Chapter 2
Groups With Given Systems of X-Permutable Subgroups

2.1 Base Concepts

X-Permutable and c-Permutable Subgroups A subgroup A of a group G is said to be permutable with a subgroup B if $AB = BA$. A subgroup A is said to be a permutable or a quasinormal subgroup of G if A is permutable with all subgroups of G. But we often meet the situation $AB \neq BA$, nevertheless there exists an element $x \in G$ such that $AB^x = B^x A$, for instance, we have the following cases:

1) Let $G = AB$ be a group. If A_p and B_p are Sylow p-subgroups of A and of B respectively, then $A_p B_p \neq B_p A_p$ in general, but G has an element x such that $A_p B_p^x = B_p^x A_p$.
2) If A and B are Hall subgroups of a soluble group G, then there exists an element $x \in G$ such that $AB^x = B^x A$. (see [89, Theorem (I, 4.11)])
3) If A and B are normally embedded subgroups (see [89, Definition (I, 7.1)]) of a soluble group, then A is permutable with some conjugate of B. (see [89, Theorem (I, 7.11)])
4) If $|G : A| = p^\alpha$ is a prime power, then for every Sylow subgroup Q of G, there is $x \in G$ such that $AQ^x = Q^x A$.

Based on the above observations, we give the following definitions.

Definition 1.1 Let A and B be subgroups of a group G, and let $\varnothing \neq X \subseteq G$. Then we say:

(1) A is *X-permutable* with B in G if there exists some $x \in X$ such that $AB^x = B^x A$;
(2) A is *completely X-permutable* (or *hereditary X-permutable)* with B in G if there exists some $x \in X \cap \langle A, B\rangle$ such that $AB^x = B^x A$;
(3) A is *conditionally permutable* (or in brevity, *c-permutable*) with B in G provided A is G-permutable with B;
(4) A is *completely c-permutable* or (*hereditary c-permutable*) with B in G provided A is complete G-permutable with B in G.

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By Chap. 1, Lemma 5.34(1), every s-quasinormal subgroup is subnormal. The following examples show that a subgroup of a group G which is c-permutable with all Sylow subgroups of G is not necessarily s-quasinormal in general even in the case when it is subnormal.

Example 1.2 Fixing some odd prime p, put $A = \langle x, y | x^{p^2} = y^p = 1, x^y = x^{1+p}\rangle$ and $L = \langle y\rangle$. Take some involution g in Aut$L$, and put $B = L \rtimes \langle g\rangle$. Consider a transitive permutation representation $\alpha : B \longrightarrow Sym(p)$ of degree p. Take the wreath product $G = A \wr_\alpha B = K \rtimes B$ of A and B with respect to α, where K is the base of $A \wr_\alpha B$. Using the terminology of [89], put $R = L^\natural$, and consider $N = N_G(R)$. It is clear that $B \subseteq N$ and $N \cap K = (N_A(L))^\natural$. Since $|A| = p^3$ and $N_A(L) \neq A$, $N_A(L)$ is an abelian group, and so $N \cap K$ is also abelian. It is clear that R is G-permutable with all Sylow subgroups of G and that R is subnormal in G. Suppose that R is permutable with all Sylow 2-subgroups of G. Then for each $x \in G$ we have $\langle g\rangle^x \subseteq N$. Consequently, the normal closure $\langle g\rangle^G$ of the subgroup $\langle g\rangle$ in G satisfies $L \subseteq \langle g\rangle^B = B \subseteq \langle g\rangle^G \subseteq N$; thus, $B^G \subseteq N$. Suppose further that $M = \{(a_1, \ldots, a_p) | a_i \in A, a_1 \ldots a_p \in A'\}$. Then $B^G = MB$ by [89, A, (18.4)]. Consequently, $M \subseteq N$. However, if $a_1 = \cdots = a_p$, then $a_1^p \subseteq A'$. Hence M has a subgroup which is isomorphic to A. This means that $N \cap K$ is not abelian. This contradiction shows that R is not permutable with some Sylow 2-subgroup of G.

Example 1.3 Let M be a subgroup of a soluble group G. Suppose that $|G : M| = p$ is a prime. Then

(i) *M completely c-permutes with all subgroups of G.* Indeed, let $T \leq G$. Let $M_1, \ldots, M_t$ and $T_1, \ldots, T_t$ be some Sylow systems of the groups M and T, respectively. Then G has Sylow systems $\Sigma = \{P_1, \ldots, P_t\}$ and $\Sigma_1 = \{Q_1, \ldots, Q_t\}$ such that $M_i = P_i \cap M$ and $T_i = Q_i \cap T$ for all $i = 1, \ldots, t$ (see Sect. 2 in [248, Chap. VI]). Moreover, the systems Σ and Σ_1 are conjugate, that is, G has an element x such that $Q_i^x = P_i$ for all $i = 1, \ldots, t$. Without loss of generality, we may assume that P_1 is a Sylow p-subgroup of G. Then $M_2 = P_2, \ldots, M_t = P_t$. Assume that $T_1^x \subseteq M_1$. Then we have

$$T^x \subseteq M_1 P_2 \ldots P_2 = M,$$

and so $T^x M = M = M T^x$.
Now let $T_1^x \nsubseteq M_1$. Since $|G : M| = p$, we have $|P_1 : M_1| = p$ and $P_1 = T_1^x M_1$. Hence,

$$T^x M = T_2^x \ldots T_t^x T_1^x M_1 M_2 \ldots M_t = T_2^x \ldots T_t^x P_1 P_2 \ldots P_t = G = M T^x.$$

(ii) *If G is supersoluble, then M is $F(G)$-permutable with every subgroup T of G.* Indeed, if $F(G) \leq M$, then $G' \leq M$. Hence M is normal in G and so it is quasinormal in G. Now assume that $F(G) \nleq M$. Then by (i), there exists an element x of G such that $MT^x = T^x M$. Then $G = MF(G)$ and hence $x = mf$ for some $m \in M$ and $f \in F(G)$. Therefore $MT^x = MT^{mf} = MT^f = T^f M$.

These examples are a motivation for introducing the following concepts.

Definition 1.4 Let A be a subgroup of a group G and $\varnothing \neq X \subseteq G$. Then we say that:

(1) A is *(completely) X-quasinormal* or *(completely) X-permutable in G* if A is (completely, respectively) X-permutable with all subgroups of G.
(2) A is *(completely) X-s-permutable in G if A is (completely, respectively) X-permutable with all Sylow subgroups of G. In particular, if $X = 1$, then an X-s-permutable subgroup is said to be s-permutable (or s-quasinormal) in G.*
(3) *A is (completely) c-permutable in G if A is (completely, respectively) c-permutable with all subgroups of G.*
(4) *A is (completely) s-c-permutable in G if A is (completely, respectively) c-permutable with all Sylow subgroups of G.*

In the following lemma we give the general properties of X-permutability.

Lemma 1.5 *Let A, B, X be subgroups of G and $K \trianglelefteq G$. Then the following statements hold:*

(1) *If A is (completely) X-permutable with B, then B is (completely) X-permutable with A.*
(2) *If A is (completely) X-permutable with B, then A^x is (completely) X^x-permutable with B^x for all $x \in G$.*
(3) *If A is (completely) X-permutable with B, then AK/K is (completely) XK/K-permutable with BK/K in G/K.*
(4) *Suppose that $K \leq A$. Then A/K is (completely) XK/K-permutable with BK/K if and only if A is (completely) X-permutable with B.*
(5) *If $A, B \leq M \leq G$ and A is completely X-permutable with B, then A is completely $(X \cap M)$- permutable with B.*
(6) *If A is (completely) X-permutable with B and $X \leq M \leq G$, then A is (completely) M-permutable with B.*
(7) *If A is X-permutable with B and $X \leq N_G(A)$, then A is permutable with B.*
(8) *If F is a quasinormal subgroup of G and A is (completely) X-permutable with B, then AF is (completely) X-permutable with B.*
(9) *If $A \leq T$, where T is a subnormal subgroup of a (solvable) group G, and A is G-permutable with all Sylow (Hall) subgroups of G, then A is T-permutable with all Sylow (Hall) subgroups of T.*
(10) *Suppose that $G = AT$ and T_1 is a subgroup of T. If A is (completely) G-permutable with T_1, then A is (completely) T-permutable with T_1.*
(11) *If A is a maximal subgroup of G, T is a minimal supplement to A in G and A is c-permutable with all subgroups of T, then $T = \langle a \rangle$ is a cyclic p-group for some prime p, and $a^p \in A$.*

Proof (1)–(3) and (5)–(8) are obvious.

(4) Suppose that A/K is (completely) XK/K-permutable with KB/K in G/K. Then there exists an element xK of XK/K (an element of $(XK/K) \cap \langle A/K, KB/K \rangle$,

respectively) such that

$$(A/K)(BK/K)^{xK} = (BK/K)^{xK}(A/K).$$

This implies that $AB^xK = AB^x = B^xA$. Clearly, $xK = hK$ for some $h \in X$. We may, therefore, assume that $x \in X$ (respectively $x \in X \cap \langle A, KB\rangle = X \cap \langle A, B\rangle$). This means that A is (completely) X-permutable with B in G. On the other hand, if A is (completely) X-permutable with B in G, then by (3), A/K is (completely) X-permutable with BK/K in G/K.

(9) Take some Sylow p-subgroup T_p of T and some Sylow subgroup G_p of G containing T_p. Pick $x \in G$ such that $AG_p^x = G_p^xA$. Then AG_p^x is a subgroup of G. Hence $AG_p^x \cap T = A(G_p^x \cap T) = (G_p^x \cap T)A$ is a subgroup of T. Since T is subnormal in G, $G_p^x \cap T$ is a Sylow p-subgroup of T. Take some element t in T such that $(G_p^x \cap T)^t = T_p$. Then $A(G_p^x \cap T) = AT_p^{t^{-1}} = T_p^{t^{-1}}A$. Similarly we can prove the second claim.

(10) Suppose that A is completely G-permutable with T_1. Then there exists some element $x \in \langle A, T_1\rangle$ such that $AT_1^x = T_1^xA$. Since $G = AT$, $x = at$ for some $a \in A$ and $t \in T$. Hence

$$AT_1^x = AatT_1t^{-1}a^{-1} = aAtT_1t^{-1}a^{-1} = a(AT_1^t)a^{-1}$$

is a subgroup of G. Hence, $AT_1^t = T_1^tA$, where $t \in T \cap \langle A, T_1\rangle$ (because $x \in \langle A, T_1\rangle$). This shows that A is completely T-permutable with T_1. If A is G-permutable with T_1, then similarly we can show that A is T-permutable with T_1.

(11) Take a maximal subgroup M of T. By (10) for some $t \in T$ we have $AM^t = M^tA$. Since T is a minimal supplement to A in G, $AM \neq G$ and so $AM^t \neq G$. Since A is a maximal subgroup of G, $M^t \leq A$. Suppose that T has some maximal subgroup M_1 which is not conjugate to M. Then as above we see that $M_1^{t_1} \leq A$ for some $t_1 \in T$. It is clear that $M^t \neq M_1^{t_1}$ and $T = \langle M^t, M_1^{t_1}\rangle \leq A$. This implies that $G = AT = A$, a contradiction. Therefore T is a primary cyclic group and $M \leq A$.

Lemma 1.6 *Suppose that $G = HT$, where H is a completely c-permutable proper subgroup of G and T is a nilpotent subgroup of G. Then G has a chain of subgroups*

$$H = T_0 \leq T_1 \leq \ldots \leq T_{t-1} \leq T_t = G$$

such that $|T_i : T_{i-1}|$ is a prime for all $i = 1, \ldots, t$.

Proof We may assume that $G \neq DH$ for any proper subgroups D of T. Let T_1 be a maximal subgroup of T. Suppose that $T_1 \leq H$. Then $|G| = \frac{|T||H|}{|T\cap H|} = \frac{|T||H|}{|T_1|}$. Since T is nilpotent, $|G : H| = |T : T_1|$ is a prime.

Now assume that $T_1 \nsubseteq H$. By hypothesis, for some element $x \in G$, we have $HT_1^x = T_1^xH$. Since $G = HT$, $x = th$, where $t \in T$ and $h \in H$. Hence $T_1^x = T_1^h$. Since $T_1 \nsubseteq H^{h^{-1}} = H$, $T_1^h \nsubseteq H$. Moreover, because $T_1^hH \leq G$, we have $(T_1^hH)^{h^{-1}} = T_1H \leq G$. It is clear that $T_1H \neq G$. Assume that $T \cap H \nsubseteq T_1$. Then

$T = T_1(T \cap H) \subseteq T_1 H$, and so $G = TH \subseteq T_1 H$, which is impossible. Hence $T \cap H \subseteq T_1$ and so $|T \cap H| = |T_1 \cap H|$. It follows that

$$|TH : T_1 H| = \frac{|T||H|}{|T \cap H|} \cdot \frac{|T_1 \cap H|}{|T_1||H|} = |T : T_1|$$

is a prime. Since $|T_1 H| < |G|$ and by hypothesis H is completely c-permutable in G, by induction on $|G|$, $T_1 H$ has a chain of subgroups

$$H = D_0 \leq D_1 \leq \ldots \leq D_{n-1} \leq D_n = HT_1$$

such that $|D_i : D_{i-1}|$ is a prime for all $i = 1, \ldots, n$. This completes the proof.

Proposition 1.7 [207]. *Let A be a proper group of a group G.*

(1) *If G is soluble and A is a completely c-permutable subgroup of G, then G has a chain of subgroups*

$$A = T_0 \leq T_1 \leq \ldots \leq T_{t-1} \leq T_t = G$$

such that $|T_i : T_{i-1}|$ is a prime for all $i = 1, \ldots, t$.

(2) *If A is subnormal in G and A c-permutes with all Sylow subgroup of G, then A/A_G is soluble.*

Proof

(1) Suppose the claim false and take a counterexample G of minimal order. Let L be any minimal normal subgroup of G. Since G is soluble, L is abelian. Suppose that $G = LA$. Then Lemma 1.6 implies (1), which contradicts the choice of G. Now assume that $LA \neq G$. Since $|LA| < |G|$ and A is a completely c-permutable subgroup of LA by Lemma 1.5(5), the choice of G implies that LA has a series

$$A = T_0 \leq T_1 \leq \ldots \leq T_{t-1} \leq T_t = LA$$

such that $|T_i : T_{i-1}|$ is a prime for all $i = 1, \ldots, t$.
Consider G/L. By Lemma 1.5(3), AL/L is a completely c-permutable subgroup of G/L, and so G/L has se series

$$AL/L = T_0/L \leq T_1/L \leq \ldots \leq T_{t-1}/L \leq T_t/L = G/L$$

such that $|T_i/L : T_{i-1}/L| = |T_i : T_{i-1}|$ is a prime for all $i = 1, \ldots, t$. Hence G has a series

$$A = T_0 \leq T_1 \leq \ldots \leq T_{t-1} \leq T_t = G$$

of subgroups with prime indexes. This completes the proof of Claim (1).

(2) By Lemma 1.5(4) we may assume that $A_G = 1$. Let $R = A^{\mathfrak{S}}$. Then, obviously, $R = R'$. Assume that A is not soluble. Then $R \neq 1$. Let $p \in \pi(G)$ and G_p be a Sylow p-subgroup of G such that $D = G_p A = AG_p$. Let Q be a Sylow q-subgroup of A, where $q \neq p$. Then evidently Q is a Sylow q-subgroup of

D. Since A is subnormal in G, A is subnormal in D and $Q^x \cap A$ is a Sylow q-subgroup of A, for all $x \in D$. Hence $L_q = \langle Q^x \mid x \in D \rangle \subseteq A$. Clearly, $L_q \trianglelefteq D$. Let L be the product of all L_q, where q runs through all prime divisors of the order of A which are different from p. Then $L \trianglelefteq D$. Since $LR/L \simeq R/L \cap R$ and D/L is a p-group, we have $R \subseteq L$. Let R_1 be the smallest normal subgroup of L with a soluble quotient. Then R_1 char $L \trianglelefteq A$, and so $R_1 \trianglelefteq A$. Since L/R_1 is a soluble group, $R_1 \subseteq R$. But since $R' = R$, $R_1 = R$ char $L \trianglelefteq D$. Therefore $R \trianglelefteq D$. Consequently, $G_p \subseteq N_G(R)$ for any $p \in \pi(G)$. It follows that $R \trianglelefteq G$. Therefore $A_G \neq 1$. This contradiction completes the proof.

X-semipermutable Let A, B be subgroups of a group G. If $AB = G$, then B is called a supplement of A in G.

Definition 1.8 Let A and B be subgroups of a group G, and let $\varnothing \neq X \subseteq G$. Then we say that *$A$ is (completely) X-semipermutable in G* if A is (completely) X-permutable with all subgroups of some supplement T of A in G.

We use $X(A)$ $(X_c(A))$ to denote the set of all supplements T of A in a group G such that A is (completely) X-permutable in G with all subgroups of T. Thus A is (completely) X-semipermutable in G if and only if $X(A) \neq \varnothing$ $(X_c(A) \neq \varnothing$, respectively).

Example 1.9 Let $G = A_5 \times C_7$, where C_7 is a group of order 7 and A_5 is the alternating group of degree 5. Let C_5 be a Sylow 5-subgroup of A_5. Let $A \simeq A_4$ be a subgroup of G with $|G : A| = 5$ and $T = C_5 \times C_7$. Then $AT = G$ and evidently A permutes with all subgroups of T. Hence A is 1-semipermutable in G. On the other hand, A is not c-permutable in G. Indeed, let P be a Sylow 3-subgroup of A_5. Then $|A_5 : N_{A_5}(P)| = 10$. Hence A is not c-permutable with $N_{A_5}(P)$.

Lemma 1.10 *Let A and X be subgroups of G. Then the following statements hold:*

(1) *If N is a permutable subgroup of G and A is X-semipermutable in G, then NA is a X-semipermutable subgroup of G.*
(2) *If $N \trianglelefteq G$, A is X-semipermutable in G and $T \in X(A)$, then AN/N is XN/N-semipermutable in G/N and $TN/N \in (XN/N)(AN/N)$.*
(3) *If A/N is XN/N-semipermutable in G/N and $T/N \in (XN/N)(A/N)$, then A is X-semipermutable in G and $T \in X(A)$.*
(4) *If A is X-semipermutable in G and $A \leq D \leq G$, $X \leq D$, then A is X-semipermutable in D.*
(5) *If A is a maximal subgroup of G, T is a minimal supplement of A in G and $T \in G(A)$, then $T = \langle a \rangle$ is a cyclic p-group, for some prime p and $a^p \in A$.*
(6) *If $T \in X(A)$ and $A \leq N_G(X)$, then $T^x \in X(A)$, for all $x \in G$.*
(7) *If A is X-semipermutable in G and $X \leq D$, then A is D-semipermutable in G.*

Proof

(1) This part follows directly from Lemma 1.5(8).
(2) It is obvious that TN/N is a supplement of AN/N in G/N. If T_1/N is a subgroup of TN/N, then $T_1/N = (T_1 \cap N)T/N = N(T_1 \cap T)/N$ and so

AN/N is XN/N-permutable with T_1/N in G/N by Lemma 1.5(3). Hence, $TN/N \in (XN/N)(AN/N)$.

(3) The proof of this part is the same as the proof in (2).
(4) This part is evident.
(5) See Lemma 1.5(11).
(6) Obviously, T^x is a supplement of A in G for any $x \in G$. Let T_1 be a subgroup of T^x. We need to prove that A is X-permutable with T_1. Since $G = AT$, we have $x = ta$, for some $a \in A, t \in T$. Hence $T^x = T^a$. Note that $T_1^{a^{-1}} \le T$ and $A = A^{a^{-1}}$. By hypothesis, for some $d \in X$, we have $A(T_1^{a^{-1}})^d = (T_1^{a^{-1}})^d A = A^a(d^{-1})^a(T_1^{a^{-1}})^a d^a = AT_1^{d^a} = T_1^{d^a}A$, where $d^a \in X$ since $A \in N_G(X)$. This shows that $T^x \in X(A)$.
(7) This part is evident.

X_m-semipermutable subgroups.

Definition 1.11 Let A be a subgroup of a group G and X a nonempty subset of G. Then we say that:

(1) A is *X_m-permutable in G* if A is X-permutable with all maximal subgroups of all Hall subgroups of G.
(2) A is *X_m-semipermutable in G* if A is X-permutable with all maximal subgroups of all Hall subgroups of some minimal supplement of A in G.

In particular, if A is 1_m-permutable (1_m-semipermutable) in G, then we say that A is m-permutable (respectively, m-semipermutable) in G.

Example 1.12 Let A be a p-group with p an odd prime and $B = D_m = \langle x, y | x^{2^{m-1}} = y^2 = 1, x^y = x^{-1}\rangle$ a dihedral group of order 2^m, where $m > 2$. Let $G = A \times B$ and $L = \langle y\rangle$. Since G is nilpotent, every maximal subgroup of any Hall subgroup of G is normal in G. Hence L is m-permutable in G. On the other hand, L is clearly not permutable with $\langle y^x\rangle$. Thus the class of the X_m-permutable subgroups is in general a broader class than the class of the X-permutable subgroups.

Example 1.13 Let B and L be the groups as in Example 1.12. Then, it is easy to see that there is a 2-group P such that $B \le P'$ and so $B \le \Phi(P)$. Therefore P is the only minimal supplement of L in P. This shows that L is X_m-semipermutable but not X-semipermutable in P. Thus the class of the X_m-semipermutable subgroups is in general a broader class than the class of the X-semipermutable subgroups.

We will use $X_m(A)$ to denote the set of all minimal supplement T of A in a group G such that A is X-permutable with all maximal subgroups of any Hall subgroup of T. Thus A is X_m-semipermutable in G if and only if $X_m(A) \neq \varnothing$.

Lemma 1.14 *Let A, T, X be subgroups of G and H be a minimal normal subgroup of G. Then:*

(1) *If either $H \le A$ or $H \le T$ and if T is a minimal supplement of A in G, then TH/H is a minimal supplement of AH/H in G/H.*

(2) *Suppose that A is X-permutable with all maximal subgroups of any Hall subgroup of T. Assume that or H is abelian or $(|H|, |T|) = 1$ or T is soluble. Then AH/H is XH/H-permutable with all maximal subgroups of any Hall subgroup of TH/H.*

Proof

(1) Suppose that $H \leq A$ and let E/H be a supplement of A/H in G/H such that $E/H \leq TH/H$. Then $E = E \cap TH = H(E \cap T)$ and so $G = AE = A(E \cap T)$. Hence $T \leq E$ and $E/H = TH/H$ is a minimal supplement of A/H in G/H. On the other hand, if $H \leq T$ and E/H is a subgroup of T/H such that $G/H = (AH/H)(E/H)$, then $G = AE$ and so $E/H = T/H$ is a minimal supplement of AH/H in G/H.
(2) Let E/H be a Hall π-subgroup of TH/H and M/H be any maximal subgroup of E/H. We prove that AH/H is XH/H-permutable with M/H. We first note that $E = E \cap TH = H(E \cap T)$, $M = M \cap TH = H(M \cap T)$ and so $|T : E \cap T|$ is a π'-integer. Therefore, if H is a π-group, then E is also a π-group and so $E \cap T$ is a Hall π-subgroup of T. On the other hand, if $(|H|, |T|) = 1$, then $H \cap E \cap T = 1$ and so in this case $E \cap T$ is also a Hall π-subgroup of T. Now we show that $M \cap T$ is a maximal subgroup of $E \cap T$. Clearly $M \cap T \neq E \cap T$. Assume that for some subgroup D of G we have $M \cap T \leq D \leq E \cap T$. Then $M = H(M \cap T) \leq HD \leq H(E \cap T) = E$ and hence or $M = DH$ or $DH = E$. If $M = DH$, then

$$D = D \cap H(M \cap T) = (M \cap T)(D \cap H) = M \cap T.$$

If $DH = E$, then,

$$D = D \cap E \leq (E \cap T) \cap H(E \cap T) \leq (E \cap T)(E \cap T \cap H) = E \cap T.$$

Therefore, $M \cap T$ is maximal in $E \cap T$ and so by the hypothesis A is X-permutable with $M \cap T$. It follows from Lemma 1.5(4) that AH/H is XH/H-permutable with $M/H = (M \cap T)H/H$.

Finally, let either T be soluble or H be a p-group where $p \in \pi'$. Then for some Hall π-subgroup T_π of T, we have $T_\pi \leq E \cap T$. Indeed, the result is evident if T is soluble. For the second case, by using $|E : H| = |(E \cap T)H : H| = |E \cap T : (E \cap T) \cap H|$ and the well known Schur-Zassenhaus Theorem, we see that $E \cap T$ has a Hall π-subgroup T_π. Since $|T : E \cap T|$ is a π'-integer, T_π is a Hall subgroup of T. Hence $T_\pi H/H = E/H = H(E \cap T_\pi)/H$ and $M/H = H(M \cap T_\pi)/H$. As above, we may prove that $M \cap T_\pi$ is a maximal subgroup of $E \cap T_\pi$ and so again AH/H is XH/H-permutable with M/H. The Lemma is proved.

Lemma 1.15 *Let A and X be subgroups of G, $H \trianglelefteq G$. Suppose that $T \in X_m(A)$ and either $H \leq A$ or $H \leq T$. Suppose also that or H is an abelian minimal normal subgroup of G or $(|H|, |T|) = 1$ or T is soluble. Then $TH/H \in (XH/H)_m(AH/H)$.*

Proof This Lemma is a direct consequence of Lemma 1.14.

2.2 Criterions of Existence and Conjugacy of Hall Subgroups

A group G is said to be π-separable if G has a normal series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_{t-1} \leq G_t = G, \tag{2.1}$$

where each index $|G_i : G_{i-1}|$ is either a π-number or a π'-number.

A group G is said to be:

(I) E_π-group provided G has a Hall π-subgroup;
(II) C_π-group provided G is a E_π-group and every two Hall π-subgroups of G are conjugate;
(III) D_π-group provided G is a C_π-group and every π-subgroup of G contained in some Hall π-subgroup of G.

The famous Schur-Zassenhaus Theorem asserts that: *If G has a normal Hall π-subgroup A, then G is an $E_{\pi'}$-group. Moreover, if either A or G/A is soluble, then A is a $C_{\pi'}$-subgroup.*

In 1928, Hall [228] proved that: *A soluble group is a D_π-group for any nonempty set π of primes.*

The most important result of the theory of π-separable groups is the following generalization of the above Hall result.

Theorem 2.1 (P. Hall [233], Čunihin [77]). *If G is a π-separable group, then G is a D_π-group.*

It is well known that the above Schur-Zassenhaus theorem, Hall theorem and Hall–Čunihin's theorem are truly fundamental results of group theory. In connection with these important results, the following two problems have naturally arisen:

Problem 2.2 Can we weaken the condition of normality for the Hall subgroup A of G so that the conclusion of the Schur-Zassenhaus Theorem is still true?

Problem 2.3 Whether we can replace the condition of normality for the members of series (2.1) by some weaker condition, for example, by permutability of the members of series (2.1) with some systems of subgroups of G.

In this section we give positive answers to the above two Problems.

A generalization of the Schur-Zassenhaus theorem.

Lemma 2.4 *Let N be a normal C_π-subgroup of G.*

(i) *If G/N is a C_π-group, then G is a C_π-group (Čunihin* [78]).
(ii) *If G/N is an E_π-group, then G is an E_π-group (Čunihin* [78]).
(iii) *If G has a nilpotent Hall π-subgroup, then G is a D_π-group* [445].
(iv) *If G has a Hall π-subgroup with cyclic Sylow subgroups, then G is a D_π-group (S. A. Rusakov* [339]).

Lemma 2.5 *Let N be a normal C_π-subgroup of G and N_π a Hall π-subgroup of N.*

(i) *If G is a C_π-group, then G/N is a C_π-group* ([233]).
(ii) *If every Sylow subgroup of N_π is cyclic and G/N is a D_π-group, then G is a D_π-group* (Shemetkov [356] or [359, IV, Theorem 18.17]).
(iii) *G is a D_π-group if and only if G/N and N are D_π-groups* (See [334]).

Lemma 2.6 (Kegel [262]). *Let A and B, be the subgroups of G such that $G \neq AB$ and $AB^x = B^x A$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

The following lemma is obvious.

Lemma 2.7 *If N is normal in G and T is a minimal supplement of N in G, then $N \cap T \leq \Phi(T)$.*

Lemma 2.8 (Knyagina, Monakhov [266]). *If H, K, and N be pairwise permutable subgroups of G and H is a Hall subgroup of G, then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Now we prove the following generalization of the Schur-Zassenhaus theorem.

Theorem 2.9 (Guo, Skiba [202]). *Let X be a normal C_π-subgroup of G and A a subgroup of G such that $|G : A|$ is a π-number. Suppose that A has a Hall π-subgroup A_0 such that either A_0 is nilpotent or every Sylow subgroup of A_0 is cyclic. Suppose that A X-permutes with every Sylow p-subgroup of G for all primes $p \in \pi$ or for all primes $p \in \pi \setminus \{q\}$ for some prime q dividing $|G : A|$. Then G is a C_π-group.*

Proof Assume that this proposition is false and let G be a counterexample of minimal order. Then $|\pi \cap \pi(G)| > 1$.

(1) *G/R is a C_π-group for any nonidentity normal subgroup R of G.*
In order to prove this assertion, in view of the choice of G, it is enough to show that the hypothesis is still true for $(G/R, AR/R, XR/R)$. First note that $|G/R : AR/R| = |G : AR|$ is a π-number, and A_0R/R is a Hall π-subgroup of AR/R since

$$|AR/R : A_0R/R| = |AR : A_0R| = |A : A \cap A_0R| = |A : A_0(A \cap R)|.$$

On the other hand, $XR/R \simeq X/X \cap R$ is a C_π-group by Lemma 2.5(i), and either $A_0R/R \simeq A_0/R \cap A_0$ is nilpotent or every Sylow subgroup of A_0R/R is cyclic. Finally, let P/R be a Sylow p-subgroup of G/R, where $p \in \pi \setminus q$. Then for some Sylow p-subgroup G_p we have $G_pR/R = P/R$. Hence AR/R XR/R-permutes with P/R by Lemma 1.5(3). Therefore the hypothesis holds for $(G/R, AR/R, XR/R)$.
(2) $X = 1$.
Indeed, if $X \neq 1$, then G/X is a C_π-group by (1). Hence G is C_π-group by Lemma 2.4(i), a contradiction.

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