefore, we make a linear stability analysis [51].
We consider a dynamical system described by the differential equation $\dot{x} = f(x)$, where x represents the dynamical variable and $f(x)$ is some nonlinear function of this variable. This system will be linearised in the vicinity of its fixed point x^*

$$\delta x = x - x^*, \tag{2.8}$$

where δx denotes a small deviation. We are interested in the time evolution of these deviations, so we derive a differential equation

$$\delta \dot{x} = \dot{x} - \dot{x}^* = f(x) - f(x^*) = f(x), \tag{2.9}$$

where we used $\dot{x}^* = f(x^*) = f(x)|_{x=x^*} = 0$. We can rewrite $f(x) = f(x^* + \delta x)$ and perform a Taylor series expansion for small deviations δx around the fixed point x^* and we find

$$\begin{aligned} \delta\dot{x} = &\ f(x^*) + \frac{d}{dx} f(x)\bigg|_{x=x^*} \delta x + \mathcal{O}(\delta x^2) \\ \approx &\ \frac{d}{dx} f(x)\bigg|_{x=x^*} \delta x. \end{aligned} \tag{2.10}$$

Equation (2.10) is a differential equation for the behaviour of a system in the vicinity of a fixed point. This problem can be solved using an exponential ansatz for the deviations $\delta x \propto e^{\Lambda t}$. More generally, in n dimensions, $\frac{d}{dx} f(x)\big|_{x=x^*}$ is replaced by the Jacobian matrix Df, so we have to deal with an eigenvalue problem of the Jacobian.
For the fixed point r_1^* (Eqs. (2.4, 2.6)) we find that it is stable for $\lambda < 0$ and becomes unstable for $\lambda > 0$.
For periodic orbits, such as limit cycles, Floquet theory answers the question as to whether a periodic state is stable or unstable, because in this case the Jacobian matrix is in general time-dependent. For the Stuart-Landau oscillator this time dependence of the Jacobian matrix vanishes, so it is possible to perform the linear stability analysis analytically.
We obtain for the supercritical case

$$\begin{pmatrix} \delta\dot{r} \\ \delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} \lambda - 3r^2 & 0 \\ 0 & 0 \end{pmatrix}\bigg|_{r^*} \begin{pmatrix} \delta r \\ \delta\phi \end{pmatrix} = \underbrace{\begin{pmatrix} -2\lambda & 0 \\ 0 & 0 \end{pmatrix}}_{M} \begin{pmatrix} \delta r \\ \delta\phi \end{pmatrix}. \tag{2.11}$$

The eigenvalues of the Matrix M are the Floquet exponents Λ:

$$\Rightarrow \quad \Lambda = \begin{cases} 0 & \text{Goldstone mode (invariance of translation along the limit cycle)} \\ -2\lambda & \end{cases}. \tag{2.12}$$

Hence the limit cycle, which exists for $\lambda > 0$ (see Eq. (2.5)), is stable.
For the subcritical Hopf normal form we have

$$\begin{aligned} \begin{pmatrix} \delta\dot{r} \\ \delta\dot{\phi} \end{pmatrix} = &\ \begin{pmatrix} \lambda + 3r^2 - 5r^4 & 0 \\ 0 & 0 \end{pmatrix}\bigg|_{r^*} \begin{pmatrix} \delta r \\ \delta\phi \end{pmatrix} \\ = &\ \underbrace{\begin{pmatrix} \lambda + 3\left(\frac{1\pm\sqrt{1+4\lambda}}{2}\right) - 5\left(\frac{1\pm\sqrt{1+4\lambda}}{2}\right)^2 & 0 \\ 0 & 0 \end{pmatrix}}_{M} \begin{pmatrix} \delta r \\ \delta\phi \end{pmatrix}, \end{aligned} \tag{2.13}$$

$$\Rightarrow \quad \Lambda = \begin{cases} 0 & \text{Goldstone mode} \\ \lambda + 3\left(\frac{1\pm\sqrt{1+4\lambda}}{2}\right) - 5\left(\frac{1\pm\sqrt{1+4\lambda}}{2}\right)^2 & \end{cases}. \tag{2.14}$$

The limit cycle with radius r_3^* (Eq. (2.7)) exists for $\lambda \in [-\frac{1}{4}, 0]$, and is unstable because the Floquet exponent

$$\Lambda = \lambda + 3\left(\frac{1-\sqrt{1+4\lambda}}{2}\right) - 5\left(\frac{1-\sqrt{1+4\lambda}}{2}\right)^2 = -1 - 4\lambda + \sqrt{1+4\lambda} \quad (2.15)$$

is always positive for $-\frac{1}{4} < \lambda < 0$ since

$$\begin{aligned} 1+4\lambda &> (1+4\lambda)^2 \\ \Leftrightarrow \quad 4 &< 8 + 16\lambda \\ \Leftrightarrow \quad 0 &< 1+4\lambda \qquad (\lambda < 0). \end{aligned} \quad (2.16)$$

However, the limit cycle with radius r_2^* (Eq. (2.7)), which exists for $\lambda \in [-\frac{1}{4}, \infty]$, is stable because the corresponding Floquet exponent

$$\Lambda = \lambda + 3\left(\frac{1+\sqrt{1+4\lambda}}{2}\right) - 5\left(\frac{1+\sqrt{1+4\lambda}}{2}\right)^2 = -1 - 4\lambda - \sqrt{1+4\lambda} \quad (2.17)$$

is negative for $\lambda > -\frac{1}{4}$.

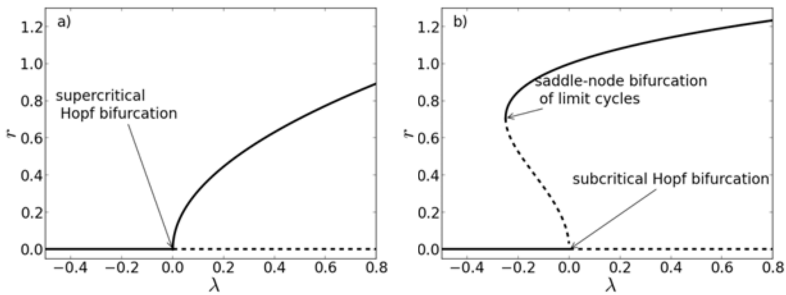

Figure 2.1: Deterministic bifurcation diagram; solid lines correspond to stable focus/limit cycle, dashed lines to unstable ones.
(a) supercritical case ($a = 1, b = 0$), see Eqs. (2.4, 2.5, 2.12),
(b) subcritical case ($a = -1, b = 1$), see Eqs. (2.6, 2.7, 2.15, 2.17).

In Fig. 2.1a) for $\lambda < 0$ we have a stable focus, which loses its stability via a supercritical Hopf bifurcation at $\lambda = 0$. For $\lambda > 0$ a stable limit cycle and an unstable focus coexist. In Fig. 2.1b) we also have a stable focus for $\lambda < 0$, which undergoes a subcritical Hopf bifurcation at $\lambda = 0$. This Hopf bifurcation gives rise to an unstable limit cycle. At $\lambda = -0.25$ a saddle-node bifurcation of limit cycles takes place and a stable limit cycle is born. The system is bistable for values of $\lambda \in [-0.25, 0]$.

2.2 Stochastic dynamics

An additional noise term in the dynamical equations leads to many new dynamical features [1, 49], e.g. noise-induced oscillations. For example, a system in the regime of a deterministically stable focus is influenced by noise in such a way that the trajectory is kicked out of the focus and shows random motion around it. Adding a random force term to Eq. (2.1), we obtain

$$\dot{z}(t) = (\lambda + i\omega_0 - a|z(t)|^2 - b|z(t)|^4)z(t) + \sqrt{2D}\xi(t). \tag{2.18}$$

$D \geq 0$ describes the strength of the fluctuations (noise strength), whereas $\xi(t) \in \mathbb{C}$ denotes the random variable. Here Gaussian white noise is used with the following properties

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi^*(t') \rangle = 2\delta(t - t'). \tag{2.19}$$

The white noise is usually used for describing an unknown perturbation, where all frequencies contribute equally. This can be shown via Fourier transform (Wiener-Khinchin theorem), where the power spectral density is a constant function [53].
For the investigations of stochastic systems we have to adapt our methods, because we are dealing with random variables. Their properties can be described by probability distributions. Therefore, the deterministic concepts like bifurcations or attractors cannot be applied, or have to be modified (see, e.g. stochastic bifurcations).
To calculate the probability distribution for our dynamical system (Eq. (2.18)), we derive the corresponding Fokker-Planck equation. This computation can be found in many textbooks, e.g. [53–55]. The Fokker-Planck equation for the amplitude r and phase ϕ (z is decomposed in polar coordinates) corresponding to Eq. (2.18) is:

$$\partial_t P = \partial_r \left(\left(-\lambda r + ar^3 + br^5 - \frac{D}{r} \right) P + D\partial_r P \right) + \partial_\phi \left(-\omega_0 P + \frac{D}{r^2}\partial_\phi P \right) \tag{2.20}$$

where $P(r, \phi)$ is the probability density. A detailed derivation of Eq. (2.20) is given in the Appendix A.
By using the spherical symmetry of the deterministic system, we neglect the derivatives with respect to the phase variable ($\partial_\phi P = 0$). Furthermore, we are only interested in the stationary behaviour of our system, therefore, we have $\partial_t P = 0$. We obtain the stationary amplitude probability distribution

$$P(r) = Nr \exp\left(\frac{r^2}{D} \left(\frac{\lambda}{2} - \frac{ar^2}{4} - \frac{br^4}{6} \right) \right), \tag{2.21}$$

where N is the normalisation constant and given by

$$N = \left(\int_0^\infty r \exp\left(\frac{r^2}{D} \left(\frac{\lambda}{2} - \frac{ar^2}{4} - \frac{br^4}{6} \right) \right) dr \right)^{-1}. \tag{2.22}$$

2.3 Stochastic bifurcations

In deterministic systems, a bifurcation is a sudden change of the behaviour of the system after varying a certain system parameter, which is the so-called bifurcation parameter. An example was shown in the beginning of this chapter, where λ is the bifurcation parameter and the system undergoes a Hopf bifurcation, which means a change from damped oscillations to self-sustained oscillations (Eq. (2.1) and Fig. 2.1).

In a system with noise (stochastic system), the deterministic concept of bifurcation theory cannot be directly applied but must be adapted. Here the noise intensity D works as the control- or bifurcation parameter. Following [3] there are two types of stochastic bifurcations: the phenomenological bifurcation (P-bifurcation) denotes a change in the shape of the probability distribution, e.g. from unimodal to a bimodal shape. The other type is the dynamical bifurcation (D-bifurcation) which occurs, when the largest Lyapunov exponent becomes positive as a function of the noise intensity. The latter will not be considered in this work.

There are many applications and investigations on stochastic bifurcations and noise-induced transitions: studies on generic models [56, 57], biological and chemical systems [58, 59] and experimental and theoretical investigations in lasers [60]. Many further examples can be found in [2]. For a generalised Van der Pol model, the stochastic P-bifurcation was investigated in [61, 62].

First, we will investigate the supercritical case ($a = 1,\ b = 0$) for our system: according to Eq. (2.21) the probability distribution is given by

$$P(r) = Nr\exp\left(-\frac{W(r)}{D}\right) \tag{2.23}$$

where $W(r)$ is the potential

$$W(r) = \frac{r^4}{4} - \frac{\lambda r^2}{2}. \tag{2.24}$$

Here we follow [61, 62] for further steps of investigation. We are looking for the extrema of the distribution

$$\frac{d}{dr}P(r) = N\exp\left(-\frac{W(r)}{D}\right)\left(1 - \frac{r}{D}\frac{d}{dr}W(r)\right) = 0. \tag{2.25}$$

A stochastic P-bifurcation takes place if the number of extrema changes. The relevant part from Eq. (2.25) reads

$$1 - \frac{r}{D}\frac{d}{dr}W(r) = 1 - \frac{r^4}{D} + \frac{\lambda r^2}{D}. \tag{2.26}$$

The extrema of the distribution are the real-valued, positive roots of Eq. (2.26). It is

$$D - r^4 + \lambda r^2 = 0 \quad \rightarrow \quad f(x) = x^2 - \lambda x - D = 0 \tag{2.27}$$

with the substitution $x = r^2$. The number of extrema changes if $f(x_c) = f'(x_c) = 0$, where x_c denotes the corresponding extremum. So $f'(x)$ reads

$$2x - \lambda = 0, \quad \rightarrow \; x_c = \frac{\lambda}{2} \tag{2.28}$$

Inserting this result into Eq. (2.27), we find that

$$D = -\frac{\lambda^2}{4}. \tag{2.29}$$

Equation (2.29) shows that a transition will only take place for negative noise intensities, but D has to be positive. That implies that a stochastic P-bifurcation of the amplitude probability distribution cannot occur in the supercritical case. This is shown in Fig. 2.2, where the number of the maxima does not change by increasing the noise intensity.

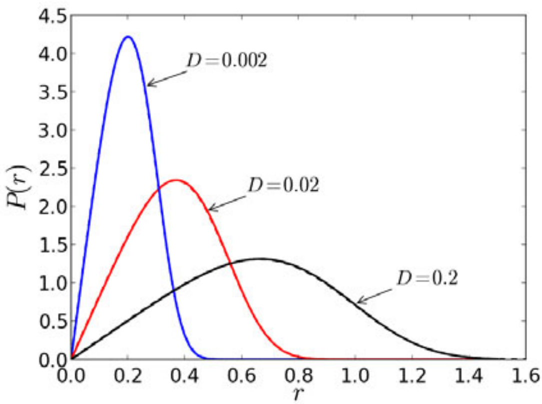

Figure 2.2: Stationary amplitude probability distribution for the supercritical case, analytically (solid, according to Eqs. (2.23, 2.24)) and numerically (dashed) calculated.
Parameters: $\omega_0 = 2\pi$, $\lambda = -0.01$.

For the subcritical normal form ($a = -1,\; b = 1$) we follow [63] for another way of the calculation. The probability distribution reads

$$P(r) = Nr \exp\left(\frac{r^2}{D}\left(\frac{\lambda}{2} + \frac{r^2}{4} - \frac{r^4}{6}\right)\right), \tag{2.30}$$

where N represents the corresponding normalisation constant (see Eq. (2.22)). We present the detailed calculation for the bifurcation lines in the Appendix B:

a condition for the variable r^2 is found

$$r^2 = -\frac{9D+\lambda}{6\lambda+2} > 0 \text{ and } r \in \mathbb{R}. \tag{2.31}$$

Equation (2.31) shows that only values of $\lambda < 0$ satisfy the condition. The bifurcation lines are represented by

$$D_{1,2} = \frac{1}{27}\left(-9\lambda - 2\left(1 \pm \sqrt{(1+3\lambda)^3}\right)\right). \tag{2.32}$$

This result can be seen in the following stochastic bifurcation diagram.

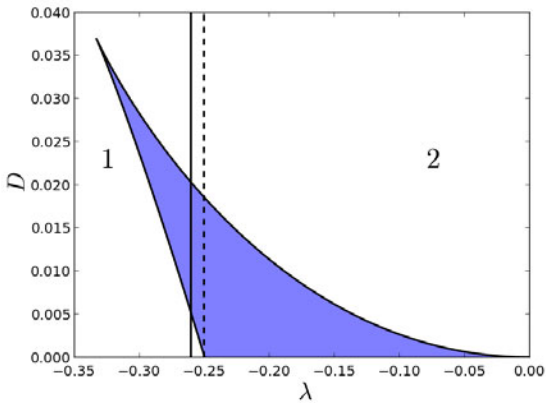

Figure 2.3: Stochastic bifurcation diagram following from Eq.(2.32) for the subcritical case ($a = -1, b = 1$). The dashed line corresponds to the boundary between deterministic monostability (region 1) and bistability (region 2). The solid line denotes the parameter line for the fixed deterministic bifurcation parameter λ. The shaded region denotes the parameter regime where the distribution has a bimodal shape. Outside of this region we have a unimodal distribution.

Figure (2.3) shows the stochastic bifurcation diagram for our noisy system: the shaded region denotes the parameter values, where the amplitude probability distribution has a bimodal shape. The dashed vertical line describes the border between the two different deterministic regimes corresponding to λ: in the left part (region 1) there is only a stable focus, whereas in the right part (region 2) the deterministic system is bistable, because additionally a stable limit cycle coexists with the stable focus. The vertical solid line denotes the λ value of interest; for the further investigations, $\lambda = -0.26$ is fixed, and we vary the noise intensity D. By crossing the solid lines to enter or to leave the shaded region (parameter regime of

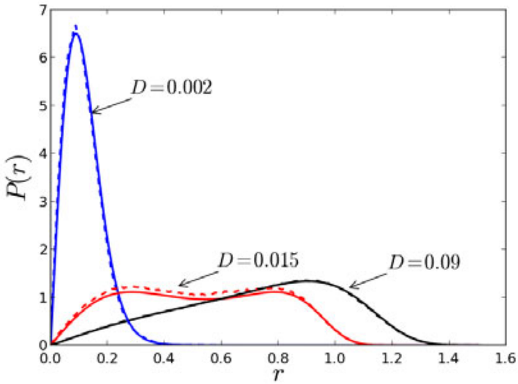

Figure 2.4: Amplitude probability distribution (Eq. (2.30)) for the subcritical case for different noise strengths, analytically (solid) and numerically (dashed) calculated.
Parameters: $\lambda = -0.26$, $\omega_0 = 2\pi$.

bimodality), we observe a stochastic P-bifurcation.

Figure 2.4 shows the probability distribution for values of the noise intensity D in and outside from the region of bimodality. Furthermore, the analytical result and the numerics for the amplitude probability distribution are shown: they are in excellent agreement.

The discussion above was carried out for additive white noise sources. An investigation for coloured noise, additive and multiplicative, can be found in [64], where a Duffing-van der Pol model is investigated.

2.4 Coherence resonance

The presence of noise is usually unwanted because of its destructive character for the deterministic dynamic. Nevertheless, noise can have a constructive effect in the interplay with the nonlinearities of a dynamical system. A very well studied effect is the stochastic resonance [4]: a weak periodic signal, which drives the dynamical system, is enhanced by the noise. The counterintuitive aspect here is that the best enhancement takes place at an intermediate non-zero noise strength.

Another phenomenon that is also related with the constructive role of the noise is coherence resonance. In contrast to stochastic resonance, there is no external periodic force present. This effect occurs from the intrinsic dynamic of the system; the noise-induced oscillations become most regular at a finite non-zero noise strength. Coherence resonance was investigated first by Gang et al. [5], who studied this phenomenon in a generic model for a SNIPER (saddle-node infinite period) bifurcation

(type I excitability). But their work was published with the title “Stochastic resonance without external periodic force”. The catchy name ”coherence resonance” was invented by Pikovsky and Kurths, who investigated the FitzHugh-Nagumo model (type II excitability), [6].
Coherence resonance was also investigated in a non-excitable system in [7, 61, 62]. [7] was a combined work with a laser experiment and theoretical modelling. In the experiment, the authors investigated resonance effects caused by noise, setting the laser close to a Hopf bifurcation. The theory was based on the estimate of a Lorentzian spectrum. This suggestion was provided by the experiment and numerical simulations of a noisy Hopf normal form. With the help of the Wiener-Khinchin theorem, the authors developed an amplitude approach to calculate measures of coherence (see section 2.4.1). They found that coherence resonance occurs only in the subcritical case. Therefore, we will not consider the supercritical case in the following investigations.
Coherence resonance can be observed also below other types of bifurcations, where characteristic signatures of noisy precursors occur [65, 66], in lasers [7, 45, 67, 68], and neural systems [6, 39, 49, 69], just to mention a few examples. We will now investigate the same theoretical model as used in [7], but our starting point for the calculations is based on the generic Eq. (2.18). Using statistical linearisation techniques, we are able to derive analytical expressions for the measures of coherence. This measures will be introduced in the following section.

2.4.1 Measures of coherence

Usually, the following three measures are used to quantify the degree of regularity of the oscillations:

- The **signal-to-noise-ratio** (SNR)

$$\beta = \frac{H}{\Delta\omega/\omega_p}. \tag{2.33}$$

  β is calculated from the power spectral density (see Fig. 2.5) and is usually used in laser physics. It was introduced by Haken, see [5].

- The normalised fluctuations of the **interspike interval** (ISI)

$$R = \frac{\sqrt{\langle t_p^2\rangle - \langle t_p\rangle^2}}{\langle t_p\rangle}. \tag{2.34}$$

  In excitable systems, it is useful to look at the time series and to determine the statistics of spikes [6]. For a non-excitable system, this measure is not used, because there are no spikes in the sense of an excitation.

- The **correlation time**

$$t_{cor} = \frac{1}{\Psi(0)} \int_0^\infty |\Psi(s)| ds, \tag{2.35}$$

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