2 Feynman-Kac formulae

In this chapter, we establish the connection between the *deterministic* EIT forward problem and the class of *reflecting diffusion processes*. We proceed along the lines of the recent paper [137] by Piiroinen and the author: We derive *Feynman-Kac formulae* in terms of these processes for the solutions to the forward problems corresponding to the continuum model and the complete electrode model, respectively. These results extend the classical Feynman-Kac formulae for elliptic boundary value problems in smooth domains and with smooth coefficients which were obtained in the 1980s and 1990s using the Feller semigroup approach and Itô stochastic calculus. In contrast to this well-studied situation, the underlying reflecting diffusion processes in this work are constructed via Dirichlet form theory, which has emerged as a powerful tool when it comes to studying boundary value problems with non-smooth coefficients, boundaries and data.

However, in general such a construction is not very convenient with regard to practical issues such as numerical simulation. Unlike the Feller semigroup approach, which uses a pointwise analysis, the Dirichlet form approach is based on *quasi-sure* analysis, implying that we are permitted to ignore certain *exceptional sets* which are not visited by the process. Therefore, processes constructed via Dirichlet form theory are in general only defined for *quasi-every* starting point in the state space, rather than for every starting point. Moreover, processes generated by divergence form operators with merely measurable coefficients do in general not belong to the class of solutions to stochastic differential equations. Rather, they can be decomposed into a local martingale and an abstract additive functional with finite quadratic variation but possibly infinite variation; or equivalently, they admit a decomposition into two processes which are semimartingales with respect to two different filtrations. Both decompositions involve processes which are only implicitly defined and are therefore not suited for numerical simulation. It will turn out, that these issues can be resolved at least in practically relevant special cases of the EIT forward problem.

We start in Section 2.1 by constructing the underlying reflecting diffusion processes via Dirichlet form theory. We show that the transition kernel densities of these processes are Hölder continuous up to the boundary so

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that we may *refine* the processes to start from every point in the state space. In Section 2.2, we provide so-called *Skorohod decompositions*, i.e., semimartingale decompositions of the reflecting diffusion processes for two practically relevant classes of conductivities, thus enabling efficient numerical simulation. Section 2.3, the derivation of the Feynman-Kac formulae, is the central part of this chapter.

2.1 Reflecting diffusion processes

In his seminal paper [52], Fukushima established a one-to-one correspondence between regular symmetric Dirichlet forms and symmetric Hunt processes which is the foundation for the construction of stochastic processes via Dirichlet form techniques. Therefore, we assume that the reader is familiar with the theory of symmetric Dirichlet forms, as elaborated for instance in the monographs [54, 115]. A concise collection of both terminology as well as fundamental results is provided in Appendix A.

Let us consider the measure space $(\overline{D}, \mathcal{B}(\overline{D}), [D]\cdot m)$ as well as the following symmetric bilinear form on $L^2(D)$:

$$\mathcal{E}(v,w) := \int_D \kappa \nabla v(x) \cdot \nabla w(x)\,\mathrm{d}x, \quad v,w \in \mathcal{D}(\mathcal{E}) := H^1(D). \tag{2.1}$$

For the particular case $\kappa \equiv 1/2$, which is of special importance, we set

$$\mathcal{E}^{\mathrm{BM}}(v,w) := \frac{1}{2}\int_D \nabla v(x) \cdot \nabla w(x)\,\mathrm{d}x, \quad v,w \in \mathcal{D}(\mathcal{E}^{\mathrm{BM}}) := H^1(D). \tag{2.2}$$

Proposition 2.1. *The pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined by (2.1) is a strongly local regular symmetric Dirichlet form on $L^2(D)$.*

Proof. First, we verify that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form on $L^2(D)$. Closedness is obvious since for all $v \in H^1(D)$ we can find positive constants c_1, c_2 such that

$$c_1 ||v||_2^2 \leq \mathcal{E}_1(v,v) \leq c_2 ||v||_2^2.$$

This follows from (1.4). To show that the unit contraction operates on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, we follow [54, Example 1.2.1, Example 1.2.3], i.e., we construct for each $\varepsilon > 0$ a differentiable function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $\phi_\varepsilon(t) = t$ for all $t \in [0,1]$, $-\varepsilon \leq \phi_\varepsilon(t) \leq 1 + \varepsilon$ for all $t \in \mathbb{R}$ and $0 \leq \phi_\varepsilon(s) - \phi_\varepsilon(t) \leq s - t$, whenever $t < s$. Given such a function we have for every $v \in H^1(D)$ that $\phi_\varepsilon(v) \in H^1(D)$ and

$$\mathcal{E}(\phi_\varepsilon(v), \phi_\varepsilon(v)) = \int_D |\phi_\varepsilon'(v(x))|^2 \kappa \nabla v \cdot \nabla v\,\mathrm{d}x \leq \mathcal{E}(v,v),$$

where the last inequality is a consequence of the property $0 \le \phi'_\varepsilon(t) \le 1$. As $\mathcal{E}$ is closed, this is equivalent to the fact that the unit contraction operates on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, cf. [54]. The function ϕ_ε can be constructed by the following standard technique: We consider the mollifier

$$\rho(x) = \begin{cases} c \cdot \exp(-(1-|x|^2)^{-1}), & |x| < 1 \\ 0, & |x| \ge 1, \end{cases}$$

where the constant c is such that $\int_{|x|<1} \rho(x)\,\mathrm{d}x = 1$. Moreover, we set $\rho_\delta(x) := \delta^{-1}\rho(\delta^{-1}x)$, $\delta > 0$, $\psi_\varepsilon(t) := ((-\varepsilon) \vee t) \wedge (1+\varepsilon)$ and define for $0 < \delta < \varepsilon$ the function

$$\phi_\varepsilon(t) := \rho_\delta * \psi_\varepsilon(t) = \int_{\mathbb{R}} \rho_\delta(t-s)\psi_\varepsilon(s)\,\mathrm{d}s.$$

The strong local property is obvious and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular by the fact that $H^1(D) \cap C(\overline{D})$ is dense in both $C(\overline{D})$ equipped with the uniform norm, as well as $H^1(D)$ equipped with the standard Sobolev norm. □

Due to Proposition 2.1 and Theorem A.24, there exist an $\mathcal{E}$-exceptional set $\mathcal{N} \subset \overline{D}$ and a conservative diffusion process $X = (\Omega, \mathcal{F}, \{X_t, t \ge 0\}, \mathbb{P}_x)$, starting from $x \in \overline{D} \backslash \mathcal{N}$. X is associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in the sense of Theorem A.24. That is, for every non-negative Borel function ϕ, the transition semigroup of X defined by

$$P_t\phi(x) := \mathbb{E}_x\phi(X_t), \quad x \in \overline{D} \backslash \mathcal{N}, \tag{2.3}$$

is a version of the strongly continuous sub-Markovian contraction semigroup $T_t\phi$ on $L^2(D)$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Without loss of generality let us assume that X is defined on the *canonical sample space* $\Omega = C([0,\infty); \overline{D})$. It is well known that the symmetric Hunt process associated with (2.2) is the *reflecting Brownian motion*. In analogy to this terminology, we call the symmetric Hunt process associated with (2.1) a *reflecting diffusion process*.

Let us briefly recall the concept of the *boundary local time* of reflecting diffusion processes, see, e.g., [69, 133, 18], which will be crucial for the subsequent derivation of the Feynman-Kac formulae. If the diffusion process is the solution to a stochastic differential equation, say the reflecting Brownian motion, then the boundary local time is given by the one-dimensional process L in the Skorohod decomposition, which prevents the sample paths from leaving $\overline{D}$, i.e.,

$$X_t = x + W_t - \frac{1}{2}\int_0^t \nu(X_s)\,\mathrm{d}L_s, \tag{2.4}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in \overline{D}$, where W is a standard d-dimensional Brownian motion. This boundary local time is a continuous non-decreasing process which increases only when $X_t \in \partial D$, namely for all $t \geq 0$ and q.e. $x \in \overline{D}$

$$L_t = \int_0^t [\partial D](X_s)\,\mathrm{d}L_s,$$

$\mathbb{P}_x$-a.s. and

$$\mathbb{E}_x \int_0^t [\partial D](X_s)\,\mathrm{d}s = 0.$$

Although the reflecting diffusion process associated with (2.1) does in general not admit a Skorohod decomposition of the form (2.4), we may still define a continuous one-dimensional process with these properties. More precisely, by the Lipschitz property of ∂D, we have that $D \cap B(x, r_D) = \{(\tilde{x}, x_d) : x_d > \gamma(\tilde{x})\} \cap B(x, r_D)$ and the Lipschitz function γ is differentiable a.e. with a bounded gradient. In particular, we have for every Borel set $B \subset \partial D \cap B(x, r_D)$ that

$$\sigma(B) = \int_{\{\tilde{x} : (\tilde{x}, \gamma(\tilde{x})) \in B\}} \left(1 + |\nabla\gamma(\tilde{x})|^2\right)^{1/2} \mathrm{d}\tilde{x}$$

and a straightforward computation yields that the Lebesgue surface measure σ is a smooth measure with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ having finite energy, i.e.,

$$\int_{\partial D} |v|\,\mathrm{d}\sigma(x) \leq c||v||_{\mathcal{E}_1} \quad \text{for all } v \in \mathcal{D}(\mathcal{E}) \cap C(\overline{D}).$$

Definition 2.2. The positive continuous additive functional of X whose Revuz measure is given by the Lebesgue surface measure σ on ∂D, i.e., the unique $L \in \mathcal{A}_c^+$ such that

$$\lim_{t \to 0+} \frac{1}{t} \int_D \mathbb{E}_x \Big\{ \int_0^t \phi(X_s)\,\mathrm{d}L_s \Big\} \psi(x)\,\mathrm{d}x = \int_{\partial D} \phi(x)\psi(x)\,\mathrm{d}\sigma(x) \qquad (2.5)$$

for all non-negative Borel functions ϕ and all α-excessive functions ψ, is called the *boundary local time* of the reflecting diffusion process X.

Remark 2.3. An equivalent construction of the boundary local time, which is, however, less convenient for our purpose, goes as follows: Set

$$L_t^\varepsilon := \varepsilon^{-1} \int_0^t [D_\varepsilon](X_s)\,\mathrm{d}s, \quad D_\varepsilon := \{x \in \overline{D} : d(x, \partial D) \leq \varepsilon\},$$

then one can show that

$$\mathbb{E}_x|L_t^\varepsilon - L_t|^2 \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly in $x \in \overline{D}$. Moreover, there exists a monotonically decreasing null sequence $(\varepsilon_k, k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} L_t^{\varepsilon_k} = L_t \quad \mathbb{P}_x\text{-a.s.}$$

for q.e. $x \in \overline{D}$, uniformly in t on any compact time interval. This is the analogue of the definition of the local time for one-dimensional diffusion processes by Itô and McKean, cf. [74].

The rest of this section is devoted to showing that the $\mathcal{E}$-exceptional set $\mathcal{N}$ is actually empty. Therefore, we consider the non-positive definite self-adjoint operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. That is, for $v \in \mathcal{D}(\mathcal{L})$ we have

$$\langle -\mathcal{L}v, w\rangle = \mathcal{E}(v, w) \qquad \text{for all } w \in \mathcal{D}(\mathcal{E}) \tag{2.6}$$

and the domain of $\mathcal{L}$ is given by

$$\mathcal{D}(\mathcal{L}) = \Big\{ v \in \mathcal{D}(\mathcal{E}) : \exists \phi \in L^2(D) \text{ s.t. } \mathcal{E}(v, w) = \int_D \phi w \,\mathrm{d}x \;\forall w \in \mathcal{D}(\mathcal{E}) \Big\}.$$

In order to *refine* the reflecting diffusion process X to start from every $x \in \overline{D}$, we exploit the connection between the strongly continuous sub-Markovian contraction semigroup $\{T_t, t \geq 0\}$ on $L^2(D)$ and the evolution system corresponding to $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, see, e.g., the monograph [134]. Namely, for every $v_0 \in L^2(D)$, the trajectory $v : (0, T) \to H^1(D)$, $v(t) = T_t v_0$ belongs to the function space

$$\{\phi \in L^2((0, T); H^1(D)) : \dot{\phi} \in L^2((0, T); H^{-1}(D))\}$$

and is the unique mild solution to the parabolic *abstract Cauchy problem*

$$\begin{aligned} &\dot{v} + \mathcal{L}v = 0 \qquad \text{in } (0, T) \\ &v(0) = v_0. \end{aligned} \tag{2.7}$$

This is equivalent to the variational formulation

$$-\int_0^T \langle v(t), w\rangle\, \dot{\varphi}(t)\,\mathrm{d}t + \int_0^T \langle \mathcal{L}v(t), w\rangle\, \varphi(t)\,\mathrm{d}t - \langle v_0, w\rangle\, \varphi(0) = 0 \tag{2.8}$$

for all $w \in H^1(D)$ and all $\varphi \in C_c^\infty([0,T))$. Moreover, T_t is known to be a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$. Therefore, by the Dunford-Pettis theorem, it can be represented as an integral operator for every $t > 0$,

$$T_t\phi(x) = \int_D p(t,x,y)\phi(y)\,\mathrm{d}y \quad \text{for every } \phi \in L^1(D), \tag{2.9}$$

where for all $t > 0$ we have $p(t,\cdot,\cdot) \in L^\infty(D \times D)$ and $p(t,\cdot,\cdot) \geq 0$ a.e. We call the function p the *transition kernel density* of X.

The following proposition adapts a well-known result for diffusion processes on $\mathbb{R}^d$, cf. [151], which follows from the famous De Giorgi-Nash-Moser theorem, to the case of reflecting diffusion processes on $\overline{D}$. The key idea of the proof is the following *extension by reflection* technique from [160, Section 2.4.3]: We extend the solution to a parabolic problem by reflection at the boundary. Then we show that this extension again solves a parabolic problem so that we can apply the interior regularity result due to De Giorgi, Nash and Moser. See also the article [129] by Nittka, where such a technique is applied to elliptic boundary value problems.

Proposition 2.4. *$p \in C^{0,\delta}((0,T] \times \overline{D} \times \overline{D})$ for some $\delta \in (0,1)$, i.e., for each fixed $0 < t_0 \leq T$, there exists a positive constant c such that*

$$|p(t_2,x_2,y_2) - p(t_1,x_1,y_1)| \leq c(\sqrt{t_2 - t_1} + |x_2 - x_1| + |y_2 - y_1|)^\delta \tag{2.10}$$

for all $t_0 \leq t_1 \leq t_2 \leq T$ and all $(x_1,y_1),(x_2,y_2) \in \overline{D} \times \overline{D}$. Moreover, the mapping $t \mapsto p(t,\cdot,\cdot)$ is analytic from $(0,\infty)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$.

Proof. First note that Nash's inequality holds for the underlying Dirichlet form $(\mathcal{E}, H^1(D))$, i.e., there exists a constant $c_1 > 0$ such that

$$||v||_2^{2+4/d} \leq c_1(\mathcal{E}(v,v) + ||v||_2^2)||v||_1^{4/d} \quad \text{for all } v \in H^1(D).$$

This is a direct consequence of the uniform ellipticity (1.4) and [15, Corollary 2.2], where Nash's inequality is shown to hold for the Dirichlet form $(\mathcal{E}^{\mathrm{BM}}, H^1(D))$ for a bounded Lipschitz domain D. Analogously to the proof of [15, Theorem 3.1], it follows thus from [29, Theorem 3.25] that the transition kernel density satisfies an Aronson type Gaussian upper bound

$$p(t,x,y) \leq c_1 t^{-d/2} \exp\Big(-\frac{|x-y|^2}{c_2 t}\Big) \tag{2.11}$$

for all $t \leq 1$ and all $(x,y) \in \overline{D} \times \overline{D}$. In particular, $\sup_{t_0 < t \leq 1} ||p(t,\cdot,\cdot)||_\infty$ is finite and hence by the interior Hölder continuity obtained from the De

Giorgi-Nash-Moser theorem, cf. [127, 151], the estimate (2.10) is true for all (x_1, y_1), (x_2, y_2) satisfying $d(x_i, \partial D), d(y_i, \partial D) > c_3$, $i = 1, 2$, for some constant $c_3 > 0$ and all $t_0 \leq t_1 \leq t_2 \leq 1$. Note that by the semigroup property the Chapman-Kolmogorov equation holds, i.e.,

$$p(t_1 + t_2, x, y) = \int_D p(t_1, x, z)\, p(t_2, z, y)\, \mathrm{d}z \tag{2.12}$$

for every pair $t_1, t_2 \geq 0$ and a.e. $x, y \in \overline{D}$. In particular, for fixed $y \in \overline{D}$ the function $v := p(\cdot, \cdot, y)$ is the unique solution to (2.7) with initial value $v_0 := p(0, \cdot, y) \in L^2(D)$. Now let $z \in \partial D$ so that by the Lipschitz property of ∂D we have after translation and rotation $B(z, r_D) \cap \overline{D} = \{(\tilde{x}, x_d) \in B(z, r_D) : x_d \geq \gamma(\tilde{x})\}$ and $B(z, r_D) \cap \partial D = \{\tilde{x} \in B(z, r_D) : x_d = \gamma(\tilde{x})\}$, where we have introduced the notation $\tilde{x} = (x_1, ..., x_{d-1})^T$. Let us furthermore introduce the one-to-one transformation $\Psi(x) := (\tilde{x}, x_d - \gamma(\tilde{x}))$ which straightens the boundary $B(z, r_D) \cap \partial D$. Ψ is a bi-Lipschitz transformation and the Jacobians of both Ψ and Ψ^{-1} are bounded with bounds that depend only on the Lipschitz constant c_D. Since v is the solution to (2.7) with appropriate initial condition, the function $\hat{v} := v(\cdot, \Psi^{-1}(\cdot))$ must satisfy the following variational formulation of the a parabolic problem in $\hat{D}(z, r_D) := \Psi(B(z, r_D) \cap \overline{D})$, namely

$$\begin{aligned}\int_0^T \dot{\varphi}(t) \int_{\hat{D}(z,r_D)} \hat{v}(t) w \,\mathrm{d}x\,\mathrm{d}t \;=\; & -\sum_{i,j=1}^{d} \int_0^T \varphi(t) \int_{\hat{D}(z,r_D)} \hat{\kappa}_{ij} \partial_i \hat{v}(t) \partial_j w \,\mathrm{d}x\,\mathrm{d}t \\ & -\varphi(0) \int_{\hat{D}(z,r_D)} \hat{v}_0 w \,\mathrm{d}x\end{aligned}$$

for all $w \in C_c^\infty(\hat{D}(z, r_D))$ and all $\varphi \in C_c^\infty([0, T))$. The coefficient $\hat{\kappa}$ is obtained via change of variables and it is bounded and uniformly elliptic by the boundedness of the Jacobians of Ψ and Ψ^{-1}, respectively. Now we use reflection at the hyperplane $\{(\tilde{y}, 0)\}$ via the mapping $\rho(x) := (\tilde{x}, -x_d)$ which yields that the function $\hat{v}(\cdot, \rho(\cdot))$ satisfies the variational formulation of a parabolic problem on $\rho(\hat{D}(z, r_D))$. Summing up both variational formulations on $\hat{D}(z, r_D)$ and on $\rho(\hat{D}(z, r_D))$, respectively, we obtain that the function

$$\check{v}(t, x) := \begin{cases} \hat{v}(t, x), & x \in \hat{D}(z, r_D) \\ \hat{v}(t, \rho(x)), & x \in \rho(\hat{D}(z, r_D)) \end{cases}$$

satisfies the variational formulation of a parabolic problem in $\hat{D}(z, r_D) \cup \rho(\hat{D}(z, r_D))$. By the interior Hölder estimate for $\check{v}$, together with the fact that we may choose $c_3 = r_D/4c_D$, we obtain thus

$$|p(t_2, x_2, \Psi^{-1}(y_2)) - p(t_1, x_1, \Psi^{-1}(y_1))| \leq c_1(\sqrt{t_2 - t_1} + |y_2 - y_1|)^{c_2}$$

for all $t_0 \leq t_1 \leq t_2 \leq 1$ and $y_1, y_2 \in \{(\tilde{x}, x_d) : |\tilde{x}| < c_3,\ x_d \in (0, r_D/4)\}$. As Ψ is bi-Lipschitz, for fixed x, the mapping $(t, y) \mapsto p(t, x, y)$ is Hölder continuous in $(t_0, 1] \times (B(z, c_3) \cap \overline{D})$ and by symmetry of the transition kernel density the same holds true for the mapping $(t, x) \mapsto p(t, x, y)$ for fixed y. Finally, the first assertion on $(t_0, 1] \times \overline{D} \times \overline{D}$ follows due to compactness of ∂D and its generalization to arbitrary $T > 0$ is obtained after repeatedly applying the Chapman-Kolmogorov equation.

The second assertion follows by the fact that the semigroup $\{T_t, t \geq 0\}$ extrapolates to a holomorphic semigroup on $L^2(D)$. More precisely, the semigroup possesses a holomorphic extension to the sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2}]$, cf., e.g., [134]. Let $0 < t_0 \leq T$ and set

$$\Sigma_\theta(t_0, T) := \{z \in \mathbb{C} : z - t_0 \in \Sigma_\theta, |z| < T\}.$$

By the Hölder continuity of p, the set $\{p(z, \cdot, \cdot) : z \in \Sigma_\theta(t_0, T)\}$ is a bounded subset of $C^{0,\delta}(\overline{D} \times \overline{D})$. Moreover, the family of functionals obtained from integration against the functions $[B_1](x)[B_2](y)$ for measurable $B_1, B_2 \subset \overline{D}$ form a separating subspace of $(C^{0,\delta}(\overline{D}, \overline{D}))'$, i.e., for $k \in C^{0,\delta}(\overline{D} \times \overline{D})$

$$\int_{D \times D} k(x, y)[B_1](x)[B_2](y)\, \mathrm{d}x\, \mathrm{d}y = 0 \quad \text{for all measurable } B_1, B_2 \subset \overline{D}$$

implies that $k \equiv 0$. As the mapping

$$z \mapsto \langle T_z[B_1], [B_2] \rangle = \int_{D \times D} p(z, x, y)[B_1](y)[B_2](x)\, \mathrm{d}x\, \mathrm{d}y$$

is holomorphic for all $z \in \Sigma_\theta$, the mapping $z \mapsto p(z, \cdot, \cdot)$ is holomorphic from $\Sigma_\theta(t_0, T)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$ by [5, Theorem 3.1]. Since t_0 and T were arbitrary, the assertion is proved. □

By [53, Theorem 2], the existence of a Hölder continuous transition kernel density ensures that we may refine the process X to start from every $x \in \overline{D}$ by identifying the strongly continuous semigroup $\{T_t, t \geq 0\}$ with the transition semigroup $\{P_t, t \geq 0\}$. In particular, if v is continuous and locally in $H^1(D)$, the Fukushima decomposition holds for every $x \in \overline{D}$, i.e.,

$$v(X_t) = v(X_0) + M_t^v + N_t^v, \quad \text{for all } t > 0, \tag{2.13}$$

$\mathbb{P}_x$-a.s., where M^v is a martingale additive functional of X having finite energy and N^v is a continuous additive functional of X having zero energy. Moreover, both M^v and N^v can be taken to be additive functionals of X in the strict sense, cf. [54, Theorem 5.2.5].

Finally, note that the 1-potential of the Lebesgue surface measure σ of ∂D is the solution to an elliptic boundary value problem on a Lipschitz domain with bounded data. By elliptic regularity theory, cf., e.g., [60], this solution is continuous, implying that the boundary local time L exists as a positive continuous additive functional in the strict sense, cf. [54, Theorem 5.1.6].

2.2 Skorohod decompositions

In this section, we derive Skorohod decompositions of the reflecting diffusion process X for two practically relevant special cases, namely local Lipschitz conductivities and isotropic piecewise constant conductivities.

The assertion of the following proposition is already covered by [55, Theorem 2.3]; we include a proof for the sake of self-containedness.

Proposition 2.5. *Let $\kappa \in C^{0,1}_{loc}(\overline{D};\mathbb{R}^{d\times d})$ be a symmetric, uniformly bounded and uniformly elliptic conductivity. Then the reflecting diffusion process X admits the following Skorohod decomposition*

$$X_t = x + \int_0^t B(X_s)\,\mathrm{d}W_s + \int_0^t \nabla\kappa(X_s)\,\mathrm{d}s - \int_0^t \kappa(X_s)\nu(X_s)\,\mathrm{d}L_s, \quad (2.14)$$

$\mathbb{P}_x$-a.s., where $B : \overline{D} \to \mathbb{R}^{d\times d}$ denotes the positive definite diffusion matrix satisfying $B^2 = 2\kappa$, W is a standard d-dimensional Brownian motion and L is the boundary local time of X.

Proof. We have shown in Section 2.1, that the Fukushima decomposition holds with a unique martingale additive functional M^v in the strict sense and a unique continuous additive functional N^v in the strict sense. Let us first compute the energy measure of M^v. For $v, w \in \mathcal{D}(\mathcal{E})$ we obtain using Lemma A.9

$$\begin{aligned}\int_D w(x)\,\mathrm{d}\mu_{\langle M^v\rangle}(x) &= \lim_{t\to 0+}\frac{1}{t}\int_D \mathbb{E}_x\left\{(v(X_t)-v(x_0))^2\right\}w(x)\,\mathrm{d}x\\ &= \lim_{t\to 0+}\frac{1}{t}\int_D (T_t v^2(x) - 2v(x)T_t v(x) + v^2(x))w(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0+} \frac{2}{t} \int_D v(x)w(x)(v(x) - T_t v(x))\,\mathrm{d}x \\
&\quad - \lim_{t\to 0+} \frac{1}{t} \int_D v^2(x)(w(x) - T_t w(x))\,\mathrm{d}x \\
&= 2\mathcal{E}(vw, v) - \mathcal{E}(v^2, w) \\
&= 2\int_D \kappa \nabla v(x) \cdot \nabla v(x)\, w(x)\,\mathrm{d}x,
\end{aligned}
$$

which yields the energy measure

$$
\mathrm{d}\mu_{\langle M^v\rangle}(x) = 2\sum_{i,j=1}^{d} \kappa_{ij}(x)\partial_i v(x)\partial_j v(x)\,\mathrm{d}x
$$

so that the predictable quadratic variation of M^v is given by

$$
\left\langle M^v \right\rangle_t = 2\int_0^t \sum_{i,j=1}^{d} \kappa_{ij}(X_s)\partial_i v(X_s)\partial_j v(X_s)\,\mathrm{d}s. \tag{2.15}
$$

Using the coordinate mappings $\phi_i(x) := x_i$, $i = 1, ..., d$, on $\overline{D}$ yields that M^ϕ is a continuous martingale additive functional in the strict sense with covariation

$$
\left\langle M^{\phi_i}, M^{\phi_j} \right\rangle_t = 2\int_0^t \kappa_{ij}(X_s)\,\mathrm{d}s,
$$

$\mathbb{P}_x$-a.s. A standard characterization of continuous martingales, cf., e.g., [72], yields that

$$
M_t^v = \int_0^t (B(X_s)\nabla v(X_s)) \cdot \mathrm{d}W_s, \tag{2.16}
$$

$\mathbb{P}_x$-a.s., where $B : \overline{D} \to \mathbb{R}^{d\times d}$ denotes the positive definite diffusion matrix satisfying $B^2 = 2\kappa$ and W is a standard d-dimensional Brownian motion.

Now let us consider the continuous additive functional N^v. Again using the coordinate mappings on $\overline{D}$, we obtain from Green's formula that

$$
\begin{aligned}
\mathcal{E}(\phi_i, w) &= \sum_{j=1}^{d} \int_D \kappa_{ij}(x)\partial_j w(x)\,\mathrm{d}x \\
&= -\sum_{j=1}^{d} \int_D \partial_j \kappa_{ij}(x) w(x)\,\mathrm{d}x + \sum_{j=1}^{d} \int_{\partial D} \kappa_{ij}(x)\nu_j(x) w(x)\,\mathrm{d}\sigma(x)
\end{aligned}
$$

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