Chapter 2
Critical Mass and Efficiency

Every gram of enriched uranium or synthesized plutonium produced in the Manhattan Project was obtained at great cost and with great difficulty, so estimating the amount of fissile material needed to make a workable nuclear weapon—the so-called critical mass—was a crucial issue for the developers of *Little Boy* and *Fat Man*. Equally important was to estimate what efficiency one might expect for a fission bomb. For various reasons, not all of the fissile material in a bomb core undergoes fission during a nuclear explosion; if the expected efficiency were to prove so low that one might just as well use a few conventional bombs to achieve the same energy release, there would be no point in taking on the massive engineering challenges involved in making nuclear weapons. In this chapter we investigate these issues.

The concept of critical mass involves two competing effects. As nuclei fission, they emit secondary neutrons. A fundamental empirical law of nuclear physics, derived in Sect. 2.1, shows that while some neutrons will cause other fissions, the remainder will reach the surface of the mass and escape. If on average more than one neutron is emitted per fission, however, we can afford to let some escape since only one is required to initiate a subsequent fission. For a small sample of material the escape probability is high; as the size of the sample increases, the escape probability declines and at some point will reach a value such that the number of neutrons that fail to escape will number enough to fission every nucleus in the mass—in theory, at least. Thus, there is a minimum size (hence mass) of material for which every nucleus will in principle be fissioned even while some neutrons escape.

The above description of critical mass should be regarded as a purely qualitative one. Technically, the important issue is known as *criticality*. Criticality is said to obtain when the number of free neutrons inside a bomb core is increasing with time. A full understanding of criticality demands familiarity with time-dependent diffusion theory. Application of diffusion theory to this problem requires understanding a concept known as the *mean free path* (MFP) for neutron travel, so this is developed in Sect. 2.1. Section 2.2 takes up a time-dependent diffusion theory treatment of criticality. Section 2.3 addresses the effect of surrounding the fissile core with a *tamper*. A tamper is a heavy metal casing which enhances weapon

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efficiency in two ways: By reflecting escaped neutrons back into the core and hence giving them another chance at causing fissions, and by briefly retarding the violent expansion of the core in order to give the chain reaction more time over which to operate. Sections 2.4 and 2.5 respectively take up the issue of bomb efficiency through analytic approximations and a numerical simulation. Section 2.6 presents an alternate treatment of untamped criticality that has an interesting historical connection, and Sect. 2.7 presents an approximate treatment of criticality for cylindrical bomb cores.

For readers interested in further sources, an excellent account of the concept of critical mass appears in Logan (1996); see also Bernstein (2002).

2.1 Neutron Mean Free Path

See Fig. 2.1. A thin slab of material of thickness s (ideally, one atomic layer) and cross-sectional area Σ is bombarded by incoming neutrons at a rate R_o neutrons/(m^{2}s).

Let the bulk density of the material be ρ gr/cm^3. In nuclear reaction calculations, however, density is usually expressed as a *number density* of nuclei in the material, that is, as the number of nuclei per cubic meter. In terms of ρ this is given by

$$n = 10^6\left(\frac{\rho N_A}{A}\right), \tag{2.1}$$

where N_A is Avogadro's number and A is the atomic weight of the material in grams per mole; the factor of 10^6 arises from converting cm^3 to m^3.

Assume that each nucleus presents a total reaction cross-section of σ square meters to the incoming neutrons. Cross-sections are usually measured in barns (bn), where 1 bn $= 10^{-28}$ m^2, a value characteristic of the physical sizes of nuclei. The first question we address is: "How many reactions will occur per second as a consequence of the bombardment rate R_o?" The volume of the slab is Σs, hence the number of nuclei contained in it will be Σsn. If each nucleus presents an effective cross-sectional area σ to the incoming neutrons, then the total area presented by all nuclei would be $\Sigma sn\sigma$. The *fraction* of the surface area of the slab that is available for reactions to occur is then $(\Sigma sn\sigma/\Sigma) = sn\sigma$. The rate of reactions R (reactions/s) can then sensibly be assumed to be the rate at which incoming particles bombard the surface area of the slab times the fraction of the surface area available for reactions:

$$\begin{pmatrix} \textit{reactions per} \\ \textit{second} \end{pmatrix} = \begin{pmatrix} \textit{incident neutron} \\ \textit{flux per second} \end{pmatrix} \begin{pmatrix} \textit{fraction of surface area} \\ \textit{occupied by cross} - \textit{section} \end{pmatrix},$$

or

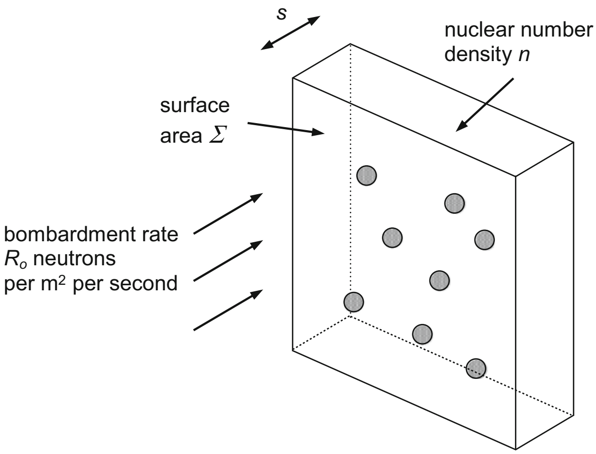

Fig. 2.1 Neutrons penetrating a thin target foil

$$R = (R_o \Sigma)\,(s\,n\,\sigma). \tag{2.2}$$

The *probability P* that an individual incident neutron precipitates a reaction is then

$$P_{react} = \frac{\left(\begin{array}{l} reactions \\ per\ \ second \end{array}\right)}{\left(\begin{array}{l} incident\ \ neutron\ \ flux \\ per\ second \end{array}\right)} = s\,n\,\sigma, \tag{2.3}$$

the same value as the fraction of the surface area available for reactions.

For the present purposes, it is more useful to work with the probability that a neutron will pass through the slab to escape out the back side:

$$P_{escape} = 1 - P_{react} = 1 - s\,n\,\sigma. \tag{2.4}$$

Now consider a block of material of macroscopic thickness x. As shown in Fig. 2.2, we can imagine this to comprise a large number of thin slabs each of thickness s placed back-to-back.

The number of slabs is x/s. If N_o neutrons are incident on the left side of the block, the number that would survive to emerge from the first thin slab would be N_oP, where P is the escape probability in (2.4). These neutrons are then incident on the second slab, and the number that would emerge unscathed from that passage would be $(N_oP)P = N_oP^2$. These neutrons would then strike the third slab, and so on. The number that survive passage through the entire block to escape from the right side would be $N_oP^{x/s}$, or

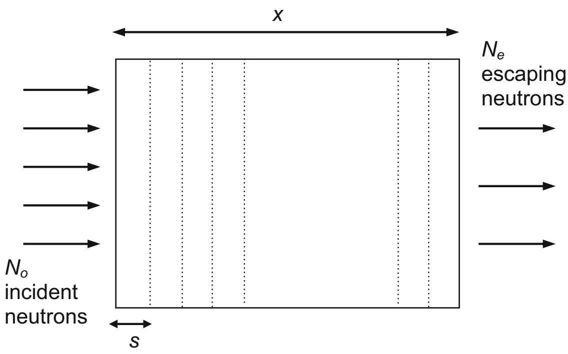

Fig. 2.2 Neutrons penetrating a thick target

$$N_{esc} = N_o(1 - sn\sigma)^{x/s}. \tag{2.5}$$

Define $z = -sn\sigma$. The number of neutrons that escape can then be written as

$$N_{esc} = N_o(1 + z)^{-\sigma n x/z} = N_o\left[(1 + z)^{1/z}\right]^{-\sigma n x}. \tag{2.6}$$

Now, ideally, s is very small, which means that $z \to 0$. The definition of the base of the natural logarithms, e, is $e = \lim\limits_{z \to 0} (1 + z)^{1/z}$, so we have

$$N_{esc} = N_o e^{-\sigma n x},$$

or

$$P_{\substack{direct\\eacape}} = \frac{N_{esc}}{N_o} = e^{-\sigma n x}. \tag{2.7}$$

Equation (2.7) is the fundamental neutron escape probability law. In words, it says that the probability that a bombarding neutron will pass through a slab of material of thickness x depends exponentially on the product of x, the number density of nuclei in the slab, and the reaction cross-section of the nuclei to incoming neutrons. If $\sigma = 0$, all of the incident particles will pass through unscathed. If $(\sigma\ n\ x) \to \infty$, *none* of the incident particles will make it through.

In practice, (2.7) is used to experimentally establish values for cross-sections by bombarding a slab of material with a known number of incident particles and then seeing how many emerge from the other side; think of (2.7) as effectively *defining* σ. Due to quantum-mechanical effects, the cross-section is *not* the geometric area of a nucleus.

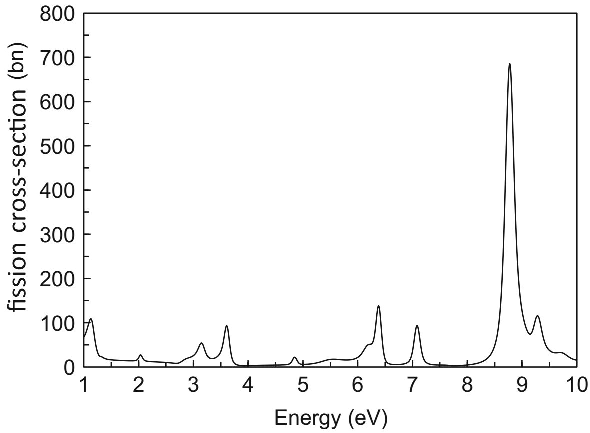

Fig. 2.3 Cross-section for the $^{235}U(n, f)$ reaction over the energy range 1–10 eV. At 0.01 eV (*off the left* end of the graph), the cross-section for this reaction is about 930 bn (Data from National Nuclear Data Center. See also Figs. 1.10 and 3.1)

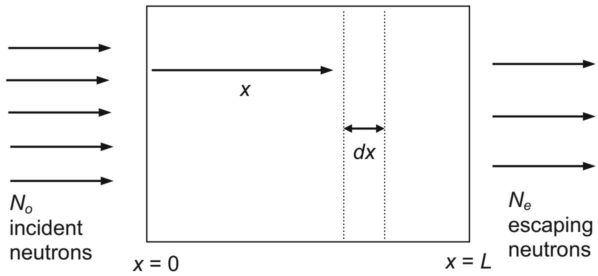

Fig. 2.4 Neutrons penetrating a target of thickness L

The total cross section had in mind here can be broken down into a sum of cross-sections for individual processes such as fission, elastic scattering, inelastic scattering, non-fission capture, etc.:

$$\sigma_{total} = \sigma_{fission} + \sigma_{\substack{elastic\\scatter}} + \sigma_{\substack{inelastic\\scatter}} + \sigma_{capture} + \dots. \tag{2.8}$$

In practice, cross-sections can depend very sensitively on the energy of the incoming neutrons; such energy-dependence plays a crucial role in the difference between how nuclear reactors and nuclear weapons function. As an example, Fig. 2.3 shows the variation of the fission cross-section for ^{235}U under neutron bombardment for neutrons in the energy range 1–10 eV; note the dramatic resonance effects at certain energies. The resonances show up even more dramatically in Fig. 3.1, which shows the fission cross-section for ^{235}U across many orders of magnitude of bombarding-neutron energy.

A very important result that derives from this escape-probability law is an expression for the *average* distance that an incident neutron will penetrate into the slab before being involved in a reaction. Look at Fig. 2.4, where we now have a slab of thickness L and where x is a coordinate for any position within the slab. Imagine also a small slice of thickness dx whose front edge is located at position x.

From (2.7), the probability that a neutron will penetrate through the entire slab to emerge from the face at $x = L$ is $P_{emerge} = e^{-\sigma nL}$. This means that the probability that a neutron will be involved in a reaction and *not* travel through to the face at $x = L$ will be $P_{react} = 1 - e^{-\sigma nL}$. It follows that if N_o neutrons are incident at the $x = 0$ face, then the number that will be consumed in reactions within the slab will be $N_{react} = N_o(1 - e^{-\sigma nL})$. We will use this result in a moment.

Also from (2.7), the number of neutrons that penetrate to distances x and $x + dx$ are given by

$$N_x = N_o e^{-\sigma n x} \tag{2.9}$$

and

$$N_{x+dx} = N_o e^{-\sigma n(x+dx)}. \tag{2.10}$$

Some of the neutrons that reach x will be involved in reactions before reaching $x + dx$, that is, $N_x > N_{x+dx}$. The number of neutrons consumed between x and $x + dx$, designated as dN_x, is given by

$$dN_x = N_x - N_{x+dx} = N_o e^{-\sigma n x}\left(1 - e^{-\sigma n\, dx}\right). \tag{2.11}$$

If dx is infinitesimal, then $(\sigma\, n\, dx)$ will be very small. This means that we can write $e^{-\sigma n\,(dx)} \sim 1 - \sigma\, n(dx)$, and hence write dN_x as

$$dN_x = N_o e^{-\sigma n x}(\sigma n\, dx), \tag{2.12}$$

a result equivalent to differentiating (2.7).

Now, these dN_x neutrons penetrated distance x into the slab before being consumed or diverted in a reaction, so the total travel distance accumulated by all of them in doing so would be $(x\, dN_x)$. The average distance that a neutron will travel before suffering a reaction is given by integrating accumulated travel distances over the length of the slab and then dividing by the number of neutrons consumed in reactions within the slab, $N_{react} = N_o(1 - e^{-\sigma nL})$ from above:

$$\langle x\rangle = \frac{1}{N_{react}}\int_0^L x\, dN_x = \frac{1}{N_o(1 - e^{-\sigma nL})}\int_0^L (N_o\sigma n) x e^{-\sigma n x} dx = \frac{1}{\sigma n}\left[\frac{1 - e^{-\sigma nL}(1 + \sigma nL)}{1 - e^{-\sigma nL}}\right]. \tag{2.13}$$

If we have a slab of infinite thickness, or, more practically, one such that the product σnL is large, then $e^{-\sigma nL}$ will be small and we will have

$$\langle x\rangle_{(\sigma n L)\, large} \rightarrow \frac{1}{\sigma n}. \tag{2.14}$$

This quantity is known as the *characteristic length* or *mean free path* for the particular reaction quantified by σ. This quantity will figure prominently throughout the remainder of this chapter. If it is computed for an individual cross section such as $\sigma_{fission}$ or $\sigma_{capture}$, one speaks of the mean free path for fission or capture. Such lengths are often designated by the symbol λ with a subscript indicating the type of reaction involved. As an example, consider fission in ^{235}U. The nuclear number density n is 4.794×10^{28} m^{-3}, and the fast-neutron cross section is $\sigma_f = 1.235$ bn $= 1.235 \times 10^{-28}$ m^2 (again averaged over the energy spectrum of fission-liberated neutrons). These numbers give $\lambda_f = 16.9$ cm, or about 6.65 in.

Finally, it should be emphasized that the derivations in this section do not apply to bombarding particles that are *charged*, in which case one has very complex ionization issues to deal with.

2.2 Critical Mass: Diffusion Theory

We now consider critical mass per se. Qualitatively, the concept of critical mass derives from the observation that some species of nuclei fission upon being struck by a bombarding neutron and consequently release secondary neutrons which can potentially go on to induce other fissions, resulting in a chain reaction. However, the development in the preceding section indicates that we can expect that a certain number of neutrons will reach the surface of the mass and escape, particularly if the mass is small. If the density of neutrons within the mass is increasing with time, *criticality* is said to obtain. Whether or not this condition is fulfilled depends on quantities such as the density of the material, its cross-section for fission, the number of neutrons emitted per fission, and the kinetic-energy spectrum of the neutrons. The number of neutrons emitted per fission is designated by the symbol ν.

A comment on ν is appropriate here. A given fission reaction will release some integer number of neutrons, which on rare occasion could in fact be zero. In carrying out calculations we will assume an operative *average* number of neutrons per fission. This will inevitably be a decimal number (see Table 2.1), but it should be borne in mind that a more advanced treatment would account for the spectrum of neutron-number emission for a given material when bombarded by neutrons of some spectrum of energies. There is almost no end to the increasingly complex levels of sophistication with which one can approach nuclear-weapons calculations.

To explore the time-dependence of the number of neutrons in a bomb core requires the use of time-dependent *diffusion theory*. In this section we use this theory to calculate the critical masses of so-called "bare" spherical assemblies of ^{235}U and ^{239}Pu, the main "active materials" used in fission weapons. The term "bare" is the technical terminology for an *untamped* core. More correctly, we compute critical *radii* which can be transformed into equivalent critical *masses* upon knowing the densities of the materials involved.

The development presented here is based on the derivation in Appendix G of a differential equation which describes the spatiotemporal behavior of the neutron number density N, that is, the number of neutrons per cubic meter within the core.

Table 2.1 Threshold critical radii and masses (untamped; $\alpha = 0$)

Quantity	Unit	^{235}U	^{239}Pu
A	gr/mol	235.04	239.05
ρ	gr/cm^3	18.71	15.6
σ_f	bn	1.235	1.800
σ_{el}	bn	4.566	4.394
ν	–	2.637	3.172
n	10^{22} cm^{-3}	4.794	3.930
$\lambda_{fission}$	cm	16.89	14.14
$\lambda_{elastic}$	cm	4.57	5.79
λ_{total}	cm	3.60	4.11
ε	–	1.467	1.090
τ	10^{-9} s	8.635	7.227
d	cm	3.52	2.99
R_O	cm	8.37	6.346
M_O	kg	45.9	16.7

The derivation in Appendix G depends upon on some material developed in Sect. 3.5; it is consequently recommended that both those sections be read in support of this one. Also, be sure not to confuse n and N; the former is the number density of fissile *nuclei* while the latter is the number density of *neutrons*; both play roles in what follows. Note also that the definition of N here differs from that in the previous section, where it represented a number of neutrons.

Before proceeding, an important limitation of this approach needs to be made clear. Following Serber (1992), I model neutron flow within a bomb core by use of a diffusion equation. A diffusion approach is appropriate if neutron scattering is isotropic. Even if this is not so, a diffusion approach will still be reasonable if neutrons suffer enough scatterings so as to effectively erase non-isotropic angular effects. Unfortunately, neither of these conditions are fulfilled in the case of a uranium core: Fast neutrons elastically scattering against uranium show a strong forward-peaked effect. Further, since the mean free path of a fast neutron in ^{235}U, about 3.6 cm, is only about half of the 8.4-cm bare critical radius (see Table 2.1), one cannot help but question the inherent accuracy of the diffusion equation developed in Appendix G. I adopt a diffusion-theory approach for a number of reasons, however. As much of the physics of this area remains classified or at least not easily accessible, we are forced to settle for an approximate model; diffusion theory has the advantage of being analytically tractable at an upper-undergraduate level. In actuality, however, we will see toward the end of this section that the predictions of diffusion theory compare very favorably with *experimentally-measured* critical masses. Also, as shown in Sect. 2.6, a comparison of critical radii as predicted by diffusion theory with those estimated from an openly-published more exact treatment shows that the two agree within about 5 % for the range of fissility parameters of interest here. We can thus be quite confident in a diffusion analysis despite its built-in approximations.

Central to any discussion of critical radius are the *fission* and *transport* mean free paths for neutrons, respectively symbolized as λ_f and λ_t. These are given by (2.14) as

$$\lambda_f = \frac{1}{\sigma_f n} \tag{2.15}$$

and

$$\lambda_t = \frac{1}{\sigma_t n}, \tag{2.16}$$

where σ_t is the so-called total or transport cross-section. If neutron scattering is isotropic (which we assume), the transport cross-section is given by the sum of the fission and elastic-scattering cross-sections:

$$\sigma_t = \sigma_f + \sigma_{el}. \tag{2.17}$$

We do not consider here the role of *in*elastic scattering, which affects the situation only indirectly in that it lowers the mean neutron velocity.[1]

For a spherical bomb core, the diffusion theory of Appendix G provides the following differential equation for the time rate of change of the neutron number density:

$$\frac{\partial N}{\partial t} = \frac{v_{neut}}{\lambda_f}(\nu - 1)N + \frac{\lambda_t v_{neut}}{3}\left(\nabla^2 N\right), \tag{2.18}$$

where v_{neut} is the average neutron velocity and the other symbols are as defined earlier. The first term on the right side of (2.18) corresponds to the growth in the

[1] Equations (2.15) and (2.16) assume that the product σnL is large; see the preceding section. For U-235, the values of the square bracket in (2.13) for $L = 10$ cm are 0.267 for $\sigma_{fiss} nL$ and 0.816 for $\sigma_{total} nL$, whereas the large-product approximation assumes that the square bracket will be equal to one. The approximation is more dramatic for the fission mean free path due to its small cross-section. It is thus somewhat surprising that diffusion theory ends up predicting critical masses in close accord with experimentally-measured values; see the discussion following Table 2.1 and Sect. 2.6. As for neglecting inelastic scattering, this is not as drastic as it may seem for a combination of reasons. What matters to the growth of the neutron population is the time τ that a neutron will typically travel before causing another fission; see (2.21). But, if one averages through the many resonance spikes in Fig. 3.1, the fission cross-section for ^{235}U (and ^{239}Pu as well) behaves approximately as $\sigma \sim 1/v_{neut}$, where v_{neut} is the neutron speed. This means that the mean free path for fission is proportional to v_{neut}, which, overall, makes τ independent of v_{neut}. Hence, if a neutron has been either elastically or inelastically scattered, the time for which it will typically travel before causing a subsequent fission is largely independent of its speed. It would then seem that one should add in the inelastic-scattering cross-section when forming the transport cross-section in (2.17). This is true, but another effect comes into play: Elastic scattering is not isotropic. This has the effect of somewhat lowering the effective value of the elastic scattering cross-section. For elements like uranium and plutonium, the two effects largely cancel each other, with the net result that (2.17) is a quite reasonable approximation. Details are given in the Appendix to Serber's *Primer*; see also Soodak et al. (1962), Chap. 3.

number of neutrons due to fissions, while the second term accounts for neutron loss by their flying out of a volume being considered.

Now, let r represent the usual spherical radial coordinate as measured from the center of the core. Upon assuming a solution for $N(t,r)$ of the form $N(t,r) = N_t(t)N_r(r)$, (2.18) can be separated as

$$\frac{1}{N_t}\left(\frac{\partial N_t}{\partial t}\right) = \left(\frac{\nu - 1}{\tau}\right) + \frac{D}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r}{\partial r}\right)\right], \tag{2.19}$$

where D is the so-called diffusion coefficient,

$$D = \frac{\lambda_t v_{neut}}{3}, \tag{2.20}$$

and where τ is the mean time that a neutron will travel before causing a fission:

$$\tau = \frac{\lambda_f}{v_{neut}}. \tag{2.21}$$

If the separation constant for (2.19) is defined as α/τ (that is, the constant to which both sides of the equation must be equal), then the solution for the time-dependent part of the neutron density emerges directly as

$$N_t(t) = N_o e^{(\alpha/\tau)\,t}, \tag{2.22}$$

where N_o represents the neutron density at the center of the core at $t = 0$. N_o would be set by whatever device is used to initiate the chain-reaction. We could have called the separation constant just α, but this form will prove more convenient for subsequent algebra. How α is determined is described following (2.31) below.

Equation (2.22) shows that the time-growth or decay (depending on the sign of α) of the neutron density is exponential. While our main concern for the present is with the *spatial* behavior of N, α will prove to be *very important* throughout this and subsequent sections. We will return to the issue of time-dependence in Sects. 2.4 and 2.5.

With the above definition of the separation constant, the radial part of (2.19) appears as

$$\left(\frac{\nu - 1}{\tau}\right) + \frac{D}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r}{\partial r}\right)\right] = \frac{\alpha}{\tau}. \tag{2.23}$$

The first and last terms in (2.23) can be combined; this is why the separation constant was defined as α/τ. On then dividing through by D, we find

$$\frac{1}{d^2} + \frac{1}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r}{\partial r}\right)\right] = 0, \tag{2.24}$$

where d is a characteristic length scale,

$$d = \sqrt{\frac{\lambda_f \lambda_t}{3(-\alpha + v - 1)}}. \tag{2.25}$$

Now define a new dimensionless coordinate x according as

$$x = \frac{r}{d}. \tag{2.26}$$

This brings (2.24) to the form

$$\frac{1}{N_r}\left[\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial N_r}{\partial x}\right)\right] = -1. \tag{2.27}$$

Aside from a normalization constant, the solution of this differential equation can easily be verified to be

$$N_r(r) = \left(\frac{\sin x}{x}\right). \tag{2.28}$$

To determine a critical radius R_C, we need a boundary condition to apply to (2.28). As explained in Appendix G, this takes the form

$$N(R_C) = -\frac{2\lambda_t}{3}\left(\frac{\partial N}{\partial r}\right)_{R_C} = -\frac{2\lambda_t}{3d}\left(\frac{\partial N}{\partial x}\right)_{R_C}. \tag{2.29}$$

On applying this to (2.28), one finds that the critical radius is given by solving the transcendental equation

$$x\cot(x) + \varepsilon x - 1 = 0, \tag{2.30}$$

where

$$\varepsilon = \frac{3d}{2\lambda_t} = \frac{1}{2}\sqrt{\frac{3\lambda_f}{\lambda_t(-\alpha + v - 1)}}. \tag{2.31}$$

With fixed values for the density and nuclear constants for some fissile material, Eqs. (2.30) and (2.31) contain two variables: the core radius r (through x) and the exponential factor α, and the two equations can be solved in two different ways. For both approaches, assume that we are working with material of "normal" density, which we designate as ρ_o. For the first approach, start by looking back to (2.22). If $\alpha = 0$, the neutron number density is neither increasing nor decreasing with time; in this case one has what is called *threshold criticality*. To determine the so-called threshold bare critical radius R_o, set $\alpha = 0$ in (2.25) and (2.31), set the density to ρ_o to determine n, λ_f, and λ_t, solve (2.30) for x, and then get r (=R_o) from (2.26). The corresponding threshold bare critical mass M_o then follows from $M_o = (4\pi/3)R_o{}^3\rho_o$. It is this mass that one usually sees referred to as *the* critical mass; this quantity will figure prominently in the discussion of bomb efficiency in Sects. 2.4 and 2.5.

The second solution begins with assuming that one has a core of some radius $r > R_o$. In this case one will find that (2.30) will be satisfied by some value of $\alpha > 0$, with α increasing as r increases. The rationale here is that since the middle term in (2.30), $\varepsilon x = 3r/2\lambda_t$, is independent of α, we can set r to some desired value; (2.30) can then be solved for x, which gives d from (2.26) and hence α from (2.25). If $\alpha > 0$, the reaction will in principle grow exponentially in time until all of the fissile material is used up, a situation known as "supercriticality."

To see why increasing the radius demands that α must increase, implicitly differentiate (2.30) to show that $d\varepsilon/dx = -(1/x)^2(1 - x^2/\sin^2 x)$. This expression demands $d\varepsilon/dx > 0$ for all values of x. From the definition of x, an increase in r (and/or in the density, for that matter) will cause x to increase. To keep (2.30) satisfied means that ε must increase, which, from (2.31), can happen only if α increases.

We come now to a very important point. This is that the condition for threshold criticality can in general be expressed as a constraint on the product ρr, where ρ is the mass density of the material and r is the core radius. The factor ε in (2.30) depends only on the cross-sections and secondary neutron number v, and so is independent of the density, Hence, for $\alpha = 0$, (2.30) will be satisfied by some unique value of x which will be characteristic of the material being considered. Since $x = r/d$ and d itself is proportional to $1/\rho$ [see (2.25)], we can equivalently say that the solution of (2.30) demands a unique value of ρr for a given combination of σ and v values. If R_o is the bare threshold critical radius for material of normal density ρ_o, then any combination of r and ρ such that $\rho r = \rho_o R_o$ will also be threshold critical, and any combination such that $\rho r > \rho_o R_o$ will be supercritical. For a sphere of material of mass M, the mass, density, and radius relate as $M \propto \rho r^3$, which means that the "criticality product" ρr can be written as $\rho r \propto M/r^2$. This relationship underlies the concept of *implosion* weapons. If a sufficiently strong implosion can be achieved, then one can get away with having less than a "normal" critical mass by starting with a sphere of material of normal density and crushing it to high density by implosion; such weapons inherently make more efficient use of available fissile material than those that depend on a non-implosive mechanism to assemble subcritical components. As described in Sect. 4.2, the implosion technique also helps to overcome predetonation issues with spontaneous fission. The key message from the present development, however, is that there is no *unique* critical mass for a given fissile material.

Table 2.1 shows calculated bare threshold critical radii and masses for U-235 and Pu-239.

Sources for the fission and elastic-scattering cross-sections appearing in the Table are given in Appendix B; the values quoted therein are used as they are averaged over the fission-energy spectra of the two nuclides. The v values were adopted from the Evaluated Nuclear Data Files (ENDF) maintained by the National Nuclear Data Center at Brookhaven National Laboratory (www.nndc.bnl.gov), and are for neutrons of energy 2 MeV, about the average energy of fission neutrons. The density for ^{235}U is (235/238) times the density of natural uranium, 18.95 gr/cm^3.

It is worth noting that the timescales involved in fission-bomb phenomena are remarkably brief: Neutrons travel for only $\tau \sim 1/100$ microsecond (=10 ns) between fissions!

Lest you think that publishing estimates of critical masses is engaging in revealing classified data, do not be alarmed; such estimates have been available in the public domain for decades. In a review article on fast reactors, Koch and Paxton (1959) quote a value of 48.7 kg for a spherical assembly of highly enriched uranium (93.9 % U-235), and 16.6 kg for a sphere of Pu-239. A 1963 publication of the United States Atomic Energy Commission, "Reactor Physics Constants," a compilation of data for nuclear engineers, lists the *experimentally determined* bare critical mass for 93.9 % U-235 as 48.8 kg, and that for Pu-239 as 16.3 kg. These values are close to those listed in Table 2.1. Estimating a critical mass is one of the *least* difficult parts of making a nuclear weapon.

Spreadsheet **CriticalityAnalytic.xls** allows users to carry out the above calculations for themselves. This spreadsheet is used for the calculations developed in this section as well as those in Sects. 2.3 and 2.4. In its simplest use—corresponding to this section—the user enters five parameters: the density, atomic weight, fission and scattering cross-sections of the core material, and the number of secondary neutrons per fission. The "Goal Seek" function then allows one to solve (2.30) and (2.31) for x (assuming $\alpha = 0$), from which the bare critical radius and mass are computed.

In practice, having available only a single critical mass of fissile material will not produce much of an explosion. The reason for this is that fissioning nuclei give rise to fission products with tremendous kinetic energies. The core consequently very rapidly—within microseconds—heats up and expands, causing its density to drop below that necessary to maintain criticality. In a core comprising only a single critical mass this will happen at the moment fissions begin, so the chain reaction will quickly fizzle as α falls below zero. To get an explosion of appreciable efficiency, one must start with more than a single critical mass of fissile material or implode an initially subcritical mass to high density before initiating the explosion. The issue of using more than one critical mass to enhance weapon efficiency is examined in more detail in Sects. 2.4 and 2.5. The effect of using a tamper is examined analytically in Sect. 2.3 and numerically in Sect. 2.5.

To determine the value of the exponential growth factor α for a core of more than one critical mass, it is necessary to solve Eqs. (2.26), (2.30), and (2.31) for α as described following (2.31) above. For the purpose of generating a seed value or simply for making quick estimates, however, an approximate value can be obtained as follows.

Equation (2.28) for the radial dependence of the neutron density appears as

$$N_r(r) = \left(\frac{\sin x}{x}\right). \tag{2.32}$$

As a *simplified* boundary condition, assume that $N_r(R_{core}) = 0$, that is, that the neutron density falls to zero at the edge of the core. This is a more restrictive condition that the true boundary condition, (2.29), and will lead to a larger bare threshold critical radius. In this case, (2.28) indicates that we must have $\sin(x) = 0$, or $R/d = \pi$. This will be the case whether a core is supercritical or just threshold critical. If we use subscripts "*core*" and "*o*" to designate a supercritical and bare-threshold core, respectively, then we must have

$$\frac{R_{core}}{d_{core}} = \frac{R_o}{d_o} \quad \Rightarrow \quad \left(\frac{R_o}{R_{core}}\right)^2 = \left(\frac{d_o}{d_{core}}\right)^2. \tag{2.33}$$

Substitute for d_o and d_{core} from (2.25), setting $\alpha = 0$ in the expression for d_o. The result can then be solved for α_{core}:

$$\alpha_{core} \sim (\nu - 1)\left[1 - (R_o/R_{core})^2\right]. \tag{2.34}$$

This result is expressed as an approximation as a reminder that it does not derive from the true boundary condition for neutron diffusion. This simplified boundary condition is explored further in Exercises 2.4 and 2.11.

As an example of how good an estimate (2.34) provides, we consider the Hiroshima *Little Boy* bomb core. We will see in the next section that this core comprised about 64-kg of ^{235}U. At a density of 18.71 gr/cm^3, this would correspond to $R_{core} = 9.347$ cm. With $R_o = 8.366$ cm and $\nu = 2.637$ from Table 2.1, (2.34) gives $\alpha_{core} \sim 0.326$. The true value for α for such a core is 0.255. The approximation is about 27 % high: not terribly accurate, but certainly in the ballpark (The *Little Boy* core was actually cylindrical, so we have taken some liberty in this example for sake of simplicity).

To close this section, it is interesting to look briefly at a famous *mis*calculation of critical mass on the part of Werner Heisenberg. At the end of World War II a number of prominent German physicists including Heisenberg were interned for 6 months in England and their conversations secretly recorded. This story is detailed in Bernstein (2001); see also Logan (1996) and Bernstein (2002). On the evening of August 6, 1945, the internees were informed that an atomic bomb had been dropped on Hiroshima and that the energy released was equivalent to about 20,000 tons of TNT (In actuality, the yield was about 13,000 tons, but this is not the problem with Heisenberg's calculation). Heisenberg then estimated the critical mass based on this number and a subtly erroneous model of the fission process.

We saw in Sect. 1.6 that complete fission of 1 kg of ^{235}U liberates energy equivalent to about 17 kt of TNT. Heisenberg predicated his estimate of the critical mass on assuming that about 1 kg of material did in fact fission. One kilogram of ^{235}U corresponds to about $\Omega \sim 2.56 \times 10^{24}$ nuclei. Assuming that on average $\nu = 2$ neutrons are liberated per fission, then the number of generations G necessary to fission the entire kilogram would be $\nu^G = \Omega$. Solving for G gives $G = \ln(\Omega)/\ln(\nu) \sim 81$, which Heisenberg rounded to 80. So far, this calculation is fine. He then argued that as neutrons fly around in the bomb core, they will randomly bounce between nuclei, traveling a mean distance λ_f before causing fissions; λ_f is the mean free path between fissions as in (2.15) above. From Table 2.1, $\lambda_f \sim 17$ cm for U-235, but, at the time, Heisenberg took $\lambda_f \sim 6$ cm. Since a random walk of G steps where each is of length λ_f will take one a distance $r \sim \lambda_f \sqrt{G}$ from the starting point, he estimated a critical radius of $r \sim (6\,cm)\sqrt{80} \sim 54$ cm. This would correspond to a mass of some 12,500 kg, roughly 13 tons! Given that only one kilogram of uranium fissioned, this would be a fantastically inefficient weapon. Such a bomb and its associated tamper, casing, and instrumentation would represent an unbearably heavy load for a World War II-era bomber.

The problem with Heisenberg's calculation was that he imagined the fission process to be created by a single neutron that randomly bounces throughout the bomb core, begetting secondary neutrons along the way. Further, his model is too stringent; there is no need for every neutron to cause a fission; many neutrons escape. In the days following August 6 Heisenberg revised his model, arriving at the diffusion theory approach described in this section.

2.3 Effect of Tamper

In the preceding section it was shown how to calculate the critical mass of a sphere of fissile material. In that development we neglected the effect of any surrounding *tamper*. In this section we develop a model to account for the presence of a tamper. The discussion here draws from the preceding section and from Serber (1992), Bernstein (2002), and especially Reed (2009).

The idea behind a tamper is to surround the fissile core with a shell of dense material, as suggested in Fig. 2.5. This serves two purposes: (1) It reduces the critical mass, and (2) It slows the inevitable expansion of the core, allowing more time for fissions to occur until the core density drops to the point where criticality no longer holds. The reduction in critical mass occurs because the tamper will reflect some escaped neutrons back into the core; indeed, the modern name for a tamper is "reflector," but I retain the historical terminology here. This effect is explored in this section. Estimating the distance over which an *untamped* core expands before criticality no longer holds is analyzed in Sect. 2.4. This slowing effect is difficult to model analytically, but can be treated approximately with a numerical model; this is done in Sect. 2.5.

The discussion here parallels that in Sect. 2.2. Neutrons that escape from the core will diffuse into the tamper. If the tamper material is not fissile, we can describe the behavior of neutrons within the tamper via (2.18) without the neutron-production term, that is, without the first term on the right side:

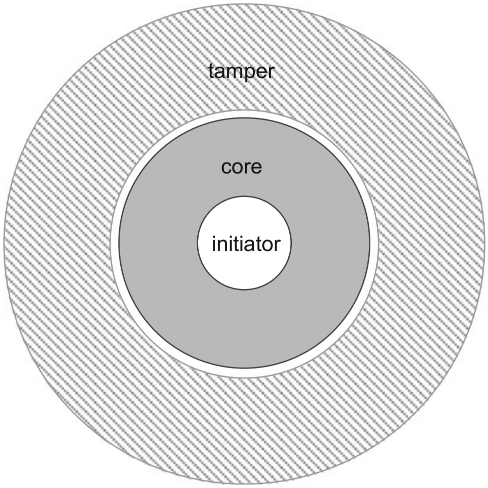

Fig. 2.5 Schematic illustration of a tamped bomb core

$$\frac{\partial N_{tamp}}{\partial t} = \frac{\lambda_{trans}^{tamp} v_{neut}}{3} \left(\nabla^2 N_{tamp}\right), \tag{2.35}$$

where N_{tamp} is the number density of neutrons within the tamper and λ_{trans}^{tamp} is their transport mean free path. v_{neut} is the average neutron speed within the tamper, which we will later assume for sake of simplicity to be the same as that within the core. We are assuming that the tamper does not capture neutrons; otherwise, we would have to add a term to (2.35) represent that effect.

Superscripts and subscripts *tamp* and *core* will be used liberally here as it will be necessary to join *tamper* physics to *core* physics via suitable boundary conditions. As was done in Sect. 2.2, take a trial solution for N_{tamp} of the form $N_{tamp}(t,r) = N_t^{tamp}(t)\, N_r^{tamp}(r)$, where $N_t^{tamp}(t)$ and $N_r^{tamp}(r)$ are respectively the time-and space dependences of N_{tamp}; r is the usual spherical radial coordinate measured from the center of the core. Upon substituting this into (2.35) we find, in analogy to (2.19),

$$\frac{1}{N_t^{tamp}} \left(\frac{\partial N_t^{tamp}}{\partial t}\right) = \left(\frac{\lambda_{trans}^{tamp} v_{neut}}{3}\right) \frac{1}{N_r^{tamp}} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial N_r^{tamp}}{\partial r}\right)\right]. \tag{2.36}$$

Define the separation constant here to be δ/τ where τ is the mean time that a neutron will travel *in the core* before causing a fission, that is, as defined in (2.21):

$$\tau = \frac{\lambda_{fiss}^{core}}{v_{neut}}. \tag{2.37}$$

While it may seem strange to invoke a *core* quantity when dealing with diffusion in the *tamper*, this choice is advantageous in that the neutron velocity v_{neut}, which we assume to be the same in both materials, will cancel out in later algebra. This choice is *not* equivalent to assuming at the outset that the core and tamper separation constants are the same, as δ may be different from the exponential factor α of Sect. 2.2. However, we will find that boundary conditions demand that they too must be equal.

This choice of separation constant renders (2.36) as

$$\frac{1}{N_t^{tamp}}\left(\frac{\partial N_t^{tamp}}{\partial t}\right) = \left(\frac{\lambda_{trans}^{tamp} v_{neut}}{3}\right)\frac{1}{N_r^{tamp}}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r^{tamp}}{\partial r}\right)\right] = \frac{\delta}{\tau}. \tag{2.38}$$

The solution of (2.38) depends on whether δ is positive, negative, or zero; the latter choice corresponds to threshold criticality in analogy to $\alpha = 0$ in Sect. 2.2. The situations of practical interest will be $\delta \geq 0$, in which case the solutions have the form

$$N_{tamp} = \begin{cases} \dfrac{A}{r} + B & (\delta = 0) \\ e^{(\delta/\tau)t}\left\{ A\dfrac{e^{r/d_{tamp}}}{r} + B\dfrac{e^{-r/d_{tamp}}}{r} \right\} & (\delta > 0), \end{cases} \tag{2.39}$$

where A and B are constants of integration (different for the two cases), and where

$$d_{tamp} = \sqrt{\frac{\lambda_{trans}^{tamp}\lambda_{fiss}^{core}}{3\delta}}. \tag{2.40}$$

The situation we now have is that the neutron density in the core is described by (2.22) and (2.28) as

$$N_{core} = A_{core} e^{(\alpha/\tau)t}\frac{\sin(r/d_{core})}{r}, \tag{2.41}$$

with d_{core} given by (2.25):

$$d_{core} = \sqrt{\frac{\lambda_{fiss}^{core}\lambda_{trans}^{core}}{3(-\alpha + \nu - 1)}}, \tag{2.42}$$

while the neutron density in the tamper is given by (2.39) and (2.40).

The question at this point is: "What boundary conditions apply in order that we have a physically reasonable solution?" Let the core have radius R_{core} and let the outer radius of the tamper be R_{tamp}; we assume that the inner edge of the tamper is snug against the core. First consider the core/tamper interface. If no neutrons are created or lost at this interface, then it follows that both the density and flux of neutrons across the interface must be continuous. That is, we must have

$$N_{core}(R_{core}) = N_{tamp}(R_{core}), \tag{2.43}$$

and, from (6.97) of Appendix G,

$$\lambda_{trans}^{core} \left(\frac{\partial N_{core}}{\partial r} \right)_{R_{core}} = \lambda_{trans}^{tamp} \left(\frac{\partial N_{tamp}}{\partial r} \right)_{R_{core}}. \tag{2.44}$$

Equation (2.44) accounts for the effect of any neutron reflectivity of the tamper via λ_{trans}^{tamp}. In writing (2.44), we have assumed that the speed of neutrons within the core and tamper is the same, and hence cancels.

In addition, we must consider what is happening at the outer edge of the tamper. If there is no "backflow" of neutrons from the outside, then the situation is analogous to the boundary condition of (2.29) that was applied to the outer edge of the untamped core:

$$N_{tamp}(R_{tamp}) = -\frac{2}{3} \lambda_{trans}^{tamp} \left(\frac{\partial N_{tamp}}{\partial r} \right)_{R_{tamp}}. \tag{2.45}$$

Applying (2.43), (2.44), and (2.45) to (2.39), (2.40), (2.41), and (2.42) results, after some algebra, in the following constraints:

$$\begin{gathered} \left[1 + \frac{2R_{thresh}\lambda_{trans}^{tamp}}{3R_{tamp}^2} - \frac{R_{thresh}}{R_{tamp}} \right] \left[\left(\frac{R_{thresh}}{d_{core}} \right) \cot \left(\frac{R_{thresh}}{d_{core}} \right) - 1 \right] \\ + \frac{\lambda_{trans}^{tamp}}{\lambda_{trans}^{core}} = 0, \quad (\delta = 0) \end{gathered} \tag{2.46}$$

and, for $\delta > 0$,

$$e^{2(x_{ct}-x_t)} \left[\frac{x_c \cot x_c - 1 - \lambda (x_{ct} - 1)}{R_{tamp} + 2\lambda_{trans}^{tamp}(x_t - 1)/3} \right] = \left[\frac{x_c \cot x_c - 1 + \lambda (x_{ct} + 1)}{R_{tamp} - 2\lambda_{trans}^{tamp}(x_t + 1)/3} \right], \tag{2.47}$$

where

$$\left. \begin{aligned} x_{ct} &= R_{core}/d_{tamp} \\ x_c &= R_{core}/d_{core} \\ x_t &= R_{tamp}/d_{tamp} \\ \lambda &= \lambda_{trans}^{tamp}/\lambda_{trans}^{core} \end{aligned} \right\}. \tag{2.48}$$

It is also necessary to demand that $\alpha = \delta$, as otherwise the fact that (2.43), (2.44), and (2.45) must also hold as a function of *time* would be violated. Some comments on these results follow.

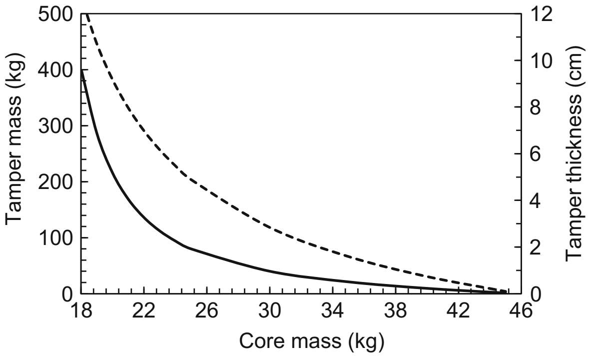

Fig. 2.6 Mass (kg; *solid curve, left scale*) and thickness (cm; *dashed curve, right scale*) of a snugly-fitting tamper of tungsten-carbide ($A = 195.84$ gr/mol, $\rho = 14.8$ gr/cm^3, $\sigma_{elastic} = 6.587$ bn) which will just render threshold critical a given core mass of pure ^{235}U. The untamped critical mass of ^{235}U is about 45.9 kg (Table 2.1)

1. Equation (2.46) corresponds to *tamped threshold criticality*, where $\alpha = \delta = 0$. Once values for the d's and λ's are given, there are two ways to use this expression:

   (a) If a core mass which is bare-threshold *sub*-critical is specified, use its radius as R_{thresh} and solve (2.46) for R_{tamp}, the tamper outer radius which will just render the core critical. The tamper mass can then be determined from the two radii; see Fig. 2.6.

   (b) If on the other hand R_{tamp} is specified, solve (2.46) for R_{thresh}, the radius of a core which would just be critical for the specified tamper outer radius. This can be a handy calculation if the size of your bomb is limited in advance by some condition such as the diameter of a missile tube.

2. To use (2.47) and (2.48): Refer to case 1(b) above, where R_{thresh} is determined for a given value of R_{tamp}. Keep R_{tamp} fixed to that value. Now choose a core radius $R_{core} > R_{thresh}$ to use in (2.47) and (2.48). This means that for the chosen value of R_{tamp}, you will have a number C (>1) of tamped threshold critical masses for your bomb core: $C = (R_{core}/R_{thresh})^3$. Then solve (2.47) numerically for δ (=α), which enters the d's and x's of (2.47) and (2.48) through (2.40) and (2.42).

The value of knowing α will become clear when the efficiency and yield calculations of Sects. 2.4 and 2.5 are developed; for the present, our main concern is with R_{thresh}.

A special-case application of (2.46) can be used to get a sense of how dramatically the presence of a tamper decreases the threshold critical mass. Suppose that the tamper is very thick, $R_{tamp} >> R_{thresh}$. In this case (2.46) reduces to

$$\left(R_{thresh}/d_{core}\right)\ \cot\left(R_{thresh}/d_{core}\right) = 1 - \left(\lambda_{trans}^{tamp}/\lambda_{trans}^{core}\right). \tag{2.49}$$

Now consider two sub-cases. The first is that the tamper is in fact a vacuum. Since empty space would have essentially zero cross-section for neutron scattering, this is equivalent to specifying $\lambda_{trans}^{tamp} = \infty$, in which case (2.49) becomes

$$(R_{thresh}/d_{core})\ \cot(R_{thresh}/d_{core}) = -\infty. \tag{2.50}$$

This can only be satisfied if

$$\left(\frac{R_{thresh}}{d_{core}}\right)_{vacuum\ tamper} = \pi. \tag{2.51}$$

The second sub-case is more realistic in that we imagine a thick tamper with a non-zero transport mean free path. For simplicity, assume that $\lambda_{trans}^{core} \sim \lambda_{trans}^{tamp}$, that is, that the neutron-scattering properties of the tamper are much like those of the core. In this case (2.49) becomes

$$(R_{thresh}/d_{core})\ \cot(R_{thresh}/d_{core}) = 0. \tag{2.52}$$

The solution here is

$$\left(\frac{R_{thresh}}{d_{core}}\right)_{\substack{thick\ tamper \\ finite\ cross-scetion}} = \frac{\pi}{2}, \tag{2.53}$$

exactly one-half the value of the vacuum-tamper case. To summarize: With an infinitely-thick tamper of finite transport mean free path, the threshold critical radius is one-half of what it would be if no tamper were present at all. A factor of two in radius means a factor of eight in mass, so the advantage of using a tamper is dramatic even aside from the issue of any retardation of core expansion. This factor of two in critical radius is predicated on an unrealistic assumption for the tamper thickness and so we cannot expect such a dramatic effect in reality, but we are about see that the effect is dramatic enough.

What sort of critical-mass reduction can one expect in practice? In a website devoted to design details of nuclear weapons, Sublette (2007) records that the Hiroshima *Little Boy* bomb used tungsten-carbide (WC) as its tamper material. Tungsten has five naturally-occurring isotopes, ^{180}W, ^{182}W, ^{183}W, ^{184}W, and ^{186}W, with abundances 0.0012, 0.265, 0.1431, 0.3064, and 0.2843, respectively. The KAERI table-of-nuclides site referenced in Appendix B gives elastic-scattering cross sections for the four most abundant of these as (in order of increasing weight) 4.369, 3.914, 4.253, and 4.253 bn. Neglecting the small abundance of ^{180}W, the abundance-weighted average of these is 4.235 bn. Adding the 2.352 bn elastic-scattering cross-section for ^{12}C gives a total of 6.587 bn; the cross-sections must be added, not averaged, since we are considering the tungsten-carbide molecules to be "single" scattering centers of atomic weight equal to the sum of the individual atomic weights for W and C, 183.84 + 12.00 = 195.84. The bulk density of tungsten-carbide is 14.8 g/cm^3. Figure 2.6 shows the tamper mass and corresponding outer radius necessary to just render critical a U-235 core of a given mass. As an example, a 25-kg core will be rendered just threshold critical when surrounded by a tamper of mass

80 kg and thickness 4.89 cm; the outer radius of the entire core/tamper assembly would be 11.7 cm.

Two spreadsheets are available for readers to run their own calculations along these lines. In **CriticalityAnalytic.xls**, users enter the core parameters for the calculations of Sect. 2.2 along with the density, atomic weight, scattering cross-section and outer radius of the tamper. The "Goal Seek" function is then used to determine the tamped threshold critical radius and mass from (2.46). Conversely, **ReflectedCore** allows the user to specify a bare-subcritical core mass and then, as in Fig. 2.6, determine the tamper mass necessary to just render the core threshold critical.

In the case of the Hiroshima *Little Boy* bomb, Sublette records that the tamper had a mass of about 311 kg and that its core comprised about 64 kg of ^{235}U in a cylindrical shape surrounded by a cylindrical WC tamper of diameter and length 13 in. (see also Coster-Mullen 2010). Assuming spherical geometry for simplicity, a 64-kg core at a density of 18.71 g/cm^3 would have an outer radius of 9.35 cm; a 311-kg tamper would then require an outer radius of about 18 cm. From Fig. 2.6, a tamper of this mass will render a core of mass ~19 kg threshold critical, so we can conclude that *Little Boy* utilized about (64/19) ~ 3.4 tamped threshold critical masses of fissile material.

Why was tungsten-carbide used as the *Little Boy* tamper material? As one of the purposes of the tamper is to briefly retard core expansion, denser tamper materials are preferable; tungsten-carbide is fairly dense and has a low neutron capture cross-section. In this sense it would seem that depleted uranium, which the Manhattan Project possessed in abundance, would be an ideal tamper material (*Depleted* is the term given to the uranium that remains after one has extracted some or all of its fissile U-235; one could equivalently say that the remains are enriched in U-238, but depleted is the preferred technical term). The reason that U-238 was not used may be that it has a fairly high spontaneous fission rate, about 675 per kilogram per second (see Sect. 4.2). Over the approximately 100 microseconds required to assemble the core of a Hiroshima gun-type bomb, a 300 kg depleted-U tamper would have a fairly high probability of suffering a spontaneous fission and hence of initiating a predetonation. Further, as discussed in Sect. 1.9, U-238 has a significant inelastic-scattering cross-section: fast neutrons striking it tend to be slowed so much that they become likely to be captured and hence lost to the possibility of being reflected back into the core. One of the best neutron-reflecting materials known is beryllium, which has a fission-spectrum averaged elastic scattering cross section of about 2.8 bn but an inelastic-scattering cross-section of only about 40 microbarns. Beryllium has an additional advantage in weapons design: for fission-energy neutrons it has a modest cross-section (~0.05 barns) for net production of neutrons via the reaction ^{9}Be $(n, 2n)$ ^{8}Be.

2.4 Estimating Bomb Efficiency: Analytic

Material in this section is adopted from Reed (2007).

In the preceding sections we examined how to estimate critical masses for bare and tamped cores of fissile material. The analysis in Sect. 2.2 revealed that the

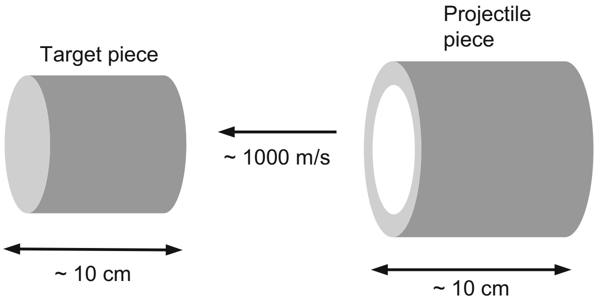

Fig. 2.7 Assembly timescale for a gun-type fission weapon

threshold bare critical mass of ^{235}U is about 46 kg. In Sect. 1.6 we saw that complete fission of 1 kg of ^{235}U liberates energy equivalent to that of about 17 kt of TNT. Given that the *Little Boy* uranium bomb that was dropped on Hiroshima used about 64 kg of ^{235}U and is estimated to have had an explosive yield of only about 13 kt, we can infer that it must have been rather inefficient. The purpose of this section is to explore what factors dictate the efficiency of a fission weapon and to show how one can estimate that efficiency.

This section is the first of several in this chapter and in Chap. 4 devoted to the question of weapon efficiency and yield. In this section these issues are examined purely analytically. The advantage of an analytic approach is that it is helpful for establishing a sense of how the efficiency depends on the parameters involved: The mass and density of the core and the values of various nuclear constants. However, conditions inside an exploding bomb core evolve very rapidly as a function of time, and this evolution cannot be fully captured with analytic approximations. To get a sense of the time-evolution of the process, one really needs to numerically integrate the core conditions as a function of time, tracking core size, expansion rate, pressure, neutron density, and energy release along the way. Such an analysis is the subject of the next section; these two sections therefore closely complement each other and should be read as a unit. Bomb efficiency and yield can also be affected by various phenomena that can trigger the chain-reaction before the weapon core has reached its fully assembled state; these issues are explored in Chap. 4.

In the present section we consider only *untamped* cores for sake of simplicity; a tamped core is simulated numerically in Sect. 2.5.

To begin, it is helpful to appreciate that the efficiency of a nuclear weapon involves three distinct time scales. The first is mechanical in nature: The time required to assemble the subcritical fissile components into a critical assembly before fission is initiated. In principle, this time can be as long as is desired, but in practice it is constrained by the occurrence of spontaneous fissions, which could lead to reaction-triggering stray neutrons during the assembly period.

What is the order of magnitude of the assembly time? In a simple "gun-type" bomb, the idea is that a "projectile" piece of fissile material is fired like a shell inside an artillery barrel toward a mating "target" piece of fissile material, as

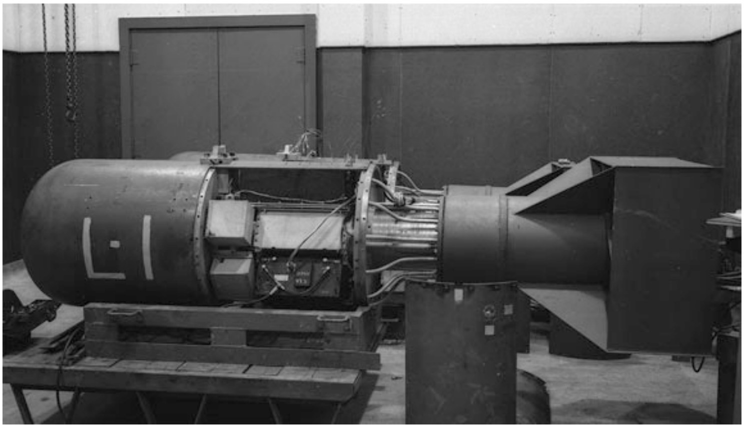

Fig. 2.8 *Little Boy* test units. *Little Boy* was 126 in. long, 28 in. in diameter, and weighed 9,700 lb when fully assembled (Sublette 2007) (Photo courtesy Alan Carr, Los Alamos National Laboratory)

sketched in Fig. 2.7. In World War II, the highest velocity that could be achieved for an artillery shell was about 1,000 m/s. If a projectile piece of length ~10 cm is shot toward a mating target piece at this speed, the time required for it to become fully engaged with the target piece from the time that the leading edge of the projectile meets the target piece will be ~ (10 cm)/(10^5 cm/s) ~ 10^{-4} s ~ 100 μs. This type of assembly mechanism was used in the Hiroshima *Little Boy* bomb, which explains its cylindrical shape as illustrated in the photograph in Fig. 2.8. As shown in the cross-sectional schematic in Fig. 2.9, the projectile piece was fired from the tail end of the bomb and traveled about 5 ft toward the nose.

As we will see in Sect. 4.2, spontaneous fission was not an issue for the *Little Boy* uranium core, but was such a problem with the *Trinity* and *Fat Man* plutonium cores as to necessitate development of the implosion mechanism for triggering those weapons. So far as the present section is concerned, however, the essential idea is that if the spontaneous fission probability can be kept negligible during the assembly time (which we assume), the efficiency of the weapon is dictated by the two other time scales.

The first of these other time scales is nuclear in nature. Once fission has been initiated, how much time is required for all of the fissile material to be consumed? This time we call $t_{fission}$. The other time scale is again mechanical. As soon as fissions have been initiated, the core will begin to expand due to the extreme gas pressure of the fission fragments. This expansion will lead after a time $t_{criticality}$ to loss of criticality, after which the reaction rate will diminish. Weapon efficiency will depend on how these times compare: If $t_{criticality} > t_{fission}$ then in principle all of the core material will undergo fission and the efficiency would be 100 %.

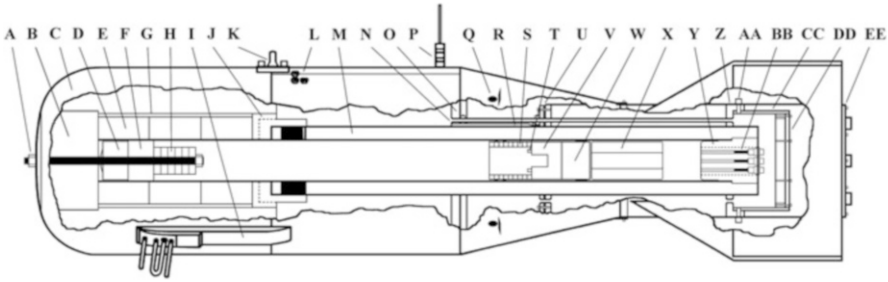

Fig. 2.9 Cross-section drawing of Y-1852 *Little Boy* showing major components. Not shown are radar units, clock box with pullout wires, barometric switches and tubing, batteries, and electrical wiring. Numbers in parentheses indicate quantity of identical components. Drawing is to scale. Copyright by and used with kind permission of John Coster-Mullen

(A) Front nose elastic locknut attached to 1-in. diameter Cd-plated draw bolt
(B) 15.125-in. diameter forged steel nose nut
(C) 28-in. diameter forged steel target case
(D) Impact-absorbing anvil with shim
(E) 13-in. diameter 3-piece WC tamper liner assembly with 6.5-in. bore
(F) 6.5-in. diameter WC tamper insert base
(G) 14-in. diameter K-46 steel WC tamper liner sleeve
(H) 4-in. diameter U-235 target insert discs (6)
(I) Yagi antenna assemblies (4)
(J) Target-case to gun-tube adapter with 4 vent slots and 6.5-in. hole
(K) Lift lug
(L) Safing/arming plugs (3)
(M) 6.5-in. bore gun
(N) 0.75-in. diameter armored tubes containing priming wiring (3)
(O) 27.25-in. diameter bulkhead plate
(P) Electrical plugs (3)
(Q) Barometric ports (8)
(R) 1-in. diameter rear alignment rods (3)
(S) 6.25-in. diameter U-235 projectile rings (9)
(T) Polonium-beryllium initiators (4)
(U) Tail tube forward plate
(V) Projectile WC filler plug
(W) Projectile steel back
(X) 2-lb Cordite powder bags (4)
(Y) Gun breech with removable inner breech plug and stationary outer bushing
(Z) Tail tube aft plate
(AA) 2.25-in. long 5/8–18 socket-head tail tube bolts (4)
(BB) Mark-15 Mod 1 electric gun primers with AN-3102-20AN receptacles (3)
(CC) 15-in. diameter armored inner tail tube
(DD) Inner armor plate bolted to 15-in. diameter armored tube
(EE) Rear plate with smoke puff tubes bolted to 17-in. diameter tail tube

Before proceeding with the detailed analysis, we pause to make a rough estimate of how much time is required to fission the entire core once the chain reaction has been initiated. In Sect. 2.2 we saw that once a neutron is emitted in a fission it will travel for only about 10 ns before causing another fission. Suppose that we have a core of mass M kilograms of fissile material of atomic weight A grams per mole. The number of nuclei N in the mass will be $N = 10^3 M N_A / A$. If ν neutrons are produced per generation, then the number of generations G that will be required to fission the entire mass will be $\nu^G = N$. At τ seconds per generation, the time to fission the entire mass will thus be $t_{fiss} \sim \tau\, G \sim \tau \ln(N)/\ln(\nu)$. For $M = 50$ kg of U-235 with $A = 235$ gr/mol, $\nu = 2.6$, and $\tau \sim 8 \times 10^{-9}$ s, $t_{fiss} \sim 0.5$ μs, an incredibly brief time. Even if only half of the neutrons cause fissions ($\nu = 1.3$), $t_{fiss} \sim 2$ μs. Such are the timescales of nuclear-weapon physics.

Once a chain reaction has been initiated, a bomb core will rapidly (within about a microsecond) heat up, melt, vaporize, and thereafter behave as an expanding gas with the expansion driven by the gas pressure in a thermodynamic $P\Delta V$ manner; we assume that the vast majority of energy liberated in fission reactions can be assumed to go into the form of kinetic energy of the fission products. Our approach to estimating yield and efficiency will be to use these concepts to establish the range of radius (and hence time) over which the core can expand before the expansion lowers the density of the fissile material to subcriticality. Some fissions will continue to happen after this time, but it is this "criticality shutdown timescale" that fundamentally sets the efficiency scale of the weapon.

On average, a neutron will cause another fission after traveling for a time given by $\tau = \lambda_f / v_{neut}$ where λ_f is the mean free path for fission and v_{neut} is the average neutron velocity; see (2.21). Inverting this, we can say that a single neutron will lead to a subsequent fission at a *rate* of $1/\tau$ per second:

$$rate\ \ of\ \ fissions\ \ per\ \ neutron = \frac{1}{\tau}. \tag{2.54}$$

The total number of fissions per second would be this rate times the number of neutrons in the core. The latter will be the product of the number density $N(t) = N_o e^{(\alpha/\tau)t}$ from (2.22) times the volume V of the core. Hence we have

$$fissions/sec = \left(\frac{N_o V}{\tau}\right) e^{(\alpha/\tau)\,t}. \tag{2.55}$$

In this expression, α is given by solving (2.25), (2.30), and (2.31) for the core at hand, and N_o is the central neutron density at $t = 0$; this will be set by the number of neutrons released by some "initiator" device. Recall that $\alpha = 0$ for threshold criticality, whereas $\alpha > 0$ for a core of more than one critical mass, an issue to which we will return shortly.

Equation (2.55) is actually more complicated than it appears because α and τ are functions of time. To appreciate this, consider a core of some general radius r and density ρ. As the core expands, r will increase while ρ decreases. The decreasing density will cause τ to increase; simultaneously, the discussion following (2.31) indicates that we can expect α to decrease. For sake of simplicity, we assume that α and τ remain constant; not accounting for changes in them will lead to overestimating the fission rate in (2.55). Since an exponential function is involved, the overestimate could be serious; indeed, we will see in Sect. 2.5 that direct use of our resulting yield formula, Eq. (2.67), can easily result in overestimating the efficiency by an order of magnitude. For the present, however, we will stick with the assumption of constant α and τ values since the purpose here is to get a sense of how the expected yield and efficiency depend in principle on the various factors involved. Section 2.5 discusses a simple refinement to (2.67) that eliminates much of the overestimate.

The time required to fission the entire core can be computed by demanding that the integral of (2.55) from time zero to time t_{fiss} to be equal to the total number of nuclei within the core, nV:

$$nV = \left(\frac{N_o V}{\tau}\right) \int_0^{t_{fiss}} e^{(\alpha/\tau)\,t} dt \quad \Rightarrow \quad t_{fiss} = \left(\frac{\tau}{\alpha}\right) \ln\left[\frac{\alpha n}{N_o}\right], \tag{2.56}$$

where it has been assumed that $e^{(\alpha/\tau)t} >> 1$ for the timescale of interest, an assumption to be investigated *a posteriori*.

What happens as the exploding core expands? Recall from Sect. 2.2 that the condition for criticality can be expressed as $\rho r \geq K$, where K is a constant characteristic of the material being used. We also saw that for a core of some mass M, $\rho r \propto M/r^2$. As the core expands the value of ρr must drop, and will eventually fall below the level needed to maintain criticality; one might call this situation "criticality shutdown," but the preferred technical term is *second criticality*.

For a single critical mass of normal-density material, second criticality will occur as soon as the expansion begins. One way to circumvent this is to provide a tamper to momentarily retard the expansion and so to give the reaction time to build up to a significant degree. Another is to start with a core of more than one critical mass of material of normal density, and this is what is assumed here. The effect of a tamper and the detailed time-evolution of $\alpha(t)$ and τ are dealt with in the following section.

To begin, assume that we have a core of C (>1) *untamped* threshold critical masses of material of normal density; the initial radius of such a core will be $r_i = C^{1/3} R_o$. We can then solve the diffusion-theory criticality Eqs. (2.30) and (2.31) for the value of α that just satisfies those equations upon setting the radius to be $C^{1/3}$ times the threshold critical radius listed in Table 2.1.

Now consider the energy released by fissions. If each fission liberates energy E_f, then the rate of energy liberation throughout the entire volume of the core will be, from (2.55),

$$\frac{dE}{dt} = \left(\frac{N_o V E_f}{\tau}\right) e^{(\alpha/\tau)\,t}. \tag{2.57}$$

Integrating this from time $t = 0$ to some general time t gives the energy liberated to that time:

$$E(t) = \left(\frac{N_o V E_f}{\tau}\right)\int_0^t e^{(\alpha/\tau)t} dt = \left(\frac{N_o V E_f}{\alpha}\right) e^{(\alpha/\tau)t}. \tag{2.58}$$

To determine the pressure within the core, we appeal to a result from thermodynamics. This is that pressure is given by $P(t) = \gamma\ U(t)$, where $U(t)$ is the *energy density* corresponding to $E(t)$: $U(t) = E(t)/V$. The value of the constant γ depends on whether gas pressure ($\gamma = 2/3$) or radiation pressure ($\gamma = 1/3$) is dominant; this issue is discussed below. Thus, the pressure will behave as

$$P(t) = \left(\frac{\gamma N_o E_f}{\alpha}\right) e^{(\alpha/\tau)t} = P_o\, e^{(\alpha/\tau)t}, \tag{2.59}$$

where $P_o = (\gamma\ N_o E_f/\alpha)$ is the central pressure at $t = 0$.

The equation of state $P(t) = \gamma\ U(t)$ deserves some comment. In the case of a gas of non-relativistic material particles each of mass m, this expression can be understood on the basis of simple kinetic theory where one considers the rate at which momentum is transferred to the walls of a container by collisions of the particles with the walls; this is covered in any freshman-level physics or chemistry text. The value of U is taken to be the total kinetic energy of all particles divided by the volume V of the container; each particle is assumed to have the same average value of the squared speed, $<v^2>$. γ emerges from this calculation as 2/3, with the factor of 2 arising from $K = m < v^2 >/2$, and the factor of 3 having its origin in the presumed isotropy of velocity components over three dimensions. To show that $\gamma = 1/3$ in the case of a gas of photons requires some background in the relativistic energy-momentum relationship of photons, but an ersatz justification for this value can be argued as follows. The non-relativistic result can be re-written as $P = \rho < v^2 >/3$ where ρ is the mass density of the gas. Photons do not have mass, but for the purposes of this quick argument we can use Einstein's famous $E = mc^2$ equation to assign the total energy of all photons an effective mass $m_{tot} = E_{tot}/c^2$. Hence the density becomes $\rho = m_{tot}/V = E_{tot}/(c^2 V)$, and so the pressure becomes $P = E_{tot} < v^2 >/(3c^2 V)$. Setting $< v^2 > = c^2$, $P = E_{tot}/3\,V$, or $P = U/3$ as advertised. In the case of a "gas" of uranium nuclei of standard density of that metal, radiation pressure dominates for per-particle energies greater than about 2 keV (see Exercise 2.14)

How does a gas of photons arise to give a radiation pressure in an exploding bomb core? Fission fragments are bare nuclei and so are highly electrically charged. As they decelerate, they naturally emit energy in the form of photons of wavelengths across the electromagnetic spectrum. Much of the energy released in a nuclear explosions in the form of gamma-rays and x-rays which ionize the surrounding air.

For simplicity, we model the bomb core as an expanding sphere of radius $r(t)$ with every atom in it moving radially outwards at speed v. Do not confuse this velocity with the average neutron speed v_{neut}, which enters into τ. If the sphere is of density $\rho(t)$ and total mass M, its total kinetic energy will be

$$K_{core} = \frac{1}{2}Mv^2 = \left(\frac{2\pi}{3}\right)\rho v^2 r^3. \tag{2.60}$$

Now invoke the work-energy theorem in its thermodynamic formulation $W = P(t)dV$, and equate the work done by the gas (or radiation) pressure in changing the core volume by dV over time dt to the change in the core's kinetic energy over that time:

$$P(t)\,\frac{dV}{dt} = \frac{dK_{core}}{dt}. \tag{2.61}$$

To formulate this explicitly, write $dK_{core}/dt = (2\pi/3)\rho r^3(2v\,dv/dt)$ from (2.60), put $dV/dt = 4\pi r^2(dr/dt)$, and incorporate (2.59) to give

$$\frac{dv}{dt} = \left(\frac{3P_o}{\rho r}\right) e^{(\alpha/\tau)\,t}. \tag{2.62}$$

To solve this for the radius of the core as a function of time we face the problem of what to do about the fact that both ρ and r are functions of time. We deal with this by means of an approximation.

Review the discussion regarding core expansion following (2.55) above. As the core expands, its density when it has any general radius r will be $\rho(r) = C\rho_o(R_o/r)^3$, and criticality will hold until such time as $\rho r = \rho_o R_o$, or, on eliminating ρ, $r = C^{1/2}R_o$. We can then define Δr, the range of radius over which criticality holds:

$$\Delta r = r_{\substack{second\\ criticlity}} - r_{initial} = \left(C^{1/2} - C^{1/3}\right)R_o, \tag{2.63}$$

a result we will use shortly.

Now, since $r_i = C^{1/3}R_o$, $(\rho r)_{\text{initial}} = C^{1/3}(\rho_o R_o)$. For $C = 2$ (for example), this gives $(\rho r)_{\text{initial}} = 1.26(\rho_o R_o)$. At second criticality we will have $(\rho r)_{\text{crit}} = (\rho_o R_o)$, so $(\rho r)_{\text{crit}}$ and $(\rho r)_{\text{initial}}$ do not differ very greatly. In view of this, we assume that the product ρr in (2.62) can be replaced with a mean value given by the average of the initial and final values of ρr:

$$\langle \rho r\rangle = \frac{1}{2}\left(1 + C^{1/3}\right)\rho_o R_o. \tag{2.64}$$

We can now integrate (2.62) from time $t = 0$ to some general time t to determine the velocity of the expanding core at that time:

$$v(t) = \left(\frac{3P_o}{\langle \rho r\rangle}\right)\int_0^t e^{(\alpha/\tau)\,t}dt = \left(\frac{3P_o\,\tau}{\langle \rho r\rangle\,\alpha}\right) e^{(\alpha/\tau)\,t}, \tag{2.65}$$

where it has again been assumed that $e^{(\alpha/\tau)t} >> 1$.

The stage is now set to compute the amount of time that the core will take to expand through the distance Δr of (2.63). Writing $v = dr/dt$ and integrating (2.65) from $r = r_i$ to $r_i + \Delta r$ for time $t = 0$ to $t_{criticality}$ gives

$$t_{crit} \sim \left(\frac{\tau}{\alpha}\right) \ln\left[\frac{\Delta r\, \alpha^2 \langle \rho r \rangle}{3 P_o \tau^2}\right] = \left(\frac{\tau}{\alpha}\right) \ln\left[\frac{\Delta r\, \alpha^3 \langle \rho r \rangle}{3 \gamma \tau^2 N_o E_f}\right], \tag{2.66}$$

again assuming $e^{(\alpha/\tau)t} >> 1$ and using $P_o = \gamma\, N_o E_f/\alpha$. Notice that we cannot determine t_{crit} without knowing the initial neutron density N_o. However, since t_{crit} depends logarithmically on N_o, the result is not terribly sensitive to the choice made for that number; presumably the *minimum* sensible value is given by assuming one initial neutron.

The energy yield Y is defined to be the energy released to time t_{crit}. From (2.58) and (2.66), this evaluates as

$$Y = \left(\frac{E_f N_o V}{\alpha}\right) \exp[(\alpha/\tau)\, t_{crit}] = \frac{\Delta r\, \alpha^2 \langle \rho r \rangle V}{3 \gamma \tau^2} = \frac{\Delta r\, \alpha^2 \langle \rho r \rangle M_{core}}{3 \gamma \tau^2 \rho}. \tag{2.67}$$

Efficiency is defined as the yield as a fraction of the energy which would be liberated if all of the nuclei in the core fissioned:

$$Efficiency = \frac{Y}{E_f\, n V} = \frac{\Delta r\, \alpha^2 \langle \rho r \rangle}{3 \gamma\, n \tau^2 E_f}. \tag{2.68}$$

Note that the yield and efficiency do not depend on the initial neutron density.

Now recall the earlier comments regarding how assuming constant values for α and τ will lead to overestimating the yield; this should be clear by examining (2.68). This tendency to overestimate will be somewhat offset by the fact that the core density ρ will drop as the core expands, so if we assume that ρ remains constant at its initial value during the expansion we would tend to underestimate the efficiency if α and τ did in fact remain constant. However, the efficiency depends on the squares of if α and τ and only on the first power of ρ, so the effects of changing α and τ will dominate over that of the changing density.

To help determine what value of γ to use, we can compute the total energy liberated to time t_{crit} as in (2.66), and then compute the average energy per particle by dividing by the number of nuclei in the core, nV. The result is

$$\begin{pmatrix} \textit{energy per nucleus} \\ \textit{at time } t_{crit} \end{pmatrix} = (\textit{efficiency}) E_f. \tag{2.69}$$

Even if the efficiency is very low, say 0.1 %, then for $E_f = 180$ MeV the energy per nucleus would be 180 keV, much higher than the ~2 keV per-particle energy where radiation pressure dominates over gas pressure. It would thus seem reasonable to take $\gamma = 1/3$ in most cases, although $\gamma = 2/3$ would be more appropriate early in the explosion process before much energy has been liberated.

Table 2.2 Criticality and efficiency parameters for $C = 1.5$, $E_f = 180$ MeV, $\gamma = 1/3$

Quantity	Unit	Physical meaning	^{235}U	^{239}Pu
$r_{initial}$	cm	Initial core radius	9.58	7.26
n	10^{22} cm^{-3}	Nuclear number density	4.794	3.930
α	–	Criticality parameter α	0.307	0.376
R_O	cm	Threshold critical radius	8.37	6.345
Δr	cm	Expansion distance to crit shutdown	0.67	0.51
Efficiency	%	Efficiency	1.03	1.29
$P(t_{crit})$	10^{15} Pa	Pressure at crit shutdown	4.73	4.87
Yield	kt	Explosive yield	12.4	5.6
t_{fiss}	μs	Time to fission all nuclei	1.67	1.12
t_{crit}	μs	Time to crit shutdown	1.54	1.04
N_o	neutron/m^3	Initial neutron density	271.8	622.9

Initial number of neutrons = 1
Secondary neutron energy = 2 MeV

Further, it can be shown by substituting (2.66) into (2.59) and (2.65) that the core velocity and pressure at the time of second criticality are given by

$$v\left(t_{crit}\right) = \frac{\alpha\,\Delta r}{\tau}, \tag{2.70}$$

and

$$P\left(t_{crit}\right) = \frac{\alpha^2\,\Delta r\,\langle\rho r\rangle}{3\,\tau^2}. \tag{2.71}$$

Curiously, this pressure does not depend on the value of γ.

Numbers for uranium and plutonium cores of $C = 1.5$ bare threshold critical masses appear in Table 2.2. Secondary neutrons are assumed to have $E = 2$ MeV, and it is assumed that the initial number of neutrons is one.

The timescales and pressures involved in the detonation process are extreme. Criticality shuts down after only 1–2 μs; a pressure of 10^{15} Pa is equivalent to about 10 *billion* atmospheres. Even though $t_{crit}/t_{fiss} \sim 0.9$, the efficiencies are low: small changes in an exponential argument lead to large changes in the results. In the case of ^{235}U, changing the initial number of neutrons to 1,000 changes the fission and criticality timescales by only about 10 %, down to 1.47 and 1.34 μs, respectively. Also, the comment following (2.56) that $e^{(\alpha/\tau)t}$ can be assumed to be much greater than unity for the timescale of interest can now be appreciated from the fact that $(\alpha/\tau)t_{crit} \sim 50$: $e^{50} \sim 10^{21}$.

Spreadsheet **CriticalityAnalytic.xls** carries out these efficiency and yield calculations for an untamped core. In addition to the parameters already entered for the calculations of the preceding two sections, the user need only additionally specify an initial number of neutrons, a value for γ, and the mass of the core. The "Goal Seek" function is then run a third time, to solve (2.30) and (2.31) for the value of α.

The spreadsheet then computes and displays quantities such as the expansion distance to second criticality, the fission and criticality timescales, the pressure within and velocity of the core at second criticality, and the efficiency and yield.

When applied to a bare 64 kg ^{235}U core ($C = 1.39$), **CriticalityAnalytic.xls** indicates that the yield will be about 6.3 kt; Eq. (2.63) indicates the core-expansion distance to second criticality is $\Delta r = 0.53$ cm. This yield figure is not directly comparable to the true ~13 kt yield of *Little Boy*, however, as that device was tamped; a more realistic simulation of *Little Boy* that incorporates a tamper is discussed in the next section.

How drastically does this analysis tend to overestimate efficiency? In Sect. 2.5 a program is described which carries out a time-dependent simulation of a tamped core. Applying this program to a ^{235}U core of mass 68.8 kg ($C = 1.5$, exactly) with *no* tamper gives a predicted yield of only about 0.29 kt, about 1/40 of the analytical result of 12.4 kt! The reason for this drastic discrepancy is explored further at the end of Sect. 2.5. In the meantime, there is a moral here: Beware of the danger of blindly applying an impressive-looking formula.

It is important to emphasize that the above calculations cannot be applied to a tamped core; that is, one cannot simply solve (2.47) and (2.48) for a core of some specified mass and tamper of some outer radius and use the value of α so obtained in the time and efficiency expressions established above. The reason for this has to do with the distance Δr through which the core expands before second criticality, Eq. (2.63) above. This expression derived from the fact that the criticality equation for the untamped case involves the density and radius of the core in the combination ρr; in the tamped case the criticality condition admits no such combination of parameters, so the subsequent calculations of criticality timescale and efficiency do not transform unaltered to using a tamped core. Efficiency in the case of a tamped core can only be established numerically.

To close this section, we compare the efficiency formula derived here to what was probably the first recorded formulation of the energy expected to be liberated by a nuclear weapon. This appeared in a document which has come to be known as the *Frisch-Peierls Memorandum*. This remarkable 7-page manuscript was prepared by Otto Frisch and Rudolf Peierls in March, 1940, to alert British government and military officials to the possibly of creating extremely powerful bombs based on utilizing a chain reaction in uranium; the title of their memo was "On the construction of a "super-bomb", based on a nuclear chain reaction in uranium." Their work was remarkably prescient: They discuss how a chain reaction could not happen in ordinary uranium, raised the possibility of bringing together two subcritical pieces of pure ^{235}U to create a supercritical mass, discussed how neutrons in cosmic radiation could be used to trigger the device, described how ^{235}U could be isolated by diffusion, and remarked that such a device would create significant radioactive fallout. Copies of the memorandum can be found in many online sites; a printed copy appears in Serber (1992). Readers are warned, however, that many reprintings contain various typographical errors. A detailed analysis of the physics involved in the memorandum is presented by Bernstein (2011), who also describes the errors.

The only mathematical formula appearing in the Frisch-Peierls memorandum is one for the expected yield of an untamped weapon. In terms of the notation of this book, this appears as

$$Y = 0.2 M_{core} (R_{core}/\tau)^2 \left(\sqrt{R_{core}/R_o} - 1 \right). \tag{2.72}$$

This looks almost completely unlike the present yield formula, (2.67). However, the latter can be transformed into (2.72) in a few steps via some sensible approximations. First, write the core volume or mass in (2.67) in terms of the core radius; also, set $\gamma = 1/3$. These manipulations give

$$Y = \frac{4\pi R_{core}^3 \alpha^2 \Delta r \langle \rho r \rangle}{3\tau^2}. \tag{2.73}$$

Now consider the product $\Delta r \langle \rho r \rangle$. From (2.63) and (2.64),

$$\Delta r \langle \rho r \rangle = \frac{1}{2} \left(C^{1/2} - C^{1/3} \right) \left(1 + C^{1/3} \right) \rho_o R_o^2. \tag{2.74}$$

In the second bracket in this expression, make the approximation that $C^{1/3} \sim 1$ to give $(1 + C^{1/3}) \sim 2$. This is reasonable as that bracket contains the *sum* of two similar quantities. We do not make this approximation within the first bracket, however, as it contains the *difference* of two similar quantities. In this case, extract a factor of $C^{1/3}$ from within the bracket and write it as $C^{1/3} = R_{core}/R_o$. The factor of $C^{1/6}$ remaining within the first bracket can then be written as $\sqrt{R_{core}/R_o}$. Thus, (2.74) becomes

$$\Delta r \langle \rho r \rangle \sim \left(\sqrt{R_{core}/R_o} - 1 \right) \rho_o R_o R_{core}. \tag{2.75}$$

On substituting this into (2.73), we can write $4\pi R_{core}^3 \rho_o / 3 = M_{core}$, and the yield becomes

$$Y \sim \alpha^2 M_{core} \left(R_o R_{core}/\tau^2 \right) \left(\sqrt{R_{core}/R_o} - 1 \right). \tag{2.76}$$

Finally, it is not unreasonable to make the approximation $R_{core} R_o \sim R_{core}^2$, and so arrive at

$$Y \sim \alpha^2 M_{core} (R_{core}/\tau)^2 \left(\sqrt{R_{core}/R_o} - 1 \right), \tag{2.77}$$

precisely the form of the Frisch-Peierls formula. They evidently took $\alpha^2 = 0.2$. On considering that we just found $\alpha_{\text{initial}} = 0.307$ for 1.5 critical masses of ^{235}U, their estimate was reasonable. Frisch and Peierls must have worked out the relevant diffusion and criticality theory “in the background” before composing their memorandum. Indeed, Peierls was a master theoretical physicist very familiar with diffusion problems; in Sect. 2.6 we will examine a formulation of criticality that he had published in the fall of 1939, several months before he teamed up with Frisch to produce their now-famous memorandum.

2.5 Estimating Bomb Efficiency: Numerical

In this section, a numerical approach to estimating weapon efficiency and yield is developed. The essential physics necessary for this development was established in the preceding three sections; what is new here is how that physics is used. The analysis presented in this section is adopted from Reed (2010).

The approach taken here is one of standard numerical integration: The parameters of a bomb core and tamper are specified, along with a timestep Δt. At each timestep the energy released from the core is computed, from which the acceleration of the core at that moment can be determined. The velocity and radius of the core can then be tracked until such time as second criticality occurs, after which the rate of fissions will drop drastically and very little additional energy will be liberated.

The integration process involves eight steps:

(i) Fundamental parameters are specified: The mass of the core, its atomic weight, initial density, and nuclear characteristics σ_f, σ_{el}, and ν. Similarly, the atomic weight, density, initial outer radius (and hence mass) and elastic-scattering cross-section of the tamper are specified. The energy release per fission E_f and gas/radiation pressure constant γ are set. A timestep Δt also needs to be chosen; this is discussed below. The initial number of neutrons also has to be specified as this value enters into the fission rate and energy release at each timestep in steps (iv) and (v) below.

(ii) Elapsed time, the speed of the core, and the total energy released are initialized to zero; the core radius is initialized according as its mass and initial density.

(iii) The exponential neutron-density growth parameter α is determined by numerical solution of (2.47) and (2.48).

(iv) The rate of fissions at a given time is computed from (2.55):

$$fissions/sec = \left(\frac{N_o V}{\tau}\right) e^{(\alpha/\tau)\, t}. \tag{2.78}$$

(v) The amount of energy released during time Δt is computed from (2.57):

$$\Delta E = \left(\frac{N_o V E_f}{\tau}\right) e^{(\alpha/\tau)\, t}(\Delta t). \tag{2.79}$$

(vi) The total energy released to time t is updated, $E(t) = E(t) + \Delta E$, and, from the discussion following (2.58), the pressure at time t is given by

$$P_{core}(t) = \frac{\gamma E(t)}{V_{core}(t)}. \tag{2.80}$$

I use the core volume in (2.80) on the rationale that the fission products which cause the gas/radiation pressure will likely largely remain within the core.

(vii) A key step is computing the change in the speed of the core over the elapsed time Δt due to the energy released during that time. In the discussion leading up to (2.61), this was approached by invoking the work-energy theorem:

$$P(t)\frac{dV_{core}}{dt} = \frac{dK_{core}}{dt}. \tag{2.81}$$

To improve the veracity of the simulation, it is desirable to account, at least in some approximate way, for the retarding effect of the tamper on the expansion of the core. To do this, I treat the dK/dt term in (2.81) as involving the speed of the core but with the mass as the sum of the core and tamper masses. The dV/dt term is taken to apply to the core only. I treat the tamper as being of constant density, which is effected by recomputing its outer radius at each step; the inner edge of the tamper is assumed to remain snug against the expanding core. With r as the radius and v the speed of the core, we have

$$\frac{\gamma E(t)}{V_{core}(t)}\left(\frac{dV_{core}}{dt}\right) = \frac{dK_{total}}{dt}$$
$$\Rightarrow \quad \frac{\gamma E(t)}{V_{core}(t)}\left(4\pi r^2 \frac{dr}{dt}\right) = \frac{1}{2}M_{total}\left(2v\frac{dv}{dt}\right),$$

from which we can compute the change in expansion speed of the core over time Δt as

$$\Delta v = \left[\frac{4\pi r^2 \gamma E(t)}{V_{core} M_{total}}\right](\Delta t). \tag{2.82}$$

With this, the expansion speed of the core and its outer radius are updated according as $v(t) = v(t) + \Delta v$ and $r(t) = r(t) + v(t)\Delta t$. The outer radius of the tamper is then adjusted on the assumption that its density and mass remain constant.

(viii) Increment time according as $t = t + \Delta t$ and return to step (iii) to begin the next timestep; continue until second criticality is reached when $\alpha = 0$. At the beginning of each timestep, update the core density to reflect its increased radius; this will concomitantly demand updating the nuclear number density of the core, its fission and transport mean-free paths, and the neutron travel-time between fissions, Eqs. (2.15), (2.16), and (2.21).

The assumption that the density of the tamper remains constant is probably not realistic. Nuclear engineers speak of the "snowplow" effect, where high-density tamper material piles up just ahead of the expanding core/tamper interface. But the point here is an order-of-magnitude pedagogical model.

This author has developed a FORTRAN program for carrying out this simulation; the code and an accompanying user manual are available at the companion website.

What of the timestep Δt ? In setting this, it is helpful to appreciate that it is not necessary to start a simulation at $t = 0$. From (2.79), little energy will be released while $(\alpha/\tau)t$ is small. An example using U-235 will help make this clear. With $\tau \sim 8.64 \times 10^{-9}$ s (Table 2.2). and, say, $\alpha \sim 0.5$, then $(\alpha/\tau) \sim 5.8 \times 10^{7}\ \text{s}^{-1}$. Starting a simulation at $t = 10^{-8}$ s should thus sacrifice no accuracy. However, the choice of a timestep Δt is a sensitive issue as the rate of energy release grows exponentially at later times. For a function of the form $y = \exp[(\alpha/\tau)t]$, the fractional change in y over a time Δt will be $dy/y = (\alpha/\tau)\ \Delta t$; to have dy/y be small suggests adopting a value of Δt no larger than the inverse of (α/τ), which is about 1.7×10^{-8} s. Consequently, all of the results described in what follows utilized a starting time of 10^{-8} s and $\Delta t = 5 \times 10^{-10}$ s; a run to a final time of 1.1 microseconds would then involve nearly 2,200 timesteps. With this value of Δt, $dy/y \sim 0.029$.

2.5.1 A Simulation of the Hiroshima Little Boy Bomb

Using the parameters for the *Little Boy* bomb given in Sect. 2.3 (64 kg core of radius 9.35 cm plus a 311 kg tungsten-carbide tamper of outer radius 18 cm), the following results were obtained with the author's program. The initial number of neutrons was set to be one.

Figure 2.10 shows the run of $\alpha(r)$ for this situation: it behaves linearly over the expansion of the core to second criticality at a radius of 12.29 cm. This represents an expansion distance of $\Delta r = 2.94$ cm from the initial core radius of 9.35 cm. As remarked earlier, for an *untamped* 64 kg core, (2.63) predicts a value for Δr of only 0.53 cm; a tamper significantly affects the expansion distance over which criticality holds.

Figures 2.11 and 2.12 show α, the core radius, the integrated energy release, and the fission rate and pressure as functions of time. While α decreases with increasing radius, the initial increase in radius is so slow that α remains close to its initial value until just before second criticality. The brevity and violence of the detonation are astonishing. The vast majority of the energy is liberated within an interval of about 0.1 μs. The pressure peaks at about 4.2×10^{15} Pa, or about 40 *billion* atmospheres, equivalent to about one-fifth of that at the center of the Sun. The fission rate peaks at about 3.5×10^{31} per second. Second criticality occurs at $t \sim 1.07$ μs, at which time

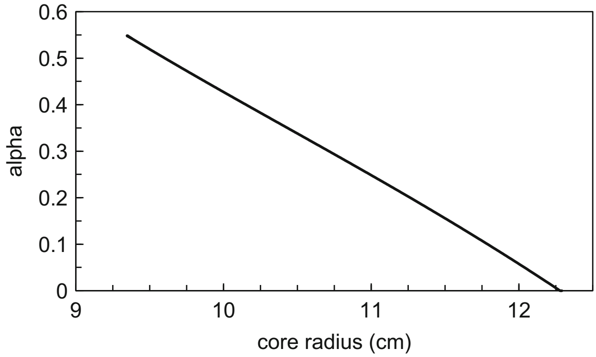

Fig. 2.10 Neutron density exponential growth parameter α vs. core radius for a simulation of the *Little Boy* bomb: 64 kg core plus 311 kg tungsten-carbide tamper. Second criticality occurs when the radius reaches 12.29 cm

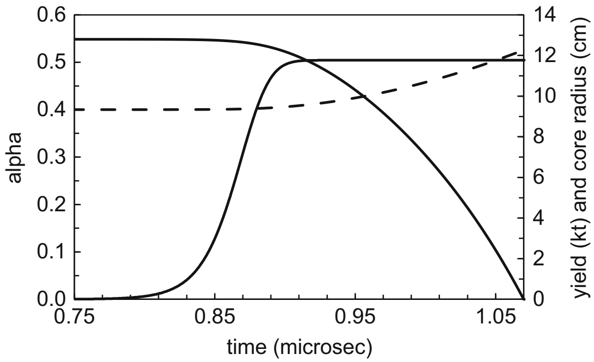

Fig. 2.11 Neutron density exponential growth parameter α (descending *solid curve, left scale*), core radius in cm (*dashed line, right scale*), and integrated energy release in kilotons (ascending *solid curve, right scale*) vs. time for a simulation of the *Little Boy* bomb. Final yield ~11.8 kt

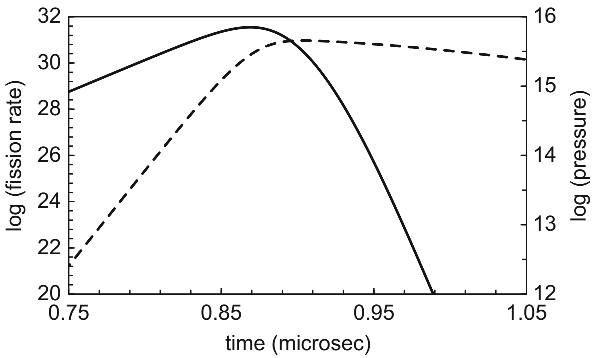

Fig. 2.12 Logarithm (base 10) of fission rate (*solid curve, left scale*) and logarithm of pressure (*dashed curve, right scale*) vs. time for a simulation of the *Little Boy* bomb

the core expansion velocity is about 270 km/s. These graphs dramatically illustrate what Robert Serber wrote in *The Los Alamos Primer*: "Since only the last few generations will release enough energy to produce much expansion, it is just possible for the reaction to occur to an interesting extent before it is stopped by the spreading of the active material."

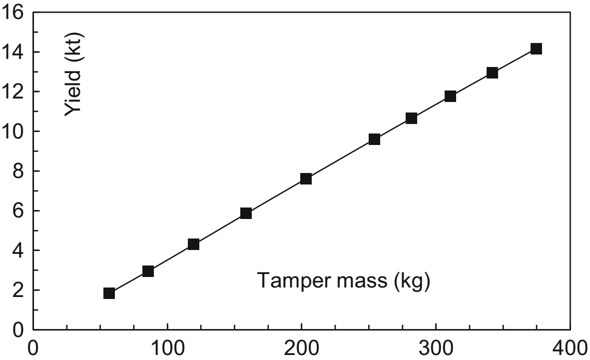

Fig. 2.13 Yield of a 64-kg U-235 core vs. mass of surrounding tungsten-carbide tamper. The *line* is interpolated. The *Little Boy* tamper mass was about 310 kg

The predicted yield of *Little Boy* from this simulation is 11.8 kt. Officially published yield estimates, are, however, quite variable. A 1952 Los Alamos report on the Hiroshima bombing, http://www.fas.org/sgp/othergov/doe/lanl/la-1398.pdf, gives a yield of 18.5 ± 5 kt. A later analysis published by Penney et al. (1970) reduced this estimate to ~12-kt, close to the present result. At a fission yield of 17.59 kt per kg of pure U-235 (180 MeV/fission), this represents an efficiency of just over 1 % for the 64-kg core. While some of this agreement must be fortuitous in view of the approximations incorporated into the present model, it is encouraging to see that it gives results of the correct order of magnitude. If the number of initial number of neutrons is increased to 100, the yield rises to 12.8 kt; 200 neutrons yields 13.0 kt.

Figure 2.13 shows how the simulated yield of a 64-kg core varies as a function of tamper mass; the points are the results of simulations for tampers of outer radii of 12, 13, ... 17, 17.5, 18, 18.5, and 19 cm. In the latter case the mass of the tamper would be about 375 kg, or just over 800 lb. A linear fit to Fig. 2.13 shows that the effect of increasing tamper can be expressed approximately as

$$\frac{d\ (Yield)}{dm_{tamp}} \sim 0.039\, \frac{kt}{kg}. \tag{2.83}$$

Of course, we would expect this curve to eventually level off to the theoretical maximum yield as the tamper mass becomes very great.

It was remarked in Sect. 2.4 that a simulation of an *untamped* U-235 core of mass 68.8 kg ($C = 1.5$ bare critical masses) results in a yield of only 0.287 kt, about 1/40 that predicted by Eq. (2.67). Why are these predictions so wildly discrepant? The culprit proves to be that in deriving (2.67), the criticality factor α was assumed to be constant. Look back to Fig. 2.11, which shows that once α begins to decline appreciably, very little additional yield occurs. In assuming that α remains constant until the core reaches second criticality, (2.67) consequently seriously overestimates the yield. Some numbers for the 68.8-kg simulation are instructive. The initial core radius in this case is 9.575 cm, and the initial value of α is 0.3062. The second-criticality radius is

10.245 cm ($\Delta r = 0.67$ cm), but by the time that the radius has expanded to only 9.607 cm (an increase of only 0.336 %), fully 90 % of the final yield has already been realized. By this point α has dropped by only about 4.5 % from its initial value, but the reaction has already begun shutting down (It is true that Fig. 2.11 is a tamped-core simulation, but the behavior of α is very similar for an untamped case).

Can (2.67) be modified to account for this problem? Here is a straightforward approach: When integrating (2.65) to determine the time to second criticality, replace the upper limit of integration $r_i + \Delta r$ with $r_i(1 + f)$, where f is the fractional increase in the core radius corresponding to that time at which you think the reaction begins shutting down; for example, for the above numbers, $f = 0.0034$ corresponds to 90 % energy release. Carrying out the integral shows that yield emerges as (2.67) except that the factor of Δr in the numerator is replaced with fr_i. For the present case of $r_i = 9.575$ cm and $f = 0.00336$, this modification predicts a yield of 0.597 kt, just twice the simulation result. There is obviously no preferred value of f to use, but this artifice removes much of the discrepancy in a straightforward way.

To close this section, a dose of perspective: Do not be *too* upset that Eq. (2.67) is not very accurate. It pertians to an untamped core, and any serious bomb-maker will incorporate a tamper. Ultimately, numerical analyses are what tell the tale of efficiency and yield. Also, treat this discrepancy as a valuable lesson. Analytic results have a compelling attractiveness and are powerful for getting a sense of how something depends on the parameters involved, but always be prepared to question the validity of underlying assumptions.

2.6 Another Look at Untamped Criticality: Just One Number

In Sect. 2.2, we saw that the criticality condition for threshold criticality ($\alpha = 0$) for an untamped core can be expressed as [Eqs. (2.30) and (2.31)]

$$x\cot(x) + \varepsilon x - 1 = 0, \tag{2.84}$$

with

$$\varepsilon = \frac{1}{2}\sqrt{\frac{3\lambda_f}{\lambda_t(\nu - 1)}} = \frac{1}{2}\sqrt{\frac{3\sigma_t}{\sigma_f(\nu - 1)}}. \tag{2.85}$$

Once the nuclear parameters σ_f, σ_{el}, and ν are set, (2.84) is solved numerically for x, from which the critical radius R follows from (2.26):

$$R = dx = \sqrt{\frac{\lambda_f \lambda_t}{3(\nu - 1)}}x = \frac{1}{n}\sqrt{\frac{1}{3\sigma_f\,\sigma_t(\nu - 1)}}x. \tag{2.86}$$

The critical radius is fundamentally set by σ_f, σ_{el}, ν, and n; our concern here will be with the first three of these variables.

Since these various quantities will be different for different fissile isotopes, it would appear that there is no general statement one can make regarding critical radii. However, σ_f, σ_{el}, and ν can be combined into one convenient dimensionless variable that dictates the critical radius in any particular case—the "just one number" of the title of this section.

As formulated, (2.84) and (2.85) are convenient in that both x and ε are dimensionless, but they are awkward in that ε is not bounded: If ν is very large, ε will approach zero, but as $\nu \to 1$, ε diverges to infinity. It would be handy to have some combination of σ_f, σ_{el}, and ν that is finitely bounded. Such a combination was developed by Peierls (1939) in a paper which was the first publication in English to explore what he termed "criticality conditions in neutron multiplication." He defined a dimensionless quantity ξ given by

$$\xi^2 = \frac{\sigma_f(\nu - 1)}{\sigma_{el} + \nu\sigma_f}. \tag{2.87}$$

For $1 \le \nu \le \infty$, $0 \le \xi \le 1$. Note that it is the elastic-scattering cross-section σ_{el} that appears in the denominator of this definition, not the transport cross-section $\sigma_t = \sigma_{el} + \sigma_f$.

If $(\nu - 1)$ is eliminated between (2.85) and (2.87), ε and ξ prove to be related as

$$\varepsilon = \sqrt{\frac{3}{4}\left(\frac{1}{\xi^2} - 1\right)}. \tag{2.88}$$

Similarly, if $(\nu - 1)$ is extracted from the definition of d in (2.86) and substituted into (2.87), then one finds

$$d = \sqrt{\frac{1}{3}\left(\frac{1}{\xi^2} - 1\right)}\lambda_t. \tag{2.89}$$

A general formulation of critical radii can now be made as follows: For a range of values of ξ between 0 and 1, (2.84) and (2.88) can be solved for x. For each solution, (2.86) and (2.89) show that the ratio of R to λ_t can be expressed purely as a function of ξ:

$$\frac{R}{\lambda_t} = x(\xi)\,d(\xi) = x(\xi)\sqrt{\frac{1}{3}\left(\frac{1}{\xi^2} - 1\right)}. \tag{2.90}$$

In other words, a graph of $x(\xi)\ d(\xi) \equiv R/\lambda_t$ vs. ξ can be used to immediately indicate the ratio of the untamped threshold critical radius to the transport mean free path for any combination of σ_f, σ_{el}, and ν values. The advantage of this approach is that the graph need only be constructed once.

Figure 2.14 shows R/λ_t as a function of ξ. For ^{235}U and ^{239}Pu, $\xi \sim 0.5084$ and 0.6221, and $R/\lambda_t \sim 2.33$ and 1.54, respectively. It is intuitively sensible that for small values of ξ (that is, for $\nu \to 1$), the critical radius will be large, and vice-versa.

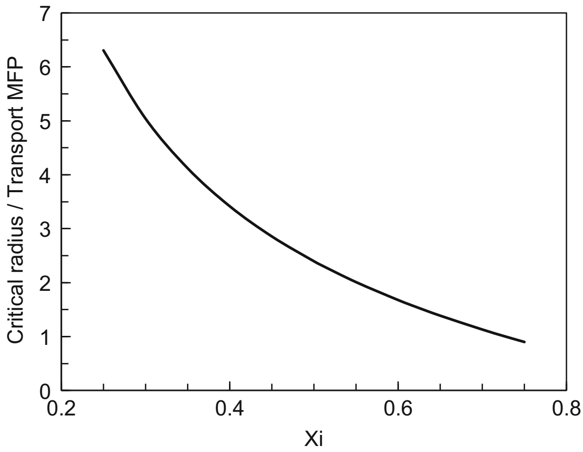

Fig. 2.14 Ratio of untamped threshold critical radius to transport mean free path as a function of Peierls' ξ parameter of (2.87)

An important aspect of Peierls' analysis is that it provides an independent check on the diffusion method of analyzing critical mass that has been used throughout this chapter. Peierls showed that his analysis led to approximate analytic solutions for the critical radius R in two limiting cases: $\xi \to 0$ and $\xi \to 1$. These are given by

$$\frac{1}{\beta R} \sim \begin{cases} 0.552\xi + 0.216\xi^2 & (\xi \to 0) \\ 0.78 - 1.02(1-\xi) & (\xi \to 1), \end{cases} \tag{2.91}$$

where

$$\beta = n\left(\sigma_{el} + \nu\sigma_f\right). \tag{2.92}$$

β is identical to the denominator of (2.87) but for a factor of the nuclear number density n.

βR can be expressed in terms of x and ξ through the following manipulations. First, from (2.88) and (2.89) we can write $d = 2\lambda_t\varepsilon/3$. With this result we can write $x = R/d$ as $x = 3R/(2\,\lambda_t\varepsilon)$. By eliminating $\sigma_f(\nu\text{-}1)$ between (2.85) and (2.87), we can show that $\lambda_t = 3/(4\beta\,\xi^{\,2}\varepsilon^{\,2})$. Substituting this result into the expression for x then shows that

$$\frac{1}{\beta R} = \frac{2\xi^2\varepsilon}{x}. \tag{2.93}$$

We can compare the results of Peierls' approach to those of diffusion analyses in much the same way as Fig. (2.14) was constructed: For a range of values of ξ between zero and one, solve (2.84) and (2.88) for x, which can be translated to $1/(\beta R)$ through (2.93) and then compared to the predictions of (2.91). Figure 2.15 shows the results of such an analysis for $0.1 \le \xi \le 0.9$. It is reassuring to see that the

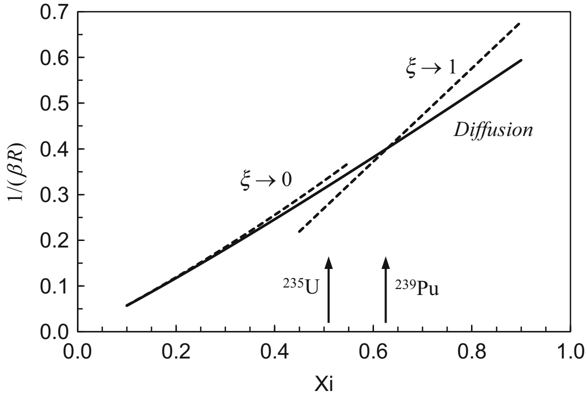

Fig. 2.15 $1/(\beta R)$ computed with Peierls' limiting expressions of (2.91) (*dashed lines*) and via diffusion analysis (*solid line*) vs. Peierls' ξ parameter of (2.87). Adapted from Reed (2008)

results of the diffusion analysis do not differ markedly from those of Peierls, This is particularly true for small values of ξ, where the core will be large and we expect diffusion theory to be accurate; curiously, the diffusion approach *overestimates* the critical radius for $\xi \to 1$. For ^{235}U, (2.91) predicts critical radii of 7.93 cm ($\xi \to 0$) and 9.57 cm ($\xi \to 1$). These radii correspond to masses of 39–69 kg, which bracket the diffusion result of 46 kg. For ^{239}Pu the Peierls-estimates masses evaluate as 13.4 and 17.0 kg, which again bracket the diffusion result of 16.7 kg.

2.7 Critical Mass of a Cylindrical Core (Optional)

In Sect. 2.4 it was pointed out that the core of the *Little Boy* bomb was cylindrical in shape. It is consequently natural to wonder how that shape affects the calculation of critical mass presented in Sect. 2.2, which was done for a spherical core.

It is difficult to analyze the situation for a cylindrical core because the boundary condition (2.29) that was used for the neutron diffusion equation in the spherical case,

$$N(R_C) = -\frac{2\lambda_t}{3}\left(\frac{\partial N}{\partial r}\right)_{R_C}, \tag{2.94}$$

is not easily generalized to the cylindrical case. However, if we are willing to admit a cruder boundary condition, much headway can be made with the cylindrical case. This is done in this section. This derivation can be considered optional as we consider only spherical cores in any subsequent section where the core geometry is relevant, such as in the analysis of predetonation in Chap. 4.

The cruder boundary condition is that the neutron density N is assumed to drop to zero at the surface for a cylinder of critical size. This situation is considered for a sphere and a cube in Exercises 2.11 and 2.4, respectively, where it is found that the critical volumes are

$$V_{sphere} = \left(\frac{4}{3}\pi^4\right)d^3 = 129.9\,d^3 \tag{2.95}$$

and

$$V_{cube} = \left(3^{3/2}\pi^3\right)d^3 = 161.1\,d^3, \tag{2.96}$$

where d is the characteristic length (2.25), which for threshold criticality ($\alpha = 0$) has the form

$$d = \sqrt{\frac{\lambda_f\,\lambda_t}{3\,(\nu - 1)}}. \tag{2.97}$$

For ^{235}U, d is about 3.5 cm.

Before beginning the formal solution, a few remarks on the diffusion equation in cylindrical coordinates are appropriate. Reactor engineers have been dealing with neutron fluxes in cylindrical geometries for decades, so the mathematics here, which involves so-called *Bessel functions*, is not new. Bessel functions show up in a number of areas of mathematical physics such as quantum mechanics (the infinite cylindrical quantum well), acoustics (vibrations of drumheads), optics (diffraction through circular apertures) and electromagnetism (waveguides). Their appearance in criticality calculations illustrates connections between very different areas of physics.

We begin with the general neutron diffusion equation of Appendix G:

$$\frac{\partial N}{\partial t} = \frac{v_{neut}}{\lambda_f}(\nu - 1)N + \frac{\lambda_t v_{neut}}{3}\left(\nabla^2 N\right). \tag{2.98}$$

The goal here is to apply this to the neutron population within a cylinder of radius R and length L as illustrated in Fig. 2.16. The bottom of the cylinder is imagined to by lying in the xy plane, with its center at $(x, y) = (0,0)$.

The separation of the diffusion equation into time and space-dependent parts proceeds as in Sect. 2.2; the temporal dependence is not of interest to us here as we seek to determine the threshold-critical condition. The spatial part of the neutron density N will be a function of the cylindrical coordinates (ρ, ϕ, z), and is assumed to be separable as

$$N_{\rho\phi z}(\rho, \phi, z) = N_\rho(\rho)N_\phi(\phi)N_z(z). \tag{2.99}$$

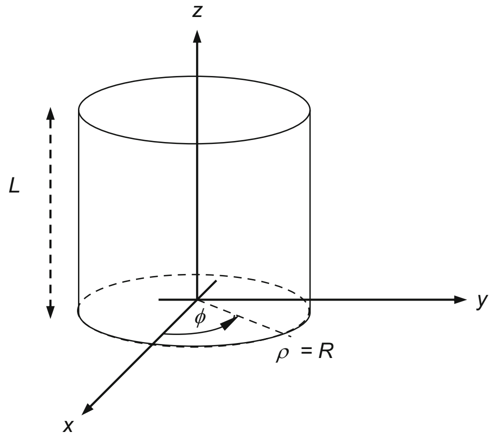

Fig. 2.16 Cylindrical core of radius R and height L

The Laplacian operator in cylindrical coordinates is

$$\nabla^2 N_{\rho\phi z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial N_{\rho\phi z}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 N_{\rho\phi z}}{\partial \phi^2} + \frac{\partial^2 N_{\rho\phi z}}{\partial z^2}. \tag{2.100}$$

On substituting (2.99) and (2.100) into (2.98) and dividing through by $N_{\rho\phi z}$, the spatial part of the diffusion equation appears, in analogy to (2.24), as

$$\frac{1}{d^2} + \frac{1}{N_\rho \rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial N_\rho}{\partial \rho}\right) + \frac{1}{\rho^2 N_\phi}\frac{\partial^2 N_\phi}{\partial \phi^2} + \frac{1}{N_z}\frac{\partial^2 N_z}{\partial z^2} = 0. \tag{2.101}$$

The solution of (2.101) proceeds as does that of any separated differential equation. First, take the z-term to the right side of the equal sign:

$$\frac{1}{d^2} + \frac{1}{N_\rho \rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial N_\rho}{\partial \rho}\right) + \frac{1}{\rho^2 N_\phi}\frac{\partial^2 N_\phi}{\partial \phi^2} = -\frac{1}{N_z}\frac{\partial^2 N_z}{\partial z^2}. \tag{2.102}$$

Since z is independent of ρ and ϕ, (2.102) can be true only if both sides are equal to a constant. This separation constant is traditionally defined to be $+k_z^2$, that is,

$$\frac{1}{N_z}\frac{\partial^2 N_z}{\partial z^2} = -k_z^2. \tag{2.103}$$

The solution of this differential equation is

$$N_z(z) = Ae^{\imath k_z z} + Be^{-\imath k_z z}, \tag{2.104}$$

a result to which we will return presently.

Return to the left side of (2.102) and equate it to $+k_z^2$. Then multiply through by ρ^2 to clear that factor from the denominator of the ϕ term, move the ϕ term to the right side, and move the resulting $k_z^2\rho^2$ term to the left side to effect another level of separation:

$$\frac{\rho}{N_\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial N_\rho}{\partial\rho}\right)+\left(\frac{1}{d^2}-k_z^2\right)\rho^2=-\frac{1}{N_\phi}\frac{\partial^2 N_\phi}{\partial\phi^2}. \tag{2.105}$$

As with (2.102), (2.105) can only be true if both sides are equal to a constant, which can be written as $+k_\phi^2$. This renders the ϕ-dependence as

$$\frac{1}{N_\phi}\frac{\partial^2 N_\phi}{\partial\phi^2}=-k_\phi^2, \tag{2.106}$$

which has the solution

$$N_\phi(\phi)=Ce^{\imath k_\phi\phi}+De^{-\imath k_\phi\phi}. \tag{2.107}$$

Now return to the left side of (2.105). Equate it to k_ϕ^2 and expand the derivative. This gives the radial dependence of the neutron density as

$$\rho^2\frac{\partial^2 N_\rho}{\partial\rho^2}+\rho\left(\frac{\partial N_\rho}{\partial\rho}\right)+\left[\left(\frac{1}{d^2}-k_z^2\right)\rho^2-k_\phi^2\right]N_\rho=0. \tag{2.108}$$

If we now define

$$\kappa^2=\left(\frac{1}{d^2}-k_z^2\right) \tag{2.109}$$

and establish the dimensionless variable

$$x=\kappa\rho, \tag{2.110}$$

(2.108) becomes

$$x^2\frac{\partial^2 N_x}{\partial x^2}+x\left(\frac{\partial N_x}{\partial x}\right)+\left(x^2-k_\phi^2\right)N_x=0. \tag{2.111}$$

(Note that x here is *not* the Cartesian-coordinate x, it is just a variable). Equation (2.111) is *Bessel's equation of argument x and order k_ϕ*. Solutions to this physically important differential equation can be found in any good textbook on mathematical physics. However, we will not need to examine the detailed solutions; our interest is in satisfying the boundary condition that at the surface of the cylinder, $N(\text{edge})=0$.

Consider first the z-direction. In (2.104), we must demand $N_z(0)=0$ and $N_z(L)=0$. The first of these demands that $A+B=0$, or $B=-A$; this gives

$$N_z(z) = A\left(e^{\imath k_z z} - e^{-\imath k_z z}\right), \tag{2.112}$$

which is equivalent to

$$N_z(z) = 2\imath A \sin\left(k_z z\right). \tag{2.113}$$

Now consider the condition $N_z(L) = 0$ applied to (2.113). This requires $\sin(k_z L) = 0$, which can only be satisfied if $k_z L$ is equal to an integer times π:

$$\sin\left(k_z L\right) = 0 \quad \Rightarrow \quad k_z = \frac{n\pi}{L}. \tag{2.114}$$

Now consider the ϕ-direction, where we have (2.107):

$$N_\phi(\phi) = C e^{\imath k_\phi \phi} + D e^{-\imath k_\phi \phi}. \tag{2.115}$$

Since there is no “edge” to the cylinder in the ϕ-direction it is not immediately obvious what we should do with this expression. But the separation constant k_ϕ does appear in the radial Eq. (2.111), so we do need to pin it down somehow.

The condition to be applied to N_ϕ arises from the fact that ϕ is a so-called *cyclic* coordinate: If the value of ϕ is changed by adding any integral multiple of 2π radians, then one has returned to the same direction from whence one began. We can express this by demanding that

$$N_\phi(\phi) = N_\phi(\phi + 2\pi), \tag{2.116}$$

or, more explicitly,

$$C e^{\imath k_\phi \phi} + D e^{-\imath k_\phi \phi} = C e^{\imath k_\phi (\phi + 2\pi)} + D e^{-\imath k_\phi (\phi + 2\pi)}. \tag{2.117}$$

This can be rewritten as

$$C e^{\imath k_\phi \phi} + D e^{-\imath k_\phi \phi} = C e^{\imath k_\phi \phi}\left(e^{2\pi \imath k_\phi}\right) + D e^{-\imath k_\phi \phi}\left(e^{-2\pi \imath k_\phi}\right). \tag{2.118}$$

This can *only* be satisfied if $e^{\pm 2\pi \imath k_\phi} = 1$, that is, if

$$\cos\left(2\pi k_\phi\right) \pm \imath \sin\left(2\pi k_\phi\right) = 1. \tag{2.119}$$

This expression will only be satisfied if

$$k_\phi = 0, 1, 2, 3, \ldots. \tag{2.120}$$

That ϕ is cyclic has led to the restriction that the *order* of our Bessel equation must be an integer.

With k_ϕ now established (at least to some extent), we can begin to get to the issue of the length and radius of a threshold-critical core. Return to the radial Eq. (2.111):

$$x^2 \frac{\partial^2 N_x}{\partial x^2} + x\left(\frac{\partial N_x}{\partial x}\right) + \left(x^2 - k_\phi^2\right) N_x = 0. \tag{2.121}$$

The length L of the core appears explicitly in k_z, which is incorporated into this expression through κ and x.

To determine when criticality is achieved, we need to know what value(s) of x will just render (2.121) satisfied for a given value of the order k_ϕ; this will dictate the critical radius ρ through (2.110). For a given choice of k_ϕ, there prove to be an infinitude of values of x that make this so; these values are known as the *zeros of Bessel's equation for order* k_ϕ and are extensively tabulated in many sources. In general, the values of the zeros increase monotonically within a given order, and the value of the m'th zero ($m = 1, 2, 3, \ldots$) also increases monotonically as a function of order number. The m'th zero for some order k is commonly designated as J_{km}; order numbers start at $k = 0$. In general, then, we will have criticality when x is equal to some zero J_{km}, or, on combining (2.109), (2.110), and (2.114), when

$$\left(\frac{1}{d^2} - \frac{n^2\pi^2}{L^2}\right)^{1/2} R = J_{km}, \tag{2.122}$$

where the radius ρ has been written as R. The volume of the core is $\pi R^2 L$. We can solve (2.122) for R and express the volume entirely in terms of L:

$$V_{crit} = \frac{\pi J_{km}^2 d^2 L^3}{\left(L^2 - n^2\pi^2 d^2\right)}. \tag{2.123}$$

The lowest possible critical volume will obtain for the lowest possible value of J_{km} and the lowest possible value for n; we can choose these independently of each other as they arose from different separation constants. As for n, the lowest acceptable value is $n = 1$; $n = 0$ would not do as it would render $N_z(z) = 0$ *everywhere* throughout the core, not just at its edge [see (2.113) and (2.114)]. The lowest-valued zero J_{km} is $J_{01} = 2.40483$, that is, the first zero for the Bessel equation of order zero. This corresponds to $k_\phi = 0$, which is physically acceptable as it renders N_ϕ equal to a constant [see (2.115)]. The minimum critical volume then becomes

$$V_{crit} = \frac{\pi J_{01}^2 d^2 L^3}{\left(L^2 - \pi^2 d^2\right)}. \tag{2.124}$$

An interesting physical consequence here is that there is a minimum length required for the denominator of (2.124) to be positively-valued:

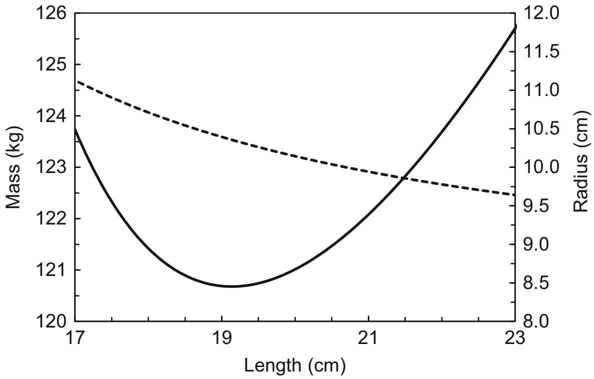

Fig. 2.17 Computed ^{235}U critical mass (*solid line, left scale*) and radius (*dashed line, right scale*) as a function of length for a cylindrical core. Note that these results hold only for the simplified boundary condition $N(\text{edge}) = 0$

$$L > \pi d. \tag{2.125}$$

This result is intuitively appealing on the rationale that if the core is not long enough, too many neutrons will escape and criticality cannot be obtained. For ^{235}U, this critical length evaluates as about 11.04 cm.

The least possible critical volume is found by determining the value of L that minimizes (2.124). This proves to be

$$\frac{\partial V_{crit}}{\partial L} = 0 \quad \Rightarrow \quad L = \sqrt{3}\pi d, \tag{2.126}$$

which, when back-substituted into (2.124) gives

$$V_{\min} = \left(\frac{3^{3/2}}{2} \pi^2 J_{01}^2 \right) d^3 = 148.3\, d^3. \tag{2.127}$$

For ^{235}U, this corresponds to a mass of about 121 kg. This result lies between those quoted at the beginning of this section for a sphere and a cube. The ratios of the critical volumes go as

$$V_{sphere} : V_{cyl} : V_{cube} = 1 : 1.142 : 1.241. \tag{2.128}$$

The penalty for using a *Little Boy*-type core instead of a sphere is thus only about a 14 % increase in mass.

Figure 2.17 shows the critical mass and cylinder radius corresponding to a given choice of L in (2.122) and (2.124) for our usual parameters for ^{235}U: $(\sigma_f, \sigma_{el}, \nu, \rho) =$ (1.235 bn, 4.566 bn, 2.637, 18.71 gr/cm^3). The minimum critical mass corresponds to a length of about 19.2 cm and a radius of about 10.3 cm—a cylinder almost as long as it is wide.

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