Chapter 2
Introduction to DEA

Data envelopment analysis (DEA), a "data-oriented" approach to evaluate the performance of a set of peer entities, has been widely used since it was first invented by Charnes. This is followed by a series of theoretical extensions. See Banker et al. [1], Charnes et al. [3], Petersen [12], Tone [14], and Cooper [6].

Our focus in this chapter is on basic DEA models for measuring the efficiency of a DMU relative to similar DMUs in order to estimate a "best practice" frontier. The initial DEA model, originally presented in Charnes et al. [2], was built on the earlier work of Farrell [10]. After that, more than 4,000 relevant articles have been published. Such rapid growth and widespread acceptance of the methodology of DEA is testimony to its strength and applicability. Researchers in a number of fields have quickly recognized that DEA is an excellent methodology for modeling operational processes, and its empirical orientation and minimization of a priori assumptions have made possible use in a number of studies involving efficient frontier estimation in the nonprofit sector, the regulated sector, and the private sector.

At present, DEA actually encompasses a variety of alternate approaches to performance evaluation. Extensions to the original CCR work have facilitated a deeper analysis of both the "multiplier side" from the dual model and the "envelopment side" from the primal model of the mathematical duality structure. Properties such as isotonicity; nonconcavity; economies of scale; piecewise linearity; discretionary, categorical variables; and ordinal relationships can also be treated through DEA.

In recent years a great variety of applications of DEA have been proposed. These DEA applications have used DMUs in various forms to evaluate the performance of such entities as hospitals, US Air Force wings, universities, cities, courts, and business firms, as well as the performance of countries, regions, etc.

This chapter will present a literature review on DEA, including the fundamental concept of DEA, frequently used DEA models, and the DMU efficiency definitions.

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2.1 Symbols and Notations

In DEA, the organization under study is called a DMU (decision-making unit). The definition of DMU is rather loose to allow flexibility in its use over a wide range of possible applications. Generically a DMU is regarded as the entity responsible for converting inputs into outputs and whose performances are to be evaluated. In managerial applications, DMUs may include banks, department stores, and supermarkets and extend to car makers, hospitals, schools, public libraries, and so on. In engineering, DMUs may take such forms as airplanes or their components such as jet engines. For the purpose of securing relative comparisons, a group of DMUs is used to evaluate each other with each DMU having a certain degree of managerial freedom in decision making.

Suppose there are n DMUs and the symbols and notations are listed as follows:

DMU_i: the ith DMU, $i = 1, 2, \cdots, n$
DMU_0: the target DMU
$\boldsymbol{x}_i = (x_{i1}, x_{i2}, \cdots, x_{ip})$: the inputs vector of DMU_i, $i = 1, 2, \cdots, n$
$\boldsymbol{x}_0 = (x_{01}, x_{02}, \cdots, x_{0p})$: the inputs vector of the target DMU_0
$\boldsymbol{y}_i = (y_{i1}, y_{i2}, \cdots, y_{iq})$: the outputs vector of DMU_i, $i = 1, 2, \cdots, n$
$\boldsymbol{y}_0 = (y_{01}, y_{02}, \cdots, y_{0q})$: the outputs vector of the target DMU_0
$\boldsymbol{u} \in R^{p\times 1}$: the vector of input weights
$\boldsymbol{v} \in R^{q\times 1}$: the vector of output weights

2.2 CCR Model

This section deals with one of the most basic DEA models named CCR model, which was initially proposed by Charnes et al. [2] in 1978:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \dfrac{\boldsymbol{v}^T\boldsymbol{y}_0}{\boldsymbol{u}^T\boldsymbol{x}_0} \\ \text{subject to:} \\ \qquad \boldsymbol{v}^T\boldsymbol{y}_j \le \boldsymbol{u}^T\boldsymbol{x}_j, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \ge 0 \\ \qquad \boldsymbol{v} \ge 0. \end{cases} \tag{2.1}$$

The constraints mean that the ratio of “virtual output” vs. “virtual input” should not exceed 1 for every DMU. The objective is to obtain the ratio of the weighted output to the weighted input weights. By virtue of the constraints, the optimal objective value θ^* is at most 1. Mathematically, the nonnegativity constraint is not sufficient for the fractional terms to have a positive value. We do not treat this assumption in explicit mathematical form at this time. Instead we put this in managerial terms by assuming that all outputs and inputs have some nonzero worth and this is to be reflected in the weights $\boldsymbol{v}$ and $\boldsymbol{u}$ being assigned some positive value.

Given the data, we measure the efficiency of each DMU once and hence need n optimizations, one for each DMU to be evaluated.

Definition 2.1 (CCR Efficiency). DMU_0 is CCR-efficient if $\theta^* = 1$ and there exists at least one optimal $\boldsymbol{u}^* > 0$ and $\boldsymbol{v}^* > 0$.

We now replace the above fractional program (FP) by the following linear program (LP):

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \boldsymbol{v}^T \boldsymbol{y}_0 \\ \text{subject to:} \\ \qquad \boldsymbol{u}^T \boldsymbol{x}_0 = 1 \\ \qquad \boldsymbol{v}^T \boldsymbol{y}_j - \boldsymbol{u}^T \boldsymbol{x}_j \leq 0, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \geq 0 \\ \qquad \boldsymbol{v} \geq 0. \end{cases} \tag{2.2}$$

Theorem 2.1. *The fractional program (2.1) is equivalent to the linear program (2.2).*

The dual problem of the linear program (2.2) is expressed with a real variable θ and a nonnegative vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of variables as follows:

$$\begin{cases} \theta = \min \theta \\ \text{subject to:} \\ \qquad \sum\limits_{j=1}^{n} x_{ij}\lambda_j \leq \theta x_{i0}, \quad i = 1, 2, \cdots, p \\ \qquad \sum\limits_{j=1}^{n} y_{rj}\lambda_j \geq y_{r0}, \quad r = 1, 2, \cdots, q \\ \qquad \lambda_j \geq 0, \quad j = 1, 2, \cdots, n. \end{cases} \tag{2.3}$$

This model (2.3) is sometimes referred to as the "Farrell model" because it is the one used in Farrell. In the economics portion of the DEA literature, it is said to conform to the assumption of "strong disposal," but the efficiency evaluation it makes ignores the presence of nonzero slacks. In the operations research portion of the DEA literature, this is referred to as "weak efficiency."

The dual model (2.3) has a feasible solution $\theta^* = 1, \lambda_0^* = 1, \lambda_j^* = 0 \ (j \neq 0)$. Hence the optimal value θ^* is not greater than 1. The optimal solution, θ^*, yields an efficiency score for a particular DMU. The process is repeated for each DMU_j, $j = 1, 2, \cdots, n$. DMUs for which $\theta^* < 1$ are inefficient, while DMUs for which $\theta^* = 1$ are boundary points.

Some boundary points may be "weakly efficient" because we have nonzero slacks. This may appear to be worrisome because alternate optima may have nonzero slacks in some solutions, but not in others. However, we can avoid being worried even in such cases by invoking the following linear program in which the slacks are taken to their maximal values:

$$\begin{cases} \max \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ \text{subject to:} \\ \qquad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta^* x_{i0}, \quad i = 1, 2, \ldots, p \\ \qquad \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{r0}, \quad r = 1, 2, \ldots, q \\ \qquad \lambda_j \geq 0, \quad j = 1, 2, \cdots, n \\ \qquad s_i^- \geq 0, \quad i = 1, 2, \cdots, p \\ \qquad s_r^+ \geq 0, \quad r = 1, 2, \cdots, q \end{cases} \tag{2.4}$$

where we note the choices of s_i^- and s_r^+ do not affect the optimal θ^*, which is determined from model (2.3).

These developments now lead to the following definitions based upon the "relative efficiency" in Definition 2.1.

Definition 2.2 (DEA Efficiency). The performance of DMU_0 is fully (100 %) efficient if and only if both (1) $\theta^* = 1$ and (2) all slacks $s_i^{-*} = s_r^{+*} = 0$.

Definition 2.3 (Weakly DEA Efficiency). The performance of DMU_0 is weakly efficient if and only if both (1) $\theta^* = 1$ and (2) $s_i^{-*} \neq 0$ and/or $s_r^{+*} \neq 0$ for some i or r in some alternate optima.

It is to be noted that the preceding development amounts to solving the following problem in two steps:

$$\begin{cases} \min \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\ \text{subject to:} \\ \qquad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta x_{i0}, \quad i = 1, 2, \cdots, p \\ \qquad \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{r0}, \quad r = 1, 2, \cdots, q \\ \qquad \lambda_j \geq 0, \quad j = 1, 2, \cdots, n \\ \qquad s_i^- \geq 0, \quad i = 1, 2, \cdots, p \\ \qquad s_r^+ \geq 0, \quad r = 1, 2, \cdots, q \end{cases} \tag{2.5}$$

where the s_i^- and s_r^+ are slack variables used to convert the inequalities in (2.3) to equivalent equations. Here, $\varepsilon > 0$ is a so-called non-Archimedean element defined to be smaller than any positive real number. This is equivalent to solving (2.3) in two stages by first minimizing θ and then fixing $\theta = \theta^*$ as in (2.4), where the slacks are to be maximized without altering the previously determined value of $\theta = \theta^*$.

Alternately, one could have started with the output side and considered instead the ratio of virtual input to output. This would reorient the objective from max to min, as in (2.1), to obtain

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \dfrac{\boldsymbol{u}^T \boldsymbol{x}_0}{\boldsymbol{v}^T \boldsymbol{y}_0} \\ \text{subject to:} \\ \qquad \boldsymbol{u}^T \boldsymbol{x}_j \le \boldsymbol{v}^T \boldsymbol{y}_j, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \ge \varepsilon > 0 \\ \qquad \boldsymbol{v} \ge \varepsilon > 0 \end{cases} \tag{2.6}$$

where $\varepsilon > 0$ is the previously defined non-Archimedean element.

Similar to model (2.2) and (2.5), the input models are as follows:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \boldsymbol{u}^T \boldsymbol{x}_0 \\ \text{subject to:} \\ \qquad \boldsymbol{v}^T \boldsymbol{y}_0 = 1 \\ \qquad \boldsymbol{u}^T \boldsymbol{x}_j - \boldsymbol{v}^T \boldsymbol{y}_j \ge 0, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \ge \varepsilon > 0 \\ \qquad \boldsymbol{v} \ge \varepsilon > 0, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \max \phi + \varepsilon \left(\sum\limits_{i=1}^{m} s_i^- + \sum\limits_{r=1}^{s} s_r^+ \right) \\ \text{subject to:} \\ \qquad \sum\limits_{j=1}^{n} x_{ij} \lambda_j + s_i^- = x_{io}, \quad i = 1, 2, \cdots, p \\ \qquad \sum\limits_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = \phi y_{ro}, \quad r = 1, 2, \cdots, q \\ \qquad \lambda_j \ge 0, \quad j = 1, 2, \cdots, n \\ \qquad s_i^- \ge 0, \quad i = 1, 2, \cdots, p \\ \qquad s_r^+ \ge 0, \quad r = 1, 2, \cdots, q \end{cases} \tag{2.8}$$

See Cooper et al. [7] for a formal development of this transformation and modification of the expression for $\varepsilon > 0$.

Table 2.1 CCR DEA model

Envelopment model	Multiplier model
Input-oriented	
$\min \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$	$\max z = \sum_{r=1}^{q} \mu_r y_{r0}$
Subject to:	Subject to:
$\sum_{j=1}^{n} x_{ij} \lambda_j + s_i^- = \theta x_{i0} \quad i = 1, 2, \cdots, p$	$\sum_{r=1}^{q} \mu_r y_{rj} - \sum_{i=1}^{q} v_i y_{ij} \le 0$
$\sum_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = y_{r0} \quad r = 1, 2, \cdots, q$	$\sum_{i=1}^{p} v_i x_{i0} = 1$
$\lambda_j \ge 0 \quad j = 1, 2, \cdots, n$	$u_r, v_i \ge \varepsilon > 0$
Output-oriented	
$\max \phi + \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$	$\min q = \sum_{i=1}^{p} v_i x_{io}$
Subject to:	Subject to:
$\sum_{j=1}^{n} x_{ij} \lambda_j + s_i^- = x_{i0} \quad i = 1, 2, \cdots, p$	$\sum_{i=1}^{p} v_i x_{ij} - \sum_{r=1}^{q} \mu_r y_{rj} \ge 0$
$\sum_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = \phi y_{r0} \quad r = 1, 2, \cdots, q$	$\sum_{r=1}^{q} \mu_r y_{r0} = 1$
$\lambda_j \ge 0 \quad j = 1, 2, \cdots, n$	$\mu_r, v_i \ge \varepsilon > 0$

Here, we use a model with an output-oriented objective as contrasted with the input orientation in (1.6). However, as before, model (1.9) is calculated in a two-stage process. First, we calculate ϕ^* by ignoring the slacks. Then we optimize the slacks by fixing ϕ^* in the following linear programming problem:

$$\begin{cases} \max \sum_{i=1}^{p} s_i^- + \sum_{r=1}^{q} s_r^+ \\ \text{subject to:} \\ \quad \sum_{j=1}^{n} x_{ij} \lambda_j + s_i^- = x_{io} \quad i = 1, 2, \cdots, p \\ \quad \sum_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = \phi^* y_{ro} \quad r = 1, 2, \cdots, q \\ \quad \lambda_j \ge 0 \quad j = 1, 2, \cdots, n. \end{cases} \tag{2.9}$$

Table 2.1 presents the CCR model in input-and-output-oriented versions, each in the form of a pair of dual linear programs.

2.3 BCC Model

The input-oriented BCC model proposed by Banker et al. [1] evaluates the efficiency of DMU_0 by solving the following linear program:

$$
\begin{cases}
\theta_B = \min\ \theta \\
\text{subject to:} \\
\qquad \sum_{j=1}^{n} x_{ij}\lambda_j \le \theta x_{i0}, \quad i = 1, 2, \cdots, p \\
\qquad \sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{r0}, \quad r = 1, 2, \cdots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \cdots, n.
\end{cases} \tag{2.10}
$$

Some boundary points may be "weakly efficient" because we have nonzero slacks. This may appear to be worrisome because alternate optima may have nonzero slacks in some solutions, but not in others. However, we can avoid being worried even in such cases by invoking the following linear program in which the slacks are taken to their maximal values:

$$
\begin{cases}
\max \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{subject to:} \\
\qquad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta^* x_{i0}, \quad i = 1, 2, \cdots, p \\
\qquad \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{r0}, \quad r = 1, 2, \cdots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \cdots, n \\
\qquad s_i^- \ge 0, \quad i = 1, 2, \cdots, p \\
\qquad s_j^+ \ge 0, \quad j = 1, 2, \cdots, q.
\end{cases} \tag{2.11}
$$

It is to be noted that the preceding development amounts to solving the following problem in two steps:

$$
\begin{cases}
\min \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
\text{subject to:} \\
\qquad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta x_{i0}, \quad i = 1, 2, \cdots, p \\
\qquad \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{r0}, \quad r = 1, 2, \cdots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \cdots, n \\
\qquad s_i^- \ge 0, \quad i = 1, 2, \cdots, p \\
\qquad s_j^+ \ge 0, \quad j = 1, 2, \cdots, q.
\end{cases} \tag{2.12}
$$

The dual multiplier form of the linear program (2.10) is expressed as

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v},v_0} \ \theta_B = \boldsymbol{v}^T \boldsymbol{y}_0 - v_0 \\ \text{subject to:} \\ \qquad \boldsymbol{u}^T \boldsymbol{x}_0 = 1 \\ \qquad \boldsymbol{v}^T \boldsymbol{y}_j - \boldsymbol{u}^T \boldsymbol{x}_j - v_0 \leq 0, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \geq 0 \\ \qquad \boldsymbol{v} \geq 0 \end{cases} \tag{2.13}$$

where $\boldsymbol{v}$ and $\boldsymbol{u}$ are vectors. The scalar v_0 may be positive or negative (or zero). The equivalent BCC fractional program is obtained from the dual program (2.13) as

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta_B = \dfrac{\boldsymbol{v}^T \boldsymbol{y}_0 - v_0}{\boldsymbol{u}^T \boldsymbol{x}_0} \\ \text{subject to:} \\ \qquad \dfrac{\boldsymbol{v}^T \boldsymbol{y}_j - v_0}{\boldsymbol{u}^T \boldsymbol{x}_j} \leq 1, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \geq 0 \\ \qquad \boldsymbol{v} \geq 0. \end{cases} \tag{2.14}$$

It is clear that a difference between the CCR and BCC models is present in the free variable v_0, which is the dual variable associated with the constraint $\sum\limits_{k=1}^{n} \lambda_k = 1$ that also does not appear in the CCR model. In the first phase, we minimize θ by model (2.10) and, in the second phase, we maximize the sum of the input excesses and output shortfalls, keeping θ^* (the optimal objective value obtained in Phase one) by model (2.11). The evaluations secured from the CCR and BCC models are also related to each other as follows. An optimal solution for (2.10) and (2.11) is represented by $\left(\theta_B^*, s^{-*}, s^{+*}\right)$, where s^{-*} and s^{+*} represent the maximal input excesses and output shortfalls, respectively. Notice that θ_B^* is not less than the optimal objective value θ^* of the CCR model, since (2.10) imposes one additional constraint, $\sum\limits_{k=1}^{n} \lambda_k = 1$, so its feasible region is a subset of feasible region for the CCR model.

Definition 2.4 (BCC Efficiency). If an optimal solution $\left(\theta_B^*, s^{-*}, s^{+*}\right)$ obtained in this two-phase process for model (2.10) satisfies $\theta_B^* = 1$ and has no slack $s^{-*} = s^{+*} = 0$, then the DMU$_0$ is called BCC-efficient; otherwise it is BCC-inefficient.

Theorem 2.2. *The improved activity* $(\theta^* \boldsymbol{x} - s^{-*}, \boldsymbol{y} + s^{+*})$ *is BCC-efficient.*

Theorem 2.3. *A DMU that has a minimum input value for any input item, or a maximum output value for any output item, is BCC-efficient.*

The output-oriented BCC model is

$$\begin{cases} \max \quad \eta \\ \text{subject to:} \\ \qquad \sum\limits_{j=1}^{n} x_{ij}\lambda_j \le x_{i0}, \quad i = 1, 2, \cdots, p \\ \qquad \sum\limits_{j=1}^{n} y_{rj}\lambda_j \ge \eta y_{r0}, \quad r = 1, 2, \cdots, q \\ \qquad \sum\limits_{k=1}^{n} \lambda_k = 1 \\ \qquad \lambda_k \ge 0, \quad k = 1, 2, \cdots, n. \end{cases} \tag{2.15}$$

The dual form associated with the above linear program (2.15) is expressed as

$$\begin{cases} \min\limits_{\boldsymbol{u},\boldsymbol{v},u_0} \ \boldsymbol{v}^T \boldsymbol{y}_0 - v_0 \\ \text{subject to:} \\ \qquad \boldsymbol{v}^T \boldsymbol{y}_0 = 1 \\ \qquad \boldsymbol{u}^T \boldsymbol{x}_j - \boldsymbol{v}^T \boldsymbol{y}_j - v_0 \ge 0, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \ge 0 \\ \qquad \boldsymbol{v} \ge 0 \end{cases} \tag{2.16}$$

where v_0 is the scalar associated with $\sum\limits_{k=1}^{n} \lambda_k = 1$ in the envelopment model. Finally, we have the equivalent (BCC) fractional programming formulation for model (2.16):

$$\begin{cases} \min\limits_{\boldsymbol{u},\boldsymbol{v},u_0} \ \dfrac{\boldsymbol{u}^T \boldsymbol{x}_0 - u_0}{\boldsymbol{v}^T \boldsymbol{y}_0} \\ \text{subject to:} \\ \qquad \dfrac{\boldsymbol{u}^T \boldsymbol{x}_j - v_0}{\boldsymbol{v}^T \boldsymbol{y}_j} \ge 1, \ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \ge 0 \\ \qquad \boldsymbol{v} \ge 0. \end{cases} \tag{2.17}$$

2.4 Additive Model

The preceding models required us to distinguish between input-oriented and output-oriented models. Now, however, we combine both orientations in a single model, called additive model proposed by Charnes et al. [3].

Let us consider a production possibility set (PPS), consisting of all convex combinations of $(x_k,\ y_k)$, $k = 1, 2, \cdots, n$. We can formulate it as

$$\text{PPS} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} = \sum_{k=1}^{n} \boldsymbol{x}_k \lambda_k,\ \boldsymbol{y} = \sum_{k=1}^{n} \boldsymbol{y}_k \lambda_k, \right.$$
$$\left. \sum_{k=1}^{n} \lambda_k = 1,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0, \cdots, \lambda_n \geq 0 \right\}.$$

Then the additive model can be given as

$$\left\{ \begin{array}{l} \max \quad \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\ \text{subject to:} \\ \qquad \sum_{k=1}^{n} x_{ki} \lambda_k = x_{0i} - s_i^-, \quad i = 1, 2 \cdots, p \\ \qquad \sum_{k=1}^{n} y_{kj} \lambda_k = y_{0j} + s_j^+, \quad j = 1, 2, \cdots, q \\ \qquad \sum_{k=1}^{n} \lambda_k = 1 \\ \qquad \lambda_k \geq 0, \quad k = 1, 2, \cdots, n \\ \qquad s_i^- \geq 0, \quad i = 1, 2, \cdots, p \\ \qquad s_j^+ \geq 0, \quad j = 1, 2, \cdots, q \end{array} \right. \tag{2.18}$$

where s_i^- and s_j^+ represent output and input slacks, respectively.

It is clear that this model considers the total slacks of inputs and output simultaneously in arriving at a point on the efficient frontier.

Definition 2.5 (ADD Efficiency). DMU$_0$ is ADD-efficient if s_i^{-*} and s_j^{+*} are zero for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$, where s_i^{-*} and s_j^{+*} are optimal solutions of (2.18).

DMU$_0$ is ADD-efficient if there is no $(x, y) \in$ PPS such that $x \leq x_0$ and $y \geq y_0$ with strict inequality holding for at least one of the components in the input or the output vector.

The dual problem to the additive model (2.18) can be expressed as follows:

$$\left\{ \begin{array}{l} \max\limits_{\boldsymbol{u},\boldsymbol{v},v_0} \ \boldsymbol{u}^T \boldsymbol{x}_0 - \boldsymbol{v}^T \boldsymbol{y}_0 + v_0 \\ \text{subject to:} \\ \qquad \boldsymbol{v}^T \boldsymbol{y}_j - \boldsymbol{u}^T \boldsymbol{x}_j - v_0 \leq 0,\ j = 1, 2, \cdots, n \\ \qquad \boldsymbol{u} \geq e \\ \qquad \boldsymbol{v} \geq e. \end{array} \right. \tag{2.19}$$

Theorem 2.4. *DMU$_0$ is ADD-efficient if and only if it is BCC-efficient.*

Theorem 2.5 (Tone [14]). *DMU$_0$ is CCR-efficient if and only if it is SBM-efficient.*

The model (2.18) uses a metric that differs from the one used in the "radial measure" model which uses what is called the ℓ_1 metric in mathematics, and the "city block metric" in operations research. It also dispenses with the need for distinguishing between an "output" and an "input" orientation as was done in the discussion leading up to (2.9) because the objective in (2.18) simultaneously maximizes outputs and minimizes inputs in the sense of vector optimizations. This can be seen by utilizing the solution to (2.18) to introduce new variables $\hat{y}_{ro}, \hat{x}_{io}$ defined as follows:

$$\begin{aligned} \hat{y}_{r0} &= y_{r0} + s_r^{+*} \geq y_{r0}, r = 1, \cdots, q \\ \hat{x}_{i0} &= x_{i0} - s_i^{-*} \leq x_{i0}, i = 1, \cdots, p. \end{aligned} \tag{2.20}$$

Now, note that the slacks are all independent of each other. Hence, an optimum is not reached until it is not possible to increase an output $\hat{y}_{ro}$ or reduce an input $\hat{x}_{io}$ without decreasing some other output or increasing some other input.

We now use the class of additive models to develop a different route to treating technical, allocative, and overall inefficiencies and their relations to each other. This can help to avoid difficulties in treating possibilities such as negative or zero profits, which are not easily treated by the ratio approaches, which are commonly used in the DEA literature. See the discussion in Cooper et al. [4, 5] from which the following development is taken. See also Chap. 8 in Cooper et al. [7].

First, we observe that we can multiply the output slacks by unit prices and the input slacks by unit costs after we have solved (1.19) and thereby accord a monetary value to this solution. Then, we can utilize (1.20) to write

$$\begin{aligned} &\sum_{r=1}^{s} p_{ro} s_r^{+*} + \sum_{i=1}^{m} c_{io} s_i^{-*} \\ &= \left(\sum_{r=1}^{s} p_{ro} \hat{y}_{ro} - \sum_{r=1}^{s} p_{ro} y_{ro} \right) + \left(\sum_{i=1}^{m} c_{io} x_{io} - \sum_{i=1}^{m} c_{io} \hat{x}_{io} \right) \\ &= \left(\sum_{r=1}^{s} p_{ro} \hat{y}_{ro} - \sum_{i=1}^{m} c_{io} \hat{x}_{io} \right) - \left(\sum_{r=1}^{s} p_{ro} y_{ro} - \sum_{i=1}^{m} c_{io} x_{io} \right). \end{aligned} \tag{2.21}$$

From the last pair of parenthesized expressions, we find that, at an optimum, the objective in (2.18) after multiplication by unit prices and costs is equal to the profit available when production is technically efficient minus the profit obtained from the observed performance. Hence, when multiplied by unit prices and costs, the solution to (2.18) provides a measure in the form of the amount of the profits lost by not performing in a technically efficient manner term by term if desired.

We can similarly develop a measure of allocative efficiency by means of the following additive model:

$$
\begin{cases}
\max \sum_{r=1}^{s} p_{r0}\hat{s}_r^{+} + \sum_{i=1}^{m} c_{i0}\hat{s}_i^{-} \\
\text{subject to:} \\
\qquad \hat{y}_{r0} = \sum_{j=1}^{n} y_{rj}\hat{\lambda}_j - \hat{s}_r^{+}, r = 1, 2, \cdots, q \\
\qquad \hat{x}_{i0} = \sum_{j=1}^{n} x_{ij}\hat{\lambda}_j + \hat{s}_i^{-}, i = 1, 2, \cdots, p \\
\qquad \sum_{j=1}^{n} \hat{\lambda}_j = 1 \\
\qquad \hat{\lambda}_j \geq 0, \quad j = 1, 2, \cdots, n.
\end{cases}
\tag{2.22}
$$

2.5 SBM Model

We now augment the additive models by introducing a measure that makes its efficiency evaluation, as effected in the objective, invariant to the units of measure used for the different inputs and outputs. That is, we would like this summary measure to assume the form of a scalar that yields the same efficiency value when distances are measured in either kilometers or miles. More generally, we want this measure to be the same when x_{ij} are replaced by $k_i x_{ij}$ and y_{rj} are replaced by $c_r y_{rj}$, where the k_i and c_r are arbitrary positive constants, $i = 1, 2 \cdots, p$, $j = 1, 2 \cdots, q$.

This property is known by names such as “dimension free” (see [13]) and “units invariant.” In this section, we introduce such a measure for additive models in the form of a single scalar called “SBM” (Slacks-Based Measure), which was introduced by Tone [14] and has the following important properties:

P1 The measure is invariant with respect to the unit of measurement of each input and output item. (Units invariant)

P2 The measure is monotone decreasing in each input and output slack. (Monotone)

In order to estimate the efficiency, we formulate the following fractional program:

$$
\begin{cases}
\min \quad \rho = \dfrac{1 - \frac{1}{p}\sum_{i=1}^{p} s_i^{-}/x_{0i}}{1 + \frac{1}{q}\sum_{j=1}^{q} s_j^{+}/y_{0j}} \\
\text{subject to:} \\
\qquad \sum_{k=1}^{n} x_{ki}\lambda_k = x_{0i} - s_i^{-}, \quad i = 1, 2 \cdots, p \\
\qquad \sum_{k=1}^{n} y_{kj}\lambda_k = y_{0j} + s_j^{+}, \quad j = 1, 2, \cdots, q \\
\qquad \lambda_k \geq 0, \quad k = 1, 2, \cdots, n \\
\qquad s_i^{-} \geq 0, \quad i = 1, 2, \cdots, p \\
\qquad s_j^{+} \geq 0, \quad j = 1, 2, \cdots, q.
\end{cases}
\tag{2.23}
$$

In this model, we assume that $\boldsymbol{x}_k \geq 0$, $k = 1, 2, \cdots, n$. If $x_{0i} = 0$, we delete the term s_i^-/x_{0i} in the objective function. If $y_{0j} \leq 0$, we replace it by a very small positive number so that the term s_i^-/x_{0i} plays a role of penalty.

It is readily verified that the objective function value ρ satisfies (PI) because the numerator and denominator are measured in the same units for every item in the objective of (2.23). It is also readily verified that an increase in either s_i^- or s_j^+, all else held constant, will decrease this objective value and, indeed, do so in a strictly monotone manner.

Furthermore, we have

$$0 \leq \rho \leq 1.$$

The formula for ρ in (2.23) can be transformed into

$$\rho = \left(\frac{1}{p} \sum_{i=1}^{p} \frac{x_{0i} - s_i^-}{x_{0i}} \right) \left(\frac{1}{q} \sum_{i=1}^{q} \frac{y_{0j} + s_j^+}{y_{0j}} \right)^{-1}. \tag{2.24}$$

The ratio $\left(x_{0i} - s_i^-\right)/x_{0i}$ evaluates the relative reduction rate of input i and, therefore, the first term corresponds to the mean proportional reduction rate of inputs or input mix inefficiencies. Similarly, in the second term, the ratio $\left(y_{0j} + s_j^+\right)/y_{0j}$ evaluates the relative proportional expansion rate of output j and $(1/q) \sum_{i=1}^{q} \left(y_{0j} + s_j^+\right)/y_{0j}$ is the mean proportional rate of output expansion. Its inverse, the second term, measures output mix inefficiency. Thus, ρ can be interpreted as the ratio of mean input and output mix inefficiencies. Further, we have the following theorem.

Theorem 2.6. *If DMU$_A$ dominates DMU$_B$ so that $\boldsymbol{x}_A \leq \boldsymbol{x}_B$ and $\boldsymbol{y}_A \geq \boldsymbol{y}_B$, then $\rho_A^* \geq \rho_B^*$.*

Definition 2.6 (SBM Efficiency). DMU$_0$ is SBM-efficient if and only if $\rho^* = 1$, where $\rho^* = 1$ is the optimal value of (2.23).

Theorem 2.7. *The optimal ρ^* in SBM model (2.23) is not greater than the optimal θ^* in CCR model (2.1).*

2.6 Russell Measure Model

We now introduce a model described as the "Russell Measure Model." Actually it was introduced and named by Färe and Lovell [8]. Their formulation is difficult to compute, however, so we turn to a more recent development due to Pastor et al. [11]. This model is given as follows:

$$
\begin{cases}
\psi = \min\limits_{\boldsymbol{\theta},\boldsymbol{\eta}} \quad \dfrac{\sum_{i=1}^{p} \theta_i / p}{\sum_{j=1}^{q} \eta_j / q} \\
\text{subject to:} \\
\qquad \sum\limits_{k=1}^{n} x_{ki}\lambda_k \le \theta_i x_{0i}, \quad i = 1, 2 \cdots, p \\
\qquad \sum\limits_{k=1}^{n} y_{kj}\lambda_k \ge \eta_j y_{0j}, \quad j = 1, 2, \cdots, q \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \cdots, n \\
\qquad 0 \le \theta_i \le 1, \quad i = 1, 2, \cdots, p \\
\qquad \eta_j \ge 1, \quad j = 1, 2, \cdots, q.
\end{cases}
\tag{2.25}
$$

Pastor et al. [11] refer to this as the "Enhanced Russell Graph Measure of Efficiency," but we shall refer to it as ERM (Enhanced Russell Measure). See Färe et al. [9] for the meaning of "graph measure." Such measures are said to be "closed," so ψ includes all inefficiencies that the model can identify. In this way we avoid limitations of the radial measures which cover only some of the input or output inefficiencies and hence measure only "weak efficiency."

The closure property is shared by SBM. In fact SBM and ERM are related as in the following theorem.

Theorem 2.8. *ERM as formulated in (2.25) and SBM as formulated in (2.23) are equivalent in that $\boldsymbol{\lambda}$ values that are optimal for one are also optimal for the other.*

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