Preface

Separation of variables is one of the oldest and most powerful methods to construct exact solutions of fundamental partial differential equations in classical and quantum physics, like the Hamilton-Jacobi equation in Newtonian mechanics, the Schrödinger equation in quantum mechanics or the wave equation. A separation of the Schrödinger equation for the hydrogen atom in spherical coordinates, for example, yields functions describing the orbital structure of the electrons, i.e. the basis for the periodic table of the elements, and is thus at the root of chemistry. Likewise, many other well known special functions which are used all over in science and technology stem from a separation of variables.

The problem to classify all coordinate systems in which this method is applicable, the so-called *separation coordinates*, was solved exhaustively by Ernest G. Kalnins & Willard Miller Jr. over 30 years ago. For this reason many experts would consider the theory of separation of variables as settled or even old-fashioned. However, in this book we argue that the above classification problem is essentially an algebraic geometric and not a differential geometric problem, i.e. governed by algebraic instead of partial differential equations. This means that Kalnins & Miller's list of separation coordinates carries a much deeper geometric structure, namely that of a projective variety equipped with the natural action of the isometry group G. From a categorical point of view the classification problem has been solved in the category of sets, but not in its natural category, the category of projective G-varieties.

It seems that, albeit obvious, this fact has so far been completely overlooked (or ignored). Virtually nothing is known about the geometry of these varieties or the topology of their quotients. The aim of the present book is to bridge this gap and to lay the foundations for a consequent algebraic geometric treatment of variable separation. By applying it to spheres, we not only give a proof of concept that the approach we propose is viable, we also demonstrate that it leads to surprising results already for this simplest family of constant curvature manifolds. Namely, we reveal a correspondence between two a priori completely unrelated objects: the space of equivalence classes of separation coordinates on the n-dimensional sphere $\mathbb{S}^n$ and the Deligne-Mumford moduli space $\bar{\mathscr{M}}_{0,n+2}(\mathbb{R})$ of stable algebraic curves of genus zero with $n+2$ marked points. Moreover, we derive a classification of separation coordinates via Stasheff polytopes from this correspondence, together with a simple and uniform construction based on the natural operad structure on the family of moduli spaces $\bar{\mathscr{M}}_{0,n}(\mathbb{R})$.

In this way we build a bridge between the theory of separation of variables, whose origins date back far into the 19th century, and most recent results in modern algebraic geometry.

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