1 The foundation: the algebraic integrability conditions

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In this chapter we translate the Nijenhuis integrability conditions for a Killing tensor on a constant curvature manifold into algebraic conditions on the corresponding algebraic curvature tensors. To this end, we substitute (0.7) into (0.2) and both into (0.3) and then use the representation theory for general linear groups to get rid of the dependence on the base point in the manifold.

Note that the algebraic curvature tensor in (0.7) is implicitly symmetrised in the first and third entry. The result of this operation is a tensor having the symmetries of an algebraic curvature tensor, but with antisymmetry replaced by symmetry.

Definition 1.1. *A* symmetrised algebraic curvature tensor *on a vector space V is an element $R \in V^* \otimes V^* \otimes V^* \otimes V^*$ satisfying the following*

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symmetries:

$$
\begin{aligned}
&S(x,w,y,z) = +S(w,x,y,z) = S(w,x,z,y) && \textit{(symmetry)}\\
&S(y,z,w,x) = S(w,x,y,z) && \textit{(pair symmetry)}\\
&S(w,x,y,z) + S(w,y,z,x) + S(w,z,x,y) = 0 && \textit{(Bianchi identity)}^{1}
\end{aligned}
$$

In subsequent computations it will be more convenient to work with the symmetrised version of algebraic curvature tensors. Actually, both representations are isomorphic.

Remark 1.2. *The space of algebraic curvature tensors on V and the space of symmetrised algebraic curvature tensors on V are isomorphic representations of* $\mathrm{GL}(V)$. *Explicitly, this isomorphism is given by*

$$S(w,x,y,z) = \tfrac{1}{\sqrt{3}}\big(R(w,y,x,z) + R(w,z,x,y)\big) \tag{1.2a}$$

$$R(w,x,y,z) = \tfrac{1}{\sqrt{3}}\big(S(w,y,x,z) - S(w,z,x,y)\big). \tag{1.2b}$$

Since the Nijenhuis torsion of K depends on K and its covariant derivative, ∇K, we need to express both in terms of the corresponding symmetrised algebraic curvature tensor S.

Lemma 1.3. *Up to a constant factor that can be neglected, we have*

$$K_x(v,w) = S(x,x,v,w) \tag{1.3a}$$

$$(\nabla_u K)_x(v,w) = 2S(x,u,v,w). \tag{1.3b}$$

Proof. Up to said factor, the expression (1.3a) for K follows from substituting (1.2b) into (0.7). For a flat space $M \subset V$ as in (0.5b), Formula (1.3b) follows trivially. So let us assume that $M \subset V$ is as

[1] Owing to the other two symmetries, the cyclic sum may be taken over any three of the four entries.

in (0.5a). Denoting the covariant derivative on V by $\hat{\nabla}$, the covariant derivative of K is then given by

$$\begin{aligned}
&(\nabla_u K)(v,w) \\
&\quad = \nabla_u\big(K(v,w)\big) - K\big(\nabla_u v, w\big) - K\big(v, \nabla_u w\big) \\
&\quad = \hat{\nabla}_u\big(S(x,x,v,w)\big) - S(x,x,\nabla_u v,w) - S(x,x,v,\nabla_u w) \\
&\quad = 2S(x,\hat{\nabla}_u x,v,w) + S(x,x,\hat{\nabla}_u v,w) + S(x,x,v,\hat{\nabla}_u w) \\
&\qquad\quad - S(x,x,\hat{\nabla}_u v - g(u,v)x,w) - S(x,x,v,\hat{\nabla}_u w - g(u,w)x) \\
&\quad = 2S(x,u,v,w).
\end{aligned}$$

For the last equality we used the fact that the Bianchi identity for S implies that $S(x,x,x,w) = 0$ and $S(x,x,v,x) = 0$. □

The proof of Proposition 0.9 is now a simple consequence of the above lemma, so we will give it here for the sake of completeness.

Proof (of Proposition 0.9). We have to show that the map defined by (1.3a) is an isomorphism between Killing tensors on $M \subset V$ and symmetrised algebraic curvature tensors on V. This map is well defined, since by (1.3b) the Killing equation for (1.3a) is equivalent to the Bianchi identity for S. For simplicity let us assume that $M \subset V$ is not flat, i.e. of the form (0.5a). To show the injectivity of the above map, suppose $S(x,x,v,w) = 0$ for all $x,v,w \in V$ with $g(x,x) = 1$ and $g(v,x) = g(w,x) = 0$. We can omit the restriction $g(v,x) = g(w,x) = 0$ due to the Bianchi identity for S. We can also omit the restriction $g(x,x) = 1$, because $S(x,x,v,w)$ is a homogeneous polynomial in x for fixed $v,w \in V$ and $\mathbb{R}M$ is open in V. From a polarisation in x we then conclude that $S = 0$. The surjectivity of the above map now follows from dimension considerations. Indeed, the dimension of the space of Killing tensors on a constant curvature manifold of dimension n is known to be

$$\frac{(n+1)n^2(n-1)}{12},$$

which happens to be the dimension of the space of algebraic curvature tensors in dimension $n+1$.[2] For a flat space $M \subset V$ as in (0.5b) the proof is analogous and will be left to the reader. $\square$

For actual computations the use of index notation is indispensable. We will write Greek indices $\alpha, \beta, \gamma, \ldots$ for local coordinates on M (ranging from 1 to n) and Latin indices $a, b, c, \ldots$ for components in V (ranging from 0 to n). We can then denote both, the inner product on V as well as the induced metric on M, by the same letter g and distinguish them only via the type of indices. Consequently, Latin indices are raised and lowered using g_{ab} and greek ones using $g_{\alpha\beta}$.

This said, we can rewrite the expressions (1.3) using $\nabla_v x^a = v^a$ as

$$K_{\alpha\beta} = S_{a_1a_2b_1b_2} x^{a_1} x^{a_2} \nabla_\alpha x^{b_1} \nabla_\beta x^{b_2} \tag{1.4a}$$

$$\nabla_\gamma K_{\alpha\beta} = 2S_{c_1c_2d_1d_2} x^{c_1} \nabla_\gamma x^{c_2} \nabla_\alpha x^{d_1} \nabla_\beta x^{d_2}, \tag{1.4b}$$

where we regard the components x^a of $x \in V$ as functions on $M \subset V$ by restriction. We are now ready to substitute (1.4) into (0.2) and then further into (0.3).

First note that in the integrability conditions (0.3) the Nijenhuis torsion (0.2) appears only antisymmetrised in its two lower indices β and γ. To simplify computations we will thus replace the Nijenhuis torsion $N^\alpha{}_{\beta\gamma}$ in the integrability conditions by the tensor

$$\bar{N}^\alpha{}_{\beta\gamma} := \tfrac{1}{2}\big(K^\alpha{}_\delta \nabla_\gamma K^\delta{}_\beta + K^\delta{}_\beta \nabla_\delta K^\alpha{}_\gamma\big), \qquad \bar{N}^\alpha{}_{[\beta\gamma]} = N^\alpha{}_{\beta\gamma}\,.$$

Together with (1.4) this can be written as

$$\begin{aligned}\bar{N}^\alpha{}_{\beta\gamma} &= S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^\alpha x^{b_1} \nabla_\delta x^{b_2} \nabla_\gamma x^{c_2} \nabla^\delta x^{d_1} \nabla_\beta x^{d_2} \\ &\quad + S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^\delta x^{b_1} \nabla_\beta x^{b_2} \nabla_\delta x^{c_2} \nabla^\alpha x^{d_1} \nabla_\gamma x^{d_2}.\end{aligned}$$

Lemma 1.4. *For a constant curvature manifold we have*

$$\nabla_\delta x^a \nabla^\delta x^b = \begin{cases} g^{ab} - x^a x^b & \text{if } M \text{ is of the form (0.5a)} \\ g^{ab} - u^a u^b & \text{if } M \text{ is of the form (0.5b).} \end{cases}$$

[2]This can be computed from the so called *hook formula*.

Proof. Let $e_1, \dots, e_n$ be a basis of T_xM and complete it with a unit normal vector $u =: e_0$ to a basis of V. Then on one hand

$$\begin{aligned}\sum_{i,j=0}^{n} g(e^i, e^j)\nabla_{e_i}x^a\nabla_{e_j}x^b &= \sum_{i,j=1}^{n} g(e^i, e^j)\nabla_{e_i}x^a\nabla_{e_j}x^b + \nabla_u x^a \nabla_u x^b \\ &= g^{\alpha\beta}\nabla_\alpha x^a \nabla_\beta x^b + u^a u^b.\end{aligned}$$

On the other hand, choosing the standard basis of V instead, the left hand side is just g^{ab}. This proves the lemma, remarking that $u = x$ if M is not flat. □

For flat M the lemma yields

$$\begin{aligned}\bar{N}^\alpha{}_{\beta\gamma} &= \bar{g}^{b_2d_1} S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1}x^{a_2}x^{c_1}\nabla^\alpha x^{b_1}\nabla_\beta x^{d_2}\nabla_\gamma x^{c_2} \\ &\quad + \bar{g}^{b_1c_2} S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1}x^{a_2}x^{c_1}\nabla^\alpha x^{d_1}\nabla_\beta x^{b_2}\nabla_\gamma x^{d_2},\end{aligned} \tag{1.5}$$

where $\bar{g} := g^{ab} - u^a u^b$. In all other cases we have

$$\begin{aligned}&\bar{N}^\alpha{}_{\beta\gamma} \\ &= \left(g^{b_2d_1} - x^{b_2}x^{d_1}\right) S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1}x^{a_2}x^{c_1}\nabla^\alpha x^{b_1}\nabla_\beta x^{d_2}\nabla_\gamma x^{c_2} \\ &+ \left(g^{b_1c_2} - x^{b_1}x^{c_2}\right) S_{a_1a_2b_1b_2} S_{c_1c_2d_1d_2} x^{a_1}x^{a_2}x^{c_1}\nabla^\alpha x^{d_1}\nabla_\beta x^{b_2}\nabla_\gamma x^{d_2}.\end{aligned}$$

But here the two subtracted terms vanish by the Bianchi identity, because they contain the terms

$$S_{a_1a_2b_1b_2}x^{a_1}x^{a_2}x^{b_2} = 0, \qquad S_{a_1a_2b_1b_2}x^{a_1}x^{a_2}x^{b_1} = 0.$$

This allows us to use (1.5) for *all* constant curvature manifolds of the form (0.5) if we define

$$\bar{g}^{ab} := \begin{cases} g^{ab} & \text{if } M \text{ is of the form (0.5a)} \\ g^{ab} - u^a u^b & \text{if } M \text{ is of the form (0.5b).} \end{cases} \tag{1.6}$$

In the case of a hyperplane $M \subset V$, the tensor $\bar{g}^{ab}$ is the pullback of the metric on M via the orthogonal projection $V \to M$ and thus degenerated. Note that we still lower and rise indices with the metric g^{ab} and not with $\bar{g}^{ab}$.

In (1.5) the lower indices b_2, d_1 respectively b_1, c_2 are contracted with $\bar{g}$. We can make use of the symmetries of $S_{a_1a_2b_1b_2}$ to bring these indices to the first position:

$$\begin{aligned}\bar{N}^{\alpha}{}_{\beta\gamma} &= \bar{g}^{b_2d_1} S_{b_2b_1a_1a_2} S_{d_1d_2c_1c_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{b_1} \nabla_{\beta} x^{d_2} \nabla_{\gamma} x^{c_2} \\ &\quad + \bar{g}^{b_1c_2} S_{b_1b_2a_1a_2} S_{c_2c_1d_1d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{d_1} \nabla_{\beta} x^{b_2} \nabla_{\gamma} x^{d_2}.\end{aligned}$$

Renaming, lowering and rising indices appropriately finally results in

$$\begin{aligned}\bar{N}_{\alpha\beta\gamma} = \bar{g}_{ij} \big(S^{i}{}_{a_2b_1b_2} S^{j}{}_{c_2d_1d_2} + S^{i}{}_{c_2b_1b_2} S^{j}{}_{d_1a_2d_2} \big) \\ x^{b_1} x^{b_2} x^{d_1} \nabla_{\alpha} x^{a_2} \nabla_{\beta} x^{c_2} \nabla_{\gamma} x^{d_2}. \quad (1.7)\end{aligned}$$

In what follows we will substitute this expression together with (1.4a) into each of the three integrability conditions (0.3) and transform them into purely algebraic integrability conditions.

1.1 Young tableaux

Throughout this chapter we will use Young tableaux as a compact means for index manipulations on tensors with many indices. The reader not familiar with this formalism may as well simply consider them as an alternative notation for symmetrisation and antisymmetrisation operators. However, we prefer Young tableaux over the more common notation using round respectively square brackets around the indices to be symmetrised, as the latter becomes confusing when several index sets are involved and even ambiguous if these sets are not disjoint. Moreover, using Young tableaux has the additional advantage that one can directly read off the symmetry class of the tensors involved. As we will basically deal with only a single type of Young tableaux, namely those of a "hook shape", we introduce them by means of examples. More details can be found in [Sch12]. For the background we refer the reader to the standard literature on representation theory of symmetric and linear groups.

Young tableaux define elements in the group algebra of the permutation group S_d. That is, a Young tableau stands for a (formal) linear combination of permutations of d objects. In our case, these objects will be certain tensor indices. For the sake of simplicity of notation we will identify a Young tableau with the group algebra element it defines. A Young tableau consisting of a single row denotes the sum of all permutations of the indices in this row. For example, using cycle notation,

$$\begin{array}{|c|c|c|}\hline a_2 & c_1 & c_2 \\ \hline\end{array} = e + (a_2c_1) + (c_1c_2) + (c_2a_2) + (a_2c_1c_2) + (c_2c_1a_2).$$

This is an element in the group algebra of the group of permutations of the indices a_2, c_1 and c_2 (or any superset). In the same way a Young tableau consisting of a single column denotes the *signed* sum of all permutations of the indices in this column, the sign being the sign of the permutation. For example,

$$\begin{array}{|c|}\hline a_1 \\ \hline b_1 \\ \hline d_2 \\ \hline\end{array} = e - (a_1b_1) - (b_1d_2) - (d_2a_1) + (a_1b_1d_2) + (d_2b_1a_1).$$

We call these *row symmetrisers* respectively *column antisymmetrisers*. The reason we define them without the usual normalisation factors is that then all numerical constants appear explicitly in our computations (although irrelevant for our concerns).

The group multiplication extends linearly to a natural product in the group algebra. A general Young tableau is then simply the product of all row symmetrisers and all column antisymmetrisers of the tableau. We will only deal with Young tableaux having a "hook shape", such as the following:

$$\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & c_1 & c_2 \\ \hline \multicolumn{1}{|c|}{b_1} \\ \cline{1-1} \multicolumn{1}{|c|}{d_2} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & c_1 & c_2 \\ \hline\end{array}\ \begin{array}{|c|}\hline a_1 \\ \hline b_1 \\ \hline d_2 \\ \hline\end{array}\,. \tag{1.8a}$$

The inversion of group elements extends linearly to an involution of the group algebra. If we consider elements in the group algebra as linear operators on the group algebra itself, this involution is the adjoint with respect to the natural inner product on the group algebra, given by defining the group elements to be an orthonormal basis. Since this operation leaves symmetrisers and antisymmetrisers invariant, it simply exchanges the order of symmetrisers and antisymmetrisers in a Young tableau. The adjoint of (1.8a) for example is

$$\begin{ytableau} a_1 & a_2 & c_1 & c_2 \\ b_1 \\ d_2 \end{ytableau}^{\,\star} = \begin{ytableau} a_1 \\ b_1 \\ d_2 \end{ytableau}\; \begin{ytableau} a_1 & a_2 & c_1 & c_2 \end{ytableau}\,. \tag{1.8b}$$

Properly scaled, Young tableaux with d boxes define projectors onto irreducible S_d-representations. A hook shaped Young tableau with p rows and q columns for example satisfies

$$\begin{ytableau} a & b & \cdots & c \\ d \\ \vdots \\ e \end{ytableau}^{\,2} = (p+q-1)(p-1)!(q-1)!\begin{ytableau} a & b & \cdots & c \\ d \\ \vdots \\ e \end{ytableau} \tag{1.9}$$

and the same formula holds for its adjoint.

The isomorphism class of the irreducible representation defined by a Young tableau is labelled by the corresponding *Young frame*, which is the Young tableau with the labels of its boxes erased. On the level of isomorphism classes, the decomposition of tensor products of irreducible representations is given by the *Littlewood-Richardson rule*. For example, according to this rule, the tensor product of a symmetric and an antisymmetric representation decomposes into two irreducible components, each of hook symmetry:

$$q\left\{\begin{ytableau} {} \\ \vdots \\ {} \end{ytableau}\right. \otimes \overbrace{\begin{ytableau} {} & \cdots & {} \end{ytableau}}^{p} \cong \overbrace{\begin{ytableau} {} & \cdots & {} \\ {} \\ \vdots \\ {} \end{ytableau}}^{p} \oplus \left.\begin{ytableau} {} & {} & \cdots & {} \\ \vdots \\ {} \end{ytableau}\right\} q. \tag{1.10}$$

The following lemma gives an explicit realisation of this decomposition in terms of orthogonal projectors.

Lemma 1.5.

$$\frac{1}{q!}\,\begin{ytableau} a_1 \\ \vdots \\ a_q \end{ytableau} \cdot \frac{1}{p!}\,\begin{ytableau} s_1 & \cdots & s_p \end{ytableau} = \frac{\frac{p}{q+1}}{(p+q)p!^2q!^2}\,\begin{ytableau} s_1 & \cdots & s_p \\ a_1 \\ \vdots \\ a_q \end{ytableau}\ \begin{ytableau} s_1 & \cdots & s_p \\ a_1 \\ \vdots \\ a_q \end{ytableau}^{\star}$$

$$+\ \frac{\frac{q}{p+1}}{(p+q)p!^2q!^2}\,\begin{ytableau} a_1 & s_1 & \cdots & s_p \\ \vdots \\ a_q \end{ytableau}^{\star}\ \begin{ytableau} a_1 & s_1 & \cdots & s_p \\ \vdots \\ a_q \end{ytableau} \qquad (1.11)$$

In particular, for $p = q = 3$*:*

$$\frac{1}{3!}\,\begin{ytableau} c_2 \\ d_2 \\ a_2 \end{ytableau} \cdot \frac{1}{3!}\,\begin{ytableau} b_2 & b_1 & d_1 \end{ytableau} = \frac{1}{2^7 3^4}\,\begin{ytableau} b_2 & b_1 & d_1 \\ c_2 \\ d_2 \\ a_2 \end{ytableau}\ \begin{ytableau} b_2 & b_1 & d_1 \\ c_2 \\ d_2 \\ a_2 \end{ytableau}^{\star} + \frac{1}{2^7 3^4}\,\begin{ytableau} c_2 & b_2 & b_1 & d_1 \\ d_2 \\ a_2 \end{ytableau}^{\star}\ \begin{ytableau} c_2 & b_2 & b_1 & d_1 \\ d_2 \\ a_2 \end{ytableau}\ . \qquad (1.12)$$

Proof. Write (1.11) as $P = P_1 + P_2$. Decomposing temporarily the hook symmetrisers on the right hand side as in (1.8) into a product of a symmetriser and an antisymmetriser and using (1.9), one easily checks that P, P_1 and P_2 are orthogonal projectors verifying $P_1P_2 = 0 = P_2P_1$, $PP_1 = P_1$ and $PP_2 = P_2$. Therefore $P_1 + P_2$ is an orthogonal projector with image $\operatorname{im} P_1 \oplus \operatorname{im} P_2 \subseteq \operatorname{im} P$. The decomposition of the isomorphism class of $\operatorname{im} P$ into irreducible components is given by (1.10). The Young frames on the right hand side are those appearing in the expression for P_1 respectively P_2. Hence they describe the isomorphism classes of $\operatorname{im} P_1$ and $\operatorname{im} P_2$. This shows that $\operatorname{im} P$ and $\operatorname{im}(P_1 + P_2) = \operatorname{im} P_1 \oplus \operatorname{im} P_2$ have the same dimension and are thus equal. This implies $P = P_1 + P_2$. $\square$

Remark 1.6. *The lemma can be interpreted as an explicit splitting of the terms in the long exact sequence*

$$0 \to \Lambda^d V \to \ldots \to S^p V \otimes \Lambda^q V \to S^{p+1} V \otimes \Lambda^{q-1} V \to \ldots \to S^d V \to 0,$$

known as the Koszul complex.

The permutation group S_d acts on d-fold covariant or contravariant tensors by permuting indices. This action extends linearly to an action of the entire group algebra. In particular, any Young tableau acts on tensors with corresponding indices. For example,

$$\begin{array}{|c|}\hline b_1 \\ \hline a_2 \\ \hline c_2 \\ \hline \end{array}\, T_{b_1 a_2 c_2} = T_{b_1 a_2 c_2} - T_{a_2 b_1 c_2} - T_{b_1 c_2 a_2} - T_{c_1 a_2 b_2} + T_{a_1 c_2 b_2} + T_{c_1 b_2 a_2}.$$

To give another example, the operator (1.8a) acts on a tensor

$$T_{b_1 b_2 d_1 d_2 a_2 c_2}$$

by an antisymmetrisation in the indices b_1, a_2, c_2 and a subsequent symmetrisation in the indices b_1, b_2, d_1, d_2. In the same way its adjoint (1.8b) acts by first symmetrising and then antisymmetrising.

1.2 The 1st integrability condition

The first integrability condition (0.3a) can be written as $\bar{N}_{[\alpha\beta\gamma]} = 0$. For the expression (1.7) this is equivalent to the vanishing of the antisymmetrisation in the upper indices a_2, c_2, d_2:

$$\begin{aligned}\bar{g}_{ij}\big(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2}\big)& \\ x^{b_1} x^{b_2} x^{d_1} \nabla_\alpha x^{[a_2} \nabla_\beta x^{c_2} \nabla_\gamma x^{d_2]} &= 0.\end{aligned}$$

Due to the symmetry of $S^j{}_{d_1 a_2 d_2}$ in a_2, d_2 the second term vanishes. If we write u, v and w for the tangent vectors ∂_α, ∂_β respectively ∂_γ and use $\nabla_u x^a = u^a$ in order to get rid of the ∇'s, we obtain the condition

$$\begin{gathered}\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]} = 0 \\ \forall x \in M, \quad \forall u, v, w \in T_x M\end{gathered} \tag{1.13}$$

on the symmetrised algebraic curvature tensor S.

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