2 Fundamentals of Elasticity

"Tradition ist nicht die Anbetung der Asche sondern die Weitergabe des Feuers."

— Gustav Mahler

This chapter provides an overview on the fundamental nature and the common methods to describe the behaviour of elastic materials, in particular rubber-like materials. First the term elasticity along with different simple forms of elastic deformation is illustrated. Secondly, mathematical models and different approaches for the description of elastic characteristics of materials are shown with the focus on rubber-like materials, being capable of large elastic deformations. The last part will deal with the experimental determination of fundamental elastic properties. The presented theory is used to evaluate the experimental data in Section 4. Most of the information in this chapter is taken from the standard literature [11–17] unless otherwise stated.

The term *elasticity* fundamentally denotes to the physical property of a material to reversibly change its shape or geometrical constitution by virtue of a force acting on it. Reversibly in this case meaning that on removal of the force, the material returns to its initial condition. There are different forms of macroscopic elasticity, which are often confused upon each other, namely stretchability and flexibility or bendability. Flexible materials are substances which show the ability to reversibly change their geometrical profile by showing rather low resistance to bending or torsional forces, but relatively high resistance to tensile (stretching) forces. Stretchable materials do not have this limitation, wherefore those kind of materials in addition can also be deformed due to rather low tensile forces; they react with rather low resistance to tensile strain. Figure 2.1 gives an

R. Moser, *Plastic Tests Plastics,* BestMasters, DOI 10.1007/978-3-658-10530-3_2,

impression of different forms of mechanical deformation.

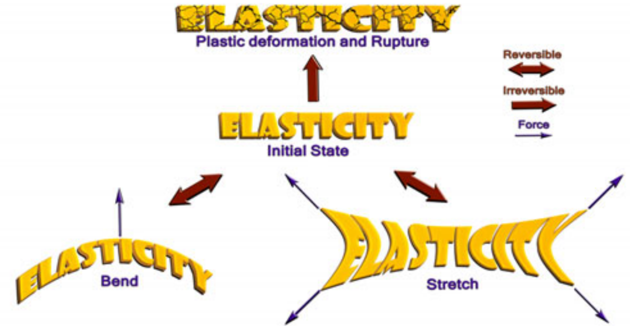


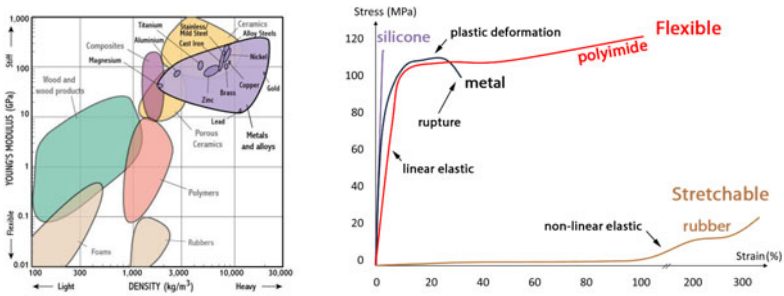
Figure 2.1 – Reversible and irreversible forms of deformation due to applied forces in elastic materials.[12]

Generally speaking, every material is elastic, or shows elastic behaviour to some extend. What discriminates commonly called elastic from inelastic materials is their degree of *reversible* deformation, in particular the amount of strain before plastic (irreversible) deformation and/or rupture occurs. Furthermore the force which is necessary to gain a significant deformation differs greatly throughout the materials. The specific deformation behaviour of a material under an applied load is distinguished by various elastic moduli, such as the Young's (or elastic) modulus and shear modulus, both applying for tensile and shear deformation respectively. Figure 2.2(a) shows the Young's moduli for different material classes.

Experimentally the elastic properties of a material are determined by tensometers capable of measuring the strain (the degree of deformation) and the opposing force while deforming a sample in a predefined way. The result of such measurements is usually presented in the form of *stress-strain* curves, which yield information about the opposing force of a material upon deformation. More precisely the average internal force per unit area, the mechanical stress σ along an axis with respect to the strain, the deformation in relation to the initial length of the material $\epsilon = \Delta l / l_0$ along

[12]Sketch inspired by [18].

this axis, is studied. This relation then yields information about characteristic mechanical material parameters. Figure 2.2(b) shows typical stress-strain relations for different kinds of materials under simple extension. This measurement method is also used in the LEGO-tensometer dealt with in the next chapter.



(a) (b)

Figure 2.2 – a) Collection of Young's moduli E for different material classes b) Qualitative stress-strain curves for silicone and metal showing a very narrow (linear) area of reversible strain followed by plastic deformation and rupture. Polyimide acts as an example for a flexible material with a high Young's modulus and a rather extensive region of plastic deformation. Rubber-like materials show a rather low Young's modulus along with a broad (non-linear) elastic range.[13]

The elastic- or Young's modulus E is an important parameter of a material informing about the stiffness of an elastic material. More precisely it terms the slope of the initial stress-strain behaviour. This regime is in most cases linear and therefore complies with Hooke's Law for one dimensional stretch as stated in Equation 2.1. This relation is often sufficient for the treatment of elastic behaviour of most solids.

$$\sigma = E \cdot \epsilon \tag{2.1}$$

As depicted by Figure 2.2(b) rubber-like materials or elastomers show a wide *non-linear* elastic range. Therefore, the latter equation does not apply for the whole elastic range. The treatment of such materials demands

[13]Sketches taken from [19] and [18].

for more sophisticated models.

In the following sections the methodology for determining strain and stress for arbitrary materials and deformations is laid out. The ”material” in this case is assumed to be of isotropic and homogeneous nature. The microscopic structure is not yet relevant at this stage, but will become important in Section 2.3. Afterwards several approaches of modelling the elastic behaviour of rubber and the theoretical basics for determining the Young’ Modulus of rubber from stress-strain measurements is presented.

2.1 The Deformation Tensor

For material tests often more complicated deformations than simple extension e.g. simple shear are used. In such cases the material deformation cannot be argued by pure geometrical considerations. To further the understanding of material deformation, the underlying theory of mechanical stress and strain with the focus on rubber-like materials is given.

Under means of external forces an arbitrary material of volume V_0 is deformed to a certain extent, therefore changing its shape and volume. In doing so, an arbitrary but fixed point inside the material in the undeformed state, indicated by the radius-vector $\vec{r}$ is shifted to a new position $\vec{r'}$ as shown in Figure 2.3. The coordinate system should remain fixed.

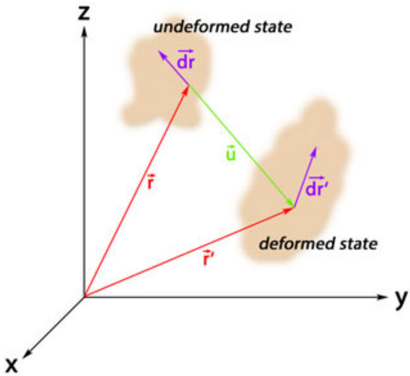


Figure 2.3 – Transformation of a point $\vec{r} \to \vec{r'}$ and infinitesimal line elements $d\vec{r} \to d\vec{r'}$ upon deformation in an arbitrary homogeneous and isotropic material. The coordinate system $\vec{e}_i \; i \,\epsilon\, \{x, y, z\}$ remains fixed during deformation.

This position shift $\vec{r} \to \vec{r'}$ can be expressed by a motion vector transformation function $\chi\left(\vec{r}, t\right)$, but it is common to denote it simply by $\vec{r'}\left(\vec{r}, t\right)$. Since stationary conditions are assumed, the time-dependence can be neglected and the point transformation can be written as

$$\vec{r'} = \chi\left(\vec{r}\right) = \vec{r'}\left(\vec{r}\right) \tag{2.2}$$

Since the deformation of the material should be determined, the transformation of an arbitrary line-element $dr \to dr'$ rather than the pure shift of points in the material is of interest.

$$\begin{aligned} d\vec{r}' &= \chi\left(\vec{r} + d\vec{r}\right) - \chi\left(\vec{r}\right) \overset{\text{TaylorExp.}}{=} \chi\left(\vec{r}\right) + \frac{\partial \chi\left(\vec{r}\right)}{\partial \vec{r}} d\vec{r} - \chi\left(\vec{r}\right) \\ &= \frac{\partial \chi\left(\vec{r}\right)}{\partial \vec{r}} d\vec{r} \overset{(2.2)}{=} \frac{\partial \vec{r'}\left(\vec{r}\right)}{\partial \vec{r}} d\vec{r} := \mathbf{F}\; d\vec{r} \end{aligned} \tag{2.3}$$

$$\rightarrow \quad F_{ij} = \frac{\partial r_i'}{\partial r_j} \tag{2.4}$$

$$r_i' = F_{ij} r_j \tag{2.5}$$

$$dr_i' = F_{ij} dr_j \tag{2.6}$$

The second order tensor $\mathbf{F}$, the *Deformation Gradient Tensor* transforms an infinitesimal line element $d\vec{r}$ as well as a point $\vec{r}$ into their deformed states $d\vec{r'}$ and $\vec{r'}$ respectively. It is important to denote that $d\vec{r'}$ is associated with an error in the order of $(d\vec{r})^2$ due to neglecting higher order terms in the Taylor expansion in Equation 2.3. The tensor $\mathbf{F}$ therefore only maps deformations in the immediate vicinity of $\vec{r}$.

To be precise $\mathbf{F}$ is a general transformation in space and can always be split up into a rotational $\mathbf{R}$ and a deformation tensor $\mathbf{D}$. The orthogonal tensor $\mathbf{R}$ is a measure of the rotation of $\vec{r}$, $\mathbf{D}$ accounts for local material deformation around $\vec{r}$. To make the transformation unique $\mathbf{D}$ has to be symmetric.

$$\mathbf{F} = \mathbf{RD} \tag{2.7}$$

The length of the the initial line element $dl = |d\vec{r}|$ and the line element in the transformed state $dl' = |d\vec{r'}|$ can be calculated from Equation 2.3.

$$\begin{aligned} dl'^2 = d\vec{r'}^2 &\overset{(2.3)}{=} (\mathbf{F}\; d\vec{r})\,(\mathbf{F}\; d\vec{r}) \\ &= d\vec{r}\,\mathbf{F}^T\mathbf{F}\; d\vec{r} := d\vec{r}\,\mathbf{C}\; d\vec{r} \end{aligned} \tag{2.8}$$

$$\rightarrow \qquad C_{jk} = F_{ij}\,F_{ik} = \left(\frac{\partial r'_i}{\partial r_j}\frac{\partial r'_i}{\partial r_k}\right) \tag{2.9}$$

$$dl'^2_i = dr_j\; C_{jk}\; dr_k \tag{2.10}$$

As shown in Equation (2.8) and (2.10) the second order tensor $\mathbf{C}$, the *Right Cauchy Green Strain Tensor* (RCGT) yields the squared length of the line element dl'^2 in relation to the square of its length dl^2 in the initial state. This relation parameter, the strain or more precisely the relative length λ^2 can be calculated as follows with $\hat{d\vec{r}}$ representing a unit vector pointing into the direction of $d\vec{r}$.

$$\lambda^2 := \frac{dl'^2}{dl^2} \overset{(2.8)}{=} \hat{d\vec{r}}\,\mathbf{C}\;\hat{d\vec{r}} \tag{2.11}$$

Since the rotation tensor $\mathbf{R}$ is orthogonal ($\mathbf{R}^T\mathbf{R} = 1$) it can be seen, that $\mathbf{C}$ only gives information about the deformation of the material and not about any rotation during motion. The relative length λ^2 therefore is a measure for the degree of material deformation.

$$\mathbf{C} \overset{(2.8)}{=} \mathbf{F}^T\mathbf{F} \overset{(2.7)}{=} \left(\mathbf{D}^T\mathbf{R}^T\right)(\mathbf{RD}) = \mathbf{D}^T\mathbf{D} \tag{2.12}$$

The length of a line element remains undeformed, stretched or compressed for $\lambda = 1$, $\lambda > 1$ and $0 < \lambda < 1$ respectively.

The tensor $\mathbf{C}$ is positive definite and symmetric, which implies that it has positive Eigenvalues $\widetilde{C}_j$. This is a very important fact, since if $\mathbf{C}$ is diagonalized, corresponding to a transformation into the principal coordinate system, the diagonal elements of $\mathbf{C}$, its Eigenvalues correlate with the squares of the principal strains ${\lambda_j}^2$ in the direction of the orthogonal principal axes, which are given by the Eigenvectors $\hat{\vec{E}}_j$ of $\mathbf{C}$. Normalisation of the Eigenvectors finally yields the principal coordinate system in which only the pure strains λ_j occur. Thus $\mathbf{C}$, with respect to the new base vectors in terms of principal coordinates can be written as

$$\begin{aligned} C_{jk_{diag}} &= \delta_{jk}\widetilde{C}_j && (2.13)\\ {\lambda_j}^2 &= \widetilde{C}_j && (2.14) \end{aligned}$$

The tensor $\mathbf{C}$ is a second rank tensor, and has as any of these three invariants I_i, which remain unchanged under rotation. They are given based on the diagonalised form of $\mathbf{C}$ and will become important in the next chapter where certain models for rubber-elasticity are discussed.

$$\begin{aligned} I_1 &= Tr(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \frac{1}{2}\left(Tr(\mathbf{C^2}) - \left(Tr(\mathbf{C^2})\right)^2\right) = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3 \\ I_3 &= Det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{2.15}$$

Often it is worthwhile to have strain parameters yielding the value 0 in the undeformed state. In such cases the relative stretch ε rather than the relative length λ is of interest. Since the λ_j are already known, the principal strains ε_j can be calculated rather easily.

$$\begin{aligned}\varepsilon_j \quad &:= \quad \frac{dl'_j - dl_j}{dl_j} = \frac{dl'_j}{dl_j} - 1 &(2.16)\\ &\overset{(2.11)}{=} \quad \lambda_j - 1 \overset{(2.14)}{=} \sqrt{\widetilde{C}_j} - 1 &(2.17)\end{aligned}$$

The length of a line element remains undeformed, stretched or compressed for $\varepsilon = 0$, $\varepsilon > 0$ and $\varepsilon < 0$ respectively.

A second approach, which is often very useful to determine the relative strains and relative lengths, is to introduce the displacement vector $\vec{u}$, as indicated in Figure 2.3, rather than to perform a pure point transformation as done above.

$$\vec{r'} = \vec{u} + \vec{r} \tag{2.18}$$

Then the RCGT transforms into the following form.

$$\begin{aligned}C_{jk} \quad &\overset{(2.10)}{=} \quad \frac{\partial r'_i}{\partial r_j}\frac{\partial r'_i}{\partial r_k}\\ &\overset{(2.18)}{=} \quad \frac{\partial(u_i + r_i)}{\partial r_j}\frac{\partial(u_i + r_i)}{\partial r_k} = \left(\frac{\partial u_i}{\partial r_j} + \delta_{ij}\right)\left(\frac{\partial u_i}{\partial r_k} + \delta_{ik}\right)\\ &= \quad \frac{\partial u_k}{\partial r_j} + \frac{\partial u_j}{\partial r_k} + \frac{\partial u_i}{\partial r_j}\frac{\partial u_i}{\partial r_k} + \delta_{jk} &(2.19)\\ &:= \quad 2U_{jk} + \delta_{jk} &(2.20)\end{aligned}$$

$$\begin{aligned}\rightarrow \mathbf{2}U \quad &= \quad \frac{\partial u_k}{\partial r_j} + \frac{\partial u_j}{\partial r_k} + \frac{\partial u_i}{\partial r_j}\frac{\partial u_i}{\partial r_k} &(2.21)\\ &= \quad \frac{1}{2}\left(\mathbf{C} - \mathbb{I}\right) &(2.22)\end{aligned}$$

This transformation reveals $\mathbf{U}$, the *Green-Lagrange Strain Tensor*, which is similarly to $\mathbf{C}$ symmetric and positive definite, therefore, as well implying positive Eigenvalues $\widetilde{U}_j$. Assuming the tensor has already been transformed into the principal coordinate system the length of the line element in the deformed state dl' can be determined as follows.

$$\begin{aligned} dl'^2 &\overset{(2.10),(2.20)}{=} dr_j \left(2U_{jk} + \delta_{jk}\right) dr_k \\ &= \left(2\widetilde{U}_j + 1\right) dl_j{}^2 \end{aligned} \tag{2.23}$$

The relative stretches ε_j and relative lengths λ_j in the direction of the principal axes can be calculated.

$$\epsilon_j \overset{(2.17)}{=} \sqrt{2\widetilde{U}_j + 1} - 1 \tag{2.24}$$

$$\lambda_j \overset{(2.11)}{=} \sqrt{2\widetilde{U}_j + 1} \tag{2.25}$$

In the case of small deformations $\varepsilon_j \to 0$, valid for most solids, also the displacement vector $\vec{u}$ tends to be small. Therefore, in Equation 2.21 differentials of second order can be neglected and the latter equations can be simplified, since $\sqrt{2\widetilde{U}_j + 1} \approx 1 + \widetilde{U}_j$. In this particular case the Eigenvalues of $\mathbf{U}$ correspond with the principal relative strains $\varepsilon_j = \widetilde{U}_j$ and the tensor $\mathbf{U}$ itself may be simplified.

$$U_{jk} \overset{(2.20)}{=} \frac{1}{2}\left(\frac{\partial u_k}{\partial r_j} + \frac{\partial u_j}{\partial r_k}\right) \tag{2.26}$$

As shown in this chapter for most deformation problems it is worthwhile to determine, according to the particular problem either the Green-Lagrange or the Cauchy-Green strain tensor. In the case of common (crystalline) solid materials, being only capable of small deformations mostly the Green-Lagrange tensor is used. In rubber-physics, where rather large deformations compared to classical solids are common, the use of the Cauchy-Green tensor is favourable, since its Eigenvalues directly yield the principal deformations λ_j, which are also valid for large deformations. In the next section the stress should be examined.

2.2 The Stress Tensor

If a body of a certain homogeneous and isotropic material is not deformed, meaning no external forces are acting on it, it remains in thermodynamic and mechanical equilibrium, implying that also the microscopic constitution is in an equilibrium state. Thus, an infinitesimal volume element dV can be considered to be representative for the whole body. In thermal equilibrium the resultant force acting on dV is zero. If the body is deformed, this equilibrium state is distorted and internal mechanical stresses within the material evolve, trying to return the body to its original shape.

The deforming force $\vec{f}$ can be expressed as a sum of all forces acting on every infinitesimal volume fraction dV of the body and can therefore be written as an integral over the force-density $\widetilde{\mathbf{F}}$[14].

$$f_i = \int \widetilde{\mathbf{F}}_i \, dV \tag{2.27}$$

These infinitesimal forces can only act at the surface of the considered volume element. Thus, the latter equation can also be expressed as an area-integral, which implies that the components of the force-density $\widetilde{\mathbf{F}}_i$ can be written as the divergence of a second order tensor σ_{ij}. This allows, by use of the Gaussian divergence theorem, to write Equation 2.27 as an area-integral, with the surface elements dA_j.

$$\widetilde{\mathbf{F}}_i \quad := \quad \frac{\partial \sigma_{ij}}{\partial r_j} = \nabla \cdot \sigma_i \tag{2.28}$$

$$f_i \quad \overset{(2.27)}{=} \quad \int \frac{\partial \sigma_{ij}}{\partial r_j} \, dV = \oiint_{\partial V} \sigma_{ij} \, dA_j \tag{2.29}$$

[14]Parameters in relation to a volume are expressed with a $\sim$

http://www.springer.com/978-3-658-10529-7

Plastic Tests Plastics
A Toy Brick Tensometer for Electromechanical Characterization of Elastomers
Moser, R.
2015, XXV, 150 p. 58 illus., Softcover
ISBN: 978-3-658-10529-7