# 2

# Extensive Measurement

## 2.1 Ratio Scales and Interval Scales

The title of this chapter refers to the measurement of the fundamental variables, they are called 'scales', entering in the equations of physics and geometry. We limit our discussion to those variables that are specified by their unit, such as mass, time, or length. The unit of mass may be one gram, or one kilogram, or one pound. It does not matter: the equations remain the same. In all such cases, the ratio of two values does not depend upon the chosen unit. For example, the ratio of the weight of Greta Garbo (G) to that of Charlie Chaplin (C) is the same whether we use grams or kilograms to measure it. If we write $\varpi_g$ for the weight in grams, and $\varpi_k$ for the weight in kilograms, we have

$$\frac{\varpi_g(G)}{\varpi_g(C)} = \frac{\varpi_k(G)}{\varpi_k(C)}. \tag{2.1}$$

This type of equation justifies the name given to this kind of measurement, which is *ratio scaling*. The variables themselves are called *ratio scales*.

This type of measurement is the most important for scientific applications, but not the only one. Temperature, for example, when measured in Celsius or Fahrenheit degrees, is not a ratio scale. It is an *interval scale.*

The concept of interval scale is governed by an equation different from (2.1), but in the same vein. Let $\tau_C$ and $\tau_F$ be two functions mapping an open interval $I$ which either contains 0 or has 0 has a lower limit point. We suppose that both functions are strictly increasing and continuous. For example, some $x$ in $I$ could represent a temperature measured on the Kelvin scale[1], while $\tau_C(x)$ and $\tau_F(x)$ are the same temperature measured on the Celsius and the Fahrenheit scales, respectively.

[1] So $\liminf\limits_{y\in I} f(y) = 0$.

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Take any four Kelvin temperatures $x$, $y$, $z$, and $w$. A possible case would be:

| | | |
|---|---|---|
| $x$ | | the maximum temperature today |
| $y$ | stand for, | the minimum temperature today |
| $z$ | respectively, | the maximum temperature yesterday |
| $w$ | | the minimum temperature yesterday. |

The corresponding *interval scale equation* is

$$\frac{\tau_C(x) - \tau_C(y)}{\tau_C(z) - \tau_C(w)} = \frac{\tau_F(x) - \tau_F(y)}{\tau_F(z) - \tau_F(w)} \qquad (\forall x, y, z, w \in I, x > y, z > w). \quad (2.2)$$

We prove later in this book that, under this equation, the two functions $\tau_C$ and $\tau_F$ are linearly related: we must have

$$\tau_C(x) = a\,\tau_F(x) + b \qquad \text{(for some constants } a > 0 \text{ and } b)$$

(see Section 3.6 and Theorem 3.6.1)[2].

While the theory described in this book is limited to ratio scales, it certainly can be extended to interval scales without undue difficulty.

## 2.2 Empirical Basis and Short History

By ‘extensive measurement’, we mean the measurement of fundamental physical variables such as mass or length by some qualitative device. An example of such a device for the measurement of mass is the *two-pan-equal-arms* balance. We discuss the case of length in detail below. Following Helmholtz (1887), this type of measurement is taken to rely on two types of empirical procedures, to be used in combination.

1. A *comparison procedure*, which is used to decide which of two objects or entities $x$ and $y$ has the greater amount of a quantity, or whether they have the same amount. For the measurement of mass, this involves placing $x$ and $y$ on the two pans of the balance and recording the state of the balance.
2. A *concatenation procedure*, which is used to merge two objects or entities. In the example of the two-pan balance, the concatenation of $x$ and $y$ would be achieved by placing $x$ and $y$ on the same pan of the balance, forming the merged object $x \oplus y$, and placing some other object $z$ on the other pan. We can then check the state of the balance to compare $x \oplus y$ to $z$ from the standpoint of the quantity.

[2] Note in passing that, on the basis of Equations (2.1) and (2.2) the names ‘ratio scale’ and ‘interval scale’ may seem inconsistent. While ‘ratio scale’ is a sensible label for Equation (2.1), ‘interval scale’ does not suggest a ratio of differences. A more appropriate name might be something like ‘ratio-interval scale.’

From a mathematical standpoint, this situation can be formalized by a triple $(\mathcal{X}, \oplus, \precsim)$ in which

$$
\begin{array}{ll}
\mathcal{X} & \text{is the set of objects to be measured} \\[1ex]
\oplus & \begin{cases} \text{stands for the concatenation procedure} \\ \text{with } x \oplus y \text{ representing the merging of } x \text{ and } y \end{cases} \\[3ex]
\precsim & \begin{cases} \text{symbolizes the comparison procedure, with } x \precsim y \\ \text{meaning that } x \text{ has, of the quantity to be measured,} \\ \text{either less than } y \text{ or the same amount as } y. \end{cases}
\end{array}
$$

The problem for the theorist is to find conditions on the triple $(\mathcal{X}, \oplus, \precsim)$, they are called *axioms*, that ensure the existence of a real valued function $m$ mapping $\mathcal{X}$ into the real numbers, such that

$$x \precsim y \iff m(x) \leq m(y) \tag{2.3}$$

$$m(x \oplus y) = m(x) + m(y). \tag{2.4}$$

So, the intuition is that the result of the comparison procedure should be represented by the order of the real numbers, and the concatenation by the addition of positive real numbers.

Another empirical example is the measurement of length via the manipulation of thin rods. The concatenation of rods $x$ and $y$ is achieved by placing the two rods along a parallel line, end to end, forming the new object $x \oplus y$ (see Figure 2.1 for a illustration of the two procedures in that case).

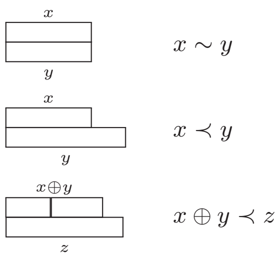


**Figure 2.1.** The measurement of length. Comparison and concatenation procedures for thin rods.

There has been a variety of axiomatizations achieving the representation (2.3)-(2.4), varying in their degree of realism (see Krantz, 1968; Krantz, Luce, Suppes, and Tversky, 1971, for reviews).

For example, as noted by Suppes (1951), the early axiomatization of Hölder (1901) does not axiomatize the relation $\precsim$, but instead treats the equivalence $\sim$ as the logical identity. Suppes's objection is that two different rods $x$ and $y$, for example, could have the same length without being identical. In Hölder's axiomatization, we would write $x = y$ instead of $x \sim y$. We shall see that this objection is not critical (cf. Definition 2.3.2 and Lemma 2.3.3).

Another weakness of Hölder's system is that the representation is consistent with the existence of infinitely large objects. This was corrected in Luce and Marley's axiomatization, which is pointedly entitled "*Extensive measurement when concatenation is restricted and maximal objects may exist*" (Luce and Marley, 1969).

This representation, as does Hölder's, still assumes the existence of infinitely small objects. While this might make sense in some situations, it does not seem suitable in the general case. This assumption is dropped in the system described in this chapter, which is due to Falmagne (1975)[3].

## 2.3 Basic Algebraic Concepts

**2.3.1 Definition.** We recall from Chapter 1 that $\mathbb{R}_{++}$ stands for the positive real numbers and we write $\mathbb{R}$ and $\mathbb{R}_+$ for the set of real numbers and the set of non-negative real numbers, respectively; $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers and the set of positive natural numbers, respectively.

A triple $(\mathcal{X}, \oplus, \precsim)$ is a *partial concatenation system* or more simply, a *system* if

1. $\mathcal{X}$ is a non empty set;
2. $\oplus$ is a not necessarily closed operation on $\mathcal{X}$;
3. $\precsim$ is a binary relation on $\mathcal{X}$.

A *representation* of a system $(\mathcal{X}, \oplus, \precsim)$ is a function $m : \mathcal{X} \to \mathbb{R}_{++}$ satisfying the two following conditions:

$$\text{(i)} \quad m \text{ is } \textit{isotone}\text{, that is} \qquad x \precsim y \iff m(x) \leq m(y) \tag{2.5}$$

$$\text{(ii)} \quad m \text{ is } \textit{additive}\text{, that is} \qquad m(x \oplus y) = m(x) + m(y)\,. \tag{2.6}$$

The isotonicity Condition (i) implies that the binary relation $\precsim$ is a 'weak order' (cf. Definition 2.3.2 and Problem 1). In addition to monotonicity and isotonicity, the various axiomatizations that have been proposed typically[4] imply that the representation $m$ also satisfies other properties, which are felt desirable. One of them is related to the uniqueness of the representation. It states that

(iii) the representation $m$ is a *ratio scale*, that is, if $m^\dagger$ is another representation satisfying (i) and (ii), then $m = \alpha\, m^\dagger$, with $\alpha > 0$.

The last property is that the objects in $\mathcal{X}$ are *regularly spaced* with respect to how much of the attribute they possess. Formally:

(iv) $m$ is *regular*, that is, $m(\mathcal{X}) = I \cap G$ for some positive interval $I$ and some subgroup $G$ of the additive real numbers.

[3] See also Falmagne (1971).

[4] But not always: see Krantz (1967, Proposition 13).

The suitability of (i)-(iii) has been discussed elsewhere (see for example, in Krantz, Luce, Suppes, and Tversky, 1971). Condition (iv) is usually not derived, but is certainly reasonable. For instance, for the measurement of mass, it would not be natural to have the range of the function $m$—the image $m(\mathcal{X})$ of the set $\mathcal{X}$ by the function $m$—containing all the numbers in the neighborhood of 210 (g, for *grams*), except that number itself.

**2.3.2 Definition.** A relation $\precsim$ on a set $\mathcal{X}$ is a *weak order* if it is

1. *transitive*, that is, for all $x$, $y$, and $z$ in $\mathcal{X}$, we have, for all $x$, $y$ and $z$ in $\mathcal{X}$,
$$x \precsim y \;\&\; y \precsim z \implies x \precsim z;$$
2. *reflexive*, that is, $x \precsim x$, for all $x$ in $\mathcal{X}$;
3. *connected*, that is, either $x \precsim y$ or $y \precsim x$, for all $x$ and $y$ in $\mathcal{X}$.

A *simple order*, or *linear order*, is an *antisymmetric* weak order, that is,

4. $x \precsim y \;\&\; y \precsim x \implies x = y, \qquad$ for all $x$ and $y$ in $\mathcal{X}$.

Suppose that $\precsim$ is a weak order on $\mathcal{X}$. The relation $\sim$ on $\mathcal{X}$ defined by

$$x \sim y \implies x \precsim y \;\&\; y \precsim x$$

is an *equivalence relation* on $\mathcal{X}$, that is, it is transitive and symmetric. For any $x \in \mathcal{X}$, the subset $\boldsymbol{x} = \{y \in \mathcal{X} \,|\, x \sim y\} \subseteq \mathcal{X}$ is a *coset* (with respect to $\sim$).

**2.3.3 Lemma.** *If $\precsim$ is a weak order on $\mathcal{X}$, and $\boldsymbol{\mathcal{X}} = \{\boldsymbol{x} \,|\, x \in \mathcal{X}\}$ is the collection of all the cosets, then the relation $\leq$ on $\boldsymbol{\mathcal{X}}$ defined by*

$$\boldsymbol{x} \leq \boldsymbol{y} \iff x \precsim y \tag{2.7}$$

*is a simple order on $\boldsymbol{\mathcal{X}}$.*

Lemma 2.3.3 shows that the transition from a weak order to a simple order is straightforward; the construction of the cosets is essentially trivial. So, there is no good reason to require that the axiomatization be constructed in terms of a weak order instead of a simple order (cf. Suppes' objection on page 13). We only deal with simply ordered systems in this chapter.

Conditions (i)-(iv) in Definition 2.3.2 capture the essentials of an extensive representation. This justifies our second definition.

**2.3.4 Definition.** A system is an *extensive system* if it has a positive, ratio scale representation which is additive, isotone, and regular. If the representation is bijective, the extensive system is called a *positive Hölder system*.

The last sentence of this definition is relevant to the so-called *Hölder Theorem* (cf. Birkhoff, 1967; Krantz, 1968), which states that any Archimedean ordered group is isomorphic to a subgroup of the additive reals. Any extensive system has an associated positive Hölder system which is obtained by forming the cosets.

For the rest of this chapter, we follow closely Falmagne (1975). In the next section, we state the main results, which consist in two theorems. Theorem 2.4.3 is the most important one in the context of this book. It is consistent with a representation onto a positive interval of real numbers or rational numbers. In Theorem 2.4.2, we also axiomatize a discrete case, consistent with a representation onto an interval of positive integers.

One aspect of these results may be emphasized. It could be argued that the existence of infinitely small objects in the empirical set is critical for the construction of a "system of standards" sufficiently refined to permit the exact measurement of extensive quantities. In this context, the proof of Theorem 2.4.3 (in the non discrete case) has the interest of being based on the construction of a system of standards involving only arbitrarily small "differences."

## 2.4 Main Results

All the conditions used in the two theorems are listed in the next definition. In Theorem 2.4.3, we assume that all these conditions hold, except the last one, discreteness. In comparison with the Hölder Theorem, this large number of conditions may be surprising. It is due to the fact that they replace very powerful conditions involving the existence of infinitely large and infinitely small objects[5].

**2.4.1 Definition.** Given a system $(\mathcal{X}, \oplus, \precsim)$, we write $xy$ to mean that the concatenation $x \oplus y$ is defined; so, we have $x \oplus y = z$ for some $z \in \mathcal{X}$.

A system $(\mathcal{X}, \oplus, \precsim)$ is

(1) *simply ordered* if $\precsim$ is antisymmetric, that is, $\precsim$ is a simple order;
(2) *conditionally closed*, that is: $xy$, $x' \prec x$ and $y' \prec y$ imply $x'y'$;
(3) *right monotonic* if whenever $xz$, $yz$, and $x \prec y$, then $x \oplus z \prec y \oplus z$;
(4) *associative* if whenever $xy$ and $(x \oplus y)z$, then $yz$, $x(y \oplus z)$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(5) *positive* if $xy$ implies $x \prec x \oplus y$;
(6) *non degenerate* if there exists $x \in \mathcal{X}$ such that $xx$ and $(x \oplus x)x$;
(7) *weakly solvable* if whenever $xy$ and $x \oplus y \prec z$ for some $z$, then there exists some $w \in \mathcal{X}$ such that $x \oplus w = z$;
(8) *commutative* if $xy$ implies $yx$, and when both hold, then $x \oplus y = y \oplus x$;
(9) *simplifiable* if whenever $xy$, $zw$, $y'w$, $xz'$, $y'y$, $zz'$, $x \oplus y = z \oplus w$, and $y' \oplus w = x \oplus z'$, then $y' \oplus y = z \oplus z'$;
(10) *trivial* if $\mathcal{X}$ only contains two elements $x$ and $x \oplus x$, with $x \prec x \oplus x$;

[5] Which are implied in the Hölder Theorem: *Any Archimedean ordered group is isomorphic to a subgroup of the additive reals.*

(11) *strongly Archimedean* if for any $x$, $y$ and $z$ in $\mathcal{X}$, the set

$$N(x, z; y) = \{n \in \mathbb{N} \,|\, x_y^n \text{ is defined and } x_y^n \prec z\}, \tag{2.8}$$

is finite, with the sequence $(x_y^n)$ being defined recursively as follows:
if $x \prec y$, then
   a) $x_y^1 = x$;
   b) if $x_y^{n-1}$ is defined, with $yx_y^{n-1}$ and there is $x' \in \mathcal{X}$ such that $xx'$ and $y \oplus x_y^{n-1} = x \oplus x'$, then $x_y^n = x'$;

(12) *(locally) discrete* if there exist $y$ and $y^*$ in $\mathcal{X}$ such that there is no $z$ in $\mathcal{X}$ such that $y \prec z \prec y^*$.

Our main results lie in the two following theorems.

**2.4.2 Theorem.** THE DISCRETE CASE. *A discrete, non trivial system is a positive Hölder system if and only if it is simply ordered, conditionally closed, right monotonic, associative, positive, non degenerate, weakly solvable, commutative, and strongly Archimedean. These conditions are independent.*

**2.4.3 Theorem.** *A non trivial system is a positive Hölder system if and only if it is simply ordered, conditionally closed, right monotonic, associative, positive, non degenerate, weakly solvable, commutative, simplifiable, and strongly Archimedean. These conditions are independent.*

Note that, except for discreteness, all the conditions of Theorem 2.4.2 appear in Theorem 2.4.3. We only sketch the proof of Theorem 2.4.2.

## 2.5 Proofs

We begin by stating three preliminary lemmas concerning a simply ordered system $(\mathcal{X}, \oplus, \leq)$, which are serviceable in the proofs of our two main results. Each lemma states that a particular subset of the conditions listed in Definition 2.4.1 implies that one of the three conditions listed below is satisfied.

[A1] Every $x \in \mathcal{X}$ is such that either $xy$ for some $y \in \mathcal{X}$, or $x = y \oplus z$ for some $y, z \in \mathcal{X}$.

[A2] If $x < y$ and $yz$, then there exists $w \in \mathcal{X}$ such that $x \oplus w = y \oplus z$. (This condition is different from weak solvability.)

[A3] If $xy$, $zw$, $y'w$, $xz'$, $y'y$, and $x \oplus y = z \oplus w$, $y' \oplus w = x \oplus z'$, then $zz'$, and $y \oplus y' = z \oplus z'$.

**2.5.1 Lemma.** *If a system is simply ordered, conditionally closed, weakly solvable, and non degenerate, it also satisfies* [A1].

Proof. Let $x$ be as in Definition 2.4.1, Part 6 (non degenerateness). Every $y \in \mathcal{X}$ is such that $x \leq x \oplus x$, or $x \oplus x \leq y$ ($\leq$ is connected). In the first case, $yx$ follows from conditional closure. In the second case $y = x \oplus z$ for some $z \in \mathcal{X}$ follows from weak solvability. □

**2.5.2 Lemma.** *If a system is simply ordered, conditionally closed, weakly solvable, commutative, and right monotonic, it also satisfies* [A2].

Proof. Suppose that $x < y$ and $yz$. Then $x \leq y$, and $xz$ obtains from conditional closure. Right monotonicity implies $x \oplus z < y \oplus z$. The result follows from weak solvability. □

**2.5.3 Lemma.** *If a system is simply ordered, conditionally closed, weakly solvable, right monotonic, and simplifiable, it also satisfies* [A3].

Proof. Suppose that $x \oplus y = z \oplus w$, $y' \oplus w = x \oplus z'$, and $y'y$. We first show that $zz'$ also holds. If $y < z$, we obtain, using successively all the conditions, $w \oplus y = y \oplus w < z \oplus w = x \oplus y$, which yields $w < x$. Similarly $y' < z'$ implies $x < w$. We conclude that either $z \leq y$ or $z' \leq y'$. Suppose that $z \leq y$, or to avoid trivialities, $z < y$. Using Lemma 2.5.2 and commutativity, we have $z \oplus k = y \oplus y'$ for some $k \in \mathcal{X}$. If $z' \leq k$, then $zz'$ follows from conditional closure. Suppose that $k < z'$. Using again conditional closure and the fact that $xz'$, we derive $xk$. We thus have $z \oplus k = y' \oplus y$, $x \oplus y = z \oplus w$, $xk$ and $y'w$, which yields $x \oplus k = y' \oplus w$. This implies that $x \oplus z' = x \oplus k$, so $k = z'$, a contradiction. The proof is similar if we assume that $z' \leq y'$. □

**Sketch of the proof of Theorem 2.4.2.** (We omit the proofs of the five preparatory Lemmas 1-5. See Problem 4 on page 44.) Note that, in the discrete case, the representation can be obtained without using simplifiability. Suppose that $(\mathcal{X}, \oplus, \leq)$ is a system satisfying all the conditions of Theorem 2.4.2.

*Lemma 1. For every $x \in \mathcal{X}$, the set $\mathcal{X}_x = \{t \in \mathcal{X} \,|\, x < t\}$ is either empty, or has a smallest element $x^* = \min \mathcal{X}_x$.*

*Lemma 2. Suppose that $x^*$ and $z^*$ exist. Then*

$$x^* z \iff xz^* \iff (x \oplus z)^* \quad \textit{exists}$$

*and in such a case*

$$x^* \oplus z = x \oplus z^* = (x \oplus z)^*.$$

*Lemma 3. The set $\mathcal{X}$ has a smallest element.*

In the sequel, we denote by $\psi$ the smallest element of $\mathcal{X}$, and we define a (possibly finite) sequence $(\psi_n)$ as follows: $\psi_0 = \psi$, $\psi_{n+1} = \psi_n^*$ if $\mathcal{X}_{\psi_n}$ is not empty.

*Lemma 4. For any non negative integers $n$, $p$, we have $\psi_n \psi_p$ if and only if $\psi_{n+p}\psi$, and when both hold, then $\psi_n \oplus \psi_p = \psi_{n+p} \oplus \psi$.*

We now proceed to the construction of a representation $m$ into $\mathbb{N}^+$. Suppose that $\psi \oplus \psi = \psi_k$ for some $k \in \mathbb{N}^+$. For any $x = \psi_n \in \mathcal{X}$, define $m(x) = n + k$. Notice that if $\psi_n^*$ exists, then $m(\psi_n^*) = n + 1 + k = m(\psi_n) + 1$.

*Lemma 5. For any $\psi_n \in \mathcal{X}$ such that $\psi_n \psi$, we have $m(\psi_n + \psi) = n + 2k$.*

We prove that $m$ is additive. Suppose that $\psi_n \psi_p$. Then

$$\begin{aligned} m(\psi_n \oplus \psi_p) &= m(\psi_{n+p} \oplus \psi) && \text{(by Lemma 4)} \\ &= n + p + 2k && \text{(by Lemma 5)} \\ &= m(\psi_n) + m(\psi_p) && \text{(by definition of } m) \end{aligned}$$

and it is not difficult to show that if $(m(\psi_n)+m(\psi_p)) \in m(\mathcal{X})$, then necessarily $\psi_n \psi_p$. (This comes from the fact that $\psi_{p+k} \in \mathcal{X}$ implies $\psi_p \psi$.)

The isotonicity of $m$ and the convexity of $m(\mathcal{X})$ in $\mathbb{N}^+$ are immediate. If not $\psi_n \psi$, then $\psi_n = \psi_p \oplus \psi$ for some $p \in \mathbb{N}^+$ and the results follow from a similar argument.

We finally turn to uniqueness. Suppose that $m^\dagger$ be another representation of $(\mathcal{X}, \oplus, \leq)$. So, $m^\dagger$ is additive. We define a constant $\alpha$ by the equation $m^\dagger(\psi) = \alpha m(\psi)$. We use induction. Suppose that $m^\dagger(\psi_i) = \alpha m(\psi_i)$ for $0 \leq i \leq n-1$. If $\psi_n \psi$, we have

$$\begin{aligned} m^\dagger(\psi_n) &= m^\dagger(\psi_n \oplus \psi) - m^\dagger(\psi) && (m^\dagger \text{ is additive}) \\ &= m^\dagger(\psi_{n-1} \oplus \psi_1) - m^\dagger(\psi) && \text{(by Lemma 4)} \\ &= m^\dagger(\psi_{n-1}) + m^\dagger(\psi_1) - m^\dagger(\psi) && (m \text{ is additive}) \\ &= \alpha m(\psi_{n-1}) + \alpha m(\psi_1) - \alpha m(\psi) && \text{(by the induction hypothesis)} \\ &= \alpha(m(\psi_{n-1}) + m(\psi_1) - m(\psi)) \\ &= \alpha m(\psi_n) \end{aligned}$$

by the additivity of $m$ and Lemma 4. Since the conditions are clearly necessary, the proof of Theorem 2.4.2 is complete. □

**Proof of Theorem 2.4.3.** Suppose that the system $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \oplus, \precsim)$ satisfies all the conditions of the theorem. As in the proof of Theorem 2.4.2, we write $\leq = \precsim$ as a reminder that $\boldsymbol{\mathcal{X}}$ is simply ordered. We suppose that $\boldsymbol{\mathcal{X}}$ is not discrete. (If it is, the sufficiency of the conditions is established by Theorem 2.4.2.) We prove that $\boldsymbol{\mathcal{X}}$ is a positive Hölder system by constructing an appropriate representation.

CASE 1. *The set $\mathcal{X}$ has a smallest element $\psi$.* We begin by constructing a strictly decreasing sequence $(x_n)$ converging toward $\psi$ (in the ordered topology). By conditional closure and non degenerateness, we have $\psi\psi$. We pick $x_0$ such that $\psi < x_0 < \psi \oplus \psi$ ($\boldsymbol{\mathcal{X}}$ is not discrete). Take $x_0'$ in $\mathcal{X}$ such that $\psi < x_0' < x_0$. Then $x_0'\psi$, with $x_0' \oplus \psi < x_0 \oplus \psi$ by right monotonicity. By weak solvability, there exists $x_0'' \in \mathcal{X}$ such that $x_0' \oplus x_0'' = x_0 \oplus \psi$. Define $x_1 = \min\{x_0', x_0''\}$. Since $x_1 < x_0$, we have $x_1\psi$. We now proceed by induction.

If $x_n\psi$, choose $x'_n$ such that $\psi < x'_n < x_n$, and with $x'_n \oplus x''_n = x_n \oplus \psi$, define $x_{n+1} = \min\{x'_n, x''_n\}$, yielding $\psi < x_{n+1} < x_n$ and $x_{n+1} \oplus x_{n+1} \leq x_n \oplus \psi$.

For convenience, we use the abbreviation $\mathcal{X}_\psi = \{x \in \mathcal{X} \mid \psi < x\}$.

*Lemma 1. Whenever $x \in \mathcal{X}_\psi$, there exists an integer $M$ such that $M < n$ implies $x_n < x$.*

*Proof.* Suppose that, for some $z \in \mathcal{X}_\psi$, we have $z < x_n$ for all integers $n$. Set $y_0 = z$. Since $x_1x_1$, we also have $y_0y_0$, with $\psi \oplus y_0 < y_0 \oplus y_0$ by right monotonicity. By weak solvability, there exists $y_1$ such that $\psi \oplus y_1 = y_0 \oplus y_0$. If $\psi < y_1 < x_1$, then $y_1y_1$ and by the same argument, there exists $y_2$ such that $\psi \oplus y_2 = y_0 \oplus y_1$. Continuing this way, let $M$ be the largest integer such that $y_{M-2}$ is defined, with $y_{M-2} < x_1$ (we use the strongly Archimedean condition here). Then $y_0y_{M-2}$, with

$$x_2 \oplus x_2 \leq \psi \oplus x_1 \leq \psi \oplus y_{M-1} = y_0 \oplus y_{M-2}$$

yielding

$$x_2 \oplus x_2 \leq y_0 \oplus y_{M-2}. \tag{2.9}$$

Since $y_0 \leq x_2$, we have $x_2 \leq y_{M-2}$ by (2.9). Similarly

$$x_3 \oplus x_3 \leq \psi \oplus x_2 \leq \psi \oplus y_{M-2} = y_0 \oplus y_{M-3}.$$

Finally, $x_M \leq y_{M-M} = y_0 = z$ and $x_n < z$ if $M < n$, a contradiction. □

We next show that the sequence $(x_n)$ converges to $\psi$ in another sense, namely: whenever $\psi < z$, then $N(\psi, z; x_n) \to \infty$ as $n \to \infty$, with $N(\psi, z; x_n)$ defined by Equation (2.8). In the sequel, we write $N(y; x) = N(\psi, y; x)$. For every $x_n$ in the sequence, we temporarily fix $n$ and define a sequence $(x_n^p)$ inductively by

(i) $x_n^1 = x_n$;
(ii) if $x_n^{p-1}$ is defined, with $x_n x_n^{p-1}$, then $x_n^p$ is defined by $\psi \oplus x_n^p = x_n \oplus x_n^{p-1}$. (Except for a change of notation, this device is not new; cf. the strongly Archimedean Condition 11 in Definition 2.4.1.)

We omit the inductive proof of the next lemma, which is immediate.

*Lemma 2. For any $p \leq n$, $x_n^p$ is defined, with $x_n^p \leq x_{n-p}$.*

*Lemma 3. For all $y \in \mathcal{X}_\psi$, we have $N(y; x_n) \to \infty$ as $n \to \infty$.*

*Proof.* This lemma is based on two simple results, the proof of which we leave to the reader.

(i) If $\psi < x < y$, then for any $z \in \mathcal{X}_\psi$, $N(x; z) \leq N(y; z)$;
(ii) $N(x_p^n; x_n) = p$ for all $n, p \in \mathbb{N}$ such that $x_p^n$ is defined.

For all $x \in \mathcal{X}_\psi$, take $k \in \mathbb{N}$ such that $x_k < x$ (Lemma 1). Then for all $n > k$, with $n = p + k$, $x_n^p$ is defined, and successively:

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