

Chapter 2

MADM with Preferences on Attribute Weights

For this type of problems, the decision makers cannot provide directly the attribute weights, but utilize a scale to compare each pair of alternatives, and then construct preference relations (in general, there are multiplicative preference relations and fuzzy preference relations). After that, some proper priority methods are used to derive the priority vectors of preference relations, from which the attribute weights can be obtained. The priority theory and methods of multiplicative preference relations have achieved fruitful research results. The investigation on the priority methods of fuzzy preference relations has also been receiving more and more attention recently. Considering the important role of the priority methods of fuzzy preference relations in solving the MADM problems in which the attribute values are interval numbers, in this chapter, we introduce mainly the priority theory and methods of fuzzy preference relations. Based on the WAA, CWA, WG and CWG operators, we also introduce some MADM methods, and illustrate these methods in detail with several practical examples.

2.1 Priority Methods for a Fuzzy Preference Relation

2.1.1 Translation Method for Priority of a Fuzzy Preference Relation

Definition 2.1 [88] Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, if

$$(b_{ik} - 0.5) + (b_{kj} - 0.5) = b_{ij} - 0.5, i, j, k = 1, 2, \dots, n \quad (2.1)$$

i.e., $b_{ij} = b_{ik} - b_{jk} + 0.5, i, j, k = 1, 2, \dots, n$, then B is called an additive consistent fuzzy preference relation.

Let G be the set of all fuzzy preference relations with n order. A n -dimension positive vector $w = (w_1, w_2, \dots, w_n)$ is called a priority vector, each of the elements

of w is the weight of an object (attribute or alternative). Let Λ be the set of all the priority vectors, where

$$\Lambda = \left\{ w = (w_1, w_2, \dots, w_n) \mid w_j > 0, j = 1, 2, \dots, n, \sum_{j=1}^n w_j = 1 \right\}$$

A priority method can be regarded as a mapping from G to Λ , denoted by $w = \Gamma(B)$, and w is the priority vector of the fuzzy preference relation B .

Definition 2.2 [93] A priority method is called of strong rank preservation, if $b_{ik} \geq b_{jk}$, then $w_i \geq w_j$, for any k , with $w_i = w_j$ if and only if $b_{ik} = b_{jk}$, for any k .

Definition 2.3 [105] A fuzzy preference relation $B = (b_{ij})_{n \times n}$ is called of rank transitivity, if the following two conditions are satisfied:

1. If $b_{ij} \geq 0.5$, then $b_{ij} \geq b_{jk}$, for all k ;
2. If $b_{ij} = 0.5$, then $b_{ij} \geq b_{jk}$, for all k , or $b_{ij} \leq b_{jk}$, for all k .

Definition 2.4 [105] Let $\Gamma(\bullet)$ be a priority method, B be any fuzzy preference relation, $w = \Gamma(B)$. If $\Psi w^T = \Gamma(\Psi B \Psi^T)$, for any permutation matrix Ψ , then the priority method $\Gamma(\bullet)$ is called of permutation invariance.

Theorem 2.1 [22] For the fuzzy preference relation $B = (b_{ij})_{n \times n}$, let

$$b_i = \sum_{j=1}^n b_{ij}, \quad i = 1, 2, \dots, n \quad (2.2)$$

which is the sum of all the elements in the i th line of B , and based on Eq. (2.2), we give the following mathematical transformation:

$$\bar{b}_{ij} = \frac{b_i - b_j}{a} + 0.5 \quad (2.3)$$

then the matrix $\bar{B} = (\bar{b}_{ij})_{n \times n}$ is called an additional consistent fuzzy preference relation.

In general, it is suitable to take $a = 2(n-1)$, which can be shown as follows [103]:

1. If we take the 0–1 scale, then the value range of the element \bar{b}_{ij} of the matrix $\bar{B} = (\bar{b}_{ij})_{n \times n}$ is $0 \leq \bar{b}_{ij} \leq 1$, and combining Eq. (2.3), we get

$$a \geq 2(n-1) \quad (2.4)$$

2. If we take the 0.1–0.9 scale, then Eq. (2.4) also holds.

It is clear that the larger the value of a , the smaller the value range of \bar{b}_{ij} derived from Eq. (2.3), and thus, the lower the closeness degree between the constructed additive consistent fuzzy preference relation and the original fuzzy preference relation (i.e., the less judgment information got from the original fuzzy preference relation). Thus, when a takes the smallest value $2(n-1)$, the additive consistent fuzzy preference relation constructed by using Eq. (2.3) can remain as much as possible the judgment information of the original fuzzy preference relation, and the deviations between the elements of these two fuzzy preference relations can also correspondingly reduce to the minimum. Obviously, this type of deviation is caused by the consistency improvement for the original fuzzy preference relation. For fuzzy preference relations with different orders, the value of a will change with the increase of the order n , and thus, it is more compatible with the practical situations. In addition, the additive consistent fuzzy preference relation derived by Eq. (2.3) is in accordance with the consistency of human decision thinking, and has good robustness (i.e., the sub-matrix derived by removing any line and the corresponding column is also an additive consistent fuzzy preference relation) and transitivity.

For the given fuzzy preference relation $B = (b_{ij})_{n \times n}$, we employ the transformation formula (2.3) to get the additive consistent fuzzy preference relation $\bar{B} = (\bar{b}_{ij})_{n \times n}$, after that, we can use the normalizing rank aggregation method to derive its priority vector.

Based on the idea above, in what follows, we introduce a formula for deriving the priority vector of a fuzzy preference relation.

Theorem 2.2 [103] Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, then we utilize Eq. (2.2) and the following mathematical transformation:

$$\bar{b}_{ij} = \frac{b_i - b_j}{2(n-1)} + 0.5 \quad (2.5)$$

to get the matrix $\bar{B} = (\bar{b}_{ij})_{n \times n}$, based on which the normalizing rank aggregation method is used to derive the priority vector, which satisfies

$$w_i = \frac{\sum_{j=1}^n \bar{b}_{ij} + \frac{n}{2} - 1}{n(n-1)}, \quad i = 1, 2, \dots, n \quad (2.6)$$

This priority method is called the translation method for priority of a fuzzy preference relation.

Proof By Eqs. (2.1), (2.2) and (2.5), we get

$$w_i = \frac{\sum_{j=1}^n \bar{b}_{ij}}{\sum_{i=1}^n \sum_{j=1}^n \bar{b}_{ij}} = \frac{\sum_{j=1}^n \bar{b}_{ij}}{\sum_{1 \leq i < j \leq n} (\bar{b}_{ij} + \bar{b}_{ji}) + 0.5n}$$

$$\begin{aligned}
&= \frac{\sum_{j=1}^n \bar{b}_{ij}}{\frac{n(n-1)}{2} + \frac{n}{2}} = \frac{\sum_{j=1}^n \bar{b}_{ij}}{\frac{n^2}{2}} = \frac{\sum_{j=1}^n \left[\frac{b_i - b_j}{2(n-1)} + 0.5 \right]}{\frac{n^2}{2}} = \frac{\sum_{j=1}^n \frac{b_i - b_j}{n-1} + n}{n^2} \\
&= \frac{b_i + \frac{n}{2} - 1}{n(n-1)} = \frac{\sum_{j=1}^n b_{ij} + \frac{n}{2} - 1}{n(n-1)}
\end{aligned}$$

Theorem 2.3 [103] The translation method is of strong rank preservation.

Proof Let $w = (w_1, w_2, \dots, w_n)$ be the priority vector of the fuzzy preference relation $B = (b_{ij})_{n \times n}$ then

$$w_i = \frac{\sum_{j=1}^n b_{ij} + \frac{n}{2} - 1}{n(n-1)}, \quad w_l = \frac{\sum_{j=1}^n b_{lj} + \frac{n}{2} - 1}{n(n-1)}$$

If $b_{ij} \geq b_{lj}$ for any j , then by the two formulas above, we can see $w_i \geq w_l$, with equality if and only $b_{ij} = b_{lj}$, for all j . Thus, the translation method is of strong rank preservation.

By Definition 2.3 and Theorem 2.3, we have

Theorem 2.4 [103] Let the fuzzy preference relation $B = (b_{ij})_{n \times n}$ be rank transitive. If $b_{ij} \geq 0.5$, then $w_i \geq w_j$; If $b_{ij} = 0.5$, then $w_i \geq w_l$ or $w_i \leq w_l$, where $w = (w_1, w_2, \dots, w_n)$ is a priority vector derived by the translation method for the fuzzy preference relation B .

Theorem 2.5 [103] The translation method has the permutation invariance.

Proof Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, and let Ψ be a permutation matrix, such that $C = (c_{ij})_{n \times n} = \Psi B \Psi^T$, $w = (w_1, w_2, \dots, w_n)$ and $v = (v_1, v_2, \dots, v_n)$ be the priority vectors derived by the translation method for B and C , respectively. Then, after the permutation, the i th line of B becomes the l th line of C , the i th column of B becomes the l th column of C , and thus

$$w_i = \frac{\sum_{j=1}^n b_{ij} + \frac{n}{2} - 1}{n(n-1)} = \frac{\sum_{j=1}^n c_{lj} + \frac{n}{2} - 1}{n(n-1)} = v_l$$

which indicates that the translation method has the permutation invariance.

According to Theorem 2.2, we can know that the translation method has the following characteristics:

1. By using Eq. (2.6), the method can directly derive the priority vector from the original fuzzy preference relation.
2. The method can not only sufficiently utilize the desirable properties and judgment information of the additive consistent fuzzy preference relation, but also needs much less calculation than that of the other existing ones.
3. The method omits many unnecessary intermediate steps, and thus, it is very convenient to be used in practical applications.

However, the translation method also has the disadvantage that the differences among the elements of the derived priority vector are somewhat small, and thus, sometimes, are not easy to differentiate them.

2.1.2 Least Variation Method for Priority of a Fuzzy Preference Relation

From the viewpoint of optimization, i.e., from the angle of the additive consistent fuzzy preference relation constructed by the priority weights approaching the original fuzzy preference relation, in what follows, we introduce a least variation method for deriving the priority vector of a fuzzy preference relation.

Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, $w = (w_1, w_2, \dots, w_n)$ be the priority vector of B , if

$$b_{ij} = w_i - w_j + 0.5, \quad i, j = 1, 2, \dots, n \quad (2.7)$$

then $b_{ij} = b_{il} - b_{jl} + 0.5$, for any $l = 1, 2, \dots, n$, and thus, $B = (b_{ij})_{n \times n}$ is an additive consistent fuzzy preference relation. If B is not an additive consistent fuzzy preference relation, then Eq. (2.7) usually does hold. As a result, we introduce a deviation element, i.e.,

$$f_{ij} = b_{ij} - (w_i - w_j + 0.5), \quad i, j = 1, 2, \dots, n$$

and construct a deviation function:

$$F(w) = \sum_{i=1}^n \sum_{j=1}^n f_{ij}^2 = \sum_{i=1}^n \sum_{j=1}^n [b_{ij} - (w_i - w_j + 0.5)]^2$$

A reasonable priority vector w^* should be determined so as to minimize $F(w)$, i.e.,

$$F(w^*) = \min_{w \in \Lambda} F(w) = \sum_{i=1}^n \sum_{j=1}^n [b_{ij} - (w_i - w_j + 0.5)]^2$$

holds, we term this approach the least variation method for deriving the priority vector of a fuzzy preference relation. The following conclusion can be obtained for $F(w)$:

Theorem 2.6 [105] Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, then the priority vector $w = (w_1, w_2, \dots, w_n)$ derived by the least variation method satisfies:

$$w_i = \frac{1}{n} \left(\sum_{j=1}^n b_{ij} + 1 - \frac{n}{2} \right), \quad i = 1, 2, \dots, n \quad (2.8)$$

Proof We construct the Lagrange function:

$$L(w, \zeta) = F(w) + \zeta \left(\sum_{j=1}^n w_j - 1 \right)$$

where ζ is the Lagrange multiplier.

Differentiating $L(w, \zeta)$ with respect to $w_i (i = 1, 2, \dots, n)$ and ζ , and setting these partial derivatives equal to zero, then

$$\sum_{j=1}^n 2[b_{ij} - (w_i - w_j + 0.5)](-1) + \zeta = 0, \quad i = 1, 2, \dots, n$$

and simplifies it as follows:

$$-2 \left[\sum_{j=1}^n b_{ij} - nw_i + 1 - \frac{n}{2} \right] + \zeta = 0, \quad i = 1, 2, \dots, n \quad (2.9)$$

Summing both the sides of Eq. (2.9) with respect to $i = 1, 2, \dots, n$, we have

$$-2 \left[\sum_{i=1}^n \sum_{j=1}^n b_{ij} - \frac{n^2}{2} \right] + \zeta = 0 \quad (2.10)$$

According to the property of fuzzy preference relation, we get

$$\sum_{i=1}^n \sum_{j=1}^n b_{ij} = \frac{n^2}{2} \quad (2.11)$$

Thus, bringing Eq. (2.10) into Eq. (2.11), it can be obtained that $\zeta = 0$. Then we combine $\zeta = 0$ with Eq. (2.9) to get Eq. (2.8), which completes the proof.

Similar to Theorems 2.3–2.5, we can derive the following result:

Theorem 2.7 [105] The least variation method is of strong rank preservation.

Theorem 2.8 [105] Let the fuzzy preference relation $B = (b_{ij})_{n \times n}$ be rank transitive. If $b_{ij} \geq 0.5$, then $w_i \geq w_j$; If $b_{ij} = 0.5$, then $w_i \geq w_j$ or $w_i \leq w_j$, where $w = (w_1, w_2, \dots, w_n)$ is a priority vector derived by the least variation method for the fuzzy preference relation B .

Theorem 2.9 [105] The least variation method has the permutation invariance.

By using Eq. (2.8), we can derive the priority vector of a fuzzy preference relation. In many practical applications, we have found that if the judgments given by the decision maker are not in accordance with the practical situation, i.e., the fuzzy preference relation constructed by the decision maker is seriously inconsistent, then the value of $\sum_{j=1}^n b_{ij}$ maybe less than $\frac{n}{2} - 1$, which results in the case that $w_i \leq 0$. In such cases, we need to return the fuzzy preference relation to the decision maker for re-evaluation or we can utilize the consistency improving method to repair the fuzzy preference relation.

2.1.3 Least Deviation Method for Priority of a Fuzzy Preference Relation

From another viewpoint of optimization, i.e., from the angle of the multiplicative consistent fuzzy preference relation constructed by the priority weights approaching the original fuzzy preference relation, in what follows, we introduce a least deviation method for deriving the priority vector of a fuzzy preference relation.

2.1.3.1 Preliminaries

In the process of MADM, the decision maker compares each pair of attributes, and provides his/her judgment (preference):

1. If the decision maker uses the 1–9 scale [98] to express his/her preferences, and constructs the multiplicative preference relation $H = (h_{ij})_{n \times n}$, which has the following properties:

$$h_{ij} \in \left[\frac{1}{9}, 9 \right], \quad h_{ji} = \frac{1}{h_{ij}}, \quad h_{ii} = 1, \quad i, j = 1, 2, \dots, n$$

If $h_{ij} = h_{ik} h_{kj}$, $i, j, k = 1, 2, \dots, n$, then $H = (h_{ij})_{n \times n}$ is called a consistent multiplicative preference relation [69, 93].

2. If the decision maker uses the 0.1–0.9 scale [98] to express his/her preferences, and constructs the fuzzy preference relation $B = (b_{ij})_{n \times n}$ which has the following properties:

$$b_{ij} \in [0.1, 0.9], \quad b_{ij} + b_{ji} = 1, \quad b_{ii} = 0.5, \quad i, j = 1, 2, \dots, n$$

If $b_{ik}b_{kj}b_{ji} = b_{ki}b_{jk}b_{ij}$, $i, j, k = 1, 2, \dots, n$, then $B = (b_{ij})_{n \times n}$ is called a multiplicative consistent fuzzy preference relation [63, 111].

Based on the multiplicative preference relation $H = (h_{ij})_{n \times n}$ we employ the transformation formula [98]:

$$b_{ij} = \frac{h_{ij}}{h_{ij} + 1}, \quad i, j = 1, 2, \dots, n$$

to get the fuzzy preference relation $B = (b_{ij})_{n \times n}$. Based on the fuzzy preference relation $B = (b_{ij})_{n \times n}$, we employ the transformation formula [98]:

$$h_{ij} = \frac{b_{ij}}{1 - b_{ij}}, \quad i, j = 1, 2, \dots, n$$

to get the multiplicative preference relation $H = (h_{ij})_{n \times n}$.

The following theorems can be proven easily:

Theorem 2.10 [98] Let $H = (h_{ij})_{n \times n}$ be a multiplicative preference relation, then the corresponding fuzzy preference relation $B = (b_{ij})_{n \times n}$ can be derived by using the following transformation formula:

$$b_{ij} = \frac{1}{1 + h_{ji}}, \quad i, j = 1, 2, \dots, n \quad (2.12)$$

Theorem 2.11 [109] Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, then the corresponding multiplicative preference relation $H = (h_{ij})_{n \times n}$ can be derived by using the following transformation formula:

$$h_{ij} = \frac{b_{ij}}{b_{ji}}, \quad i, j = 1, 2, \dots, n \quad (2.13)$$

Theorem 2.12 [109] If $H = (h_{ij})_{n \times n}$ is a consistent multiplicative preference relation, then the fuzzy preference relation $B = (b_{ij})_{n \times n}$ derived by using Eq. (2.12) is a multiplicative fuzzy preference relation.

Theorem 2.13 [109] If $B = (b_{ij})_{n \times n}$ is a multiplicative fuzzy preference relation, then the multiplicative preference relation $H = (h_{ij})_{n \times n}$ derived from Eq. (2.13) is a consistent multiplicative preference relation.

Definition 2.5 [109] Let $B = (b_{ij})_{n \times n}$ be a fuzzy preference relation, then $H = (h_{ij})_{n \times n}$ is called the transformation matrix of B , where $h_{ij} = \frac{b_{ij}}{b_{ji}}$, $i, j = 1, 2, \dots, n$.

Since Eqs. (2.12) and (2.13) establish a close relation between two different types of preference information, and thus, they has great theoretical importance and wide application potential.

2.1.3.2 Main Results

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ be the priority vector of the multiplicative preference relation $H = (h_{ij})_{n \times n}$, where $\gamma_j > 0$, $j = 1, 2, \dots, n$, and $\sum_{j=1}^n \gamma_j = 1$, then when H is a consistent multiplicative preference relation, we have $h_{ij} = \frac{\gamma_i}{\gamma_j}$, $i, j = 1, 2, \dots, n$. If we combine $h_{ij} = \frac{\gamma_i}{\gamma_j}$ with Eq. (2.12), then $b_{ij} = \frac{\gamma_i}{\gamma_i + \gamma_j}$, $i, j = 1, 2, \dots, n$. If we bring $b_{ij} = \frac{\gamma_i}{\gamma_i + \gamma_j}$ into $b_{ik}b_{kj}b_{ji} = b_{ki}b_{jk}b_{ij}$, $i, j, k = 1, 2, \dots, n$, this equality holds, i.e., $B = (b_{ij})_{n \times n}$ is a multiplicative consistent fuzzy preference relation. Therefore, if we let $w = (w_1, w_2, \dots, w_n)$ be the priority vector of the fuzzy preference relation B , where $w_j > 0$, $j = 1, 2, \dots, n$, and $\sum_{j=1}^n w_j = 1$, then when B is a multiplicative consistent fuzzy preference relation, we have $b_{ij} = \frac{w_i}{w_i + w_j}$, $i, j = 1, 2, \dots, n$, i.e., $(1 - b_{ij})w_i = b_{ij}w_j$, $i, j = 1, 2, \dots, n$. Since $b_{ij} + b_{ji} = 1$, then

$$b_{ji}w_i = b_{ij}w_j, \quad i, j = 1, 2, \dots, n \quad (2.14)$$

i.e.,

$$w_i = \frac{b_{ij}}{b_{ji}} w_j, \quad i, j = 1, 2, \dots, n \quad (2.15)$$

or

$$\frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} = \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j} = 1, \quad i, j = 1, 2, \dots, n \quad (2.16)$$

Combining Eq. (2.15) and $\sum_{j=1}^n w_j = 1$, we get the exact solution to the priority vector of the multiplicative consistent fuzzy preference relation B :

$$w = \left(\frac{1}{\sum_{i=1}^n \frac{b_{i1}}{b_{1i}}}, \frac{1}{\sum_{i=1}^n \frac{b_{i2}}{b_{2i}}}, \dots, \frac{1}{\sum_{i=1}^n \frac{b_{in}}{b_{ni}}} \right) \quad (2.17)$$

Considering that the fuzzy preference relation provided by the decision maker in the decision making process is usually inconsistent, Eq. (2.16) generally does not hold. As a result, we introduce the deviation factor:

$$f_{ij} = \frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} + \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j} - 2, \quad i, j = 1, 2, \dots, n \quad (2.18)$$

and construct the deviation function:

$$F(w) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} + \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j} - 2 \right), \quad i, j = 1, 2, \dots, n \quad (2.19)$$

Obviously, a reasonable priority vector w^* should be determined so as to minimize $F(w)$, i.e.,

$$\begin{cases} \min F(w) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} + \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j} - 2 \right) \\ \text{s.t. } w_j > 0, \quad j = 1, 2, \dots, n, \quad \sum_{j=1}^n w_j = 1 \end{cases} \quad (2.20)$$

We term this approach the least deviation method for deriving the priority vector of a fuzzy preference relation [150]. The following conclusion can be obtained for $F(w)$:

Theorem 2.14 [150] The least deviation function $F(w)$ has a unique minimum point w^* , which is also the unique solution of the following set of equations in Λ :

$$\sum_{j=1}^n \frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} = \sum_{j=1}^n \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j}, \quad i, j = 1, 2, \dots, n \quad (2.21)$$

where Λ is defined as in Sect. 2.1.1.

Proof (1) **(Existence)** Since Λ is a bounded vector space, $F(w)$ is a continuous function in Λ , for any $w \in \Lambda$, and

$$\frac{b_{ij}}{b_{ji}} \frac{w_j}{w_i} + \frac{b_{ji}}{b_{ij}} \frac{w_i}{w_j} \geq 2 \sqrt{\frac{b_{ij}}{b_{ji}} \frac{b_{ji}}{b_{ij}} \frac{w_j}{w_i} \frac{w_i}{w_j}} = 2, \quad \text{for any } i, j$$



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