Chapter 2
Local Transformation Equations and Essential Parameters

Abstract Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, throughout. As said in Chap. 1, transformation equations $x_i' = f_i(x; a_1, \dots, a_r)$, $i = 1, \dots, n$, which are local, analytic diffeomorphisms of $\mathbb{K}^n$ parametrized by a finite number r of real or complex numbers $a_1, \dots, a_r$, constitute the archetypal objects of Lie's theory. The preliminary question is to decide whether the f_i really depend upon *all* parameters, and also, to get rid of superfluous parameters, if there are any.

Locally in a neighborhood of a fixed x_0, one expands $f_i(x; a) = \sum_{\alpha \in \mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x - x_0)^\alpha$ in power series and one looks at the *infinite coefficient mapping* $\mathsf{U}_\infty : a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha \in \mathbb{N}^n}^{1 \leqslant i \leqslant n}$ from $\mathbb{K}^r$ to $\mathbb{K}^\infty$, which is expected to faithfully describe the dependence with respect to a in question. If ρ_∞ denotes the maximal, generic and locally constant rank of this map, with of course $0 \leqslant \rho_\infty \leqslant r$, then the answer says that locally in a neighborhood of a generic a_0, there exist both a local change of parameters $a \mapsto \big(\mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a)\big) =: \mathsf{u}$ decreasing the number of parameters from r down to ρ_∞, and new transformation equations:

$$x_i' = g_i\big(x; \mathsf{u}_1, \dots, \mathsf{u}_{\rho_\infty}\big) \qquad (i=1 \cdots n)$$

depending *only* upon ρ_∞ parameters which give again the old ones:

$$g_i\big(x; \mathsf{u}(a)\big) \equiv f_i(x; a) \qquad (i=1 \cdots n).$$

At the end of this brief chapter, before giving a precise introduction to the local Lie group axioms, we present an example due to Engel which shows that the axiom of inverse cannot be deduced from the axiom of composition, contrary to one of Lie's *Idées fixes*.

2.1 Generic Rank of the Infinite Coefficient Mapping

Thus, we consider local transformation equations:

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$$x_i' = f_i(x_1, \dots, x_n; a_1, \dots, a_r) \qquad (i=1 \cdots n).$$

We want to illustrate how the principle of free generic relocalization described above on p. 4 helps to get rid of superfluous parameters a_k. We assume that the f_i are defined and analytic for x belonging to a certain (unnamed, connected) domain of $\mathbb{K}^n$ and for a belonging to some domain of $\mathbb{K}^r$.

Expanding the f_i of $x_i' = f_i(x;a)$ in power series with respect to $x - x_0$ in some neighborhood of a point x_0:

$$f_i(x;a) = \sum_{\alpha \in \mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha,$$

we get an infinite number of analytic functions $\mathscr{U}_\alpha^i = \mathscr{U}_\alpha^i(a)$ of the parameters that are defined in some uniform domain of $\mathbb{K}^r$. Intuitively, this infinite collection of coefficient functions $\mathscr{U}_\alpha^i(a)$ should show how $f(x;a)$ depends on a.

To make this claim precise, we thus consider the map:

$$\mathsf{U}_\infty : \quad \mathbb{K}^r \ni a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha \in \mathbb{N}^n}^{1 \leqslant i \leqslant n} \in \mathbb{K}^\infty.$$

For the convenience of applying standard differential calculus in finite dimensions, we simultaneously consider all of its κ-th truncations:

$$\mathsf{U}_\kappa : \quad \mathbb{K}^r \ni a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{|\alpha| \leqslant \kappa}^{1 \leqslant i \leqslant n} \in \mathbb{K}^{n\frac{(n+\kappa)!}{n!\,\kappa!}},$$

where $\frac{(n+\kappa)!}{n!\,\kappa!}$ is the number of multiindices $\alpha \in \mathbb{N}^n$ whose length $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfies the upper bound $|\alpha| \leqslant \kappa$. We call U_κ, U_∞ the *(in)finite coefficient mapping(s)* of $x_i' = f_i(x;a)$.

The *Jacobian matrix* of U_κ is the $r \times \big(n\,\frac{(n+\kappa)!}{n!\,\kappa!}\big)$ matrix:

$$\left(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\right)_{1 \leqslant j \leqslant r}^{|\alpha| \leqslant \kappa,\, 1 \leqslant i \leqslant n},$$

its r rows being indexed by the partial derivatives. The *generic rank* of U_κ is the largest integer $\rho_\kappa \leqslant r$ such that there is a $\rho_\kappa \times \rho_\kappa$ minor of $\operatorname{Jac} \mathsf{U}_\kappa$ which does not vanish identically, but all $(\rho_\kappa + 1) \times (\rho_\kappa + 1)$ minors do vanish identically. The uniqueness principle for analytic functions then insures that the common zero-set of all $\rho_\kappa \times \rho_\kappa$ minors is a *proper* closed analytic subset D_κ (of the unnamed domain where the $\mathscr{U}_\alpha^i$ are defined), so it is stratified by a finite number of submanifolds of codimension $\geqslant 1$ ([8, 2, 3, 5]), and in particular, it has empty interior, hence it is intuitively "thin".

So the set of parameters a at which there is a least one $\rho_\kappa \times \rho_\kappa$ minor of $\operatorname{Jac} \mathsf{U}_\kappa$ which does not vanish is open and *dense*. Consequently, "for a generic point a", the map U_κ is of rank $\geqslant \rho_\kappa$ at every point a' sufficiently close to a (since the corresponding $\rho_\kappa \times \rho_\kappa$ minor does not vanish in a neighborhood of a), and because all $(\rho_\kappa + 1) \times (\rho_\kappa + 1)$ minors of $\operatorname{Jac} \mathsf{U}_\kappa$ were assumed to vanish identically, the map

U_κ happens to be in fact of *constant* rank U_κ in a (small) neighborhood of every such generic a.

Insuring constancy of a rank is one important instance of why free relocalization is useful: a majority of theorems of the differential calculus and of the classical theory of (partial) differential equations hold under specific local constancy assumptions.

As κ increases, the number of columns of $\operatorname{Jac} U_\kappa$ increases, hence $\rho_{\kappa_1} \leqslant \rho_{\kappa_2}$ for $\kappa_1 \leqslant \kappa_2$. Since $\rho_\kappa \leqslant r$ is bounded, the generic rank of U_κ becomes constant for all $\kappa \geqslant \kappa_0$ bigger than some sufficiently large κ_0. Thus, let $\rho_\infty \leqslant r$ denote this maximal possible generic rank.

Definition 2.1. The parameters $(a_1, \ldots, a_r)$ of given point transformation equations $x_i' = f_i(x;a)$ are called *essential* if, after expanding $f_i(x;a) = \sum_{\alpha\in\mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha$ in power series at some x_0, the generic rank ρ_∞ of the coefficient mapping $a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha\in\mathbb{N}^n}^{1\leqslant i\leqslant n}$ is maximal, equal to the number r of parameters: $\rho_\infty = r$.

Without entering into technical details, we make a remark. It is a consequence of the principle of analytic continuation and of some reasonings with power series that the *same* maximal rank ρ_∞ is enjoyed by the coefficient mapping $a \mapsto \big(\mathscr{U}'^i_\alpha(a)\big)_{\alpha\in\mathbb{N}^n}^{1\leqslant i\leqslant n}$ for the expansion of $f_i(x;a) = \sum_{\alpha\in\mathbb{N}^n} \mathscr{U}'^i_\alpha(a)\,(x-x_0')^\alpha$ at another, arbitrary point x_0'. Also, one can prove that ρ_∞ is independent of the choice of coordinates x_i and of parameters a_k. These two facts will not be needed, and the interested reader is referred to [9] for proofs of quite similar statements holding true in the context of *Cauchy-Riemann geometry*.

2.2 Quantitative Criterion for the Number of Superfluous Parameters

It is not very practical to compute the generic rank of the infinite Jacobian matrix $\operatorname{Jac}\mathsf{U}_\infty$. To check essentiality of parameters in concrete situations, a helpful criterion due to Lie is **(iii)** below.

Theorem 2.1. *The following three conditions are equivalent:*

(i) *In the transformation equations*

$$x_i' = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r) = \sum_{\alpha\in\mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha \qquad (i=1\cdots n),$$

the parameters $a_1, \ldots, a_r$ are not *essential.*

(ii) *(By definition) The generic rank ρ_∞ of the infinite Jacobian matrix:*

$$\operatorname{Jac}\mathsf{U}_\infty(a) = \Big(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\Big)_{1\leqslant j\leqslant r}^{\alpha\in\mathbb{N}^n,\,1\leqslant i\leqslant n}$$

is strictly less than r.

(iii) *Locally in a neighborhood of* every (x_0, a_0), *there exists a not identically zero analytic vector field on the parameter space:*

$$\mathscr{T} = \sum_{k=1}^{n} \tau_k(a) \frac{\partial}{\partial a_k}$$

which annihilates all the $f_i(x;a)$:

$$0 \equiv \mathscr{T} f_i = \sum_{k=1}^{n} \tau_k \frac{\partial f_i}{\partial a_k} = \sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^{r} \tau_k(a) \frac{\partial \mathscr{U}_\alpha^i}{\partial a_k}(a)\,(x - x_0)^\alpha \qquad (i = 1 \cdots n).$$

More generally, if ρ_∞ denotes the generic rank of the infinite coefficient mapping:

$$\mathsf{U}_\infty : \qquad a \longmapsto \big(\mathscr{U}_\alpha^i(a)\big)_{\alpha \in \mathbb{N}^n}^{1 \leqslant i \leqslant n},$$

then locally in a neighborhood of every (x_0, a_0), *there exist exactly $r - \rho_\infty$, and no more, analytic vector fields:*

$$\mathscr{T}_\mu = \sum_{k=1}^{n} \tau_{\mu k}(a) \frac{\partial}{\partial a_k} \qquad (\mu = 1 \cdots r - \rho_\infty),$$

with the property that the dimension of $\operatorname{Span}\big(\mathscr{T}_1\big|_a, \dots, \mathscr{T}_{r-\rho_\infty}\big|_a\big)$ *is equal to $r - \rho_\infty$ at every parameter a at which the rank of U_∞ is maximal, equal to ρ_∞, such that the derivations $\mathscr{T}_\mu$ all annihilate the $f_i(x;a)$:*

$$0 \equiv \mathscr{T}_\mu f_i = \sum_{k=1}^{r} \tau_{\mu k}(a) \frac{\partial f_i}{\partial a_k}(x;a) \qquad (i = 1 \cdots n;\ \mu = 1 \cdots r - \rho_\infty).$$

Proof. Just by the chosen definition, we have **(i)** $\Longleftrightarrow$ **(ii)**. Next, suppose that condition **(iii)** holds, in which the coefficients $\tau_k(a)$ of the concerned nonzero derivation $\mathscr{T}$ are locally defined. Recalling that the Jacobian matrix $\operatorname{Jac} \mathsf{U}_\infty$ has r rows and an infinite number of columns, we then see that the n annihilation equations $0 \equiv \mathscr{T} f_i$, when rewritten in matrix form as:

$$0 \equiv \big(\tau_1(a), \dots, \tau_r(a)\big) \left(\frac{\partial \mathscr{U}_\alpha^i}{\partial a_j}(a)\right)_{1 \leqslant j \leqslant r}^{\alpha \in \mathbb{N}^n,\, 1 \leqslant i \leqslant n}$$

just say that the transpose of $\operatorname{Jac} \mathsf{U}_\infty(a)$ has nonzero kernel at each a where the vector $\mathscr{T}\big|_a = \big(\tau_1(a), \dots, \tau_r(a)\big)$ is nonzero. Consequently, $\operatorname{Jac} \mathsf{U}_\infty$ has rank strictly less than r locally in a neighborhood of every a_0, hence in the whole a-domain. So **(iii)** $\Rightarrow$ **(ii)**.

Conversely, assume that the generic rank ρ_∞ of $\operatorname{Jac} \mathsf{U}_\infty$ is $< r$. Then there exist $\rho_\infty < r$ "basic" coefficient functions $\mathscr{U}_{\alpha(1)}^{i(1)}, \dots, \mathscr{U}_{\alpha(\rho_\infty)}^{i(\rho_\infty)}$ (there can be several choices) such that the generic rank of the extracted map $a \mapsto \big(\mathscr{U}_{\alpha(l)}^{i(l)}\big)_{1 \leqslant l \leqslant \rho_\infty}$ equals ρ_∞ already. We abbreviate:

$$\mathsf{u}_l(a) := \mathscr{U}^{i(l)}_{\alpha(l)}(a) \qquad (l=1\cdots\rho_\infty).$$

The goal is to find vectorial local analytic solutions $\big(\tau_1(a),\dots,\tau_r(a)\big)$ to the infinite number of linear equations:

$$0 \equiv \tau_1(a)\,\frac{\partial \mathscr{U}^i_\alpha(a)}{\partial a_1}(a) + \cdots + \tau_r(a)\,\frac{\partial \mathscr{U}^i_\alpha(a)}{\partial a_r}(a) \qquad (i=1\cdots n;\ \alpha\in\mathbb{N}^n).$$

To begin with, we look for solutions of the finite, extracted linear system of ρ_∞ equations with the r unknowns $\tau_k(a)$:

$$\begin{cases} 0 \equiv \tau_1(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_1}(a) + \cdots + \tau_{\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty}}(a) + \cdots + \tau_r(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_r}(a) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \equiv \tau_1(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_1}(a) + \cdots + \tau_{\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty}}(a) + \cdots + \tau_r(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_r}(a). \end{cases}$$

After possibly renumbering the variables $(a_1,\dots,a_r)$, we can assume that the left $\rho_\infty\times\rho_\infty$ minor of this system:

$$\Delta(a) := \det\left(\frac{\partial \mathsf{u}_l}{\partial a_m}(a)\right)^{1\leqslant l\leqslant\rho_\infty}_{1\leqslant m\leqslant\rho_\infty}$$

does not vanish identically. However, it can vanish at some points, and while endeavoring to solve the above linear system by an application of the classical Cramer rule, the necessary division by the determinant $\Delta(a)$ introduces poles that are undesirable, for we want the $\tau_k(a)$ to be analytic. So, for any μ with $1\leqslant\mu\leqslant r-\rho_\infty$, we look for a solution (rewritten as a derivation) in the specific form:

$$\mathscr{T}_\mu := -\Delta(a)\,\frac{\partial}{\partial a_{\rho_\infty+\mu}} + \sum_{1\leqslant k\leqslant\rho_\infty}\tau_{\mu k}(a)\,\frac{\partial}{\partial a_k} \qquad (\mu=1\cdots r-\rho_\infty),$$

in which we introduce in advance a factor $\Delta(a)$ designed to compensate the unavoidable division by $\Delta(a)$. Indeed, such a $\mathscr{T}_\mu$ will annihilate the u_l:

$$0 \equiv \mathscr{T}_\mu\,\mathsf{u}_1 \equiv \cdots \equiv \mathscr{T}_\mu\,\mathsf{u}_{\rho_\infty}$$

if and only its coefficients are solutions of the linear system:

$$\begin{cases} \Delta(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty+\mu}}(a) \equiv \tau_{\mu 1}(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_1}(a) + \cdots + \tau_{\mu\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty}}(a) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty+\mu}}(a) \equiv \tau_{\mu 1}(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_1}(a) + \cdots + \tau_{\mu\rho_\infty}(a)\,\dfrac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty}}(a). \end{cases}$$

Cramer's rule then yields the unique solution:

$$\tau_{\mu k}(a) = \frac{1}{\Delta(a)} \begin{vmatrix} \frac{\partial \mathsf{u}_1}{\partial a_1}(a) & \cdots & \Delta(a)\,\frac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty+\mu}}(a) & \cdots & \frac{\partial \mathsf{u}_1}{\partial a_{\rho_\infty}}(a) \\ \cdots\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots \\ \frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_1}(a) & \cdots & \Delta(a)\,\frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty+\mu}}(a) & \cdots & \frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_{\rho_\infty}}(a) \end{vmatrix} \qquad {\scriptstyle (k=1\cdots r)},$$

where, as predicted, the overall factor $\Delta(a)$ of the k-th column compensates the division by the determinant $\Delta(a)$ of this system, so that the $\mathscr{T}_\mu$ indeed all have *analytic* coefficients.

Clearly, $\mathscr{T}_1, \dots, \mathscr{T}_{r-\rho_\infty}$ are linearly independent at all (generic) points a where $\Delta(a) \neq 0$. It remains to show that the $\mathscr{T}_\mu$ also annihilate all other coefficient functions $\mathscr{U}_\alpha^i$.

Thus, let $(i,\alpha) \neq (i(1),\alpha(1)), \dots, (i(\rho_\infty),\alpha(\rho_\infty))$. By the very definition of ρ_∞, the generic rank of any $r \times (1+\rho_\infty)$ extracted subJacobian matrix:

$$\begin{pmatrix} \frac{\partial \mathscr{U}_\alpha^i}{\partial a_1}(a) & \frac{\partial \mathsf{u}_1}{\partial a_1}(a) & \cdots & \frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_1}(a) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathscr{U}_\alpha^i}{\partial a_r}(a) & \frac{\partial \mathsf{u}_1}{\partial a_r}(a) & \cdots & \frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_r}(a) \end{pmatrix}$$

must always be equal to ρ_∞, the generic rank of its last ρ_∞ columns. Consequently, in a neighborhood of every point a at which its top right $\rho_\infty \times \rho_\infty$ minor $\Delta(a)$ does not vanish, the first column is a certain linear combination:

$$\frac{\partial \mathscr{U}_i^{\,\alpha}}{\partial a_k}(a) = \lambda_1(a)\,\frac{\partial \mathsf{u}_1}{\partial a_k}(a) + \cdots + \lambda_{\rho_\infty}(a)\,\frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_k}(a) \qquad {\scriptstyle (i=1\cdots n;\ \alpha\in\mathbb{N}^n;\ k=1\cdots r)}$$

of the last ρ_∞ columns in question, where, again thanks to an appropriate application of Cramer's rule, the coefficients $\lambda_l(a)$ are *analytic* in the concerned generic neighborhood, for their denominator $\Delta(a)$ is $\neq 0$ there. It then follows immediately by appropriate scalar multiplication and summation that:

$$\begin{aligned} \sum_{k=1}^{r} \tau_{\mu k}(a)\,\frac{\partial \mathscr{U}_\alpha^i}{\partial a_k}(a) &\equiv \lambda_1(a) \sum_{k=1}^{r} \tau_{\mu k}(a)\,\frac{\partial \mathsf{u}_1}{\partial a_k}(a) + \cdots + \lambda_{\rho_\infty}(a) \sum_{k=1}^{r} \tau_{\mu k}(a)\,\frac{\partial \mathsf{u}_{\rho_\infty}}{\partial a_k}(a) \\ &\equiv \lambda_1(a)\,\mathscr{T}_\mu\,\mathsf{u}_1 + \cdots + \lambda_{\rho_\infty}(a)\,\mathscr{T}_\mu\,\mathsf{u}_{\rho_\infty} \\ &\equiv 0 \qquad\qquad {\scriptstyle (i=1\cdots n;\ \alpha\in\mathbb{N}^n)}. \end{aligned}$$

But since these analytic equations hold on the dense open set where $\Delta(a) \neq 0$, we deduce by continuity that the equations:

$$0 \equiv \mathscr{T}_1 \, \mathscr{U}_\alpha^i \equiv \cdots \equiv \mathscr{T}_{r-\rho_\infty} \, \mathscr{U}_\alpha^i \qquad \scriptstyle (i=1\cdots n;\; \alpha\in\mathbb{N}^n)$$

do hold *everywhere*, as desired. In conclusion, we have shown the implication **(ii)** $\Rightarrow$ **(iii)**, and simultaneously, we have established the last part of the theorem. □

Corollary 2.1. *Locally in a neighborhood of every generic point a_0 at which the infinite coefficient mapping $a \mapsto \mathsf{U}_\infty(a)$ has maximal, locally constant rank equal to its generic rank ρ_∞, there exist both a local change of parameters $a \mapsto \big(\mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a)\big) =: \mathsf{u}$ decreasing the number of parameters from r down to ρ_∞, and new transformation equations:*

$$x_i' = g_i\big(x;\, \mathsf{u}_1, \dots, \mathsf{u}_{\rho_\infty}\big) \qquad \scriptstyle (i=1\cdots n)$$

depending only *upon ρ_∞ parameters which give again the old ones:*

$$g_i\big(x;\, \mathsf{u}(a)\big) \equiv f_i(x;\, a) \qquad \scriptstyle (i=1\cdots n).$$

Proof. Choose ρ_∞ coefficients $\mathscr{U}_{\alpha(l)}^{i(l)}(a) =: \mathsf{u}_l(a)$, $1 \leqslant l \leqslant \rho_\infty$, with $\Delta(a) := \det\big(\frac{\partial \mathsf{u}_l(a)}{\partial a_m}(a)\big)_{1\leqslant m\leqslant\rho_\infty}^{1\leqslant l\leqslant\rho_\infty} \not\equiv 0$ as in the proof of the theorem. Locally in some small neighborhood of any a^0 with $\Delta(a_0) \neq 0$, the infinite coefficient map U_∞ has constant rank ρ_∞, hence the constant rank theorem provides, for every (i,α), a certain function $\mathscr{V}_\alpha^i$ of ρ_∞ variables such that:

$$\mathscr{U}_\alpha^i(a) \equiv \mathscr{V}_\alpha^i\big(\mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a)\big).$$

Thus, we can work out the power series expansion:

$$\begin{aligned} f_i(x;a) &= \textstyle\sum_{\alpha\in\mathbb{N}^n} \mathscr{U}_\alpha^i(a)\,(x-x_0)^\alpha \\ &= \textstyle\sum_{\alpha\in\mathbb{N}^n} \mathscr{V}_\alpha^i(\mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a))\,(x-x_0)^\alpha \\ &=: g_i(x, \mathsf{u}_1(a), \dots, \mathsf{u}_{\rho_\infty}(a)) \end{aligned}$$

which yields the natural candidate for $g_i(x;\mathsf{u})$. Lastly, one may verify that any Cauchy estimate for the growth decrease of $\mathscr{U}_\alpha^i(a)$ as $|\alpha| \to \infty$ insures a similar Cauchy estimate for the growth decrease of $b \mapsto \mathscr{V}_\alpha^i(\mathsf{u})$, whence each g_i is analytic, and in fact, termwise substitution was legitimate. □

Definition 2.2. The transformation equations $x_i' = f_i(x_1, \dots, x_n; a_1, \dots, a_r)$, $i = 1, \dots, n$, are called *r-term* if all the parameters $(a_1, \dots, a_r)$ are essential.

2.3 The Axiom of Inverse and Engel's Counterexample

Every analytic diffeomorphism of an n-times extended space permutes all the points in a certain differentiable, invertible way. Although they act on a set of infinite cardinality, diffeomorphisms can thus be regarded as a kind of analog of the *substitutions* on a finite set. In fact, in the years 1873–80, Lie's *Idée fixe* was to build, in the geometric realm of n-dimensional continua, a counterpart of the Galois theory of substitutions of roots of algebraic equations ([6]).

As above, let $x' = f(x; a_1, \dots, a_r) =: f_a(x)$ be a family of (local) analytic diffeomorphisms parametrized by a finite number r of parameters. For Lie, the basic, single group axiom should just require that such a family be *closed under composition*, namely that one always has $f_a\big(f_b(x)\big) \equiv f_c(x)$ for some c depending on a and on b. More details on this definition will given in the next chapter, but at present, we ask whether one can really economize the other two group axioms: existence of an identity element and existence of inverses.

Lemma 2.1. *If H is any subset of some abstract group G with* $\operatorname{Card} H < \infty$ *which is closed under group multiplication:*

$$h_1 h_2 \in H \qquad \text{whenever} \qquad h_1, h_2 \in H,$$

then H contains the identity element e of G and every $h \in H$ has an inverse in H, so that H itself is a true subgroup of G.

Proof. Indeed, picking arbitrary $h \in H$, the infinite sequence $h, h^2, h^3, \dots, h^k, \dots$ of elements of the finite set H must become eventually periodic: $h^a = h^{a+n}$ for some $a \geqslant 1$ and for some $n \geqslant 1$, whence $e = h^n$, so $e \in H$ and h^{n-1} is the inverse of h. □

For more than thirteen years, Lie was convinced that a purely similar property should also hold with $G = \mathsf{Diff}_n$ being the (infinite continuous pseudo)group of analytic diffeomorphisms and with $H \subset \mathsf{Diff}_n$ being any continuous family closed under composition. We quote a characteristic excerpt of [7], pp. 444–445.

> As is known, one shows in the theory of substitutions that the permutations of a group can be ordered into pairwise inverse couples of permutations. Now, since the distinction between a permutation group and a transformation group only lies in the fact that the former contains a finite and the latter an infinite number of operations, it is natural to presume that the transformations of a transformation group can also be ordered into pairs of inverse transformations. In previous works, I came to the conclusion that this should actually be the case. But because in the course of my investigations in question, certain *implicit* hypotheses have been made about the nature of the functions appearing, I think that it is necessary to *expressly add the requirement that the transformations of the group can be ordered into pairs of inverse transformations*. In any case, I conjecture that this is a necessary consequence of my original definition of the concept [BEGRIFF] of transformation group. However, it has been impossible for me to prove this in general.

In his first year working with Lie (1884), Engel proposed the following counterexample. Consider the family of transformation equations:

$$x' = \zeta x,$$

where $x, x' \in \mathbb{C}$ and the parameter $\zeta \in \mathbb{C}$ is restricted to $|\zeta| < 1$. Of course, this family is closed under any composition, say: $x' = \zeta_1 x$ and $x'' = \zeta_2 x' = \zeta_1 \zeta_2 x$, with indeed $|\zeta_2 \zeta_1| < 1$ when $|\zeta_1|, |\zeta_2| < 1$, but neither the identity element nor any inverse transformation belongs to the family. However, the requirement $|\zeta| < 1$ here is too artificial: in fact the family trivially extends as the complete group $\big(x' = \zeta x\big)_{\zeta \in \mathbb{C}}$ of dilations of the line. Engel's idea was to appeal to a Riemann map ω having $\{|\zeta| = 1\}$ as a boundary of nonextendability. The map used by Engel is the following.[1] (Translator's note: In the treatise [4], this example is presented at the end of Chap. 9, *see* below p. 179.) Let od_k denote the number of odd divisors (including 1) of any integer $k \geqslant 1$. The theory of holomorphic functions in one complex variables yields the following.

Lemma 2.2. *The infinite series:*

$$\omega(a) := \sum_{\nu \geqslant 1} \frac{a^\nu}{1 - a^{2\nu}} = \sum_{\nu \geqslant 1} \big(a^\nu + a^{3\nu} + a^{5\nu} + a^{7\nu} + \cdots\big) = \sum_{k \geqslant 1} \mathsf{od}_k \, a^k$$

converges absolutely in every open disc $\Delta_\rho = \{z \in \mathbb{C} : \ |z| < \rho\}$ of radius $\rho < 1$ and defines a univalent holomorphic function $\Delta \to \mathbb{C}$ from the unit disc $\Delta := \{|z| < 1\}$ to $\mathbb{C}$ which does not *extend holomorphically across any point of the unit circle $\partial\Delta := \{|z| = 1\}$.*

In fact, any other similar Riemann biholomorphic map $\zeta \longmapsto \omega(\zeta) =: \lambda$ from the unit disc Δ onto some simply connected domain $\Lambda := \omega(\Delta)$ having fractal boundary which is not a Jordan curve, e.g. the Von Koch Snowflake Island, would do the job.[2] (Translator's note: A concise presentation of Carathéodory's theory may be found in Chap. 17 of [10].) Denote then by $\lambda \longmapsto \chi(\lambda) =: \zeta$ the inverse of such a map and consider the family of transformation equations:

$$\big(x' = \chi(\lambda)\, x\big)_{\lambda \in \Lambda}.$$

By construction, $|\chi(\lambda)| < 1$ for every $\lambda \in \Lambda$. Any composition of $x' = \chi(\lambda_1)\, x$ and of $x'' = \chi(\lambda_2)\, x'$ is of the form $x'' = \chi(\lambda)\, x$, with the uniquely defined parameter $\lambda := \omega\big(\chi(\lambda_1)\, \chi(\lambda_2)\big)$, hence the group composition axiom is satisfied. However, there is again no identity element, and again, no transformation has an inverse. Furthermore, crucially (and lastly), there does not exist any extension of the family to a larger domain $\widetilde{\Lambda} \supset \Lambda$ together with a holomorphic extension $\widetilde{\chi}$ of χ to $\widetilde{\Lambda}$ so that $\widetilde{\chi}\big(\widetilde{\Lambda}\big)$ contains a neighborhood of $\{1\}$ (in order to include the identity) or *a fortiori* a neighborhood of $\overline{\Delta}$ (in order to include inverses of transformations $x' = \chi(\lambda)\, x$ with $\lambda \in \Lambda$ close to $\partial\Lambda$).

Observation

In Vol. I of the *Theorie der Transformationsgruppen*, this example appears only in Chap. 9, on pp. 163–165, and it is written in small characters. In fact, Lie still believed that a deep analogy with substitution groups should come out as a theorem. Hence *the structure of the first nine chapters insist on setting aside*, whenever possible, *the two axioms of existence of identity element and of existence of inverses*. To do justice to this great treatise, we shall translate in Chap. 9 how Master Lie managed to produce Theorem 26 on p. 177, which he considered to provide the sought analogy with finite group theory, after taking Engel's counterexample into account.

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