Preface

The central challenge of any mathematical theory is to provide reasonable classifications and constructive descriptions of the investigating objects that are most useful in diverse applications. At the same time, we realize that the purpose is to support new methods of investigation, which, in the end, constitute ideological riches of the given theory. For example, the development of the theory of finite non-simple groups in the past 50 years has clearly shown this tendency. Although the theory of finite groups has never been lacking in general methods, ideas, or unsolved problems, the large body of results has inevitably brought us to the point of needing to develop new methods to systemize the material.

One example of such systemizing is the idea of Gaschüts that the use of some given classes of groups, called saturated formations, is convenient for investigating the inner structure of finite groups. The theory of formation initially found wide application in the research of finite groups and infinite groups, as reflected in a series of classical monographs. The first is the famous book of Huppert [248]. The book, which described research in the structure of finite groups, attracted the attention of many specialists in algebra. The investigation involving saturated and partially saturated (ω-saturated or soluble ω-saturated) formations became one of the predominate directions of development within the contemporary theory of group classes, which indubitably testifies to its actual importance. Following the publication of monographs by L. A. Shemetkov [359], L. A. Shemetkov and A. N.Skiba [366], and Doerk and Hawkes [89], the direction of theoretical development was shaped by pithy and harmonious theory. The subsequent monographs by W. Guo [135] and A. Ballester-Bolinches and L. M. Ezquerro [34] reflect the modern status of the theory of formations. At the same time, it is necessary to mention that the swiftly augmented flow of articles about the theory of finite groups, particularly in the past 20 years, has brought to the forefront the necessity of further analysis and development of new methods of research. Additional developments have included the formation properties of the classes of all soluble (π-soluble) groups, all supersoluble (π-supersoluble) groups, all nilpotent (π-nilpotent) groups, all quasinilpotent groups, and other classes of finite groups with broad application in modern finite group theory research.

The first aim of this book is to introduce the subsequent development and application of the theory of classes of finite groups. Note also that many important results on the theory of classes of groups obtained during recent years are not well reflected in the monographs formerly cited, for example, the theory of subgroup functors, the theory of X-permutable subgroups, the theory of quasi-F-groups, the theory of F-cohypercentres for Fitting classes, and the theory of the algebra of formations, which is related to the research of semigroups and the lattices of formations. To eliminate such information deficiency is the second aim of this book. The third aim is to systemize and unify the results obtained during recent years and related research in various classes of non-simple groups.

The largest term $Z_\infty(G)$ of the upper central series of a finite group G is called the hypercenter of G. In fact, $Z_\infty(G)$ is the largest normal subgroup of G such that every chief factor H/K of G below $Z_\infty(G)$ is central; that is, $C_G(H/K) = G$. This elementary observation allows us to define the formation analog of the hypercenter. The $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G is the largest normal subgroup of G such that every chief factor H/K of G below $Z_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-central in G, that is, $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$.

The $\mathfrak{F}$-hypercenter and $\mathfrak{F}$-hypercentral subgroups (that are, the normal subgroups contained in $Z_{\mathfrak{F}}(G)$) have great influence on the structure of groups, and so have been investigated by a large number of researchers. Nevertheless, there are still some open problems concerning the $\mathfrak{F}$-hypercenter. The main goal of Chap. 1 is a further study of the $\mathfrak{F}$-hypercenter and the generalized $\mathfrak{F}$-hypercenter of a group. As a result, in Sect. 5 of Chap. 1, we provide solutions to some of these problems, in particular, the solution to Baer–Shemetkov's problem of the description of the subgroup $Z_{\mathfrak{F}}(G)$ as the intersection of all $\mathcal{F}$-maximal subgroups of G, and the solution to Agrawal's problem regarding the intersection of all maximal supersoluble subgroups.

In the first section of Chap. 1, we collect some base results on saturated and solubly saturated formations that are used in our proofs.

In Sect. 2 of Chap. 1, we study the $\mathfrak{F}$-hypercentral subgroups in detail. The theorems we prove in this section are used in many other sections of this book and develop some known results of many authors (e.g., Gaschütz [112]; Huppert [89 IV, 6.15]; Shemetkov [357]; Selkin [344]; Ballester-Balinches [24]; Skiba [384] and others).

In Sect. 3 of Chap. 1, we consider applications of the theory of generalized quasinilpotent groups. Recall that a group G is said to be quasinilpotent if for every chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K (see [250, p. 124]). We note that for every central chief factor H/K, an element of G induces trivial automorphism on H/K; thus, we can say that a group G is quasinilpotent if for every noncentral chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K. In a general case, we say that G is a quasi-$\mathfrak{F}$-group if for every $\mathfrak{F}$-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is an inner automorphism. The theory of quasinilpotent groups is well represented in the book [250]. The theory of the quasi-$\mathfrak{F}$-groups, which covers the theory of quasinilpotent groups, was developed by W. Guo and A. N. Skiba [188, 189, 192]. In this section of Chap. 1, we give the complete theory of quasi-$\mathfrak{F}$-groups and consider some related applications.

In Sect. 4 of Chap. 1, we explain further applications of the results of the theoretical work described in Sect. 2. In particular, we study the groups G with factorization $G = AB$ such that $A \cap B \leq Z_{\mathfrak{F}}(A) \cap Z_{\mathfrak{F}}(B)$. Such an approach to studying groups allows us to generalize some results of Baer [23], Friesen [108], Wielandt [447], Kegel [264], Doerk [87] concerning factorizations of groups.

If for subgroups A and B of a group G we have $AB = BA$, then A, B are said to be permutable or A is said to be permutable with B. In this case, AB is a subgroup of G. Hence, the permutability of subgroups is one of the important relations among subgroups. The research of permutability in finite and infinite groups is still of broad interest. In fact, such a direction for group theory may go back to the classic works [231] and [324]. Hall [231] proved that group G is soluble if and only if it has at least one Sylow system, that is, a complete set of pairwise permutable Sylow subgroups. Ore [324] introduced to the mathematical practice the so-called quasinormal subgroups, that are, the subgroups which permute with all subgroups of the overall group. Increasing interest in that subject was characteristic of the 1960s and 70s (e.g., Huppert [246, 247]; Kegel [263]; Thompson [403]; Deskins [85], Čunihin [79]; Stonehewer [396]; Ito and Szé [252]; Maier and Schmid [307] and others). The results obtained during that period were well-written in the books [32, 79, 89, 248, 276, 343, 450].

Some new ideas about the research of permutable subgroups are described in Chaps. 2–4 of this book.

We often encounter a situation in which $AB \neq BA$ for subgroups A and B of a group G, but there exists an element $x \in X \subseteq G$ such that $AB^x = B^x A$. In this case, we say that A is X-permutable with B. The well-known examples are any two Hall subgroups of a soluble group and any two normal embedded subgroups of a group, which are G-permutable (see [89], Chap. 1). The concept of X-permutability of subgroups is extraordinarily useful when researching the problems of classifying groups, and the terminology of X-permutability provides a beautiful way to describe group classes. The second chapter in that monograph is devoted to the detailed analysis of the properties of X-permutable subgroups and their application to solve a series of open problems in group theory. The first of such problems concerns the generalization of the well-known Schur–Zassenhaus theorem.

The Schur–Zassenhaus theorem asserts that *If G has a normal Hall π-subgroup A, then G is an $E_{\pi'}$-group (that is, G has a Hall π'-group). Moreover, if either A or G/A is soluble, then G is a $C_{\pi'}$-subgroup (that is, any two Hall π'-subgroups of G are conjugate).*

Recall that a group G is said to be π-separable if G has a chief series

$$1 = G_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G, \qquad (*)$$

such that each index $|H_i : H_{i-1}|$ is either a π-number or a π'-number.

The most interesting generalization of the Schur–Zassenhaus theorem is the following classical result of P. Hall and S. A. Čunihin: Any π-separable group is a

D_π-group (that is, G is a C_π-group and any π-subgroup of G is contained in some Hall π-subgroup of G).

It is well known that the above Schur–Zassenhaus theorem and the Hall–Čunihin theorem are truly fundamental results of group theory. In connection with these important results, the following two problems have naturally arisen: (I) *Does the conclusion of the Schur–Zassenhaus theorem hold if the Hall subgroup A of G is not normal? In other words, can we weaken the condition of normality for the Hall subgroup A of G so that the conclusion of the Schur–Zassenhaus theorem is still true?* (II) *Can we replace the condition of normality for the members of series* (*) by some weaker condition? For example, can we replace the condition of normality by permutability of the members of series (*) with some systems of subgroups of G?

In Sect. 2 of Chap. 2, we provide a positive answer to these two problems based on a careful analysis of the properties of X-permutability. The results obtained are nontrivial generalizations of the Schur–Zassenhaus theorem and the Hall–Čunihin theorem.

Another open problem, which is solved in Chap. 2, is to describe the group in which every subgroup can be written as an intersection of the subgroups of prime power indexes. Note that the structure of groups in which every subgroup can be written as an intersection of pairwise coprime prime power indexes is known (see [450], Chap. 6). The problem is completely solved in Sect. 3 of Chap. 2. The resolution of this problem was found in the close connection with Johnson's problem [256] about the description of the group in which every meet-irreducible subgroup has a prime-power index. Partially this problem was solved in [450]. The complete resolution of Johnson's problem is also introduced in this section. In the remaining sections of Chap. 2, we discuss some new characteristics for many classes of finite groups.

For two subgroups A and B of a group G, if $G = AB$, then B is said to be a supplement of A in G. More often we consider the supplements with some restrictions on $A \cap B$. For example, we often encounter the situation in which $A \cap B = 1$. In this case, B is called a complement of A in G. If B is also normal in G, then B is called a normal complement of A in G.

In [324], Ore considered the subgroup H of the group G with the following (Ore's) condition: G has normal subgroups N and M such that $HM = G$ and $H \cap M \leq N \leq H$. It is clear that if a subgroup H of G is either normal in G or has a normal complement in G, then H satisfies Ore's condition. Note also that if H is either complemented in G or is normal in G, then H satisfies the following (Ballester-Bolinches–Guo–Wang's) condition: G has a subgroup M and a normal subgroup N such that $HM = G$ and $H \cap M \leq N \leq H$ [45, 435].

In the past 20 years, a large number of research publications have involved finding and applying other generalized complemented and generalized normally complemented subgroups. The papers of A. N. Skiba [383], W. Guo [141] and B. Li [277] have made the greatest impact on this direction of research.

Note that, at present, a large number of interesting and profound ideas and theorems in this direction that have accumulated require systemization and unification. The aim of Chap. 3 is to solve this non-simple problem. The main tool for solving this problem is the method of subgroup functors.

If a group G has a chain of subgroups

$$H = H_0 < H_1 < \ldots < H_n = G,$$

where H_{i-1} is a maximal subgroup of H_i, $i = 1, 2, \ldots, n$, then this chain is called a maximal (H, G)-chain, and the number n is said to be the length of the chain. In this case, H is said to be an n-maximal subgroup of G. If $K < H \leq G$ and K is a maximal subgroup of H, then (K, H) is said to be a maximal pair in G.

In Sect. 1 of Chap. 4, we construct the original theory of Σ-embedded subgroups. The basic tool of this theory is the concept of covering and avoiding subgroup pairs. Suppose that $A \leq G$ and $K \leq H \leq G$. Then we say that A covers the pair (K, H) if $AH = AK$; and A avoids (K, H) if $A \cap H = A \cap K$. Let A be a subgroup of G and $\Sigma = \{G_0 \leq G_1 \leq \ldots \leq G_n\}$ be some subgroup series of G. Then we say that A is Σ-embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$, for some i. We show that the description of many important classes of groups may be based on the concept of the Σ-embedded subgroup.

In Sect. 2 of Chap. 4, we investigate the classes of groups with distinct restrictions on maximal, 2-maximal and 3-maximal subgroups. In particular, we provide complete descriptions of the groups in which any maximal subgroup, 2-maximal subgroups and 3-maximal subgroups are pair-permutable. In Sect. 3 of Chap. 4, we describe the group in which every maximal chain of G of length 3 contains a proper subnormal entry and there exists at least one maximal chain of G of length 2 that contains no proper subnormal entry. In Sect. 4, we establish the theory of $\hat{\theta}$-pairs for maximal subgroups of a finite group and introduce some new characterizations of the structure of finite groups.

The application of formations and Fitting classes to the investigation of group theory is inconceivable without the benefit of detailed research of formations and Fitting classes, themselves. This circumstance leads to the necessity of developing methods for building and investigating formations and Fitting classes. In particular, this leads to the investigation of algebraic systems connected with operations on the set of formations and Fitting classes. Important roles are played by the operations of the product of the classes (which lead to the semigroups of all formations and Fitting classes), and the operations of generation and intersection of classes of groups (which lead to research of the lattices of all formations and Fitting classes). Many difficult problems have arisen in this direction of research. We provide solutions to some of these problems in Chap. 5.

A formation $\mathfrak{F}$ is said to be ω-solubly saturated or ω-composition if for every $p \in \omega$, $\mathfrak{F}$ contains every group G with $G/\Phi(O_p(G)) \in \mathfrak{F}$. If there is a group G such that $\mathfrak{F}$ coincides with the intersection of all ω-composition formations containing G, then $\mathfrak{F}$ is called a one-generated ω-composition formation. Since one-generated ω-composition formations are compact elements of the algebraic lattice of all ω-composition formations, it is important to study problems associated with one-generated ω-composition formations. This circumstance has predetermined a wide range of interest in studying formations of this kind.

In Sects. 2 and 3 of Chap. 5, on the basis of the results in Chap. 1 and some technical results of Sect. 1 of Chap. 5, we provide solutions to the following two problems concerning factorizations of one-generated ω-composition formations: (I) Letting $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be a one-generated ω-composition formation, we consider whether it is true that $\mathfrak{M}$ is an ω-composition formation if $\mathfrak{H} \neq \mathfrak{F}$ (see Problem 18 in [389]); (II) Describe non-cancellated factorizations of one-generated ω-composition formations ([389], Problem 21).

Some special cases of these two problems have been discussed in the papers of many authors (see, in particular, Skiba [377, 380, 382, 388]; Rizhik and Skiba [331]; Vishnevskaya [426]; Jahad [257]; Guo and Shum [170]; Guo and Skiba [186]; Ballester-Bolinches et al. [28, 30, 31]; Ballester-Bolinches and Perez-Ramos [43]; Vorob'ev [429]; Elovikov [95, 96]).

In Chap. 5, we give complete solutions to these two problems.

Considerable space in Chap. 5 is devoted to methods for constructing and analyzing the lattices of formations. In Sect. 4 of Chap. 5, based on a detailed study of the compact elements of the lattice of the formations, we provide a description of the formation in which the lattice of the subformations is a Boolean lattice. This provides solutions for two of the problems proposed in the book of A. N. Skiba [381, p. 192].

In Sect. 5 of Chap. 5, we provide a negative answer to the problem posed by O. V. Melnikov, regarding whether any graduated formation is a Baer-local formation.

Sections 6 and 7 of Chap. 5 are devoted to the subsequent development of the research methods of groups by means of the theory of formations and Fitting classes. In particular, we describe the formations and Fitting class as determined by the properties of Hall subgroups, and find new characterizations for diverse classes of finite subgroups (covering subgroups, injectors, etc.).

Sections 8 and 9 of Chap. 5 are devoted to introduce the theory of $\mathfrak{F}$-cohypercentre for Fitting classes and ω-local Fitting classes.

In Sect. 10 of Chap. 5, we provide a negative answer to the Doerk-Howkes-Shemetkov problem [363], which considered whether it was proved that if the formation $\mathfrak{F}$ is hereditary or soluble or saturated, then the class $(G^{\mathfrak{F}}|G$ is a group) is not necessarily closed under subdirect products in general.

In the last section of each chapter, we add some additional information and present some open problems.

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