

Chapter 2

Literature Review

2.1 Introduction

This chapter briefly reviews the development of traffic flow theory from the perspectives of macroscopic, mesoscopic, microscopic, and stochastic approaches. Some representative models and methods are summarized. Particularly, characteristics of the stochastic traffic flow modeling will be emphasized and analyzed in detail. Since headway/spacing are the fundamental parameters highly correlated with traffic flow rate, density and speed, then probabilistic modeling of headway/spacing distributions will be categorized such as univariate distribution, compositional distribution, mixed distribution, and random matrix theory based approach. The joint distribution of headway/spacing/velocity can be regarded as a basis for stochastic modeling of traffic state evolutions. The historical development of traffic flow theory forms a solid theoretical background for the whole book.

2.2 Historical Development of Traffic Flow Theory

2.2.1 Macroscopic Modeling

The study of macroscopic continuum traffic flows began with the well-known Lighthill-Whitham-Richards (LWR) model or kinematic wave model which was proposed independently by Lighthill and Whitham (1955); Richards (1956). The model assumes that a discrete flow of vehicles can be approximated by a continuous flow. And then, vehicle dynamics can be described by the spatial vehicle density $\rho(x, t)$ as a function of location x and time t . As a result, many theoretical and numerical methods were developed to study this property based on the hyperbolic partial differential equation (PDE) type traffic flow model.

The conservation law of vehicles in an arbitrary stretch $[x_1, x_2]$ of the road, over an arbitrary time interval $[t_1, t_2]$, is written as:

$$\int_{x_1}^{x_2} \rho(x, t_2) dx - \int_{x_1}^{x_2} \rho(x, t_1) dx = \int_{t_1}^{t_2} \rho(x_1, t) v(x_1, t) dt - \int_{t_1}^{t_2} \rho(x_2, t) v(x_2, t) dt \quad (2.1)$$

where $\rho(x, t)$ (veh/km) is the mean density at location x and time t , $v(x, t)$ (km/h) is the mean velocity at location x and time t .

However, weak solutions of Eq. (2.1) are not unique, and not all weak solutions capture the physics of traffic flow correctly (Jabari and Liu 2012). In order to derive the conservation law of traffic flow, Prigogine and Herman (1971) defined the phase space density as a production of $\rho(x, t)$ and the velocity distribution $\tilde{P}(v|x, t)$, i.e.,

$$\tilde{\rho}(x, t, v) \triangleq \rho(x, t) \tilde{P}(v|x, t) \quad (2.2)$$

where $\tilde{\rho}(x, t, v)$ is the phase space density function, that is, the joint PDF with respect to v at location x and time t , the velocity distribution satisfies $\int_0^\infty \tilde{P}(v|x, t) dv = 1$, $\int_0^\infty v \tilde{P}(v|x, t) dv = v(x, t)$.

Let $\{(x, v) \mapsto (y, u)\}$ denotes the transition rate from state (x, v) to (y, u) , where x and y represent locations, v and u represent velocities. Then the backward Kolmogorov type master equation that depicts the evolution of phase space density is

$$\begin{aligned} \frac{d\tilde{\rho}(x, t, v)}{dt} = & \int_0^\infty \int_{-\infty}^\infty \{(y, u) \mapsto (x, v)\} \tilde{\rho}(y, t, u) dy du \\ & - \int_0^\infty \int_{-\infty}^\infty \{(x, v) \mapsto (y, u)\} \tilde{\rho}(x, t, v) dy du \end{aligned} \quad (2.3)$$

Suppose when $t \rightarrow \infty$, the phase space density tends to a steady state, i.e.,

$$\lim_{t \rightarrow \infty} \frac{d\tilde{\rho}(x, t, v)}{dt} = 0 \quad (2.4)$$

Expand Eq. (2.4) to the total derivative

$$\frac{\partial \tilde{\rho}(x, t, v)}{\partial t} + \frac{dx}{dt} \frac{\partial \tilde{\rho}(x, t, v)}{\partial x} + \frac{dv}{dt} \frac{\partial \tilde{\rho}(x, t, v)}{\partial v} = 0 \quad (2.5)$$

Integrate Eq. (2.5) with respect to v , we have

$$\int_0^\infty \frac{\partial \tilde{\rho}(x, t, v)}{\partial t} dv + \frac{\partial \rho(x, t)}{\partial x} \int_0^\infty v \tilde{P}(v|x, t) dv + \rho(x, t) \int_0^\infty \frac{\partial \tilde{P}(v|x, t)}{\partial v} \frac{dv}{dt} dv = 0 \quad (2.6)$$

Simplify Eq. (2.5), the reduced form is

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial \rho(x, t)}{\partial x} v(x, t) + \frac{\partial v(x, t)}{\partial x} \rho(x, t) = 0 \quad (2.7)$$

or

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial \rho(x, t) v(x, t)}{\partial x} = 0 \quad (2.8)$$

Incorporate the equilibrium flow function $q_e(\rho)$, we have the hyperbolic PDE of LWR model as

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{dq_e(\rho)}{d\rho} \frac{\partial \rho(x, t)}{\partial x} = 0 \quad (2.9)$$

The cell transmission model (CTM) was proposed by Daganzo (1994, 1995a) as a direct discretization of LWR model to simulate traffic flow evolutions using the Godunov Scheme (Lebacque 1996), in which the flow rate was modeled as a function of density with a triangular or trapezoidal form. Various modifications of the CTM model had been proposed in last two decades. For example, CTM was extended to model network traffic flow with general fundamental diagrams (Daganzo 1995b). Lags were introduced to formulate the lagged cell transmission model (LCTM) that adopted a nonconcave fundamental diagram, in the fact that, the forward wave speed was larger than the backward wave speed (Daganzo 1999). Recently, the original LCTM was modified by Szeto (2008) as the enhanced LCTM (ELCTM) to guarantee that the nonnegative densities would not be greater than the jam density. To validate the parameters online by loop detectors data, a switching mode model (SMM) was formulated, in which the evolution of traffic density switched among different sets of linear difference equations (Muñoz et al. 2003, 2006). The asymmetric cell transmission model (ACTM) was applied in optimal freeway ramp metering by Gomes and Horowitz (2006); Gomes et al. (2008). The cell-based dynamic traffic assignment formulation was further developed for networks. Lo et al. (2001); Lo and Szeto (2002); Boel and Mihaylova (2006) proposed the compositional CTM afterwards.

To model the evolutions of velocity more accurately, higher-order terms of density and velocity were incorporated. The higher-order density gradient type dynamic equation has the following expression

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{v_e(\rho) - v}{\tau} - \frac{c_0^2}{\rho} \frac{\partial \rho}{\partial x} \quad (2.10)$$

where $v = v(x, t)$, $\rho = \rho(x, t)$, $v_e(\rho)$ is the equilibrium velocity-density function, τ is the relaxation time, $c_0 \geq 0$ is the substitution variable with the same unit as velocity, $\{v + c_0, v - c_0\}$ are the characteristic velocities. $v + c_0$ is larger than the macroscopic velocity of traffic flow, so that this model was criticized by Daganzo (1995c).

In order to overcome the problem, higher-order velocity gradient type dynamic equation was derived as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{v_e(\rho) - v}{\tau} + c_0 \frac{\partial v}{\partial x} \quad (2.11)$$

where characteristic velocities are not larger than the macroscopic velocity of traffic flow, i.e., $\{v, v - c_0\} \leq v$, so there are no reverse movements (Daganzo 1995c).

Helbing et al. (2009) proposed a general form for the higher-order model. They regarded that it was not contradictory when the characteristic velocity was larger than the macroscopic velocity. The general form is

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{v_e(\rho) - v}{\tau} - \frac{1}{\rho} \left(\frac{\partial P_1}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial P_2}{\partial v} \frac{\partial v}{\partial x} \right) \quad (2.12)$$

where $P_1 = P_1(\rho, v)$ and $P_2 = P_2(\rho, v)$ are pressure terms, satisfying $\partial_\rho P_1 \leq 0$, $\partial_v P_2 \leq 0$.

Analogously, we propose the following general macroscopic traffic flow model

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{v_e(\rho) - v}{\tau} + k_1 v'_e(\rho) \frac{\xi}{\tau} \frac{\partial \rho}{\partial x} + k_2 \frac{\eta}{\tau} \frac{\partial v}{\partial x} \quad (2.13)$$

where k_1, k_2 are the weighted coefficients for the density gradient term and the velocity gradient term, $\xi = \xi(\rho, v)$ and $\eta = \eta(\rho, v)$ reflect the anticipative and adaptive driving behaviors, respectively.

We obtain the general simultaneous PDEs

$$\frac{\partial \rho}{\partial t} + v_e(\rho) \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0 \quad (2.14)$$

$$\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial t} = \frac{v_e(\rho) - v}{\tau} + k_1 v'_e(\rho) \frac{\xi}{\tau} \frac{\partial \rho}{\partial x} + k_2 \frac{\eta}{\tau} \frac{\partial v}{\partial x} \quad (2.15)$$

The analytical linear stability condition for this general model is as following (refer to the detailed derivation in Appendix A)

$$\rho_0 v'_e(\rho_0) + k_1 \frac{\xi}{\tau} + k_2 \frac{\eta}{\tau} \geq 0 \quad (2.16)$$

Characteristic velocities are $v + c_\pm$, where

$$c_\pm = -\frac{|\eta|}{2\tau} \mp \sqrt{\rho v'_e(\rho) \frac{\xi}{\tau} + \left(\frac{\eta}{2\tau}\right)^2} \quad (2.17)$$

2.2.2 Mesoscopic Modeling

Common mesoscopic models belong to three categories (Hoogendoorn and Bovy 2001): time-headway distribution models, cluster models, and gas-kinetic models.

Table 2.1 Typical macroscopic traffic flow models

Models	k_1	k_2	ξ	η	Characteristic velocities $v + c_{\pm}$
Payne (1971)	1	0	$\frac{1}{2\rho}$	—	$c_{\pm} = \mp \sqrt{\frac{ v'_e(\rho) }{2\tau}}$
Whitham (1974)	1	0	$-\frac{\mu}{\rho v'_e(\rho)}$	—	$c_{\pm} = \mp \sqrt{\mu/\tau}$
Phillips (1979)	1	0	$-\frac{\tau\Theta_0(1-\rho/\rho_j)}{\rho v'_e(\rho)}$	—	$c_{\pm} = \mp \sqrt{\Theta_0(1-\rho/\rho_j)}$
Zhang (1998)	1	0	$-\rho v'_e(\rho)\tau$	—	$c_{\pm} = \pm \rho v'_e(\rho)$
Aw and Rascle (2000) Greenberg (2001)	0	1	—	$\tau\gamma\rho^\gamma$	$c_+ = -\gamma\rho^\gamma \leq 0, c_- = 0$
Zhang (2002)	0	1	—	$-c(\rho)\tau$	$c_+ = c(\rho) \leq 0, c_- = 0$
Jiang et al. (2001, 2002)	0	1	—	$c_0\tau$	$c_+ = c_0 \leq 0, c_- = 0$
Xue and Dai (2003)	0	1	—	$-t_r\rho v'_e(\rho)$	$c_+ = \frac{t_r}{\tau}\rho v'_e(\rho) \leq 0, c_- = 0$
Helbing and Johansson (2009)	1	1	$-\frac{\tau\partial_\rho P_1}{v'_e(\rho)\rho}$	$-\frac{\tau\partial_v P_2}{\rho}$	$c_{\pm} = \frac{\partial_v P_2}{2\rho} \mp \sqrt{\partial_\rho P_1 + (\frac{\partial_v P_2}{2\rho})^2}$

The book will discuss headway distribution models in Sect. 2.3. Due to the space limitation, cluster models will be omitted. We will briefly show the idea of gas-kinetic model (Table 2.1). Prigogine and Herman (1971) proposed the following Boltzmann equation

$$\frac{d_v \tilde{\rho}}{dt} = \frac{d\tilde{\rho}}{dt} + v \frac{d\tilde{\rho}}{dx} = \left(\frac{d\tilde{\rho}}{dt} \right)_{\text{acc}} + \left(\frac{d\tilde{\rho}}{dt} \right)_{\text{int}} \quad (2.18)$$

Acceleration behaviors can be modeled by the relaxation process that transforms from velocity distribution $\tilde{P}(v; x, t)$ to the expected velocity distribution $\tilde{P}_0(v)$

$$\left(\frac{d\tilde{\rho}}{dt} \right)_{\text{acc}} = \frac{\rho(x, t)}{\tau(\rho(x, t))} \left[\tilde{P}_0(v) - \tilde{P}(v|x, t) \right] \quad (2.19)$$

Interactions among vehicles are

$$\left(\frac{d\tilde{\rho}}{dt} \right)_{\text{int}} = (1 - p(\rho))\rho(x, t) [\bar{v}(x, t) - v] \tilde{\rho}(x, t, v) \quad (2.20)$$

where

$$\bar{v}(x, t) = \int_0^\infty v \tilde{P}(v|x, t) dv = \int_0^\infty v \frac{\tilde{\rho}(x, t, v)}{\rho(x, t)} dv \quad (2.21)$$

Recently, mesoscopic traffic simulation has been attracted more efforts on the operations of dynamic traffic systems, e.g., CONTRAM (Leonard et al. 1989), DYNASMART (Jayakrishnan et al. 1994), FASTLANE (Gawron 1998), DYNAMIT (Ben-Akiva et al. 2010), INTEGRATION (van Aerde and Rakha 2002), MEZZO

(Burghout et al. 2006), DYNAMEQ (Snelder 2009), dynaCHINA (Lin and Song 2006), etc.

2.2.3 Microscopic Modeling

Microscopic traffic flow model utilizes the *Lagrangian* method to study traffic flow dynamics by describing one vehicular trajectory or interactions among multiple vehicles. Microscopic modeling can be divided into car-following model and lane changing model. This section only reviews the theoretical development of car-following models. Common car-following models include: stimulus response model, safe distance or behavioral model, psychological-physical/action point model, artificial intelligence-based model, cellular automaton (CA), etc.

The updating equations of velocity and location are

$$\begin{cases} v_n(t + \Delta t) = v_n(t) + \dot{v}_n(t)\Delta t \\ x_n(t + \Delta t) = x_n(t) + v_n(t)\Delta t + \frac{1}{2}\dot{v}_n(t)(\Delta t)^2 \end{cases} \quad (2.22)$$

where $x_n(t)$ is the location of the n th vehicle at time t , $v_n(t) = \dot{x}_n(t)$ is the velocity of the n th vehicle at time t , Δt is the updating time step.

The general form of acceleration equation is

$$a_n = f(x_{n-1}, x_n, v_{n-1}, v_n), \quad n \in \mathbb{N}^+ \quad (2.23)$$

Table 2.2 shows the long-term evolution of typical microscopic car-following models. We can expand Eq. (2.23) to the scenario of a multiple-car-following model as

$$a_n = f(x_n, \dots, x_{n-m+1}; v_n, \dots, v_{n-m+1}), \quad m, n \in \mathbb{N}^+ \quad (2.24)$$

where m is the number of vehicles that influence the n th vehicle.

In the past decade, some multi-anticipative car-following models were proposed to enhance the stability of traffic flow. One approach assumed that the individual location and velocity information could be shared among different vehicles to simulate multi-anticipative behaviors via inter-vehicle communications (Li and Wang 2007). Other approaches emphasized the multi-anticipative behaviors of human drivers and tried to model the actions of drivers via simulation models (Treiber et al. 2006a). Numerical experiments showed that multi-vehicle interactions generally enlarged the stable region of traffic flow. For example, based on the extended optimal velocity model (OVM) and full velocity difference model (FVDM), Lenz et al. (1999) showed that the stability of traffic flow was improved by taking into account relative velocities of vehicles. Results indicated that the multi-anticipative behavior enlarged the linear stability region. On the contrary, the human reaction or manipulation delays might lead to the instability of traffic flow. Appendix B applies perturbation

Table 2.2 Typical microscopic traffic flow models

Models	Acceleration equations ^a	Parameters
Pipes (1953)	$a_n(t + \tau) = c(v_n(t) - v_{n-1}(t))$	c
Gazis et al. (1961)	$a_n(t) = cv_n^m(t) \frac{v_n(t) - v_{n-1}(t)}{(x_n(t) - x_{n-1}(t))^l}$	c, m, l
Newell (1961)	$a_n(t) = c(x_n(t) - x_{n-1}(t))^l$	c
	$a(t) = \frac{1}{\tau} [v_{\text{opt}}(x_n(t) - x_{n-1}(t)) - v_{n-1}(t)]^b$	c, d
Bierley (1963)	$a_n(t) = \alpha(v_n(t) - v_{n-1}(t)) + \beta(x_n(t) - x_{n-1}(t))^l$	α, β
Sultan et al. (2004)	$a_n(t + \tau) = cv_n^m(t) \frac{\Delta v_n(t)}{\Delta x_n^l(t)} + k_1 a_{n-1}(t) + k_2 a_n(t)$	c, k_1, k_2
Bando et al. (1995)	$a_n(t) = c[v_{\text{opt}}(x_n(t) - x_{n-1}(t)) - v_n(t)]^c$	c, s_{safe}
Helbing and Tilch (1998)	$a_n(t) = c[v_{\text{opt}}(\Delta x_n(t)) - v_n(t)] + \lambda H \Delta v_n(t)^d$	c, λ
Jiang et al. (2001)	$a_n(t) = c[v_{\text{opt}}(\Delta x_n(t)) - v_n(t)] + \lambda \Delta v_n(t)$	c, λ
Treiber et al. (2000)	$a_n(t) = a_{\text{max}} \left[1 - \left(\frac{v_n}{v_{\text{max}}} \right)^\delta - \left(\frac{s^*(v_n, \Delta v_n)}{s_n} \right)^2 \right]^e$	δ, T_n, s_n

^a $\Delta x_n(t) = \Delta x_{n,n-1}(t) = x_n(t) - x_{n-1}(t)$, $\Delta v_n(t) = \Delta v_{n,n-1}(t) = v_n(t) - v_{n-1}(t)$

^b $v_{\text{opt}}(x_n(t) - x_{n-1}(t)) = v_{\text{free}} \left[1 - \exp \left(-\frac{c}{v_{\text{free}}} (x_n(t) - x_{n-1}(t)) - d \right) \right]$

^c $v_{\text{opt}}(x_n(t) - x_{n-1}(t)) = \frac{v_{\text{free}}}{2} [\tanh(X_n(t) - x_{n-1}(t) - S_{\text{safe}}) + \tanh(S_{\text{safe}})]$

^d $H = H(-\Delta v_n(t))$ is Heaviside function

^e $s^*(v_n, \Delta v_n) = s_{\text{min}} + \max\{T_n v_n + v_n \Delta v_n / (2\sqrt{a_{\text{max}} b_n}), 0\}$

analysis (PA) to derive the critical linear stability condition for multi-car-following models.

However, the common problem of typical models listed in Table 2.2 is to define a deterministic acceleration equation. In field applications, vehicles are influenced by many stochastic internal and external factors that are not taken into consideration in the classic deterministic acceleration equation based models. How to depict the stochastic characteristics of driving behaviors and time-varying traffic flow states more accurately will be discussed in Chap. 4 by using the Markov model based on headway/spacing distributions.

2.2.4 Stochastic Modeling

Road traffic flow is influenced by various random factors, including both external factors such as the weather, and internal factors such as transportation facilities, vehicle characteristics, driver behaviors, etc. These stochastic factors make the deterministic approaches difficult to accurately estimate or predict dynamic traffic evolutions. To overcome this problem, numerous stochastic approaches were developed for continuous traffic flow modeling. In this study, they are divided into the following four categories:

- Macroscopic traffic flow modeling by the randomization of the first-order conservation law and/or higher-order momentum equations;

- Microscopic traffic flow modeling by the randomization of driving behaviors and/or mixed traffic flows;
- Fundamental diagram and the corresponding phase transitions by the randomization of relationships among flow, density, and velocity;
- Transportation reliability studies on the randomization of road capacity and/or travel time distribution.

In summary, the four categories of stochastic approaches can be generally written as

$$\Theta(x + \Delta x, t + \Delta t) = f(\Theta(x, t), \Delta x, \Delta t) + \epsilon(x, t) \quad (2.25)$$

where $\Theta(x, t)$ is the vector of traffic states at location x and time t , $f(\cdot)$ is the traffic state evolution function, $\epsilon(x, t)$ is noise function.

First, in macroscopic modeling, Boel and Mihaylova (2006) proposed a stochastic compositional model for freeway traffic flows, where the randomness was reflected in the probability distributions of sending and receiving functions, also in the well-defined noise term of speed adaptation rules. Sumalee et al. (2011) proposed a first-order macroscopic stochastic cell transmission model (SCTM), each operational mode of which was formulated as a discrete time bilinear stochastic system to model traffic density of freeway segments in stochastic demand and supply. However, this approach still assumed a deterministic FD with a second-order wide-sense stationary (WSS) noisy disturbance. Wang et al. (2005, 2006, 2009b) presented a general stochastic macroscopic traffic flow model of freeway stretches based on a traffic state estimator using extended Kalman filtering and developed the freeway network state monitoring software (i.e., REal-time motorway Network trAffic State Surveillance, RENAISSANCE).

Secondly, in microscopic modeling, Wagner (2011) proposed a time-discrete stochastic harmonic oscillator for car-following based on the deterministic acceleration Eq. (2.23), i.e., $a_n = f(x_{n-1}, x_n, v_{n-1}, v_n) + \epsilon$, $n \in \mathbb{N}^+$ where ϵ is the noise term. Huang et al. (2001) proposed a stochastic CA model by incorporating braking probability, occurrence, and dissipation probability. Some other approaches include Nagel and Schreckenberg (1992); Zhu et al. (2007). Since microscopic car-following model is highly correlated with headway/spacing distributions, we will further discuss a Markov model to depict headway/spacing evolutions in Chap. 4.

Thirdly, in stochastic FD and its phase transition analysis, the classic assumption is the existence of deterministic functions of flow-density and speed-density. FD has been the foundation of traffic flow theory and transportation engineering. According to the definition by Edie (1961), in the $t \sim x$ vehicular trajectory diagram, we have

$$\rho = \sum_{i=1}^n \frac{T_i}{|A|}, \quad q = \sum_{i=1}^n \frac{D_i}{|A|}, \quad v = \frac{q}{\rho} = \frac{\sum_{i=1}^n D_i}{\sum_{i=1}^n T_i} \quad (2.26)$$

where $|A|$ is the area of an arbitrary region A , T_i , and D_i are the travel time and distance for the i th vehicle in A .

Treiber et al. (2006b) investigated the adaptation of headways in car-following models as a function of the local velocity variance to study the scattering features in flow-density plot. Ngoduy (2011) argued that the widely scattering flow-density relationship might be caused by the random variations in driving behavior. The distribution features and probabilistic boundaries estimation method will be further discussed in Chap. 5.

At last, in transportation reliability studies, Brilon et al. (2005) pointed out that the concept of stochastic capacities was more realistic and more useful than the traditional concept of deterministic capacity. In the last decade, many efforts were made to identify the characteristics of traffic flow breakdown and its occurrence condition (Evans et al. 2001; Kerner and Klenov 2006; Kesting et al. 2010; Smilowitz and Daganzo 2002). Usually, traffic breakdown phenomena can be triggered by external disturbances or internal perturbations. This book only considers the latter one that has been widely observed and validated when studying the features of oscillations (Banks 2006; Del Castillo 2001; Jost and Nagel 2003; Kerner and Klenov 2006; Kim and Zhang 2008; Lu and Skabardonis 2007; Son et al. 2004; Wang et al. 2007). Many approaches can be used in traffic flow breakdown phenomena. For example, Bassan et al. (2006) used the mathematical property of log periodic oscillations (LPO) to model traffic density over time. Habib-Mattar et al. (2009) developed a density-versus-time model to describe traffic breakdown, it was found that density increased sharply toward the peak period and then decreased and increased again toward the breakdown. Since density cannot be directly measured in field, many researchers tend to study the relationship between traffic flow breakdown probability with the upstream flow. In general, the probability shows an increasing sigmoid curve in terms of the upstream flow, where Weibull distribution is commonly incorporated to formulate the curve (Banks 2006; Brilon et al. 2005; Chow et al. 2009; Lorenz and Elefteriadou 2001; Mahnke and Kühne 2007). Chen and Zhou (2010) proposed the α -reliable mean-excess traffic equilibrium (METE) model that explicitly considered both reliability and unreliability aspects of travel time variability in the route choice decision process. Stochastic capacity is highly correlated with traffic flow breakdown probability, and the analytical derivation of phase transition will be present in Chap. 6.

Differing from the above four kinds of stochastic approaches, Mahnke et al. (2001, 2005); Mahnke and Kühne (2007); Mahnke and Pieret (1997) applied stochastic process to the dynamic mechanism of traffic congestion occurrence and dissipation based on time-varying probability distributions, by using master equation of statistical physics to analyze the phase transitions and nucleation phenomena in jam queues. The general form of master equation is (Chowdhury et al. 2010; van Kampen 2007)

$$\frac{\partial \mathcal{P}(z, t)}{\partial t} = \int [\omega_{z'z} \mathcal{P}(z', t) - \omega_{zz'} \mathcal{P}(z, t)] dz' \quad (2.27)$$

where z and z' are continuous state variables, t is continuous time, $\mathcal{P}(z, t)$ is the probability of being in state z at time t , $\omega_{zz'}$ is the transition rate from state z to z' , satisfying $\int \omega_{zz'} dz' = \int \omega_{z'z} dz' = 1$.

In stochastic traffic flow modeling, traffic states are usually represented by discrete variables. Let \mathcal{S} be an arbitrary discrete state, the probability that traffic belongs to state \mathcal{S} at time t is $\mathcal{P}_{\mathcal{S}}(t)$. The evolution of traffic states can be described by the following discrete master equation

$$\frac{d\mathcal{P}_{\mathcal{S}}(t)}{dt} = \sum_{\mathcal{S}'} \omega_{\mathcal{S}'\mathcal{S}} \mathcal{P}_{\mathcal{S}'}(t) - \sum_{\mathcal{S}'} \omega_{\mathcal{S}\mathcal{S}'} \mathcal{P}_{\mathcal{S}}(t) \quad (2.28)$$

where $\omega_{\mathcal{S}\mathcal{S}'}$ is the transition rate from state \mathcal{S} to \mathcal{S}' , satisfying $\sum_{\mathcal{S}'} \omega_{\mathcal{S}\mathcal{S}'} = \sum_{\mathcal{S}'} \omega_{\mathcal{S}'\mathcal{S}} = 1$, the first term on the right side is the transition rate from one state \mathcal{S}' to the current state \mathcal{S} , the second term is the transition rate from the current state \mathcal{S} to another state \mathcal{S}' .

Furthermore, when traffic state and time are both discrete, suppose traffic state is \mathcal{S} at time t , according to the master equation in Eq. (2.28), at time $t + \Delta t$, the probability that traffic still belongs to state \mathcal{S} is

$$\mathcal{P}_{\mathcal{S}}(t + \Delta t) = \left(1 - \sum_{\mathcal{S}' \neq \mathcal{S}} \omega_{\mathcal{S}\mathcal{S}'} \Delta t\right) \mathcal{P}_{\mathcal{S}}(t) + \sum_{\mathcal{S}' \neq \mathcal{S}} \omega_{\mathcal{S}'\mathcal{S}} \Delta t \mathcal{P}_{\mathcal{S}'}(t) \quad (2.29)$$

Regard the formation and dissipation of a jam queue as a Markov process, and assume at most one vehicle can join the jam queue in a differentiation δt , i.e., the probability that two or more vehicles join the jam queue in δt is $o(\delta t)$, then the master equation of the jam queue length distribution is

$$\begin{aligned} \frac{\partial \mathcal{P}(n, t)}{\partial t} = & \omega_+(n-1) \mathcal{P}(n-1, t) + \omega_-(n+1) \mathcal{P}(n+1, t) \\ & - [\omega_+(n) \mathcal{P}(n, t) + \omega_-(n) \mathcal{P}(n, t)], \quad n \in \mathbb{N}^+ \end{aligned} \quad (2.30)$$

where $\omega_+(n)$ and $\omega_-(n)$ are the transition rates for a jam queue length changes of $\{n \mapsto n+1\}$ and $\{n \mapsto n-1\}$ vehicles, respectively.

Mahnke et al. (2001, 2005); Mahnke and Kühne (2007); Mahnke and Pieret (1997) studied the occurrence and evolution of jam queues in a homogeneous circle road with periodic boundary conditions, and defined the joining and leaving rates as

$$\omega_+(n) = \frac{v_{\text{opt}}(\Delta x_{\text{free}}(n)) - v_{\text{opt}}(\Delta x_{\text{cong}}(n))}{\Delta x_{\text{free}}(n) - \Delta x_{\text{cong}}(n)}, \quad \omega_-(n) = \frac{1}{\tau_{\text{out}}} \quad (2.31)$$

where the joining rate is calculated by the OVM, the optimal free-flow velocity is $v_{\text{opt}}(\Delta x_{\text{free}}(n))$, the congested optimal velocity is $v_{\text{opt}}(\Delta x_{\text{cong}}(n))$, and $\Delta x_{\text{cong}}(n) = \Delta x_{\text{cong}}$. Then we get the jam queue length $L_{\text{cong}} = n \Delta x_{\text{cong}}$ and the free-flow length $L_{\text{free}} = L - L_{\text{cong}} = n \Delta x_{\text{cong}}$, L is the length of the circle road. τ_{out} is the mean waiting time for a vehicle to leaving a jam queue.

Furthermore, Kühne et al. (2002) applied the master equation-based nucleation model to traffic flow breakdown phenomenon in ramping bottlenecks, and formulated the Fokker-Planck equation that described the jam queue evolutions.

2.3 Probabilistic Headway/Spacing Distributions

2.3.1 Simple Univariable Distributions

- **Negative exponential distribution**

Let h denote the random variable of headway. Negative exponential distribution describes the interarrival time as Poisson process. Events occur continuously and independently at a constant average rate. It is appropriate when traffic flow rate and density are small.

The PDF is

$$f(h|\lambda) = \lambda e^{-\lambda h}, \quad h \geq 0 \quad (2.32)$$

where λ is the rate parameter. It indicates aggressive driving behaviors when λ is relatively large, while it indicates timid driving behaviors when λ is relatively small.

The CDF is

$$F(h|\lambda) = 1 - e^{-\lambda h}, \quad h \geq 0 \quad (2.33)$$

The expectation and variance of headway are

$$E[h] = \frac{1}{\lambda}, \quad \text{Var}[h] = \frac{1}{\lambda^2} \quad (2.34)$$

Define the maximum likelihood function as

$$\ell(\lambda) \triangleq \lambda^n \exp\left(-\lambda \sum_{i=1}^n h_i\right) \quad (2.35)$$

Let $\partial \ell(\lambda) / \partial \lambda = 0$, we have the following maximum likelihood estimator (MLE)

$$\hat{\lambda} = \frac{1}{\bar{h}} \quad (2.36)$$

where $\bar{h} = \frac{1}{n} \sum_{i=1}^n h_i$, $\{h_1, \dots, h_n\}$ are samples.

• Shifted exponential distribution

Since the probability of headway near zero is large in negative exponential distribution, to avoid the extremely short headways, the PDF is shifted rightwards with a deterministic positive value.

The PDF is

$$f(h|\lambda, h_0) = \lambda e^{-\lambda(h-h_0)}, \quad h \geq h_0 \quad (2.37)$$

where h_0 is the translation parameter.

The CDF is

$$F(h|\lambda, h_0) = 1 - e^{-\lambda(h-h_0)}, \quad h \geq h_0 \quad (2.38)$$

The expectation and variance of headway are

$$E[h] = \frac{1}{\lambda} + h_0, \quad \text{Var}[h] = \frac{1}{\lambda^2} \quad (2.39)$$

Define the maximum likelihood function as

$$\ell(\lambda, h_0) \triangleq \lambda^n \exp\left(-\lambda \sum_{i=1}^n (h_i - h_0)\right) \quad (2.40)$$

Let $\partial \ell(\lambda, h_0) / \partial \lambda = 0$, we have the MLEs

$$\hat{h}_0 = \min\{h_1, \dots, h_n\}, \quad \hat{\lambda} = \frac{1}{\bar{h} - \hat{h}_0} \quad (2.41)$$

• Gamma distribution (Pearson type III distribution)

Gamma distribution is a two-parameter family of continuous probability distributions used to model headway commonly.

The PDF is

$$f(h|\alpha, \beta, h_0) = \frac{(h - h_0)^{\alpha-1} e^{-(h-h_0)/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad h \geq h_0 \quad (2.42)$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function with a shape parameter α and a scale parameter β .

The CDF is

$$F(h|\lambda, h_0) = \frac{\gamma(\alpha, (h - h_0)/\beta)}{\Gamma(\alpha)}, \quad h \geq h_0 \quad (2.43)$$

where $\gamma(\alpha, (h - h_0)/\beta) = \int_0^{(h-h_0)/\beta} t^{\alpha-1} e^{-t} dt$.

The expectation and variance are

$$E[h] = \alpha\beta + h_0, \quad \text{Var}[h] = \alpha\beta^2 \quad (2.44)$$

There is no closed-form solution for α and β . They can be numerically approximated by using Newton's method, method of moments, etc. However, Gamma distribution cannot be suitable to depict headway distribution when the shape parameter is larger than 1 because the bell-like shape gives low probability to short headways.

• Shifted lognormal distribution

If $\log(h - h_0)$ follows the normal distribution $\mathcal{N}(\mu_h, \sigma_h^2)$, then h belongs to shifted lognormal distribution. The lognormal relation holds if the change in a headway during a small time interval is a random proportion of the headway at the start of the interval, and the mean and the variance of the headway remain constant over time (Luttinen 1996). Shifted lognormal distribution is widely used in headway/spacing modeling in both scenarios of continuous and interrupted transportation facilities. It is also closely related to car-following models as well.

The PDF is

$$f(h|\mu_h, \sigma_h, h_0) = \frac{1}{\sqrt{2\pi}\sigma_h(h - h_0)} \exp\left(-\frac{(\log(h - h_0) - \mu_h)^2}{2\sigma_h^2}\right), \quad h \geq h_0 \quad (2.45)$$

where μ_h, σ_h are the mean and standard deviation, respectively, of the headway's natural logarithm

The CDF is

$$F(h|\mu_h, \sigma_h, h_0) = \Phi\left(\frac{\log(h - h_0) - \mu_h}{\sigma_h}\right), \quad h \geq h_0 \quad (2.46)$$

where $\Phi(\cdot)$ is the standard normal distribution function.

The expectation and variance of headway are

$$E[h] = \exp\left(\mu_h + \sigma_h^2/2\right) + h_0, \quad \text{Var}[h] = \exp\left(2\mu_h + \sigma_h^2\right) \left(\exp(\sigma_h^2) - 1\right) \quad (2.47)$$

Define the maximum likelihood function as

$$\ell(\mu_h, \sigma_h, h_0) \triangleq \prod_{i=1}^n f(h_i|\mu_h, \sigma_h, h_0) \quad (2.48)$$

Let

$$\frac{\partial \ell(\hat{\mu}_h, \hat{\sigma}_h, \hat{h}_0)}{\partial \hat{\mu}_h} = \frac{\partial \ell(\hat{\mu}_h, \hat{\sigma}_h, \hat{h}_0)}{\partial \hat{\sigma}_h} = \frac{\partial \ell(\hat{\mu}_h, \hat{\sigma}_h, \hat{h}_0)}{\partial \hat{h}_0} = 0$$

we have the following MLEs

$$\hat{h}_0 = \min\{h_1, \dots, h_n\} \quad (2.49a)$$

$$\hat{\mu}_h = \frac{1}{n} \sum_{i=1}^n \log(h_i - \hat{h}_0) \quad (2.49b)$$

$$\hat{\sigma}_h = \frac{1}{n} \sum_{i=1}^n \left(\log(h_i - \hat{h}_0) \right)^2 - \left(\frac{1}{n} \sum_{i=1}^n \log(h_i - \hat{h}_0) \right)^2 \quad (2.49c)$$

2.3.2 Compositional Distributions

The simple univariable distributions are incapable of describing both sharp peak and long tail properties of headway/spacing. Because of the coexistence of two main traffic flow states, i.e., free-flow headways and car-following headways, whose distributions are significantly different. So it's better to incorporate the compositional distribution functions.

The compositional PDF of headway is defined as $f(h)$, i.e.,

$$f(h) = \varphi p(h) + (1 - \varphi)q(h) \quad (2.50)$$

where $0 \leq \varphi < 1$ is the proportion of constrained headways in car-following states, $p(h)$ is the PDF of constrained headways, $q(h)$ is the PDF of free-flow headways.

According to the convolution formula, we have

$$q(h) = p^*(\lambda)^{-1} \lambda e^{-\lambda h} \int_0^h p(z) dz \quad (2.51)$$

where $p^*(\lambda)$ is the Laplace transform of $p(h)$, i.e.

$$p^*(\lambda) = \int_0^\infty e^{-\lambda h} p(h) dh \quad (2.52)$$

- **Hyperexponential distribution**

Schuhl (1955) first applied Hyperexponential distribution to model headway, known as Schuhl's (composite exponential) distribution.

The PDFs are

$$p(h) = \lambda_p e^{-\lambda_p(h-h_0)}, \quad q(h) = \lambda_q e^{-\lambda_q(h-h_0)} \quad (2.53)$$

where λ_p, λ_q are parameters of exponential distributions for car-following and free-flow states, respectively.

- **Hyperlang distribution**

Dawson and Chimini (1968) suggested the Erlang-distribution as a model for car-following headways, i.e.,

$$p(h) = \frac{\lambda_p^k (h - h_0)^{k-1} e^{-\lambda_p(h-h_0,p)}}{(k-1)!}, \quad q(h) = \lambda_q e^{-\lambda_q(h-h_0,q)} \quad (2.54)$$

where $k \in \mathbb{N}^+$, λ_p, λ_q are parameters of car-following and free-flow distributions, respectively. $h_{0,p}, h_{0,q}$ are the translation parameters. Luttinen (1996) pointed out that the hyperlang distribution has an exponential tail, and the shape of the PDF is similar to empirical headway distributions.

2.3.3 Mixed Distributions

It is found that many stationary distribution models could fit the empirical data of free flow but not congested flow. One way to solve this problem is to use the mixed headway distribution models. For example, in the M3 model (Cowan 1975), the headways of free-driving vehicles and those leader-following vehicles were assumed to follow different PDFs. But the calibration of mixed models is usually tedious, if we want to fit the empirical distributions with a high accuracy.

- **Semi-Poisson distributions**

Buckley (1968) proposed the semi-Poisson model that described the fluctuations in car-following states, conjectured there was a zone of emptiness in front of each vehicle, and compared it with Gamma distribution, shifted Gamma distribution, exponential distribution by using field measurements. Wasielewski (1974) applied non-parametric method to calculate constrained headway in semi-Poisson distribution.

The PDF of semi-Poisson distribution is

$$f(h) = \varphi p(h) + (1 - \varphi) \frac{\int_0^h p(h) dt}{p^*(\theta)} \theta e^{-\theta h}, \quad h \geq 0, \quad \theta > 0 \quad (2.55)$$

where $p^*(\theta) = \int_0^\infty e^{-\theta h} p(h) dh$ is the Laplace transform of $p(h)$.

Semi-Poisson distribution defines two common headway distributions for car-following scenarios, i.e.,

Gamma distribution and its Laplace transform

$$p(h) = \frac{h^{\alpha-1} e^{-h/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad q^*(\lambda) = (1 + \lambda\beta)^{-\alpha} \quad (2.56)$$

Gaussian distribution and its Laplace transform

$$p(h) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(h-\mu)^2}{2\sigma^2}}, \quad p^*(\lambda) = e^{(\sigma^2/2 - \mu)\lambda} \quad (2.57)$$

where μ and σ are the expectation and standard deviation of Gaussian distribution.

Branston (1976) proposed a mixed model of a generalized queuing model and semi-Poisson model. The PDF for free-flow headway is

$$q(h) = \lambda e^{-\lambda h} \int_0^h p(z) e^{\lambda z} dz \quad (2.58)$$

where $1/\lambda$ is the average headway. Assume headway follows lognormal distribution in car-following mode as

$$p(h) = \frac{1}{\sqrt{2\pi}\sigma h} e^{-\frac{(\log h - \mu)^2}{2\sigma^2}} \quad (2.59)$$

then the PDF for mixed headway is

$$f(h) = \varphi p(h) + (1 - \varphi) \lambda e^{-\lambda h} \int_0^h p(z) e^{\lambda z} dz \quad (2.60)$$

2.3.4 Random Matrix Model

Krbálex and Sěba (2001) showed that traffic data from different sources belonged to a class of random matrix distributions. Abul-Magd (2006) applied the random matrix theory (RMT) to the car-parking problem and adopted a Coulomb gas model that associated coordinates of gas particles with the eigenvalues of a random matrix, in which the Wigner surmise for Gaussian unitary ensemble (GUE) was given by

$$P(\varsigma) = \frac{32}{\pi^2} \varsigma^2 e^{-4\varsigma^2/\pi}, \quad \varsigma \geq 0 \quad (2.61)$$

where ς is the space gap, i.e. spacing minus vehicle length.

Abul-Magd (2007) pointed out that RMT modeled the Hamiltonians of chaotic systems as members of an ensemble of random matrices that depended only on the symmetry properties of the system and GUE modeled systems violating time reversal symmetry. In the phase transition from the free-flow state to the congested state, the GUE of RMT is

$$P_{\text{GUE}}(\varsigma) = \frac{32\varsigma^2}{\pi^2\bar{\varsigma}^3} e^{-4\varsigma^2/\pi\bar{\varsigma}^2} \quad (2.62)$$

where $\bar{\varsigma}$ is the mean space gap.

Space gaps in free-flow traffic are uncorrelated and follow the Poisson distribution

$$P_{\text{Poisson}}(\varsigma) = \frac{1}{\bar{\varsigma}} e^{-\varsigma/\bar{\varsigma}} \quad (2.63)$$

More recently, models based on headway/spacing distributions and state transitions received more interests. For example, Jin et al. (2009) revealed that the distributions of departure headways at each position in a queue approximately followed a certain lognormal distribution except the first one by using video data collected from four intersections in Beijing. Wang et al. (2009a) estimated a cellular automation model for spacing distribution.

2.4 Summary

This chapter reviews the history of traffic flow theory from the perspectives of macroscopic, mesoscopic, microscopic modeling approaches, and summaries their scopes of applications. This chapter focuses on the stochastic traffic flow modeling method and the headway/spacing probabilistic distributions. Based on the overview of the state-of-the-art traffic flow model, the historical development of traffic flow modeling approaches are summarized as a literature foundation for the other chapters.

Stochastic Evolutions of Dynamic Traffic Flow
Modeling and Applications

Chen, X.M.; Li, L.; Shi, Q.

2015, XIX, 193 p. 65 illus., 56 illus. in color., Hardcover

ISBN: 978-3-662-44571-6