Chapter 2
A BEC System with Dimensions $N = 2, 3$: Ground State Solutions

Abstract As introduced in Chap. 1, we study the ground state solutions of system (1.2) in the entire space $\mathbb{R}^N$ with $N = 2, 3$. Precisely, motivated by Sirakov's previous work, we prove some uniqueness results of positive (ground state) solutions for the special case $\lambda_1 = \lambda_2$. These give partial answers to Sirakov's conjecture. For the general case $\lambda_1 \neq \lambda_2$, we prove a sharp result on the parameter range for the existence of ground state solutions. The asymptotic behaviors of ground state solutions can be investigated as a corollary. We also prove a nonexistence result about positive solutions. These results answer partially some open questions raised by Ambrosetti, Colorado and Sirakov. Our proof is mainly applying asymptotic analysis together with the classical bifurcation theory.

2.1 Main Results

Consider the following system with cubic nonlinearities which arises as mathematical models from nonlinear optics and Bose-Einstein condensates (BEC):

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta v u^2, \quad x \in \mathbb{R}^N, \\ u \geq 0, v \geq 0 \text{ in } \mathbb{R}^N, \\ u, v \to 0 \text{ as } |x| \to +\infty, \end{cases} \tag{2.1}$$

where $N = 2, 3$, $\mu_1, \mu_2, \lambda_1, \lambda_2 > 0$, and $\beta \neq 0$ is a coupling constant. As pointed out in Chap. 1, system (2.1) has two semi-trivial solutions $(\omega_1, 0)$ and $(0, \omega_2)$, where ω_i is the unique positive radially symmetric solution of the following scalar equation (see [60])

$$-\Delta u + \lambda_i u = \mu_i u^3, \;\; u > 0, \;\; u \in H^1(\mathbb{R}^N), \tag{2.2}$$

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and the corresponding least energy is

$$B_i := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla\omega_i|^2 + \lambda_i\omega_i^2)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N} \mu_i\omega_i^4\,\mathrm{d}x. \tag{2.3}$$

Clearly there holds

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_i u^2)\,\mathrm{d}x \geq 2\sqrt{B_i}\left(\int_{\mathbb{R}^N} \mu_i u^4\,\mathrm{d}x\right)^{1/2}, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.4}$$

Denote $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for convenience. Recalling that $E_\beta, \mathcal{N}_\beta, A_\beta$ are defined in (1.4), (1.5) and (1.6) respectively, and the ground state solution is defined in Definition 1.2. Sirakov [80] proved the following interesting result in 2007.

Theorem A ([80, Theorem 1]) *Suppose* $\lambda_1 = \lambda_2 = \lambda$.

(i) *For* $\beta \in (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty)$, A_β *is attained by the couple* $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$, *where* (k_β, l_β) *satisfies* $\mu_1 k + \beta l = 1$ *and* $\mu_2 l + \beta k = 1$, *and* ω_0 *is the unique positive radially symmetric solution of* (2.2) *with* $\mu_i = 1$. *That is,* $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$ *is a ground state solution of* (2.1).

(ii) *For* $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ *and* $\mu_1 \neq \mu_2$, (2.1) *has no nontrivial nonnegative solutions.*

Then Sirakov [80] raised a conjecture: For spatial dimensions $N = 1, 2, 3$, the couple $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$ is the unique positive solution to (2.1) up to a translation. Recently, Wei and Yao [89, Theorem 4.2] proved this conjecture in case $\beta > \max\{\mu_1, \mu_2\}$, and [89, Theorem 1.1] proved it in the case where $0 < \beta < \min\{\mu_1, \mu_2\}$ and $N = 1$. For $N = 2, 3$, this conjecture was also proved in [89, Theorem 4.1] for $0 < \beta < \beta'$, where β' is a small constant. That is, whether Sirakov's conjecture holds or not in the remaining case where $N = 2, 3$ and $\beta \in (\beta', \min\{\mu_1, \mu_2\})$ remains open. Here we can give a partial answer.

Theorem 2.1 *Assume that* $\lambda_1 = \lambda_2$ *and* $\mu_1 \neq \mu_2$. *Then there exists small* $\delta > 0$ *such that for any* $\beta \in (\min\{\mu_1, \mu_2\} - \delta,\ \min\{\mu_1, \mu_2\})$, *the couple* $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$ *is the unique positive solution of system* (2.1) *up to a translation.*

The proof of Theorem 2.1 mainly use the classical bifurcation theory. Clearly, neither [89, Theorem 4.1] nor Theorem 2.1 proves Sirakov's conjecture for all $\beta \in (0, \min\{\mu_1, \mu_2\})$. A slightly weaker but natural question is: for $\beta \in (0, \min\{\mu_1, \mu_2\})$, whether the *ground state* solutions are unique or not, up to a translation? Here we can give a positive answer to this question.

Theorem 2.2 *Assume* $\lambda_1 = \lambda_2$. *Then for any* $\beta \in (0, \min\{\mu_1, \mu_2\})$, *the couple* $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$ *is the unique ground state solution of system* (2.1) *up to a translation.*

Clearly there is a fully symmetric case where $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2 = \beta$ remaining. In this case, it is easy to check that $((2\beta)^{-1/2}\cos\theta\omega_0, (2\beta)^{-1/2}\sin\theta\omega_0)$ is a positive solution of (2.1) for any $\theta \in (0, \frac{\pi}{2})$. In fact, Wei and Yao [89, Theorem 1.2] proved that

$$\mathfrak{S} := \left\{((2\beta)^{-1/2}\cos\theta\omega_0, (2\beta)^{-1/2}\sin\theta\omega_0) : \theta \in (0, \frac{\pi}{2})\right\} \tag{2.5}$$

contains all positive solutions of (2.1) for $N = 1$. Here we can prove a stronger result for $N = 2, 3$.

Theorem 2.3 *Let $N = 2, 3$, $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2 = \beta$. Assume that (u, v) be a nontrivial solution of* (2.1) *with $u > 0$. Then $v = Cu$ for some constant $C \neq 0$. In particular, the set $\mathfrak{S}$ contains all positive solutions of* (2.1)*, and* (2.1) *has no semi-nodal solutions (namely one component of the solution positive and the other one sign-changing; the definition will be given Chap. 3).*

Now let us consider the general case $\lambda_1 \neq \lambda_2$. Clearly in this case, system (2.1) has *no* nontrivial solutions (u, v) satisfying $u/v \equiv$ constant. This fact makes the general case $\lambda_1 \neq \lambda_2$ much more delicate comparing to the symmetric case $\lambda_1 = \lambda_2$. Recalling ω_i in (2.2)–(2.3), we define two constants

$$\beta_1 := \inf_{\phi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \lambda_2\phi^2)}{\int_{\mathbb{R}^N} \omega_1^2\phi^2}, \tag{2.6}$$

$$\beta_2 := \inf_{\phi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla\phi|^2 + \lambda_1\phi^2)}{\int_{\mathbb{R}^N} \omega_2^2\phi^2}. \tag{2.7}$$

These two constants were first introduced by Ambrosetti and Colorado [6] in 2006, where the reader can find the significance of these two constants. Furthermore, they proved the following result on the existence of positive (ground state) solutions.

Theorem B ([6, Theorems 1 and 2])

(i) *System* (2.1) *has a positive radially symmetric solution (U_β, V_β) for any $0 < \beta < \min\{\beta_1, \beta_2\}$.*
(ii) *For any $\beta > \max\{\beta_1, \beta_2\}$, system* (2.1) *has a positive radially symmetric ground state solution (U_β, V_β) with*

$$E_\beta(U_\beta, V_\beta) = A_\beta < \min\{B_1, B_2\}. \tag{2.8}$$

In the same paper, Ambrosetti and Colorado also suspected that (U_β, V_β) obtained in Theorem B-(i) are also ground state solutions (see [6, Remark 5]). Recently, Ikoma and Tanaka [57] answered this question partially.

Theorem C (see [57, Propositions 2.3 and 2.5, Remark 2.6]) *For any* $0 < \beta < \min\{\beta_1, \beta_2, \sqrt{\mu_1\mu_2}\}$, (U_β, V_β) *obtained in Theorem B-(i) is a ground state solution with*

$$E_\beta(U_\beta, V_\beta) = A_\beta > \max\{B_1, B_2\}. \tag{2.9}$$

Comparing to Theorem B, $\beta < \sqrt{\mu_1\mu_2}$ is assumed in Theorem C because, as pointed out in Chap. 1, that whether $\mathcal{N}_\beta$ is a natural constraint of E_β for $\beta \ge \sqrt{\mu_1\mu_2}$ is *unknown* (see Proposition A).

A basic question is: What are the optimal ranges of the parameter β for the existence of positive (ground state) solutions? This is an open question raised by Sirakov in [80, Remark 4]. Here we are interested in the question concerning *ground state* solutions. In other words, let us define

$$\begin{aligned} \bar{\beta}_1 &:= \sup\{\beta' > 0 \mid (2.1) \text{ has a ground state solution for all } 0 < \beta < \beta'\}, \\ \bar{\beta}_2 &:= \inf\{\beta' > 0 \mid (2.1) \text{ has a ground state solution for all } \beta > \beta'\}. \end{aligned} \tag{2.10}$$

Then both $(0, \bar{\beta}_1)$ and $(\bar{\beta}_2, +\infty)$ are the optimal ranges for the existence of ground state solutions. Our question is: What are the optimal constants $\bar{\beta}_i,\ i = 1, 2$?

Remark that, when $\lambda_1 = \lambda_2$, Theorem A answered this question completely. Since $\beta_1 = \mu_1$ and $\beta_2 = \mu_2$ (the proof will be given later), we see that β_i are the optimal constants for the existence of ground state solutions, namely $\bar{\beta}_1 = \min\{\beta_1, \beta_2\}$ and $\bar{\beta}_2 = \max\{\beta_1, \beta_2\}$ for $\lambda_1 = \lambda_2$. Indeed, β_i are also the optimal constants for the existence of nontrivial positive solutions for this case $\lambda_1 = \lambda_2$.

A natural question that people are interested in is: For the general case $\lambda_1 \ne \lambda_2$, do we still have $\bar{\beta}_1 = \min\{\beta_1, \beta_2\}$ or $\bar{\beta}_2 = \max\{\beta_1, \beta_2\}$? If so, then β_i are the optimal constants for the existence of ground state solutions for all cases, that is, β_i might have much deeper significance comparing to those pointed out in [6]. Define

$$\begin{aligned} H_r &:= \{(u, v) \in H : u, v \text{ are both radially symmetric}\}, \\ \mathcal{N}_\beta^* &:= \mathcal{N}_\beta \cap H_r, \quad A_\beta^* := \inf_{(u,v)\in\mathcal{N}_\beta^*} E_\beta(u, v). \end{aligned}$$

Without loss of generality, we may assume that $\lambda_1 < \lambda_2$. The following results were proved by Sirakov [80].

Proposition A ([80, Propositions 1.1]) *If* A_β *(resp.* A_β^**) is attained by a couple* $(u, v) \in \mathcal{N}_\beta$ *(resp.* $(u, v) \in \mathcal{N}_\beta^*$*), then* (u, v) *is a critical point of* E_β*, provided* $\beta < \sqrt{\mu_1\mu_2}$.

Proposition A indicates that, when $\beta < \sqrt{\mu_1\mu_2}$, the existence of ground state solutions is equivalent to that A_β is attained. As pointed out in Chap. 1, whether this conclusion holds or not for the remaining case $\beta \ge \sqrt{\mu_1\mu_2}$ remains open.

Theorem D ([80, Theorem 2.2]) *Suppose* $\lambda_1 < \lambda_2$.

(i) *There exists* $\beta' > 0$ *such that for any fixed* $\beta \in (0, \beta')$, (2.1) *has a positive ground state solution* (u, v) *with* $E_\beta(u, v) = A_\beta = A_\beta^*$.

(ii) *System* (2.1) *has no nonnegative nontrivial solutions for any* $\beta \in [\mu_2, \mu_1]$.
(iii) *For any* $\beta \in [\mu_2, \sqrt{\mu_1\mu_2})$, *neither* A_β *nor* A^*_β *is attained.*
(iv) *There exists* $\beta'' > 0$ *such that for any* $\beta > \beta''$, (2.1) *has a positive ground state solution* (u, v) *with* $E_\beta(u, v) = A_\beta = A^*_\beta$.

Here we have the following result, which gives the first partial answer to the question of existing optimal constants for the existence of ground state solutions for the general case $\lambda_1 < \lambda_2$ and so improves Theorem D.

Theorem 2.4 *Let* $\lambda_1 < \lambda_2$ *and* $\mu_1 \geq \mu_2$. *Then* $\beta_2 < \mu_2 \leq \mu_1 < \beta_1$, *and*

(i) *there exists small* $\delta > 0$ *such that* (2.1) *has no nonnegative nontrivial solutions for any* $\beta \in (\mu_2 - \delta,\ \mu_1 + \delta)$*;*
(ii) *for any* $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$, *neither* A_β *nor* A^*_β *is attained, namely* (2.1) *has no ground state solutions. Therefore, by Theorem C, it follows that* β_2 *is an optimal constant for the existence of ground state solutions;*
(iii) $(U_\beta, V_\beta) \to (0, \omega_2)$ *strongly in* H *as* $\beta \uparrow \beta_2$, *where* (U_β, V_β) *is in Theorems B and C;*
(iv) *there exists small* $\delta_1 > 0$ *such that for any* $\beta \in (\beta_2 - \delta_1, \beta_2)$, *the ground state solution of* (2.1) *is unique up to a translation.*

Remark 2.1 Theorem 2.4-(i) and (ii) indicate that (2.1) has *no* ground state solutions for any $\beta \in [\beta_2, \mu_1 + \delta)$.

Remark 2.2 Sirakov [80] gave the precise defintion of β', but *no* information whether β' is an optimal constant or not. Obviously, in the case $\mu_2 \leq \mu_1$, Theorem 2.4 improves Theorem D, and $\beta' \leq \beta_2$ must holds. Besides, we obtain the uniqueness and asymptotic behaviors of ground state solutions as $\beta \uparrow \beta_2$, namely (U_β, V_β) is unique and must converges to the semi-trivial solution $(0, \omega_2)$. It is known that $(0, \omega_2)$ is a semi-trivial solution for all β. Thus we can treat $(0, \omega_2; \beta)$ as a trivial branch of solutions for system (2.1). Our result indicates that, β_2 is actually a bifurcation point, and $(U_\beta, V_\beta; \beta)$ is a nontrivial branch of solutions arising from the trivial branch $(0, \omega_2; \beta)$ at the bifurcation point β_2. This gives a partial answer to an open question raised by Ambrosetti and Corolado [7].

Remark 2.3 If we assume $\lambda_1 > \lambda_2$ and $\mu_1 \leq \mu_2$, we can get a similar theorem. Hence, in the case where $\lambda_1 \neq \lambda_2$ and $(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \leq 0$, we have $\bar{\beta}_1 = \min\{\beta_1, \beta_2\}$ and so $(0, \min\{\beta_1, \beta_2\})$ is an optimal range for the existence of ground state solutions. This seems to be the first result on this aspect in general case $\lambda_1 \neq \lambda_2$.

Remark 2.4 Theorems 2.1 and 2.4 were published in a joint work with Zou [35], and Theorem 2.2 was published in another joint work with Zou [33]. We remark that Theorem 2.3 is *new* and we did not write it in any articles in the past. On the other hand, recently we proved a non-existence result of nontrivial positive bounded solutions to a more general system (i.e. system (2.1) can be seen as a special case of it) in the half space $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x = (x_1, \ldots, x_N),\ x_N > 0\}$, where $N \geq 2$ can be arbitrary large and so this system can be of *supercritical* growth; see [28] for details.

We will prove Theorems 2.1 and 2.3 in Sect. 2.2. In Sect. 2.3 we give the proof of Theorem 2.4. Theorem 2.2 will be proved in Chap. 4, where we will study system (1.2) in the critical dimension case $N = 4$. We give some notations here. Throughout this chapter, we denote the norm of $L^p(\mathbb{R}^N)$ by $|u|_p = (\int_{\mathbb{R}^N} |u|^p \,dx)^{\frac{1}{p}}$. Define

$$\|u\|_{\lambda_i}^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda_i |u|^2\right) dx, \quad i = 1, 2$$

as norms of $H^1(\mathbb{R}^N)$. The norm of H is defined by $\|(u, v)\|^2 := \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2$.

2.2 Uniqueness of Positive Solutions

First we give the proof of Theorem 2.3 via a simple observation.

Proof (Proof of Theorem 2.3) Let $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \beta$. Assume that (u, v) is a nontrivial solution of system (2.1) with $u > 0$. By elliptic estimates we see that $u, v \in H^1(\mathbb{R}^N)$. Define $\psi = v/u$. Since $\Delta u + Pu = 0$ and $\Delta v + Pv = 0$, where $P = \beta u^2 + \beta v^2 - \lambda$, we easily conclude that $\nabla \cdot (u^2 \nabla \psi) = 0$. Define cut-off functions $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_R(x) = \begin{cases} 1, & x \in B_R(0), \\ 0, & x \notin B_{2R}(0), \end{cases} \quad \text{with } |\nabla \varphi| \le \frac{C}{R},$$

where C is independent of R. Then

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \varphi_R^2 \psi \nabla \cdot (u^2 \nabla \psi) dx = -\int_{\mathbb{R}^N} \nabla(\varphi_R^2 \psi) \cdot (u^2 \nabla \psi) dx \\ &= -\int_{\mathbb{R}^N} \varphi_R^2 u^2 |\nabla \psi|^2 dx - 2\int_{\mathbb{R}^N} \varphi_R \psi u^2 \nabla \varphi_R \nabla \psi \, dx, \end{aligned}$$

and so

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi_R^2 u^2 |\nabla \psi|^2 dx &\le 2\int_{\mathbb{R}^N} |\varphi_R u \nabla \psi||\psi u \nabla \varphi_R| dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} \varphi_R^2 u^2 |\nabla \psi|^2 dx + 2\int_{\mathbb{R}^N} v^2 |\nabla \varphi_R|^2 dx. \end{aligned}$$

Consequently,

$$\begin{aligned}\int_{|x|\le R} u^2|\nabla\psi|^2\mathrm{d}x &\le \int_{\mathbb{R}^N}\varphi_R^2u^2|\nabla\psi|^2\mathrm{d}x \le 4\int_{\mathbb{R}^N}v^2|\nabla\varphi_R|^2\mathrm{d}x\\ &\le \frac{4C^2}{R^2}\int_{\mathbb{R}^N}v^2\mathrm{d}x\to 0 \quad \text{as } R\to+\infty.\end{aligned}$$

Therefore, $\int_{\mathbb{R}^N} u^2|\nabla\psi|^2\mathrm{d}x = 0$, namely ψ is a non-zero constant. Thus $v = Cu$ for some constant $C \neq 0$. In particular, v does not change sign, and so Theorem 2.3 follows immediately. □

Now let us turn to the proof of Theorem 2.1. Recall that $H_r^1(\mathbb{R}^N)$ is a subspace of $H^1(\mathbb{R}^N)$ that consists of radially symmetric functions. Let ω be the unique radially symmetric positive solution of

$$-\Delta u + u = u^3,\ \ u>0,\ \ u\in H^1(\mathbb{R}^N), \tag{2.11}$$

and the corresponding least energy is

$$B := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\omega|^2+\omega^2)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N}\omega^4\,\mathrm{d}x. \tag{2.12}$$

Then it is easy to check that

$$\omega_i(x) = \sqrt{\lambda_i/\mu_i}\,\omega\left(\sqrt{\lambda_i}x\right),\quad B_i = \frac{1}{4}\int_{\mathbb{R}^N}\mu_i\omega_i^4\,\mathrm{d}x = \mu_i^{-1}\lambda_i^{2-N/2}B. \tag{2.13}$$

The following result was proved by Dancer and Wei [47].

Lemma 2.1 ([47, Lemma 2.3]) *When $\beta=\beta_1$, $(u,v)=(\omega_1,0)$ or $\beta=\beta_2$, $(u,v)=(0,\omega_2)$, the following linearized problem*

$$\begin{cases}\Delta\varphi-\lambda_1\varphi+3\mu_1u^2\varphi+\beta v^2\varphi+2\beta uv\phi=0, & x\in\mathbb{R}^N,\\ \Delta\phi-\lambda_2\phi+3\mu_2v^2\phi+\beta u^2\phi+2\beta uv\varphi=0, & x\in\mathbb{R}^N,\\ \varphi,\phi\in H_r^1(\mathbb{R}^N)\end{cases}$$

has exactly a one-dimensional set of solutions.

Now we can give the proof of Theorem 2.1.

Proof (Proof of Theorem 2.1) Let $\lambda_1=\lambda_2$, and without of loss of generality, we assume $\mu_1<\mu_2$. Assume by contradiction that there exists $\beta^n\uparrow\mu_1$ as $n\to\infty$, such that (2.1) has a nontrivial nonnegative solution (u_n,v_n) for $\beta=\beta^n$ with

$$\inf_{y\in\mathbb{R}^N}\left\|(u_n(\cdot+y), v_n(\cdot+y)) - (\sqrt{k_{\beta^n}}\omega_0, \sqrt{l_{\beta^n}}\omega_0)\right\| > 0, \quad \forall n \in \mathbb{N}. \tag{2.14}$$

The strong maximum principle gives $u_n, v_n > 0$. By [20], we see that, when $\beta > 0$, any positive solution of (2.1) is radially symmetric decreasing up to a translation. Therefore, we may assume that u_n, v_n are radially symmetric decreasing.

Step 1. We prove that $\|u_n\|_{L^\infty(\mathbb{R}^N)} + \|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$, where C is a positive constant independent of n.

It is known that

$$-\Delta u \ge \mu_i u^3, \quad u(x) \ge 0, \quad x \in \mathbb{R}^N$$

has no nontrivial solutions if $N \le 3$. Therefore, this conclusion may follow from a well-known blow up procedure. Since this argument is standard now, we omit the details here, which can be seen in the proof of [47, Lemma 2.4].

Step 2. We show that, for any small $\varepsilon > 0$, there exists $R > 0$ such that

$$u_n(x) + v_n(x) \le \varepsilon, \quad \forall\, |x| \ge R, \ \forall n \in \mathbb{N}. \tag{2.15}$$

The details of this proof can also be seen in the proof of [47, Lemma 2.4]. However, since this argument is not trivial, we would like to give the details here for the reader's convenience.

Recalling that u_n, v_n are radially symmetric decreasing, we write $u_n(|x|) = u_n(x)$ and $v_n(|x|) = v_n(x)$ for convenience. Assume that there exists small $\varepsilon > 0$ and $r_n \to +\infty$ such that $u_n(r_n)+v_n(r_n) = \varepsilon$. Define $(\bar{u}_n(r), \bar{v}_n(r)) = (u_n(r+r_n), v_n(r+r_n))$, then

$$\begin{cases} -\bar{u}_n'' - \frac{N-1}{r+r_n}\bar{u}_n' = -\lambda_1\bar{u}_n + \mu_1\bar{u}_n^3 + \beta^n\bar{u}_n\bar{v}_n^2, & r > -r_n, \\ -\bar{v}_n'' - \frac{N-1}{r+r_n}\bar{v}_n' = -\lambda_2\bar{v}_n + \mu_2\bar{v}_n^3 + \beta^n\bar{v}_n\bar{u}_n^2, & r > -r_n. \end{cases}$$

By elliptic estimates and up to a subsequence, we may assume that $(\bar{u}_n, \bar{v}_n) \to (u, v)$ uniformly in every compact subset of $\mathbb{R}$ as $n \to \infty$, where u, v satisfy

$$\begin{cases} -u'' = -\lambda_1 u + \mu_1 u^3 + \mu_1 u v^2, & r \in \mathbb{R}, \\ -v'' = -\lambda_2 v + \mu_2 v^3 + \mu_1 v u^2, & r \in \mathbb{R}, \end{cases}$$

and $u(0) + v(0) = \varepsilon$, $u, v \ge 0$ are bounded. Since u_n, v_n are both decreasing on $[0, \infty)$, it follows that u, v are both non-increasing on $\mathbb{R}$. Then u, v have limit u_+, v_+ at $+\infty$ and limit u_-, v_- at $-\infty$. Thus, (u_+, v_+) and (u_-, v_-) both satisfy

$$\lambda_1 u = \mu_1 u^3 + \mu_1 u v^2, \quad \lambda_2 v = \mu_2 v^3 + \mu_1 v u^2.$$

Since $\varepsilon > 0$ is small, we have $u_+ = v_+ = 0$ by $u_+ + v_+ \le \varepsilon$. Since $u_- + v_- \ge \varepsilon$, we may assume that $u_- > 0$, then $\lambda_1 = \mu_1 u_-^2 + \mu_1 v_-^2$. Recall that u and v are non-increasing on $\mathbb{R}$, we see that $u(-\lambda_1 + \mu_1 u^2 + \mu_1 v^2) \le 0$ on $\mathbb{R}$ and $u(-\lambda_1 + \mu_1 u^2 + \mu_1 v^2) < 0$ on $[0, +\infty)$, which implies that $u'' \ge 0$ on $\mathbb{R}$ and $u'' > 0$ on

$[0, +\infty)$. That is, u is convex on $\mathbb{R}$ and strictly convex on $[0, +\infty)$. This contradicts with $0 \le u \le C$, which has been obtained in Step 1. This completes the proof of Step 2.

Step 3. We prove that $\{(u_n, v_n)\}_n$ are uniformly bounded in H.

By (2.15), there exists sufficiently large $R > 0$ such that

$$\max\left\{\mu_1 u_n^2(x) + \beta^n v_n^2(x),\ \mu_2 v_n^2(x) + \beta^n u_n^2(x)\right\} \le \frac{\lambda_1}{2}, \quad \forall\, |x| \ge R,\ \forall\, n \in \mathbb{N}.$$

Since (u_n, v_n) satisfies (2.1), we derive

$$-\Delta u_n(x) + \frac{\lambda_1}{2} u_n(x) \le 0, \quad -\Delta v_n(x) + \frac{\lambda_1}{2} v_n(x) \le 0, \quad \forall\, |x| \ge R,\ \forall\, n \in \mathbb{N}.$$

Then by a comparison principle, there exists $C > 0$ independent of n such that

$$u_n(x),\ v_n(x) \le C e^{-\sqrt{\frac{\lambda_1}{2}}|x|}, \quad \forall\, |x| \ge R,\ \forall\, n \in \mathbb{N}. \tag{2.16}$$

Define $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}$. Combining Step 1 with (2.16), it is easily seen that

$$\|u_n\|_{\lambda_1}^2 = \int\limits_{B(0,R)} (\mu_1 u_n^4 + \beta^n u_n^2 v_n^2) + \int\limits_{\mathbb{R}^N \setminus B(0,R)} (\mu_1 u_n^4 + \beta^n u_n^2 v_n^2) \le C,$$

where C is independent of n. Similarly, $\|v_n\|_{\lambda_2}^2 \le C$ for C independent of n.

Step 4. We complete the proof via the bifurcation theory.

By Step 3, up to a subsequence, we may assume that $(u_n, v_n) \to (u, v)$ weakly in H and strongly in $L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N)$ as $\beta^n \uparrow \mu_1$, where $u, v \ge 0$. Then $E'_{\mu_1}(u, v) = 0$ and

$$\lim_{n\to\infty} \|u_n\|_{\lambda_1}^2 = \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (\mu_1 u_n^4 + \beta^n u_n^2 v_n^2) = \int\limits_{\mathbb{R}^N} (\mu_1 u^4 + \mu_1 u^2 v^2) = \|u\|_{\lambda_1}^2,$$

namely $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. Similarly, $v_n \to v$ strongly in $H^1(\mathbb{R}^N)$. By Theorem A and (2.13), it easily follows that

$$\begin{aligned} E_{\mu_1}(u, v) &= \lim_{n\to\infty} E_{\beta^n}(u_n, v_n) \ge \lim_{n\to\infty} E_{\beta^n}(\sqrt{k_{\beta^n}}\omega_0, \sqrt{l_{\beta^n}}\omega_0) \\ &= E_{\mu_1}(\omega_1, 0) = B_1 > E_{\mu_1}(0, \omega_2) = B_2 > 0. \end{aligned} \tag{2.17}$$

Therefore, $(u, v) \ne (0, 0)$. Theorem A-(ii) implies that $u \equiv 0$ or $v \equiv 0$. If $u \equiv 0$, then $v = \omega_2$, a contradiction with (2.17). Hence, $(u, v) = (\omega_1, 0)$. Since $\lambda_1 = \lambda_2$, we have

$$\beta_1 \le \frac{\int_{\mathbb{R}^N} (|\nabla \omega_1|^2 + \lambda_1 \omega_1^2)}{\int_{\mathbb{R}^N} \omega_1^4} = \mu_1.$$

Moreover, for any $\phi \in H^1(\mathbb{R}^N)\backslash\{0\}$, there holds

$$\frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + \lambda_1 \phi^2)}{\int_{\mathbb{R}^N} \omega_1^2 \phi^2} \ge \mu_1 \frac{2\sqrt{B_1}\left(\int_{\mathbb{R}^N} \mu_1 \phi^4\right)^{1/2}}{\left(\int_{\mathbb{R}^N} \mu_1 \omega_1^4\right)^{1/2}\left(\int_{\mathbb{R}^N} \mu_1 \phi^4\right)^{1/2}} = \mu_1.$$

So $\mu_1 = \beta_1$ and $(u_n, v_n, \beta^n) \to (\omega_1, 0, \beta_1)$ is a bifurcation from $(\omega_1, 0, \beta_1)$. By (2.14), $(\sqrt{k_{\beta^n}}\omega_0, \sqrt{l_{\beta^n}}\omega_0, \beta^n) \to (\omega_1, 0, \beta_1)$ is another bifurcation from $(\omega_1, 0, \beta_1)$. By Lemma 2.1, this is a bifurcation from a simple eigenvalue, hence there cannot be two different bifurcations (see [44, 45] or [13, Lemma 3.1]), that is, we get a contradiction. Therefore, there exists small $\delta > 0$ such that for $\mu_1 - \delta < \beta < \mu_1$, $(\sqrt{k_\beta}\omega_0, \sqrt{l_\beta}\omega_0)$ is the unique positive solution to (2.1) up to a translation. □

2.3 Optimal Parameter Range

In this section, we give the proof of Theorem 2.4. In the sequel we assume that $\lambda_1 < \lambda_2$ and $\mu_1 \ge \mu_2$. Then (2.13) gives $B_1 < B_2$.

Lemma 2.2 *System* (2.1) *has no nontrivial nonnegative solutions for any* $\beta \in [\mu_2, \mu_1]$.

Proof This result has been pointed out in Theorem D by Sirakov [80], and the proof is very simple. In fact, assume that (2.1) has a nontrivial nonnegative solution (u, v) for some $\beta \in [\mu_2, \mu_1]$, then by the strong maximum principle, we have $u > 0$ and $v > 0$. Multiply the equation for u in (2.1) by v, the equation for v by u, and integrate over $\mathbb{R}^N$, which yields

$$\int_{\mathbb{R}^N} uv[(\lambda_2 - \lambda_1) + (\mu_1 - \beta)u^2 + (\beta - \mu_2)v^2] = 0,$$

a contradiction. □

Remark 2.5 Theorem D-(iii) is a trivial corollary of Lemma 2.2 and Proposition A. As we will see in the following, the proof of Theorem 2.4 (i)–(ii) seem much more delicate.

Lemma 2.3 $\beta_2 < \mu_2 \le \sqrt{\mu_1\mu_2} \le \mu_1 < \beta_1$.

Proof By (2.7) we have

$$\beta_2 \le \frac{\int\limits_{\mathbb{R}^N} (|\nabla\omega_2|^2 + \lambda_1\omega_2^2)}{\int\limits_{\mathbb{R}^N} \omega_2^4} < \frac{\int\limits_{\mathbb{R}^N} (|\nabla\omega_2|^2 + \lambda_2\omega_2^2)}{\int\limits_{\mathbb{R}^N} \omega_2^4} = \mu_2.$$

On the other hand, it is easy to prove the existence of $\phi_1 \in H^1(\mathbb{R}^N)\backslash\{0\}$such that

$$\beta_1 = \frac{\int\limits_{\mathbb{R}^N} (|\nabla\phi_1|^2 + \lambda_2\phi_1^2)}{\int\limits_{\mathbb{R}^N} \omega_1^2\phi_1^2}.$$

Then by (2.4) and Hölder inequality, we conclude

$$\beta_1 > \frac{\int\limits_{\mathbb{R}^N} (|\nabla\phi_1|^2 + \lambda_1\phi_1^2)}{\int\limits_{\mathbb{R}^N} \omega_1^2\phi_1^2} \ge \mu_1 \frac{2\sqrt{B_1}\left(\int\limits_{\mathbb{R}^N} \mu_1\phi_1^4\right)^{1/2}}{\left(\int\limits_{\mathbb{R}^N} \mu_1\omega_1^4\right)^{1/2}\left(\int\limits_{\mathbb{R}^N} \mu_1\phi_1^4\right)^{1/2}} = \mu_1.$$

This completes the proof. □

Lemma 2.4 *For any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$, there holds $A_\beta \le A_\beta^* \le B_2$. Moreover, $A_{\beta_2} = A_{\beta_2}^* = B_2$.*

Proof Fix any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$. As before, it is easy to prove the existence of $\phi_2 \in H_r^1(\mathbb{R}^N)\backslash\{0\}$such that

$$\beta_2 = J(\phi_2) := \frac{\int\limits_{\mathbb{R}^N} (|\nabla\phi_2|^2 + \lambda_1\phi_2^2)}{\int\limits_{\mathbb{R}^N} \omega_2^2\phi_2^2}, \quad \|\phi_2\|_{\lambda_1} = 1.$$

In fact, ϕ_2 is the first eigenfunction of the following eigenvalue problem

$$-\Delta\phi + \lambda_1\phi = \tau\omega_2^2\phi, \quad \phi \in H^1(\mathbb{R}^N),$$

with the first eigenvalue $\tau_1 = \beta_2$. Hence, for any other $\phi \in H^1(\mathbb{R}^N)$ such that $J(\phi) = \beta_2$, there holds $\phi = C\phi_2$ for some constant C.

If $J(\phi) \le \beta$ for any $\phi \in H_r^1(\mathbb{R}^N)\backslash\{0\}$, then

$$\frac{\int\limits_{\mathbb{R}^N} (|\nabla\phi|^2 + \lambda_1\phi^2)}{(\int\limits_{\mathbb{R}^N} \phi^4)^{1/2}} \le \beta \left(\int\limits_{\mathbb{R}^N} \omega_2^4\right)^{1/2} \le C, \quad \forall\phi \in H_r^1(\mathbb{R}^N)\backslash\{0\}. \tag{2.18}$$

However, a classical result in [16] proved that the following equation

$$-\Delta u + \lambda_1 u = u^3, \quad u \in H^1(\mathbb{R}^N)$$

has infinitely many sign-changing radially symmetric solutions with energy tending to $+\infty$, which implies that (2.18) cannot hold. Therefore, we may take some $\phi_0 \in H_r^1(\mathbb{R}^N)$ such that $\|\phi_0\|_{\lambda_1} = 1$ and $J(\phi_0) > \beta$. Define

$$u_l := (1-l)\phi_0 + l\phi_2, \quad 0 \le l \le 1.$$

Then $u_l \not\equiv 0$ is radially symmetric for any $0 \le l \le 1$, $J(u_0) = J(\phi_0) > \beta$ and $J(u_1) = J(\phi_2) = \beta_2$. Therefore, there exists $0 < l_0 \le 1$ such that

$$J(u_l) > \beta, \quad \forall\, 0 \le l < l_0; \quad J(u_{l_0}) = \beta. \tag{2.19}$$

Now we let $l \in (0, l_0)$. Note that $(\sqrt{t_l}\, u_l, \sqrt{s_l}\omega_2) \in \mathcal{N}_\beta^*$ for some $t_l, s_l > 0$ is equivalent to $t_l, s_l > 0$ satisfying

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} (|\nabla u_l|^2 + \lambda_1 u_l^2) = t_l \int\limits_{\mathbb{R}^N} \mu_1 u_l^4 + s_l \int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2, \\ &\int\limits_{\mathbb{R}^N} (|\nabla\omega_2|^2 + \lambda_2\omega_2^2) = s_l \int\limits_{\mathbb{R}^N} \mu_2\omega_2^4 + t_l \int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2 = \int\limits_{\mathbb{R}^N} \mu_2\omega_2^4, \end{aligned}$$

that is,

$$t_l = \frac{\int\limits_{\mathbb{R}^N} \mu_2\omega_2^4 \left[\int\limits_{\mathbb{R}^N} (|\nabla u_l|^2 + \lambda_1 u_l^2) - \int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2\right]}{\int\limits_{\mathbb{R}^N} \mu_1 u_l^4 \int\limits_{\mathbb{R}^N} \mu_2\omega_2^4 - \left[\int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2\right]^2},$$

$$s_l = \frac{\int\limits_{\mathbb{R}^N} \mu_1 u_l^4 \int\limits_{\mathbb{R}^N} \mu_2\omega_2^4 - \int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2 \int\limits_{\mathbb{R}^N} (|\nabla u_l|^2 + \lambda_1 u_l^2)}{\int\limits_{\mathbb{R}^N} \mu_1 u_l^4 \int\limits_{\mathbb{R}^N} \mu_2\omega_2^4 - \left[\int\limits_{\mathbb{R}^N} \beta\omega_2^2 u_l^2\right]^2}.$$

Since $\beta < \sqrt{\mu_1\mu_2}$, we see from Hölder inequality and (2.19) that $t_l > 0$ for all $l \in (0, l_0)$. Since $u_l \to u_{l_0}$ strongly in $H^1(\mathbb{R}^N)$ as $l \uparrow l_0$, it is easy to see from (2.19) that

$$\lim_{l \uparrow l_0} (t_l, s_l) = (0, 1).$$

So $s_l > 0$ for $l_0 - l > 0$ small enough. Recalling that $u_l, \omega_2 \in H_r^1(\mathbb{R}^N)$, we see that $(\sqrt{t_l}\, u_l, \sqrt{s_l}\omega_2) \in \mathcal{N}_\beta^*$ for $l_0 - l > 0$ small enough and then

$$A_\beta \le A_\beta^* \le \lim_{l \uparrow l_0} E_\beta(\sqrt{t_l} u_l, \sqrt{s_l}\omega_2) = E_\beta(0, \omega_2) = B_2.$$

Therefore, $A_\beta \le A_\beta^* \le B_2$ for any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$.

To finish the proof, it suffices to show that $A_{\beta_2} \ge B_2$. Since $\beta_2 < \sqrt{\mu_1\mu_2}$ by Lemma 2.3, for any $(u, v) \in \mathcal{N}_{\beta_2}$ and any $0 < \beta < \beta_2$, it is easy to prove the existence of $t_\beta, s_\beta > 0$ such that $(\sqrt{t_\beta} u, \sqrt{s_\beta} v) \in \mathcal{N}_\beta$ and

$$\lim_{\beta \uparrow \beta_2} (t_\beta, s_\beta) = (1, 1).$$

Then

$$\limsup_{\beta \uparrow \beta_2} A_\beta \le \limsup_{\beta \uparrow \beta_2} E_\beta(\sqrt{t_\beta} u, \sqrt{s_\beta} v) = E_{\beta_2}(u, v), \quad \forall\, (u, v) \in \mathcal{N}_{\beta_2},$$

and so

$$\limsup_{\beta \uparrow \beta_2} A_\beta \le A_{\beta_2}. \tag{2.20}$$

Consequently, Theorem C gives $A_{\beta_2} \ge B_2$. This completes the proof. □

Lemma 2.5 *Assume that* $\beta_0 \in (0, \sqrt{\mu_1\mu_2})$ *and there exists* $(u_0, v_0) \in \mathcal{N}_{\beta_0}^*$ *such that* $A_{\beta_0}^* = E_{\beta_0}(u_0, v_0)$. *Then*

$$A_\beta^* < A_{\beta_0}^* \quad \text{for any } \beta - \beta_0 > 0 \text{ small enough.}$$

Proof Under assumptions in the lemma, it is easy to prove the existence of $t_\beta, s_\beta > 0$ such that $(\sqrt{t_\beta} u_0, \sqrt{s_\beta} v_0) \in \mathcal{N}_\beta^*$ for any $\beta - \beta_0 > 0$ small enough, and $(t_\beta, s_\beta) \to (1, 1)$ as $\beta \to \beta_0$. On the other hand, we note that $(|u_0|, |v_0|) \in \mathcal{N}_{\beta_0}^*$ and $A_{\beta_0}^* = E_{\beta_0}(|u_0|, |v_0|)$. Then by Proposition A one deduces that $(|u_0|, |v_0|)$ is a nontrivial solution of (2.1). Using the strong maximum principle we have $|u_0| > 0,\ |v_0| > 0$, namely $\int_{\mathbb{R}^N} u_0^2 v_0^2 \,dx > 0$. Therefore,

$$A_\beta^* \le E_\beta(\sqrt{t_\beta} u_0, \sqrt{s_\beta} v_0) = \frac{t_\beta}{4} \int\limits_{\mathbb{R}^N} (|\nabla u_0|^2 + \lambda_1 u_0^2) + \frac{s_\beta}{4} \int\limits_{\mathbb{R}^N} (|\nabla v_0|^2 + \lambda_2 v_0^2)$$

$$
\begin{aligned}
&= \frac{t_\beta}{4}\int_{\mathbb{R}^N}(\mu_1 u_0^4 + \beta_0 u_0^2 v_0^2) + \frac{s_\beta}{4}\int_{\mathbb{R}^N}(\mu_2 v_0^4 + \beta_0 u_0^2 v_0^2)\\
&< \frac{t_\beta}{4}\int_{\mathbb{R}^N}(\mu_1 u_0^4 + \beta u_0^2 v_0^2) + \frac{s_\beta}{4}\int_{\mathbb{R}^N}(\mu_2 v_0^4 + \beta u_0^2 v_0^2)\\
&= \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + \lambda_1 u_0^2) + \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla v_0|^2 + \lambda_2 v_0^2)\\
&= E_{\beta_0}(u_0, v_0) = A_{\beta_0}^*, \quad \text{for any } \beta - \beta_0 > 0 \text{ small enough,}
\end{aligned}
$$

which completes the proof. □

Lemma 2.6 *Let $\beta_2 < \beta < \sqrt{\mu_1\mu_2}$. If $A_\beta^* < B_2$, then (2.1) has a positive solution $(u, v) \in \mathcal{N}_\beta^*$ such that $E_\beta(u, v) = A_\beta^*$.*

Proof Fix any $\beta_2 < \beta < \sqrt{\mu_1\mu_2}$. Recall that $A_\beta^* \geq A_\beta > 0$. In this proof, we will drop the subscript β for convenience. Note that E is coercive and bounded from below on $\mathcal{N}^*$. Then by the Ekeland variational principle (cf. [81]), there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^*$ satisfying

$$E(u_n, v_n) \leq A^* + \frac{1}{n}, \tag{2.21}$$

$$E(u', v') \geq E(u_n, v_n) - \frac{1}{n}\|(u_n, v_n) - (u', v')\|, \quad \forall (u', v') \in \mathcal{N}^*. \tag{2.22}$$

Clearly $\{(u_n, v_n)\}$ is bounded in H. Up to a subsequence, we may assume that $(u_n, v_n) \to (u, v)$ weakly in H and strongly in $L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N)$. Then

$$
\begin{aligned}
B_2 > A^* &= \frac{1}{4}\lim_{n\to\infty}\int_{\mathbb{R}^N}(\mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4)\\
&= \frac{1}{4}\int_{\mathbb{R}^N}(\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4) > 0,
\end{aligned} \tag{2.23}
$$

so $(u, v) \neq (0, 0)$.

Step 1. We show that both $u \not\equiv 0$ and $v \not\equiv 0$.

If $u \equiv 0$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + \lambda_1 u_n^2) = \lim_{n\to\infty}\int_{\mathbb{R}^N}(\mu_1 u_n^4 + \beta u_n^2 v_n^2) = 0,$$

namely $u_n \to 0$ strongly in $H^1(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N}(|\nabla v_n|^2 + \lambda_2 v_n^2) = \int_{\mathbb{R}^N}(\mu_2 v_n^4 + \beta u_n^2 v_n^2)$, by Fatou Lemma and (2.4) we have

$$2\sqrt{B_2}\left(\int_{\mathbb{R}^N} \mu_2 v^4\right)^{1/2} \leq \int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda_2 v^2) \leq \int_{\mathbb{R}^N} \mu_2 v^4,$$

and so $\int_{\mathbb{R}^N} \mu_2 v^4 \geq 4B_2$, a contradiction with (2.23). Hence, $u \not\equiv 0$.

If $v \equiv 0$, similarly we have that $v_n \to 0$ strongly in $H^1(\mathbb{R}^N)$. Define

$$\widetilde{v}_n = \frac{v_n}{|v_n|_4}.$$

Then

$$\|\widetilde{v}_n\|_{\lambda_2}^2 = \mu_2 |v_n|_4^2 + \beta \int_{\mathbb{R}^N} \widetilde{v}_n^2 u_n^2 \leq \mu_2 |v_n|_4^2 + \beta |u_n|_4^2, \tag{2.24}$$

that is, $\widetilde{v}_n$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, $\widetilde{v}_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^N)$. Since $\widetilde{v}_n$ is radially symmetric, we also have $\widetilde{v}_n \to \phi$ strongly in $L^4(\mathbb{R}^N)$, that is, $|\phi|_4 = 1$ and so $\phi \neq 0$. By (2.24), (2.4), Fatou Lemma and Hölder inequality, one has that

$$\begin{aligned} 2\sqrt{B_2}\left(\int_{\mathbb{R}^N} \mu_2 \phi^4\right)^{1/2} &\leq \|\phi\|_{\lambda_2}^2 \leq \beta \int_{\mathbb{R}^N} u^2 \phi^2 \\ &\leq \frac{\beta}{\sqrt{\mu_1 \mu_2}} \left(\int_{\mathbb{R}^N} \mu_1 u^4\right)^{1/2} \left(\int_{\mathbb{R}^N} \mu_2 \phi^4\right)^{1/2}, \end{aligned}$$

that is, $\int_{\mathbb{R}^N} \mu_1 u^4 \geq \frac{\mu_1 \mu_2}{\beta^2} 4B_2 > 4B_2$, a contradiction with (2.23). Hence, $v \not\equiv 0$.

Step 2. We show that $E|'_{H_r}(u_n, v_n) \to 0$ as $n \to \infty$.

By Step 1, there exists $C_2 > C_1 > 0$ such that

$$C_1 \leq \int_{\mathbb{R}^N} u_n^4 \, \mathrm{d}x, \int_{\mathbb{R}^N} v_n^4 \, \mathrm{d}x \leq C_2, \quad \forall n \in \mathbb{N}. \tag{2.25}$$

Thanks to (2.25), the following procedure is a standard argument. For any $(\varphi, \phi) \in H_r$ with $\|\varphi\|, \|\phi\| \leq 1$ and each $n \in \mathbb{N}$, we define h_n and $g_n : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} h_n(t, s, l) &= \int_{\mathbb{R}^N} |\nabla(u_n + t\varphi + \frac{s}{2} u_n)|^2 + \lambda_1 \int_{\mathbb{R}^N} |u_n + t\varphi + \frac{s}{2} u_n|^2 \\ &- \mu_1 \int_{\mathbb{R}^N} |u_n + t\varphi + \frac{s}{2} u_n|^4 - \beta \int_{\mathbb{R}^N} |u_n + t\varphi + \frac{s}{2} u_n|^2 |v_n + t\phi + \frac{l}{2} v_n|^2, \end{aligned} \tag{2.26}$$

and

$$g_n(t,s,l) = \int\limits_{\mathbb{R}^N} |\nabla(v_n + t\phi + \frac{l}{2}v_n)|^2 + \lambda_2 \int\limits_{\mathbb{R}^N} |v_n + t\phi + \frac{l}{2}v_n|^2$$
$$- \mu_2 \int\limits_{\mathbb{R}^N} |v_n + t\phi + \frac{l}{2}v_n|^4 - \beta \int\limits_{\mathbb{R}^N} |u_n + t\varphi + \frac{s}{2}u_n|^2 |v_n + t\phi + \frac{l}{2}v_n|^2. \quad (2.27)$$

Let $\mathbf{0} = (0,0,0)$. Then $h_n, g_n \in C^1(\mathbb{R}^3, \mathbb{R})$ and $h_n(\mathbf{0}) = g_n(\mathbf{0}) = 0$. Define the matrix

$$F_n := \begin{pmatrix} \frac{\partial h_n}{\partial s}(\mathbf{0}) & \frac{\partial h_n}{\partial l}(\mathbf{0}) \\ \frac{\partial g_n}{\partial s}(\mathbf{0}) & \frac{\partial g_n}{\partial l}(\mathbf{0}) \end{pmatrix}.$$

Then we see from (2.25) that

$$\begin{aligned} \det(F_n) &= \mu_1\mu_2 \int\limits_{\mathbb{R}^N} u_n^4 \,\mathrm{d}x \int\limits_{\mathbb{R}^N} v_n^4 \,\mathrm{d}x - \beta^2 \left(\int\limits_{\mathbb{R}^N} u_n^2 v_n^2 \,\mathrm{d}x \right)^2 \\ &\geq (\mu_1\mu_2 - \beta^2) \int\limits_{\mathbb{R}^N} u_n^4 \,\mathrm{d}x \int\limits_{\mathbb{R}^N} v_n^4 \,\mathrm{d}x \geq C > 0, \end{aligned} \quad (2.28)$$

where C is independent of n. By the implicit function theorem, functions $s_n(t)$ and $l_n(t)$ are well defined and class C^1 on some interval $(-\delta_n, \delta_n)$ for $\delta_n > 0$. Moreover, $s_n(0) = l_n(0) = 0$ and

$$h_n(t, s_n(t), l_n(t)) \equiv 0, \quad g_n(t, s_n(t), l_n(t)) \equiv 0, \quad t \in (-\delta_n, \delta_n).$$

This implies that

$$\begin{cases} s_n'(0) = \frac{1}{\det(F_n)} \left(\frac{\partial g_n}{\partial t}(\mathbf{0}) \frac{\partial h_n}{\partial l}(\mathbf{0}) - \frac{\partial g_n}{\partial l}(\mathbf{0}) \frac{\partial h_n}{\partial t}(\mathbf{0}) \right), \\ l_n'(0) = \frac{1}{\det(F_n)} \left(\frac{\partial g_n}{\partial s}(\mathbf{0}) \frac{\partial h_n}{\partial t}(\mathbf{0}) - \frac{\partial g_n}{\partial t}(\mathbf{0}) \frac{\partial h_n}{\partial s}(\mathbf{0}) \right). \end{cases}$$

On the other hand, since $\{(u_n, v_n)\}$ is bounded in H, we have

$$\left| \frac{\partial h_n}{\partial t}(\mathbf{0}) \right| = 2 \left| \int\limits_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + \lambda_1 u_n \varphi - 2\mu_1 u_n^3 \varphi - \beta u_n v_n^2 \varphi - \beta u_n^2 v_n \phi) \right| \leq C,$$

where C is independent of n. Similarly, $|\frac{\partial g_n}{\partial t}(\mathbf{0})| \leq C$. From (2.25) we also have

$$\left|\frac{\partial h_n}{\partial s}(\mathbf{0})\right|, \quad \left|\frac{\partial h_n}{\partial l}(\mathbf{0})\right|, \quad \left|\frac{\partial g_n}{\partial s}(\mathbf{0})\right|, \quad \left|\frac{\partial g_n}{\partial l}(\mathbf{0})\right| \leq C.$$

Hence, combining these with (2.28), we conclude

$$|s_n'(0)|, \quad |l_n'(0)| \leq C, \tag{2.29}$$

where C is independent of n.

Define $\varphi_{n,t} := u_n + t\varphi + \frac{s_n(t)}{2}u_n$ and $\phi_{n,t} := v_n + t\phi + \frac{l_n(t)}{2}v_n$, then $(\varphi_{n,t}, \phi_{n,t}) \in \mathcal{N}^*$ for $t \in (-\delta_n, \delta_n)$. It follows from (2.22) that

$$E(\varphi_{n,t}, \phi_{n,t}) - E(u_n, v_n) \geq -\frac{1}{n}\left\|\left(t\varphi + \frac{s_n(t)}{2}u_n, t\phi + \frac{l_n(t)}{2}v_n\right)\right\|. \tag{2.30}$$

Note that $E|'_{H_r}(u_n, v_n)(u_n, 0) = E|'_{H_r}(u_n, v_n)(0, v_n) = 0$. By Taylor Expansion we have

$$\begin{aligned} &E(\varphi_{n,t}, \phi_{n,t}) - E(u_n, v_n) \\ &= E|'_{H_r}(u_n, v_n)\left(t\varphi + \frac{s_n(t)}{2}u_n, t\phi + \frac{l_n(t)}{2}v_n\right) + r(n,t) \\ &= tE|'_{H_r}(u_n, v_n)(\varphi, \phi) + r(n,t), \end{aligned} \tag{2.31}$$

where $r(n,t) = o(\|(t\varphi + \frac{s_n(t)}{2}u_n, t\phi + \frac{l_n(t)}{2}v_n)\|)$ as $t \to 0$. By (2.29) we see that

$$\limsup_{t\to 0}\left\|\left(\varphi + \frac{s_n(t)}{2t}u_n, \phi + \frac{l_n(t)}{2t}v_n\right)\right\| \leq C, \tag{2.32}$$

where C is independent of n. Hence, $r(n,t) = o(t)$. By (2.30), (2.31), (2.32) and letting $t \to 0$, we get that

$$\left|E|'_{H_r}(u_n, v_n)(\varphi, \phi)\right| \leq \frac{C}{n},$$

where C is independent of n. Thus,

$$\lim_{n\to+\infty} E|'_{H_r}(u_n, v_n) = 0. \tag{2.33}$$

Step 3. We show that $(|u|, |v|)$ is a positive solution of system (2.1) such that $E(|u|, |v|) = A^*$.

By Step 2 we have $E|'_{H_r}(u, v) = 0$ and so $(u, v) \in \mathcal{N}^*$. Then we see from (2.23) that $E(u, v) = A^*$. Therefore, $(|u|, |v|) \in \mathcal{N}^*$ and $E(|u|, |v|) = A^*$. Repeating the proof of [80, Proposition 1.1], we have $E|'_{H_r}(|u|, |v|) = 0$. Then by Palais's symmetric criticality principle [75], we see that $E'(|u|, |v|) = 0$. Finally, the maximum principle gives $|u|, |v| > 0$. □

Now we are in the position to prove Theorem 2.4.

Proof (Proof of Theorem 2.4) (ii) Suppose by contradiction that there exists $\beta_0 \in [\beta_2, \sqrt{\mu_1\mu_2})$ and $(U, V) \in \mathcal{N}_{\beta_0}$ such that $E_{\beta_0}(U, V) = A_{\beta_0}$. By replacing (U, V) by $(|U|, |V|)$ if necessary, we may assume $U, V \geq 0$. Lemma 2.4 gives $A_{\beta_0} \leq B_2$. By Proposition A, we see that (U, V) is a positive ground state solution of (2.1) for $\beta = \beta_0$. By [20] again, we may assume that U and V are both radially symmetric, namely $(U, V) \in \mathcal{N}^*_{\beta_0}$ and so

$$A^*_{\beta_0} \leq E_{\beta_0}(U, V) = A_{\beta_0} \leq A^*_{\beta_0},$$

which means that $E_{\beta_0}(U, V) = A^*_{\beta_0} \leq B_2$. Define

$$\beta^* = \sup\Big\{\beta' \in [\beta_0, \sqrt{\mu_1\mu_2}] : \text{for all } \beta \in [\beta_0, \beta'],\ (2.1) \text{ has a positive solution } (u_\beta, v_\beta) \in \mathcal{N}^*_\beta \text{ with } \mathrm{E}_\beta(\mathrm{u}_\beta, \mathrm{v}_\beta) = \mathrm{A}^*_\beta\Big\}.$$

Lemmas 2.5 and 2.6 indicate that $\beta^* > \beta_0$. Assume by contradiction that $\beta^* < \sqrt{\mu_1\mu_2}$. Then for any $\beta \in (\beta_0, \beta^*)$, (2.1) has a positive solution (u_β, v_β) with $E_\beta(u_\beta, v_\beta) = A^*_\beta < B_2$. Moreover, A^*_β is strictly decreasing with respect to $\beta \in (\beta_0, \beta^*)$. Up to a subsequence, $(u_\beta, v_\beta) \to (u, v)$ weakly in H and strongly in $L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N)$ as $\beta \uparrow \beta^*$, and $u, v \geq 0$. Then $E'_{\beta^*}(u, v) = 0$ and

$$\lim_{\beta\to\beta^*} \|u_\beta\|^2_{\lambda_1} = \lim_{\beta\to\beta^*} \int_{\mathbb{R}^N} (\mu_1 u^4_\beta + \beta u^2_\beta v^2_\beta) = \int_{\mathbb{R}^N} (\mu_1 u^4 + \beta u^2 v^2) = \|u\|^2_{\lambda_1},$$

that is, $u_\beta \to u$ strongly in $H^1(\mathbb{R}^N)$. Similarly, $v_\beta \to v$ strongly in $H^1(\mathbb{R}^N)$. Then it is easy to prove that $E_{\beta^*}(u, v) = \lim_{\beta\to\beta^*} A^*_\beta > 0$, and so $(u, v) \neq (0, 0)$. If $u \equiv 0$, then $v = \omega_2$. As before, we define

$$\widetilde{u}_\beta = \frac{u_\beta}{|u_\beta|_4}.$$

Similarly as (2.24), we see that $\widetilde{u}_\beta$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, $\widetilde{u}_\beta \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^N)$. Since $\widetilde{u}_\beta$ is radially symmetric, we also have $\widetilde{u}_\beta \to \phi$ strongly in $L^4(\mathbb{R}^N)$, namely $|\phi|_4 = 1$ and $\phi \geq 0$. Since $-\Delta u_\beta + \lambda_1 u_\beta = \mu_1 u^3_\beta + \beta u_\beta v^2_\beta$, letting $\beta \to \beta^*$ we see that

$$-\Delta\phi + \lambda_1\phi = \beta^*\omega_2^2\phi, \quad \phi \geq 0 \ \text{ in } \mathbb{R}^N, \tag{2.34}$$

which implies that $\beta^* = \beta_2$, a contradiction. If $v \equiv 0$, then we may also prove that $\beta^* = \beta_1$, a contradiction with Lemma 2.3. Therefore, $u \neq 0$ and $v \neq 0$, namely $(u, v) \in \mathcal{N}^*_{\beta^*}$ and so $E_{\beta^*}(u, v) \geq A^*_{\beta^*}$. On the other hand, by the same proof of (2.20), we have

$$\limsup_{\beta \uparrow \beta^*} A^*_\beta \leq A^*_{\beta^*} \leq E_{\beta^*}(u, v) = \lim_{\beta \uparrow \beta^*} A^*_\beta.$$

Therefore, $E_{\beta^*}(u, v) = A^*_{\beta^*} = \lim_{\beta \to \beta^*} A^*_\beta < B_2$. By Lemmas 2.5 and 2.6 again, there exists $0 < \varepsilon < \sqrt{\mu_1\mu_2} - \beta^*$ such that for any $\beta \in [\beta^*, \beta^* + \varepsilon]$, (2.1) has a positive solution (u_β, v_β) with $E_\beta(u_\beta, v_\beta) = A^*_\beta$, which contradicts with the definition of β^*. Therefore, $\beta^* = \sqrt{\mu_1\mu_2}$. Then by repeating the argument above, we see that (2.1) has a positive solution $(u_{\sqrt{\mu_1\mu_2}}, v_{\sqrt{\mu_1\mu_2}})$ for $\beta = \sqrt{\mu_1\mu_2}$, which contradicts with Lemma 2.2. Therefore, for any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$, A_β is not attained, that is, (2.1) has no ground state solutions for any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$. This proof also implies that A^*_β is not attained for any $\beta \in [\beta_2, \sqrt{\mu_1\mu_2})$.

(iii) Let (U_β, V_β) be in Theorems B and C. By (2.20) and Theorem C we have

$$\lim_{\beta \uparrow \beta_2} A_\beta = A_{\beta_2} = B_2. \tag{2.35}$$

Assume that there exists a sequence $\beta^n \uparrow \beta_2$ as $n \to \infty$ such that

$$\liminf_{n \to +\infty} \|(U_{\beta^n}, V_{\beta^n}) - (0, \omega_2)\| > 0. \tag{2.36}$$

Up to a subsequence, we may assume that $(U_{\beta^n}, V_{\beta^n}) \to (U, V)$ weakly in H and strongly in $L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N)$, where $U, V \geq 0$. Similarly, we can prove that $(U_{\beta^n}, V_{\beta^n}) \to (U, V)$ strongly in H, $E'_{\beta_2}(U, V) = 0$ and $E_{\beta_2}(U, V) = A_{\beta_2} = B_2$. Since A_{β_2} is not attained, we have $U \equiv 0$ or $V \equiv 0$. If $V \equiv 0$, then $U \not\equiv 0$. Since $U \geq 0$ is radially symmetric, we see that $U = \omega_1$ and then $E_{\beta_2}(U, V) = E_{\beta_2}(\omega_1, 0) = B_1 < B_2$, a contradiction. Similarly, if $U \equiv 0$, then $V = \omega_2$, which contradicts with (2.36). Therefore, $(U_\beta, V_\beta) \to (0, \omega_2)$ strongly in H as $\beta \uparrow \beta_2$. (i) For $\mu_2 \leq \beta \leq \mu_1$, this result has been proved in Lemma 2.2. Assume that there exists $\beta^n \uparrow \mu_2$ as $n \to \infty$ such that (2.1) has a nonnegative nontrivial solution (u_n, v_n) for $\beta = \beta^n$. By the strong maximum principle, we see that $u_n, v_n > 0$. By [20] again, we may assume that u_n, v_n are radially symmetric decreasing. By a similar argument as the proof of Theorem 2.1, we can prove that $(u_n, v_n) \to (u, v)$ strongly in H and $E'_{\mu_2}(u, v) = 0$. By Remark 2.6, there holds $E_{\beta^n}(u_n, v_n) \geq A^*_{\beta^n} = B_2$ for n sufficiently large, and so $E_{\mu_2}(u, v) \geq B_2 > 0$, that is, $(u, v) \neq (0, 0)$. If $u \equiv 0$, then $v \not\equiv 0$. Since $v \geq 0$ is radially symmetric, we see that $v = \omega_2$. By a similar argument as the proof of (ii), we see that $\mu_2 = \beta_2$, a contradiction with Lemma 2.3. If $v \equiv 0$, then we may prove that $\mu_2 = \beta_1$, also a contradiction. Therefore, $u \not\equiv 0$ and $v \not\equiv 0$, namely (u, v) is a nonnegative nontrivial solution of (2.1) with $\beta = \mu_2$, a contradiction with Lemma 2.2. Therefore, there exists small $\delta_0 > 0$ such that (2.1) has no nonnegative nontrivial solutions for any $\beta \in (\mu_2 - \delta_0, \mu_1]$. Define

$$A'_\beta := \inf_{(u,v)\in\mathcal{N}'_\beta} E_\beta(u, v),$$

where $\mathcal{N}'_\beta := \{(u, v) \in H\backslash\{(0, 0)\} : E'_\beta(u, v)(u, v) = 0\}$. Then it is easy to prove that

$$A'_\beta = \inf_{(u,v)\in H\backslash\{(0,0)\}} \max_{t>0} E_\beta(tu, tv).$$

This implies that $A'_\beta > 0$ is non-increasing with respect to β and so

$$\liminf_{\beta\downarrow\mu_1} A'_\beta \geq A'_{\beta_1} > 0.$$

Assume by contradiction that there exists $\beta^n \downarrow \mu_1$ as $n \to \infty$ such that (2.1) has a nonnegative nontrivial solution (u_n, v_n) for $\beta = \beta^n$. Then repeating the proof above, we may prove that $(u_n, v_n) \to (u, v)$ strongly in H and

$$E_{\mu_1}(u, v) = \lim_{\beta^n\downarrow\mu_1} E_{\beta^n}(u_n, v_n) \geq \liminf_{\beta^n\downarrow\mu_1} A'_{\beta^n} > 0.$$

That is, $(u, v) \neq (0, 0)$. Repeating the proof above, we get a contradiction with Lemma 2.2. Therefore, there exists small $\delta \in (0, \delta_0]$ such that (2.1) has no nonnegative nontrivial solutions for any $\beta \in (\mu_2 - \delta, \mu_1 + \delta)$.

(iv) Note that for $\beta < \sqrt{\mu_1\mu_2}$, by a similar argument as Step 3 in the proof of Lemma 2.6, we may assume that all ground state solutions are positive radially symmetric. Assume that there exists $\beta^n \uparrow \beta_2$ as $n \to \infty$ such that (2.1) has a positive radially symmetric ground state solution (u_n, v_n) for $\beta = \beta^n$ with

$$\|(u_n, v_n) - (U_{\beta^n}, V_{\beta^n})\| > 0, \quad \forall n \in \mathbb{N}. \tag{2.37}$$

By a similar argument as the proof of Theorem 2.1, we can prove that $(u_n, v_n) \to (u, v)$ strongly in H and $E'_{\beta_2}(u, v) = 0$. By (2.35), we have

$$E_{\beta_2}(u, v) = \lim_{n\to\infty} E_{\beta^n}(u_n, v_n) = \lim_{n\to\infty} A_{\beta^n} = A_{\beta_2} = B_2,$$

then the proof of (iii) implies that $(u, v) = (0, \omega_2)$, and so (u_n, v_n, β^n) is a bifurcation from $(0, \omega_2, \beta_2)$. Combining (iii) and Lemma 2.1, we get a contradiction just as in the proof of Theorem 2.1. Therefore, there exists small $\delta_1 > 0$ such that for any $\beta \in (\beta_2 - \delta_1, \beta_2)$, (U_β, V_β) is the unique ground state solution of (2.1) up to a translation. This completes the proof. □

Remark 2.6 Let $\lambda_1 < \lambda_2$ and $\mu_1 \geq \mu_2$. Lemmas 2.4, 2.6 and Theorem 2.4-(i) imply that

$$A^*_\beta \equiv B_2, \quad \text{for } \beta \in [\beta_2, \sqrt{\mu_1\mu_2}).$$

Then $(\sqrt{t_l}u_l, \sqrt{s_l}\omega_2) \in \mathcal{N}^*_\beta$ constructed in the proof of Lemma 2.4 is indeed a minimizing sequence of A^*_β as $l \to l_0$.

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