Chapter 2
Modified Fourier Series and Rayleigh-Ritz Method

Although the governing equations and associated boundary equations for laminated beams, plates and shells presented in Chap. 1 show the possibility of seeking their exact solutions of vibration, however, it is commonly believed that very few exact solutions are possible for plate and shell vibration problems. For instance, an exact solution is available only for rectangular plates which are simply supported along, at least, one pair of opposite edges, and one has to resort to an approximate solution for other boundary conditions (Zhang and Li 2009). It is important for engineering applications to have available approaches that give accurate solutions for cases that cannot be solved accurately.

In recent decades, many accurate and efficient experimental and computational methods have been developed for the vibration analysis of laminated beams, plates and shells, such as the scaled down models and similitude theory, Ritz method, differential quadrature method (DQM), Galerkin method, wave propagation approach, multiquadric radial basis function method, meshless method, finite element method (FEM), discrete singular convolution approach (DSC), etc. It should be stressed that most of these methods were applied firstly to isotropic structures, and were subsequently extended to study the dynamic behaviors of the anisotropic and laminated composite ones. However, it appears that most of the existing methods are only suitable for a particular type of boundary conditions which typically require constant modifications of the solution procedures to adapt to different boundary cases. Therefore, the use of the existing solution procedures will result in very tedious calculations and be easily inundated with various boundary conditions in practical applications due to the fact that the boundary conditions of a beam, plate or shell may not always be classical in nature, a variety of possible boundary restraining cases, including classical boundary conditions, elastic restraints and their combinations can be encountered in practice. For example, even just considering the four simplest classical boundary conditions (i.e., F, SD, S and C), one should realize that there can constitute 256 combinations of different boundary conditions for a thin shell (four edges) or unsymmetrically laminated thin plate. Furthermore, the possible combinations of classical boundary conditions of a general thick open shell or unsymmetrically laminated thick plate can be as many as 331,776 types. The finite element method (FEM) has dominated engineering

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computations since its invention and its application has expanded to a variety of engineering fields. The FEM overcomes the difficulties in dealing with various boundary conditions, however, there still exist some drawbacks due to its mesh-based interpolation. For instance, it suffers heavily from mesh distortion in large deformation and intensive remeshing requirements in dealing with the structures with complex geometries and discontinuities. In addition, the computational demands increase with structural and material complexity and with analysis frequency range (Price et al. 1998; Liew et al. 2011). It is necessary and of great significance to develop a unified, efficient and accurate method which is capable of universally dealing with laminated beams, plates and shells with general boundary conditions.

The present chapter deals with a unified modified Fourier series method which is capable of universally dealing with laminated beams, plates and shells with general boundary conditions. The accurate modified Fourier series solutions of isotropic, anisotropic and laminated beams, plates and shells can be obtained by using both strong and weak form solution procedures as described in the following sections.

2.1 Modified Fourier Series

For vibration problems of beams, plates and shells, the admissible functions are often expressed in the form of Fourier series expansions because of their orthogonality and completeness, as well as their excellent stability in numerical calculations. Furthermore, vibrations are naturally expressible as waves, which are normally described by Fourier series (Li 2000). However, the conventional Fourier series expression will generally has a convergence problem along the boundary edges except for a few simple boundary conditions, thus limiting the applications of Fourier series to only a few ideal boundary conditions. Mathematically, when the displacements of a shell (2D) are periodically extended as standard Fourier series onto the entire α–β surface, discontinuities potentially exist in original displacements and their derivatives at the edges. In such case, the Fourier series expansions cannot be differentiated term-by-term, and thus the solution may not converge or converge slowly. Recognizing the fact that the convergence rate for the Fourier series expansion of a periodic function is directly related to its smoothness, Li (2000, 2002) proposed a modified Fourier series method for the vibration analysis of isotropic Euler Bernoulli beams with general elastic boundary conditions.

In this book, this method is further developed and extended to the vibration analysis of laminated composite beams, plates and shells with general boundary conditions and arbitrary lamination schemes, aiming to provide a unified and reasonable accurate alternative to other analytical and numerical techniques. The method will be briefly explained in this section for the completeness of the book.

2.1.1 Traditional Fourier Series Solutions

To fully illustrate the basic idea of the modified Fourier series method, we consider the longitude and transverse vibrations of a classical straight beam with length L, uniform thickness h and width b as shown in Fig. 2.1. The two-dimensional rectangular coordinate system (x, z) is used to describe the geometry dimensions and deformations of the beam, in which co-ordinates along the axial and thickness directions are represented by x and z, respectively.

Letting $\alpha = x$, $A = 1$, according to Eqs. (1.7), (1.14) and (1.28), the governing equations for free vibration of a generally laminated composite beam are obtained as:

$$\begin{aligned} A_{11}\frac{\partial^2 u}{\partial x^2} - B_{11}\frac{\partial^3 w}{\partial x^3} &= -\omega^2 I_0 u \\ B_{11}\frac{\partial^3 u}{\partial x^3} - D_{11}\frac{\partial^4 w}{\partial x^4} &= -\omega^2 I_0 w \end{aligned} \tag{2.1a, b}$$

where ω represent the natural frequencies of the beam. Suppose the classical beam considered here is made from isotropic materials, therefore, the B_{11} terms become zero. In such case, the longitude and transverse vibrations of the beam are decoupled. Subsequently, Eq. (2.1) is rewritten as:

$$\begin{aligned} A_{11}\frac{\partial^2 u}{\partial x^2} &= -\omega^2 I_0 u \\ D_{11}\frac{\partial^4 w}{\partial x^4} &= \omega^2 I_0 w \end{aligned} \tag{2.2a, b}$$

The solution of Eq. (2.2) is often desired to be expanded in the form of either Fourier sine series or Fourier cosine series. Take the transverse vibration problem for example (Eq. 2.2b), mathematically, the displacement $w(x)$ can be expanded as Fourier series only contains the cosine terms by making the even extension of $w(x)$ from the interval [0, L] onto the interval [$-L$, 0], as shown in Fig. 2.2 (Xu 2011):

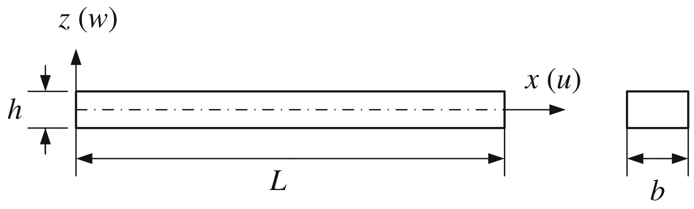

Fig. 2.1 Notations of a classical straight beam

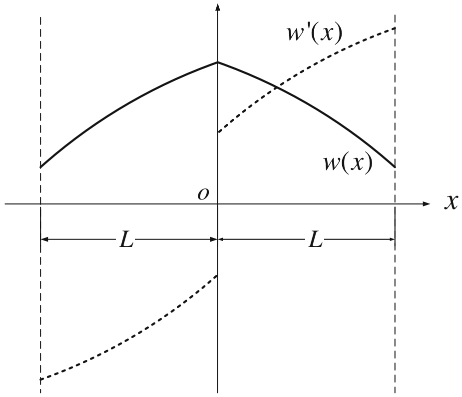

Fig. 2.2 An illustration of the possible discontinuities of the displacement at the ends

$$w(x) = \sum_{m=0}^{\infty} A_m \cos \lambda_m x \quad 0 \leq x \leq L \tag{2.3}$$

where A_m are the expansion coefficients, $\lambda_m = m\pi/L$. According to Eq. (2.2b), it is obvious that the transverse displacement $w(x)$ is required to have up to the fourth-derivative ($w''''(x)$). The Fourier cosine series is able to correctly converge to $w(x)$ at any point over [0, L]. However, its first-derivative $w'(x)$ and third- derivative $w'''(x)$ are odd functions over [$-L$, L] leading to a jump at end locations (see Fig. 2.2). Thus, their Fourier series expansions (sine series) will accordingly have a convergence problem due to the discontinuity at end points. Moreover, the displacement function $w(x)$ of the beam given in Eq. (2.3) may not be differentiated term-by-term. The reasons are given below (Tolstov 1976):

Theorem 1 *Let $f(x)$ be a continuous function defined on* [0, L] *with an absolutely integrable derivative, and let $f(x)$ be expanded in Fourier sine series*

$$f(x) = \sum_{m=1}^{\infty} a_m \sin \lambda_m x, \quad 0 < x < L \quad (\lambda_m = m\pi/L) \tag{2.4}$$

then

$$f'(x) = \frac{f(L) - f(0)}{L} + \sum_{m=1}^{\infty} \left(\frac{2}{L} [(-1)^m f(L) - f(0)] + a_m \lambda_m \right) \cos \lambda_m x \tag{2.5}$$

Apparently, when $f(L) = f(0) = 0$,

$$f'(x) = \sum_{m=1}^{\infty} a_m \lambda_m \cos \lambda_m x \tag{2.6}$$

The theorem reveals that a sine series can be differentiated term-by-term only if $f(L) = f(0) = 0$.

Theorem 2 *Let f(x) be a continuous function defined on* [0, *L*] *with an absolutely integrable derivative, and let f(x) be expanded in Fourier cosine series*

$$f(x) = \sum_{m=0}^{\infty} b_m \cos \lambda_m x, \quad 0 < x < L \quad (\lambda_m = m\pi/L) \tag{2.7}$$

then

$$f'(x) = -\sum_{m=1}^{\infty} b_m \lambda_m \sin \lambda_m x \tag{2.8}$$

The theorem reveals that a cosine series can always be differentiated term-by-term.

Theorem 3 *Let f(x) be a continuous function of period 2L, which has n derivatives, where* $n-1$ *derivatives are continuous and the mth derivative is absolutely integrable* (*the mth derivative may not exist at certain points*). *Then, the Fourier series of all m derivatives can be obtained by term-by-term differentiation of the Fourier series of f(x), where all the series, except possibly the last, converge to the corresponding derivatives. Moreover, the Fourier coefficients of the function f(x) satisfy the relations*

$$\lim_{n\to\infty} a_m \lambda_m^n = \lim_{n\to\infty} b_m \lambda_m^n = 0 \tag{2.9}$$

With these in mind, for the cases when the beam is elastically supported, we have

$$w'(x) = -\sum_{m=1}^{\infty} \lambda_m A_m \sin \lambda_m x, \quad 0 < x < L \tag{2.10}$$

$$\begin{aligned} w''(x) = {} & \frac{w'(L) - w'(0)}{L} \\ & + \sum_{m=1}^{\infty} \left(\frac{2}{L}[(-1)^m w'(L) - w'(0)] - A_m \lambda_m^2 \right) \cos \lambda_m x, \quad 0 \le x \le L \end{aligned} \tag{2.11}$$

$$w'''(x) = -\sum_{m=1}^{\infty} \left(\frac{2}{L}[(-1)^m w'(L) - w'(0)]\lambda_m - A_m \lambda_m^3 \right) \sin \lambda_m x, \quad 0 < x < L \tag{2.12}$$

$$\begin{aligned} w''''(x) = {} & \frac{w'''(L) - w'''(0)}{L} \\ & + \sum_{m=1}^{\infty} \begin{pmatrix} \frac{2}{L}[(-1)^m w'''(L) - w'''(0)] \\ -\frac{2}{L}[(-1)^m w'(L) - w'(0)]\lambda_m^2 + A_m \lambda_m^4 \end{pmatrix} \cos \lambda_m x, \quad 0 \le x \le L \end{aligned} \tag{2.13}$$

and the Fourier coefficient A_m satisfies

$$\lim_{m\to\infty} A_m\lambda_m = 0 \tag{2.14}$$

Combining Eqs. (2.2b) and (2.13) results in

$$\begin{aligned} &D_{11}\frac{w'''(L)-w'''(0)}{L} + \sum_{m=1}^{\infty}\left(\frac{2D_{11}}{L}[(-1)^m w'''(L)-w'''(0)]\right)\cos\lambda_m x \\ &\quad + \sum_{m=1}^{\infty}\left(\frac{2D_{11}}{L}[(-1)^m w'(L)-w'(0)]\lambda_m^2\right. \\ &\qquad \left. + \left(D_{11}\lambda_m^4-\omega^2 I_0\right)A_m\right)\cos\lambda_m x = \omega^2 I_0 \end{aligned} \tag{2.15}$$

Obviously, it is a big challenge to obtain the natural frequencies and determine the expansion coefficients from Eq. (2.15).

Alternatively, one may prefer to expand the beam displacement $w(x)$ in the form of Fourier sine series. In such case

$$w(x) = \sum_{m=1}^{\infty} A_m \sin\lambda_m x, \quad 0<x<L \tag{2.16}$$

Then

$$\begin{aligned} w'(x) &= \frac{w(L)-w(0)}{L} \\ &\quad + \sum_{m=1}^{\infty}\left(\frac{2}{L}[(-1)^m w(L)-w(0)] + A_m\lambda_m\right)\cos\lambda_m x, \quad 0\le x\le L \end{aligned} \tag{2.17}$$

$$w''(x) = -\sum_{m=1}^{\infty}\left(\frac{2}{L}[(-1)^m w(L)-w(0)]\lambda_m + A_m\lambda_m^2\right)\sin\lambda_m x, \quad 0<x<L \tag{2.18}$$

$$\begin{aligned} w'''(x) &= \frac{w''(L)-w''(0)}{L} \\ &\quad + \sum_{m=1}^{\infty}\begin{pmatrix} \frac{2}{L}[(-1)^m w''(L)-w''(0)] \\ -\frac{2}{L}[(-1)^m w(L)-w(0)]\lambda_m^2 - A_m\lambda_m^3 \end{pmatrix}\cos\lambda_m x, \quad 0\le x\le L \end{aligned} \tag{2.19}$$

$$w''''(x) = -\sum_{m=1}^{\infty}\begin{pmatrix} \frac{2}{L}[(-1)^m w''(L)-w''(0)]\lambda_m \\ -\frac{2}{L}[(-1)^m w(L)-w(0)]\lambda_m^3 - A_m\lambda_m^4 \end{pmatrix}\sin\lambda_m x, \quad 0<x<L \tag{2.20}$$

and the Fourier coefficient A_m satisfies

$$\lim_{m\to\infty} A_m = 0 \tag{2.21}$$

Combining Eqs. (2.2b) and (2.20) results in

$$\begin{aligned}[(-1)^m w''(L) - w''(0)]\lambda_m - [(-1)^m w(L) \\ - w(0)]\lambda_m^3 = \frac{A_m L}{2D_{11}}\left(\omega^2 I_0 - D_{11}\lambda_m^4\right)\end{aligned} \tag{2.22}$$

and

$$A_m = \frac{2D_{11}\lambda_m}{L\left(D_{11}\lambda_m^4 - \omega^2 I\right)_0}\begin{pmatrix}[(-1)^m w''(L) - w''(0)] \\ -[(-1)^m w(L) - w(0)]\lambda_{mm}^2\end{pmatrix} \tag{2.23}$$

Mathematically, the natural frequencies are simply obtained by requiring the determinant of the coefficient matrix to vanish (Wang and Lin 1996). Such a procedure involves solving a non-linear equation, which may not always be an easy job numerically (Li 2000).

In conclusion, a beam with simply supported boundary conditions, the Fourier sine series can be used to determine the vibrations of the beam readily due to the fact that all the required derivatives of the displacement function can be directly obtained from the Fourier sine series through term-by-term differentiation. For other boundary conditions, however, a Fourier series tends to become slow converged, if it converges at all, and its derivatives may not be so easily obtained (Li 2000). In order to overcome these difficulties and satisfy the general boundary conditions, a modified Fourier series method was proposed by Li (2000), in which several supplementary terms are introduced into the Fourier series expansion to remove any potential discontinuities of the original displacements and their derivatives throughout the entire solution domain including the boundaries and then to effectively enhance the convergence of the results. This modified Fourier series method is briefly illustrated in following section.

2.1.2 One-Dimensional Modified Fourier Series Solutions

Unlike in the traditional Fourier methods, the transverse displacement $w(x)$ of the beam is expanded into a standard Fourier cosine series plus an sufficiently smooth auxiliary polynomial function defined over [0, L] as:

$$w(x) = W(x) + P(x), \quad \text{and} \quad W(x) = \sum_{m=0}^{\infty} A_m \cos \lambda_m x \tag{2.24}$$

where A_m are the expansion coefficients, $\lambda_m = m\pi/L$. The sufficiently smooth auxiliary polynomial function $P(x)$ is selected to remove all the discontinuities potentially associated with the first-order and third-order derivatives at the boundaries. By setting

$$\begin{aligned} P'(0) = w'(0) = \varsigma_{10} \quad P'(L) = w'(L) = \varsigma_{11} \\ P'''(0) = w'''(0) = \varsigma_{30} \quad P'''(L) = w'''(L) = \varsigma_{31} \end{aligned} \tag{2.25}$$

Such requirements can be readily satisfied by choosing simple polynomials as follows (Zhang and Li 2009; Du 2009):

$$P(x) = \begin{bmatrix} P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{bmatrix}^T \begin{bmatrix} \varsigma_{10} \\ \varsigma_{11} \\ \varsigma_{30} \\ \varsigma_{31} \end{bmatrix} \tag{2.26a}$$

and

$$\begin{bmatrix} P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{bmatrix} = \begin{bmatrix} \frac{9L}{4\pi}\sin\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi}\sin\left(\frac{3\pi x}{2L}\right) \\ -\frac{9L}{4\pi}\cos\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi}\cos\left(\frac{3\pi x}{2L}\right) \\ \frac{L^3}{\pi^3}\sin\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3}\sin\left(\frac{3\pi x}{2L}\right) \\ -\frac{L^3}{\pi^3}\cos\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3}\cos\left(\frac{3\pi x}{2L}\right) \end{bmatrix} \tag{2.26b}$$

It should be pointed out that in actual calculation, the boundary values ζ_{10}, ζ_{11}, ζ_{30} and ζ_{31} can be treated as undetermined coefficient associated with the auxiliary polynomial function and solved in a strong form solution procedures or a weak form one such as Ritz method. It is easy to verify that

$$\begin{aligned} P'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \varsigma_{10} \\ \varsigma_{11} \\ \varsigma_{30} \\ \varsigma_{31} \end{bmatrix} \quad P'(L) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \varsigma_{10} \\ \varsigma_{11} \\ \varsigma_{30} \\ \varsigma_{31} \end{bmatrix} \\ P'''(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \varsigma_{10} \\ \varsigma_{11} \\ \varsigma_{30} \\ \varsigma_{31} \end{bmatrix} \quad P'''(L) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} \varsigma_{10} \\ \varsigma_{11} \\ \varsigma_{30} \\ \varsigma_{31} \end{bmatrix} \end{aligned} \tag{2.27}$$

so that

$$W'(0) = W'(L) = 0 \quad W'''(0) = W'''(L) = 0 \tag{2.28}$$

Essentially, $W(x)$ represents a residual beam displacement which is continuous over [0, L] and has zero-slopes at the both ends as shown in Fig. 2.3. Apparently,

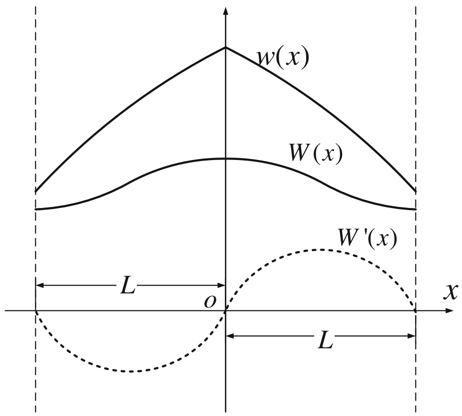

Fig. 2.3 An illustration of the modified Fourier method (Xu 2010)

the cosine series representation of $W(x)$ is able to converge correctly to the function itself and its first derivative at every point (including the boundaries) on the beam. Analogously, discontinuities potentially associated with the third-order derivative can be removed as well. In addition, the residual beam displacement $W(x)$ has at least three continuous derivatives, then all the required differentiations can be simply carried out term-by-term basically. In such case, we have

$$w'(x) = -\sum_{m=1}^{\infty} \lambda_m A_m \sin \lambda_m x + P'(x) \tag{2.29}$$

$$w''(x) = -\sum_{m=1}^{\infty} A_m \lambda_m^2 \cos \lambda_m x + P''(x) \tag{2.30}$$

$$w'''(x) = \sum_{m=1}^{\infty} A_m \lambda_m^3 \sin \lambda_m x + P'''(x) \tag{2.31}$$

$$w''''(x) = \sum_{m=1}^{\infty} A_m \lambda_m^4 \cos \lambda_m x + P''''(x) \tag{2.32}$$

and the Fourier coefficient A_m satisfies

$$\lim_{m \to \infty} A_m \lambda_m^4 = 0 \tag{2.33}$$

Comparing Eq. (2.33) with (2.21), it can be found that the modified Fourier series solution converges at a much faster speed. It should be stressed that the form of auxiliary polynomial function given in Eq. (2.26a, b) should be understood as a continuous function that satisfies Eq. (2.25), its form is not a concern with respect to the convergence of the series solution (Li 2004). Actually, any function satisfies

Eq. (2.25) such as polynomials and trigonometric functions can be used. Combining Eqs. (2.2b) and (2.32) obtains

$$\sum_{m=1}^{\infty} A_m \lambda_m^4 \cos \lambda_m x + P''''(x) = \frac{\omega^2 I_0}{D_{11}} \left(\sum_{m=0}^{\infty} A_m \cos \lambda_m x + P(x) \right) \tag{2.34}$$

In order to derive the constraint equations for the unknown Fourier coefficients, the auxiliary polynomial function $P(x)$ and its four-order derivative $P''''(x)$ in Eq. (2.32) are expanded into Fourier cosine series, namely

$$\begin{aligned} P(x) &= \sum_{m=0}^{\infty} B_m \cos \lambda_m x \\ P''''(x) &= \sum_{m=0}^{\infty} C_m \cos \lambda_m x \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} B_m &= \frac{\int_0^L P(x) \cos \lambda_m x dx}{\int_0^L (\cos \lambda_m x)^2 dx} \\ C_m &= \frac{\int_0^L P''''(x) \cos \lambda_m x dx}{\int_0^L (\cos \lambda_m x)^2 dx} \end{aligned} \tag{2.36}$$

Substituting Eq. (2.35) into Eq. (2.34), we have

$$C_0 + \sum_{m=1}^{\infty} \left(A_m \lambda_m^4 + C_m \right) \cos \lambda_m x = \sum_{m=0}^{\infty} \rho_D \omega^2 (A_m + B_m) \cos \lambda_m x \tag{2.37}$$

where $\rho_D = I_0/D_{11}$, Therefore

$$\begin{aligned} & C_0 - \rho_D \omega^2 (A_0 + B_0) = 0 \\ & A_m \lambda_m^4 + C_m - \rho_D \omega^2 (A_m + B_m) = 0 \quad m = 1, 2, \ldots \end{aligned} \tag{2.38}$$

According to Eq. (1.29), the general boundary conditions for the beam can be written as

$$\begin{aligned} k_{x0}^w w(0) &= D_{11} w'''(0) \quad k_{x1}^w w(L) = -D_{11} w'''(L) \\ K_{x0}^w w'(0) &= D_{11} w''(0) \quad K_{x1}^w w'(L) = -D_{11} w''(L) \end{aligned} \tag{2.39}$$

Substituting Eq. (2.24) into Eq. (2.39) the boundary conditions of the beam can be rewritten as

$$\begin{aligned}
&k_{x0}^{w}\left(\sum_{m=0}^{\infty}A_m+P(0)\right)=D_{11}P'''(0)\\
&k_{x1}^{w}\left(\sum_{m=0}^{\infty}(-1)^mA_m+P(L)\right)=-D_{11}P'''(L)\\
&K_{x0}^{w}P'(0)=D_{11}\left(\sum_{m=0}^{\infty}-\lambda_m^2A_m+P''(0)\right)\\
&K_{x1}^{w}P'(L)=-D_{11}\left(\sum_{m=0}^{\infty}(-1)^{m+1}\lambda_m^2A_m+P''(L)\right)
\end{aligned}\tag{2.40}$$

The natural frequencies and mode shapes of the beam can now be easily determined by solving Eq. (2.38) with boundary condition equations Eq. (2.40), the more detail solution procedure will be given in Sect. 2.2.

Alternatively, the transverse displacement of the beam can also be expanded into a modified Fourier sine series. In that case, the auxiliary polynomial function $P(x)$ is selected to remove all the discontinuities potentially associated with the original displacement and its second-order derivative at the boundaries. Namely, the transverse displacement $w(x)$ of the beam should be expanded into a standard Fourier sine series plus a sufficiently smooth auxiliary polynomial function defined over $[0, L]$ as:

$$w(x)=W(x)+P(x),\quad \text{where}\quad W(x)=\sum_{m=0}^{\infty}A_m\sin\lambda_m x \tag{2.41}$$

and

$$\begin{aligned}
P(0)=w(0)=\varsigma_{00}\qquad P(L)=w(L)=\varsigma_{01}\\
P''(0)=w''(0)=\varsigma_{20}\qquad P''(L)=w''(L)=\varsigma_{21}
\end{aligned}\tag{2.42}$$

Similarly, Eqs. (2.17)–(2.20) can be rewritten as

$$w'(x)=\sum_{m=1}^{\infty}A_m\lambda_m\cos\lambda_m x+P'(x) \tag{2.43}$$

$$w''(x)=-\sum_{m=1}^{\infty}A_m\lambda_m^2\sin\lambda_m x+P''(x) \tag{2.44}$$

$$w'''(x) = -\sum_{m=1}^{\infty} A_m \lambda_m^3 \cos \lambda_m x + P'''(x) \tag{2.45}$$

$$w''''(x) = \sum_{m=1}^{\infty} A_m \lambda_m^4 \sin \lambda_m x + P''''(x) \tag{2.46}$$

The solution procedure is the same as those of the modified Fourier cosine series.

2.1.3 Two-Dimensional Modified Fourier Series Solutions

Using the modified Fourier series technique, in the manner similar to that described earlier, two-dimensional modified Fourier series solutions for laminated plates and shells are presented in this section. For the sake of completeness, we consider the free vibration analysis of a moderately thick. Generally laminated rectangular plate (where a and b denote its length and width) with general boundary conditions (see Fig. 2.4), the solution procedure is given step-by-step as follows.

Substituting $\alpha = x$, $\beta = y$, $A = B = 1$ and $R_\alpha = R_\beta = \infty$ into Eq. (1.59), the governing equations of the plate are written as:

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = -\omega^2 (I_0 u + I_1 \phi_x) \\
&\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = -\omega^2 \left(I_0 v + I_1 \phi_y\right) \\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\, \omega^2 I_0 w \\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = -\, \omega^2 (I_1 u + I_2 \phi_x) \\
&\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = -\, \omega^2 \left(I_1 v + I_2 \phi_\beta\right)
\end{aligned} \tag{2.47a–e}$$

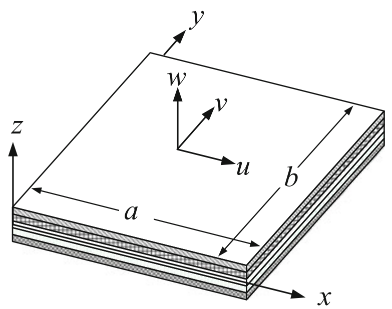

Fig. 2.4 A generally laminated moderately thick rectangular plate

Similarly, substituting $\alpha = x$, $\beta = y$, $A = B = 1$ and $R_\alpha = R_\beta = \infty$ into Eqs. (1.34), (1.46) and (1.47) and then substituting these three equations into Eq. (2.47) yields

$$\left(\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & 0 & 0 & M_{14} & 0 \\ 0 & M_{22} & 0 & 0 & M_{25} \\ 0 & 0 & M_{33} & 0 & 0 \\ M_{41} & 0 & 0 & M_{44} & 0 \\ 0 & M_{52} & 0 & 0 & M_{55} \end{bmatrix} \right) \begin{bmatrix} u \\ v \\ w \\ \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.48}$$

where the coefficients of the linear operator $L(L_{ij} = L_{ji}, M_{ij} = M_{ji})$ are given below:

$$\begin{aligned}
L_{11} &= A_{11}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{\partial^2}{\partial x \partial y} + A_{66}\frac{\partial^2}{\partial y^2} \\
L_{12} &= A_{16}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2}{\partial x \partial y} + A_{26}\frac{\partial^2}{\partial y^2} \\
L_{13} &= 0 \quad L_{23} = 0
\end{aligned}$$

$$\begin{aligned}
L_{14} &= B_{11}\frac{\partial^2}{\partial x^2} + 2B_{16}\frac{\partial^2}{\partial x \partial y} + B_{66}\frac{\partial^2}{\partial y^2} \\
L_{15} &= B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66})\frac{\partial^2}{\partial x \partial y} + B_{26}\frac{\partial^2}{\partial y^2} \\
L_{22} &= A_{66}\frac{\partial^2}{\partial x^2} + 2A_{26}\frac{\partial^2}{\partial x \partial y} + A_{22}\frac{\partial^2}{\partial y^2} \\
L_{24} &= B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66})\frac{\partial^2}{\partial x \partial y} + B_{26}\frac{\partial^2}{\partial y^2} \\
L_{25} &= B_{66}\frac{\partial^2}{\partial x^2} + 2B_{26}\frac{\partial^2}{\partial x \partial y} + B_{22}\frac{\partial^2}{\partial y^2} \\
L_{33} &= -A_{55}\frac{\partial^2}{\partial x^2} - 2A_{45}\frac{\partial^2}{\partial x \partial y} - A_{44}\frac{\partial^2}{\partial y^2} \\
L_{34} &= -A_{55}\frac{\partial}{\partial x} - A_{45}\frac{\partial}{\partial y} \\
L_{35} &= -A_{45}\frac{\partial}{\partial x} - A_{44}\frac{\partial}{\partial y} \\
L_{44} &= D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{\partial^2}{\partial x \partial y} + D_{66}\frac{\partial^2}{\partial y^2} - A_{55} \\
L_{45} &= D_{16}\frac{\partial^2}{\partial x^2} + (D_{12} + D_{66})\frac{\partial^2}{\partial x \partial y} + D_{26}\frac{\partial^2}{\partial y^2} - A_{45} \\
L_{55} &= D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{\partial^2}{\partial x \partial y} + D_{22}\frac{\partial^2}{\partial y^2} - A_{44} \\
M_{11} &= M_{22} = M_{33} = -I_0 \\
M_{14} &= M_{25} = -I_1 \\
M_{44} &= M_{55} = -I_2
\end{aligned} \tag{2.49}$$

and the general boundary conditions of the plate are:

$$\begin{aligned}
&x=0:\begin{cases} N_x = k_{x0}^u u \\ N_{xy} = k_{x0}^v v \\ Q_x = k_{x0}^w w \\ M_x = K_{x0}^x \phi_x \\ M_{xy} = K_{x0}^y \phi_y \end{cases} \quad x=a:\begin{cases} N_x = -k_{x1}^u u \\ N_{xy} = -k_{x1}^v v \\ Q_x = -k_{x1}^w w \\ M_x = -K_{x1}^x \phi_x \\ M_{xy} = -K_{x1}^y \phi_y \end{cases} \\
&y=0:\begin{cases} N_{yx} = k_{y0}^u u \\ N_y = k_{y0}^v v \\ Q_y = k_{y0}^w w \\ M_{yx} = K_{y0}^x \phi_x \\ M_y = K_{y0}^y \phi_y \end{cases} \quad y=\mathrm{b}:\begin{cases} N_{yx} = -k_{y1}^u u \\ N_y = -k_{y1}^v v \\ Q_y = -k_{y1}^w w \\ M_{yx} = -K_{y1}^x \phi_x \\ M_y = -K_{y1}^y \phi_y \end{cases}
\end{aligned} \tag{2.50}$$

Taking the plate displacement component $u(x, y)$ for example, it can be expanded into a standard double Fourier cosine series plus two sufficiently smooth auxiliary polynomial functions defined over $[0, a] \times [0, b]$ as

$$u(x,y) = U(x,y) + P_x(x,y) + P_y(x,y) \tag{2.51a}$$

and

$$U(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos \lambda_m x \cos \lambda_n y \tag{2.51b}$$

where A_{mn} are the expansion coefficients. $\lambda_m = m\pi/a$ and $\lambda_n = n\pi/b$. $P_x(x, y)$ and $P_y(x, y)$ denote the auxiliary polynomial functions introduced to ensure and accelerate the convergence of the series expansion of the displacement $u(x, y)$. According to Eq. (2.49), it is obvious that each of the displacements and rotation components of the plate is required to have up to the second derivatives. Therefore, the auxiliary polynomial functions $P_x(x, y)$ and $P_y(x, y)$ are selected to remove all the discontinuities potentially associated with the first-order derivatives at the boundaries. By setting

$$\begin{aligned}
\frac{\partial P_x(0,y)}{\partial x} &= \frac{\partial u(0,y)}{\partial x} = \xi_{x0}(y) \\
\frac{\partial P_x(a,y)}{\partial x} &= \frac{\partial u(a,y)}{\partial x} = \xi_{x1}(y) \\
\frac{\partial P_y(x,0)}{\partial y} &= \frac{\partial u(x,0)}{\partial y} = \xi_{y0}(x) \\
\frac{\partial P_y(x,b)}{\partial y} &= \frac{\partial u(x,b)}{\partial y} = \xi_{y1}(x)
\end{aligned} \tag{2.52}$$

where $\xi_{x0}(y)$, $\xi_{x1}(y)$, $\xi_{y0}(x)$ and $\xi_{y1}(x)$ are the unknown boundary derivatives at boundaries $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively. They can be expanded in the form of Fourier cosine series as

$$\begin{aligned} \xi_{x0}(y) &= \sum_{n=0}^{\infty} a_{1n} \cos \lambda_n y \\ \xi_{x1}(y) &= \sum_{n=0}^{\infty} a_{2n} \cos \lambda_n y \\ \xi_{y0}(x) &= \sum_{m=0}^{\infty} b_{1m} \cos \lambda_m x \\ \xi_{y1}(x) &= \sum_{m=0}^{\infty} b_{2m} \cos \lambda_m x \end{aligned} \tag{2.53}$$

where a_{1n}, a_{2n}, b_{1m} and b_{2m} are the expansion coefficients. The requirements of Eq. (2.52) can be readily satisfied by choosing the auxiliary polynomial functions $P_x(x, y)$ and $P_y(x, y)$ as follows (Du 2009):

$$P_x(x,y) = \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix}^T \begin{bmatrix} \xi_{x0}(y) \\ \xi_{x1}(y) \end{bmatrix}, \quad \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix} = \begin{bmatrix} x(x/a-1)^2 \\ x^2(x/a-1)/a \end{bmatrix} \tag{2.54a}$$

and

$$P_y(x,y) = \begin{bmatrix} P_1(y) \\ P_2(y) \end{bmatrix}^T \begin{bmatrix} \xi_{y0}(x) \\ \xi_{y1}(x) \end{bmatrix}, \quad \begin{bmatrix} P_1(y) \\ P_2(y) \end{bmatrix} = \begin{bmatrix} y(y/b-1)^2 \\ y^2(y/b-1)/b \end{bmatrix} \tag{2.54b}$$

It is easy to verify that

$$\begin{aligned} \frac{\partial P_x(0,y)}{\partial x} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \xi_{x0}(y) \\ \xi_{x1}(y) \end{bmatrix} \\ \frac{\partial P_x(a,y)}{\partial x} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} \xi_{x0}(y) \\ \xi_{x1}(y) \end{bmatrix} \\ \frac{\partial P_y(x,0)}{\partial y} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \xi_{y0}(x) \\ \xi_{y1}(x) \end{bmatrix} \\ \frac{\partial P_y(x,b)}{\partial y} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} \xi_{y0}(x) \\ \xi_{y1}(x) \end{bmatrix} \end{aligned} \tag{2.55}$$

so that

$$\begin{aligned} \frac{\partial U(0,y)}{\partial x} &= \frac{\partial U(a,y)}{\partial x} = 0 \\ \frac{\partial U(x,0)}{\partial y} &= \frac{\partial U(x,b)}{\partial y} = 0 \end{aligned} \tag{2.56}$$

Essentially, $U(x, y)$ represents a residual plate displacement which is continuous over $[0, a] \times [0, b]$ and has zero-slopes at the four boundaries. Apparently, the cosine series representation of $U(x, y)$ is able to converge correctly to the function itself and its first derivative at every point on the plate. Moreover, all the required differentiations of the residual plate displacement $U(x, y)$ can be simply carried out term-by-term. Thus, the plate displacement function of component $u(x, y)$ can be rewritten as

$$\begin{aligned} u(x,y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\cos\lambda_m x\cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{\infty} a_{ln}P_l(x)\cos\lambda_n y \\ &+ \sum_{l=1}^{2}\sum_{m=0}^{\infty} b_{lm}P_l(y)\cos\lambda_m x \end{aligned} \tag{2.57}$$

Similarly, the other displacements and rotation components of the plate can be expanded as the two-dimensional modified Fourier series as

$$\begin{aligned} v(x,y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\cos\lambda_m x\cos\lambda_n y + \sum_{l=0}^{2}\sum_{n=0}^{\infty} c_{ln}P_l(x)\cos\lambda_n y \\ &+ \sum_{l=0}^{2}\sum_{m=0}^{\infty} d_{lm}P_l(y)\cos\lambda_m x \\ w(x,y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\cos\lambda_m x\cos\lambda_n y + \sum_{l=0}^{2}\sum_{n=0}^{\infty} e_{ln}P_l(x)\cos\lambda_n y \\ &+ \sum_{l=0}^{2}\sum_{m=0}^{\infty} f_{lm}P_l(y)\cos\lambda_m x \\ \phi_x(x,y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn}\cos\lambda_m x\cos\lambda_n y + \sum_{l=0}^{2}\sum_{n=0}^{\infty} g_{ln}P_l(x)\cos\lambda_n y \\ &+ \sum_{l=0}^{2}\sum_{m=0}^{\infty} h_{lm}P_l(y)\cos\lambda_m x \\ \phi_y(x,y) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{mn}\cos\lambda_m x\cos\lambda_n y + \sum_{l=0}^{2}\sum_{n=0}^{\infty} i_{ln}P_l(x)\cos\lambda_n y \\ &+ \sum_{l=0}^{2}\sum_{m=0}^{\infty} j_{lm}P_l(y)\cos\lambda_m x \end{aligned} \tag{2.58}$$

where B_{mn}, C_{mn}, D_{mn} and E_{mn} are the standard Fourier series expansion coefficients. c_{ln}, d_{lm}, e_{ln}, f_{lm} g_{ln}, h_{lm}, i_{ln} and j_{lm} are the corresponding supplement coefficients.

2.2 Strong Form Solution Procedure

With the modified Fourier series, vibration of isotropic, anisotropic and laminated beams, plates and shells can be obtained by using the strong form solution procedures as described below. Taking the previously studied laminated rectangular plate for example, the solution procedure is given step-by-step as follows.

In the actual calculations, all the five infinite modified Fourier series expressions given in Eqs. (2.57) and (2.58) need to be truncated as finite series to obtain the results with acceptable accuracy due to the limited speed, the capacity and the numerical accuracy of computers. Unless otherwise stressed, the involving terms in all the plate displacements and rotation components are uniformly taken as $m \in [0, M]$ and $n \in [0, N]$. Thus, the modified Fourier series expressions presented in Eqs. (2.57) and (2.58) can be rewritten in the matrix form as:

$$\begin{aligned} u(x,y) &= \mathbf{H}_{xy}\mathbf{A} + \mathbf{H}_{x}\mathbf{a} + \mathbf{H}_{y}\mathbf{b} \\ v(x,y) &= \mathbf{H}_{xy}\mathbf{B} + \mathbf{H}_{x}\mathbf{c} + \mathbf{H}_{y}\mathbf{d} \\ w(x,y) &= \mathbf{H}_{xy}\mathbf{C} + \mathbf{H}_{x}\mathbf{e} + \mathbf{H}_{y}\mathbf{f} \\ \phi_x(x,y) &= \mathbf{H}_{xy}\mathbf{D} + \mathbf{H}_{x}\mathbf{g} + \mathbf{H}_{y}\mathbf{h} \\ \phi_y(x,y) &= \mathbf{H}_{xy}\mathbf{E} + \mathbf{H}_{x}\mathbf{i} + \mathbf{H}_{y}\mathbf{j} \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} \mathbf{H}_{xy} &= [\cos\lambda_0 x \cos\lambda_0 y, \ldots, \cos\lambda_m x \cos\lambda_n y, \ldots, \cos\lambda_M x \cos\lambda_N y] \\ \mathbf{H}_{x} &= [P_1(x)\cos\lambda_0 y, \ldots, P_l(x)\cos\lambda_n y, \ldots, P_2(x)\cos\lambda_N y] \\ \mathbf{H}_{y} &= [P_1(y)\cos\lambda_0 x, \ldots, P_l(y)\cos\lambda_m x, \ldots, P_2(y)\cos\lambda_M x] \end{aligned} \tag{2.60}$$

and

$$\begin{aligned} &\mathbf{A} = [A_{00}, \ldots, A_{mn}, \ldots, A_{MN}]^T \quad \mathbf{a} = [a_{10}, \ldots, a_{ln}, \ldots, a_{2N}]^T \\ &\mathbf{b} = [b_{10}, \ldots, b_{lm}, \ldots, b_{2M}]^T \\ &\mathbf{B} = [B_{00}, \ldots, B_{mn}, \ldots, B_{MN}]^T \quad \mathbf{c} = [c_{10}, \ldots, c_{ln}, \ldots, c_{2N}]^T \\ &\mathbf{d} = [d_{10}, \ldots, d_{lm}, \ldots, d_{2M}]^T \\ &\mathbf{C} = [C_{00}, \ldots, C_{mn}, \ldots, C_{MN}]^T \quad \mathbf{e} = [e_{10}, \ldots, e_{ln}, \ldots, e_{2N}]^T \\ &\mathbf{f} = [f_{10}, \ldots, f_{lm}, \ldots, f_{2M}]^T \\ &\mathbf{D} = [D_{00}, \ldots, D_{mn}, \ldots, D_{MN}]^T \quad \mathbf{g} = [g_{10}, \ldots, g_{ln}, \ldots, g_{2N}]^T \\ &\mathbf{h} = [h_{10}, \ldots, h_{lm}, \ldots, h_{2M}]^T \\ &\mathbf{E} = [E_{00}, \ldots, E_{mn}, \ldots, E_{MN}]^T \quad \mathbf{i} = [i_{10}, \ldots, i_{ln}, \ldots, i_{2N}]^T \\ &\mathbf{j} = [j_{10}, \ldots, j_{lm}, \ldots, j_{2M}]^T \end{aligned} \tag{2.61}$$

Superscript T represents the transposition operator. Substituting Eq. (2.59) into Eq. (2.48) results in

$$\mathbf{L}_{xy}\boldsymbol{\Gamma}_{xy} + \mathbf{L}_x\boldsymbol{\Gamma}_x + \mathbf{L}_y\boldsymbol{\Gamma}_y - \omega^2\left(\mathbf{M}_{xy}\boldsymbol{\Gamma}_{xy} + \mathbf{M}_x\boldsymbol{\Gamma}_x + \mathbf{M}_y\boldsymbol{\Gamma}_y\right) = \mathbf{0} \tag{2.62}$$

where

$$\mathbf{L}_i = \begin{bmatrix} L_{11}\mathbf{H}_i & L_{12}\mathbf{H}_i & L_{13}\mathbf{H}_i & L_{14}\mathbf{H}_i & L_{15}\mathbf{H}_i \\ L_{21}\mathbf{H}_i & L_{22}\mathbf{H}_i & L_{23}\mathbf{H}_i & L_{24}\mathbf{H}_i & L_{25}\mathbf{H}_i \\ L_{31}\mathbf{H}_i & L_{32}\mathbf{H}_i & L_{33}\mathbf{H}_i & L_{34}\mathbf{H}_i & L_{35}\mathbf{H}_i \\ L_{41}\mathbf{H}_i & L_{42}\mathbf{H}_i & L_{43}\mathbf{H}_i & L_{44}\mathbf{H}_i & L_{45}\mathbf{H}_i \\ L_{51}\mathbf{H}_i & L_{52}\mathbf{H}_i & L_{53}\mathbf{H}_i & L_{54}\mathbf{H}_i & L_{55}\mathbf{H}_i \end{bmatrix}, \quad (i = xy, x, y) \tag{2.63}$$

$$\mathbf{M}_i = \begin{bmatrix} M_{11}\mathbf{H}_i & \mathbf{0} & \mathbf{0} & M_{14}\mathbf{H}_i & \mathbf{0} \\ \mathbf{0} & M_{22}\mathbf{H}_i & \mathbf{0} & \mathbf{0} & M_{25}\mathbf{H}_i \\ \mathbf{0} & \mathbf{0} & M_{33}\mathbf{H}_i & \mathbf{0} & \mathbf{0} \\ M_{41}\mathbf{H}_i & \mathbf{0} & \mathbf{0} & M_{44}\mathbf{H}_i & \mathbf{0} \\ \mathbf{0} & M_{52}\mathbf{H}_i & \mathbf{0} & \mathbf{0} & M_{55}\mathbf{H}_i \end{bmatrix}, \quad (i = xy, x, y) \tag{2.64}$$

$$\boldsymbol{\Gamma}_{xy} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \\ \mathbf{E} \end{bmatrix}, \quad \boldsymbol{\Gamma}_x = \begin{bmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{e} \\ \mathbf{g} \\ \mathbf{i} \end{bmatrix}, \quad \boldsymbol{\Gamma}_y = \begin{bmatrix} \mathbf{b} \\ \mathbf{d} \\ \mathbf{f} \\ \mathbf{h} \\ \mathbf{j} \end{bmatrix} \tag{2.65}$$

In the same way, substituting Eq. (2.59) into Eq. (2.50), the general boundary conditions of the plate can be rewritten as

$$\begin{aligned} x = 0 &: \mathbf{L}_{xy}^{x0}\boldsymbol{\Gamma}_{xy} + \mathbf{L}_x^{x0}\boldsymbol{\Gamma}_x + \mathbf{L}_y^{x0}\boldsymbol{\Gamma}_y = \mathbf{0} \\ x = a &: \mathbf{L}_{xy}^{x1}\boldsymbol{\Gamma}_{xy} + \mathbf{L}_x^{x1}\boldsymbol{\Gamma}_x + \mathbf{L}_y^{x1}\boldsymbol{\Gamma}_y = \mathbf{0} \end{aligned} \tag{2.66a}$$

$$\begin{aligned} y = 0 &: \mathbf{L}_{xy}^{y0}\boldsymbol{\Gamma}_{xy} + \mathbf{L}_x^{y0}\boldsymbol{\Gamma}_x + \mathbf{L}_y^{y0}\boldsymbol{\Gamma}_y = \mathbf{0} \\ y = b &: \mathbf{L}_{xy}^{y1}\boldsymbol{\Gamma}_{xy} + \mathbf{L}_x^{y1}\boldsymbol{\Gamma}_x + \mathbf{L}_y^{y1}\boldsymbol{\Gamma}_y = \mathbf{0} \end{aligned} \tag{2.66b}$$

in which

$$\mathbf{L}_i^j = \begin{bmatrix} L_{11}^j\mathbf{H}_i & L_{12}^j\mathbf{H}_i & L_{13}^j\mathbf{H}_i & L_{14}^j\mathbf{H}_i & L_{15}^j\mathbf{H}_i \\ L_{21}^j\mathbf{H}_i & L_{22}^j\mathbf{H}_i & L_{23}^j\mathbf{H}_i & L_{24}^j\mathbf{H}_i & L_{25}^j\mathbf{H}_i \\ L_{31}^j\mathbf{H}_i & L_{32}^j\mathbf{H}_i & L_{33}^j\mathbf{H}_i & L_{34}^j\mathbf{H}_i & L_{35}^j\mathbf{H}_i \\ L_{41}^j\mathbf{H}_i & L_{42}^j\mathbf{H}_i & L_{43}^j\mathbf{H}_i & L_{44}^j\mathbf{H}_i & L_{45}^j\mathbf{H}_i \\ L_{25}^j\mathbf{H}_i & L_{52}^j\mathbf{H}_i & L_{53}^j\mathbf{H}_i & L_{54}^j\mathbf{H}_i & L_{55}^j\mathbf{H}_i \end{bmatrix}, \quad \begin{pmatrix} i = xy, x, y \\ j = x0, x1, \\ \quad y0, y1 \end{pmatrix} \tag{2.67}$$

the coefficients of the linear operator in Eq. (2.67) are given below:

$$\begin{array}{ll}
L_{11}^{x0} = A_{11}\frac{\partial}{\partial x} + A_{16}\frac{\partial}{\partial y} - k_{x0}^{u}, & L_{12}^{x0} = A_{12}\frac{\partial}{\partial y} + A_{16}\frac{\partial}{\partial x} \\
L_{13}^{x0} = L_{31}^{x0} = L_{43}^{x0} = 0, & L_{14}^{x0} = B_{11}\frac{\partial}{\partial x} + B_{16}\frac{\partial}{\partial y} = L_{41}^{x0} \\
L_{15}^{x0} = B_{12}\frac{\partial}{\partial y} + B_{16}\frac{\partial}{\partial x}, & L_{21}^{x0} = A_{16}\frac{\partial}{\partial x} + A_{66}\frac{\partial}{\partial y} \\
L_{22}^{x0} = A_{26}\frac{\partial}{\partial y} + A_{66}\frac{\partial}{\partial x} - k_{x0}^{v}, & L_{23}^{x0} = L_{32}^{x0} = L_{53}^{x0} = 0 \\
L_{24}^{x0} = B_{16}\frac{\partial}{\partial x} + B_{66}\frac{\partial}{\partial y}, & L_{25}^{x0} = B_{26}\frac{\partial}{\partial y} + B_{66}\frac{\partial}{\partial x} = L_{52}^{x0} \\
L_{33}^{x0} = A_{45}\frac{\partial}{\partial y} + A_{55}\frac{\partial}{\partial x} - k_{x0}^{w}, & L_{34}^{x0} = A_{55}, L_{35}^{x0} = A_{45} \\
L_{42}^{x0} = B_{12}\frac{\partial}{\partial y} + B_{16}\frac{\partial}{\partial x}, & L_{44}^{x0} = D_{11}\frac{\partial}{\partial x} + D_{16}\frac{\partial}{\partial y} - K_{x0}^{x} \\
L_{45}^{x0} = D_{12}\frac{\partial}{\partial y} + D_{16}\frac{\partial}{\partial x}, & L_{51}^{x0} = B_{16}\frac{\partial}{\partial x} + B_{66}\frac{\partial}{\partial y} \\
L_{54}^{x0} = D_{16}\frac{\partial}{\partial x} + D_{66}\frac{\partial}{\partial y}, & L_{55}^{x0} = D_{26}\frac{\partial}{\partial y} + D_{66}\frac{\partial}{\partial x} - K_{x0}^{y}
\end{array} \tag{2.68}$$

and

$$\begin{array}{ll}
L_{11}^{y0} = A_{16}\frac{\partial}{\partial x} + A_{66}\frac{\partial}{\partial y} - k_{y0}^{u}, & L_{12}^{y0} = A_{26}\frac{\partial}{\partial y} + A_{66}\frac{\partial}{\partial x} \\
L_{13}^{y0} = L_{31}^{y0} = L_{43}^{y0} = 0, & L_{14}^{y0} = B_{16}\frac{\partial}{\partial x} + B_{66}\frac{\partial}{\partial y} = L_{41}^{y0} \\
L_{15}^{y0} = B_{26}\frac{\partial}{\partial y} + B_{66}\frac{\partial}{\partial x}, & L_{21}^{y0} = A_{12}\frac{\partial}{\partial x} + A_{26}\frac{\partial}{\partial y} \\
L_{22}^{y0} = A_{22}\frac{\partial}{\partial y} + A_{26}\frac{\partial}{\partial x} - k_{y0}^{v}, & L_{23}^{y0} = L_{32}^{y0} = L_{53}^{y0} = 0 \\
L_{24}^{y0} = B_{12}\frac{\partial}{\partial x} + B_{26}\frac{\partial}{\partial y}, & L_{25}^{y0} = B_{22}\frac{\partial}{\partial y} + B_{26}\frac{\partial}{\partial x} = L_{52}^{y0} \\
L_{33}^{y0} = A_{44}\frac{\partial}{\partial y} + A_{45}\frac{\partial}{\partial x} - k_{y0}^{w}, & L_{34}^{y0} = A_{45}, \quad L_{35}^{y0} = A_{44} \\
L_{42}^{y0} = B_{26}\frac{\partial}{\partial y} + B_{66}\frac{\partial}{\partial x}, & L_{44}^{y0} = D_{16}\frac{\partial}{\partial x} + D_{66}\frac{\partial}{\partial y} - K_{y0}^{x} \\
L_{45}^{y0} = D_{26}\frac{\partial}{\partial y} + D_{66}\frac{\partial}{\partial x}, & L_{51}^{y0} = B_{12}\frac{\partial}{\partial x} + B_{26}\frac{\partial}{\partial y} \\
L_{54}^{y0} = D_{12}\frac{\partial}{\partial x} + D_{26}\frac{\partial}{\partial y}, & L_{55}^{y0} = D_{22}\frac{\partial}{\partial y} + D_{26}\frac{\partial}{\partial x} - K_{y0}^{y}
\end{array} \tag{2.69}$$

And for $j = x1$ and $j = y1$, we have:

$$\begin{array}{ll}
L_{ij}^{x1} = L_{ij}^{x0}, (i \neq j); & L_{ij}^{y1} = L_{ij}^{y0}, (i \neq j) \\
L_{11}^{x1} = A_{11}\frac{\partial}{\partial x} + A_{16}\frac{\partial}{\partial y} + k_{x1}^{u}, & L_{11}^{y1} = A_{16}\frac{\partial}{\partial x} + A_{66}\frac{\partial}{\partial y} + k_{y1}^{u} \\
L_{22}^{x1} = A_{26}\frac{\partial}{\partial y} + A_{66}\frac{\partial}{\partial x} + k_{x1}^{v}, & L_{22}^{y1} = A_{22}\frac{\partial}{\partial y} + A_{26}\frac{\partial}{\partial x} + k_{y1}^{v} \\
L_{33}^{x1} = A_{45}\frac{\partial}{\partial y} + A_{55}\frac{\partial}{\partial x} + k_{x1}^{w}, & L_{33}^{y1} = A_{44}\frac{\partial}{\partial y} + A_{45}\frac{\partial}{\partial x} + k_{y1}^{w} \\
L_{44}^{x1} = D_{11}\frac{\partial}{\partial x} + D_{16}\frac{\partial}{\partial y} + K_{x1}^{x}, & L_{44}^{y1} = D_{16}\frac{\partial}{\partial x} + D_{66}\frac{\partial}{\partial y} + K_{y1}^{x} \\
L_{55}^{x1} = D_{26}\frac{\partial}{\partial y} + D_{66}\frac{\partial}{\partial x} + K_{x1}^{y}, & L_{55}^{y1} = D_{22}\frac{\partial}{\partial y} + D_{26}\frac{\partial}{\partial x} + K_{y1}^{y}
\end{array} \tag{2.70}$$

In order to derive the constraint equations for the unknown Fourier coefficients, all the sine terms, the auxiliary polynomial functions and their derivatives in Eqs. (2.62) and (2.66a, b) will be expanded into Fourier cosine series. Letting

$$\begin{aligned}
\mathbf{C}_{xy} &= [\cos\lambda_0 x \cos\lambda_0 y, \ldots, \cos\lambda_m x \cos\lambda_n y, \ldots, \cos\lambda_M x \cos\lambda_N y]^T \\
\mathbf{C}_{x} &= [\cos\lambda_0 x, \ldots, \cos\lambda_m x, \ldots, \cos\lambda_M x]^T \\
\mathbf{C}_{y} &= [\cos\lambda_0 y, \ldots, \cos\lambda_n x, \ldots, \cos\lambda_N y]^T
\end{aligned} \tag{2.71}$$

Multiplying Eq. (2.62) with $\mathbf{C}_{xy}$ in the left side and integrating it from 0 to a and 0 to b separately with respect to x and y obtains

$$\overline{\mathbf{L}}_{xy}\boldsymbol{\Gamma}_{xy} + \begin{bmatrix} \overline{\mathbf{L}}_x & \overline{\mathbf{L}}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}_x \\ \boldsymbol{\Gamma}_y \end{bmatrix} - \omega^2 \left(\overline{\mathbf{M}}_{xy}\boldsymbol{\Gamma}_{xy} + \begin{bmatrix} \overline{\mathbf{M}}_x & \overline{\mathbf{M}}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}_x \\ \boldsymbol{\Gamma}_y \end{bmatrix} \right) = \mathbf{0} \tag{2.72}$$

where

$$\begin{array}{ll} \overline{\mathbf{L}}_{xy} = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{L}_{xy}dydx, & \overline{\mathbf{M}}_{xy} = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{M}_{xy}dydx \\ \overline{\mathbf{L}}_x = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{L}_x dydx, & \overline{\mathbf{M}}_x = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{M}_x dydx \\ \overline{\mathbf{L}}_y = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{L}_y dydx, & \overline{\mathbf{M}}_y = \int_0^a \int_0^b \mathbf{C}_{xy}\mathbf{M}_y dydx \end{array} \tag{2.73}$$

Similarly, multiplying Eq. (2.66a) with $\mathbf{C}_y$ in the left side then integrating it from 0 to b with respect to y, and multiplying Eq. (2.66b) with $\mathbf{C}_x$ in the left side then integrating it from 0 to a with respect to x, we have

$$\begin{array}{l} x = 0 : \overline{\mathbf{L}}_{xy}^{x0}\boldsymbol{\Gamma}_{xy} + \overline{\mathbf{L}}_x^{x0}\boldsymbol{\Gamma}_x + \overline{\mathbf{L}}_y^{x0}\boldsymbol{\Gamma}_y = \mathbf{0} \\ x = a : \overline{\mathbf{L}}_{xy}^{x1}\boldsymbol{\Gamma}_{xy} + \overline{\mathbf{L}}_x^{x1}\boldsymbol{\Gamma}_x + \overline{\mathbf{L}}_y^{x1}\boldsymbol{\Gamma}_y = \mathbf{0} \end{array} \tag{2.74a}$$

$$\begin{array}{l} y = 0 : \overline{\mathbf{L}}_{xy}^{y0}\boldsymbol{\Gamma}_{xy} + \overline{\mathbf{L}}_x^{y0}\boldsymbol{\Gamma}_x + \overline{\mathbf{L}}_y^{y0}\boldsymbol{\Gamma}_y = \mathbf{0} \\ y = b : \overline{\mathbf{L}}_{xy}^{y1}\boldsymbol{\Gamma}_{xy} + \overline{\mathbf{L}}_x^{y1}\boldsymbol{\Gamma}_x + \overline{\mathbf{L}}_y^{y1}\boldsymbol{\Gamma}_y = \mathbf{0} \end{array} \tag{2.74b}$$

where

$$\begin{array}{ll} \overline{\mathbf{L}}_{xy}^{x0} = \int_0^b \mathbf{C}_y\mathbf{L}_{xy}^{x0}dy, & \overline{\mathbf{L}}_{xy}^{x1} = \int_0^b \mathbf{C}_y\mathbf{L}_{xy}^{x1}dy \\ \overline{\mathbf{L}}_x^{x0} = \int_0^b \mathbf{C}_y\mathbf{L}_x^{x0}dy, & \overline{\mathbf{L}}_x^{x1} = \int_0^b \mathbf{C}_y\mathbf{L}_x^{x1}dy \\ \overline{\mathbf{L}}_y^{x0} = \int_0^b \mathbf{C}_y\mathbf{L}_y^{x0}dy, & \overline{\mathbf{L}}_y^{x1} = \int_0^b \mathbf{C}_y\mathbf{L}_y^{x1}dy \end{array} \tag{2.75a}$$

$$\begin{array}{ll} \overline{\mathbf{L}}_{xy}^{y0} = \int_0^a \mathbf{C}_x\mathbf{L}_{xy}^{y0}dx, & \overline{\mathbf{L}}_{xy}^{y1} = \int_0^a \mathbf{C}_x\mathbf{L}_{xy}^{y1}dx \\ \overline{\mathbf{L}}_x^{y0} = \int_0^a \mathbf{C}_x\mathbf{L}_x^{y0}dx, & \overline{\mathbf{L}}_x^{y1} = \int_0^a \mathbf{C}_x\mathbf{L}_x^{y1}dx \\ \overline{\mathbf{L}}_y^{y0} = \int_0^a \mathbf{C}_x\mathbf{L}_y^{y0}dx, & \overline{\mathbf{L}}_y^{y1} = \int_0^a \mathbf{C}_x\mathbf{L}_y^{y1}dx \end{array} \tag{2.75a}$$

Thus, Eq. (2.74a, b) can be rewritten as

$$\begin{bmatrix} \boldsymbol{\Gamma}_x \\ \boldsymbol{\Gamma}_x \end{bmatrix} = - \begin{bmatrix} \overline{\mathbf{L}}_x^{x0} & \overline{\mathbf{L}}_y^{x0} \\ \overline{\mathbf{L}}_x^{x1} & \overline{\mathbf{L}}_y^{x1} \\ \overline{\mathbf{L}}_x^{y0} & \overline{\mathbf{L}}_y^{y0} \\ \overline{\mathbf{L}}_x^{y1} & \overline{\mathbf{L}}_y^{y1} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{L}}_{xy}^{x0} \\ \overline{\mathbf{L}}_{xy}^{x1} \\ \overline{\mathbf{L}}_{xy}^{y0} \\ \overline{\mathbf{L}}_{xy}^{y1} \end{bmatrix} \boldsymbol{\Gamma}_{xy} \tag{2.76}$$

Finally, combine Eqs. (2.72) and (2.76) results in

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\boldsymbol{\Gamma}_{xy}=\mathbf{0} \tag{2.77}$$

where $\mathbf{K}$ is the stiffness matrix for the plate, and $\mathbf{M}$ is the mass matrix. They are defined as

$$\mathbf{K}=\overline{\mathbf{L}}_{xy}-\left[\overline{\mathbf{L}}_x \quad \overline{\mathbf{L}}_y\right]\begin{bmatrix}\overline{\mathbf{L}}_x^{x0} & \overline{\mathbf{L}}_y^{x0}\\ \overline{\mathbf{L}}_x^{x1} & \overline{\mathbf{L}}_y^{x1}\\ \overline{\mathbf{L}}_x^{y0} & \overline{\mathbf{L}}_y^{y0}\\ \overline{\mathbf{L}}_x^{y1} & \overline{\mathbf{L}}_y^{y1}\end{bmatrix}^{-1}\begin{bmatrix}\overline{\mathbf{L}}_{xy}^{x0}\\ \overline{\mathbf{L}}_{xy}^{x1}\\ \overline{\mathbf{L}}_{xy}^{y0}\\ \overline{\mathbf{L}}_{xy}^{y1}\end{bmatrix} \tag{2.78a}$$

$$\mathbf{M}=\overline{\mathbf{M}}_{xy}-\left[\overline{\mathbf{M}}_x \quad \overline{\mathbf{M}}_y\right]\begin{bmatrix}\overline{\mathbf{L}}_x^{x0} & \overline{\mathbf{L}}_y^{x0}\\ \overline{\mathbf{L}}_x^{x1} & \overline{\mathbf{L}}_y^{x1}\\ \overline{\mathbf{L}}_x^{y0} & \overline{\mathbf{L}}_y^{y0}\\ \overline{\mathbf{L}}_x^{y1} & \overline{\mathbf{L}}_y^{y1}\end{bmatrix}^{-1}\begin{bmatrix}\overline{\mathbf{L}}_{xy}^{x0}\\ \overline{\mathbf{L}}_{xy}^{x1}\\ \overline{\mathbf{L}}_{xy}^{y0}\\ \overline{\mathbf{L}}_{xy}^{y1}\end{bmatrix} \tag{2.78b}$$

Mathematically, Eq. (2.77) represents a generalized eigenvalue problem from which all the natural frequencies and modes of the plate can be determined easily by solving the standard characteristic equation. Once the coefficient eigenvector $\boldsymbol{\Gamma}_{xy}$ is determined for a given frequency, the corresponding supplement coefficient eigenvectors $\boldsymbol{\Gamma}_x$ and $\boldsymbol{\Gamma}_y$ can be obtained. Then the displacements and rotation components of the plate can be determined by substituting these coefficients into Eqs. (2.57) and (2.58). Thus, the corresponding mode shape of the plate can be directly constructed from the determined displacement functions. Although Eq. (2.77) represents the free vibration of laminated rectangular plates, by summing the loading vector $\mathbf{F}$ on the right side of Eq. (2.77), thus, the characteristic equation for the forced vibration is readily obtained. Similarly, the present formulation can be readily applied to static analysis of laminated plates with general boundary conditions by letting $\omega = 0$ and summing the loading vector $\mathbf{F}$ on the right side of Eq. (2.77).

Although the modified Fourier series solution procedure derived herein is focused on rectangular plates, it can readily be used for other laminated structures, such as beams, cylindrical shells, conical shells, spherical shells and shallow shell, etc. The method described in this section is believed to include two main advantages: first, it is a general method which can be used to determinate the static, bending, free and forced vibration behaviors of laminated pates with arbitrary boundary conditions accurately; second, the proposed method offers an easy analysis operation for the entire restraining conditions and the change of boundary conditions from one case to another is as easy as changing structure parameters without the need of making any change to the solution procedure or modifying the basic functions as often required in other methods.

Instead of seeking a solution in strong form solution procedure as described in the previous paragraphs, all the expansion coefficients can be treated equally and independently as the generalized coordinates and solved directly from the weak form solution procedure such as Rayleigh–Ritz technique, which is the focus of the next section.

2.3 Rayleigh-Ritz Method (Weak Form Solution Procedure)

In a variety of vibration problems, exact solutions always unable to be obtained, in such cases, one has to employ approximate method. In this regard, many methods exist. Among them, the minimization of energy approaches such as the Rayleigh-Ritz method, the variational integral method and the Galerkin method are widely used in the vibration analysis of continuous systems due to the reliability of their results and efficiency in modeling and solution procedure. In this section, we focus on the Rayleigh-Ritz method. The modified Fourier series version of Rayleigh-Ritz method is presented as follows.

In the Rayleigh-Ritz method, a displacement field associated with undetermined coefficients is assumed firstly. The displacement field is then substituted into the Lagrangian energy functional (i.e., $\Pi = T - U + W$). Then the undetermined coefficients in the displacement field are determined by finding the stationary value of the energy functional, namely, minimizing the total expression of the Lagrangian energy function by taking its derivatives with respect to the undetermined coefficients and making them equal to zero. Finally, a series of equations related to corresponding coefficients can be achieved and summed up in matrix form as a standard characteristic equation. And the desired frequencies and modes of the structure can be determined easily by solving the standard characteristic equation (Qatu 2004; Reddy 2002).

The constructing of appropriate admissible displacement field is of crucial importance in the Rayleigh–Ritz procedure because the accuracy of the solution will usually depend upon how well the actual displacement can be faithfully represented by it. For vibration analysis of laminated beams, plates and shells, the admissible displacement field is often expressed in terms of beam functions under the same boundary conditions. Thus, a specially customized set of beam functions is required for each type of boundary conditions. As a result, the use of the existing solution procedures will result in very tedious calculations and be easily inundated with various boundary conditions because even only considering the classical (homogeneous) cases, one will have a total of hundreds of different combinations. Instead of the beam functions, one may also use other forms of admissible functions such as orthogonal polynomials. However, the higher order polynomials tend to become numerically unstable due to the computer round-off errors. This numerical difficulty can be avoided by expressing the displacement functions in the form of a Fourier series expansion because Fourier functions constitute a complete set and exhibit an excellent numerical stability. However, the conventional Fourier series expression will generally have a convergence problem along the boundary edges and cannot be differentiated term-by-term except for a few simple boundary conditions (see Sect. 2.1.1). These difficulties can be overcame by using the modified Fourier series. A weak form solution procedure which combining the modified Fourier series and the Rayleigh-Ritz method is given below step-by-step.

Taking the previous studied moderately thick laminated rectangular plate (Fig. 2.4) for example, letting

$$\begin{aligned} \mathbf{G}^u &= [\mathbf{A}, \quad \mathbf{a}, \quad \mathbf{b}]^T \\ \mathbf{G}^v &= [\mathbf{B}, \quad \mathbf{c}, \quad \mathbf{d}]^T \\ \mathbf{G}^w &= [\mathbf{C}, \quad \mathbf{e}, \quad \mathbf{f}]^T \\ \mathbf{G}^x &= [\mathbf{D}, \quad \mathbf{g}, \quad \mathbf{h}]^T \\ \mathbf{G}^y &= [\mathbf{E}, \quad \mathbf{i}, \quad \mathbf{j}]^T \\ \mathbf{H} &= [\mathbf{H}_{xy}, \mathbf{H}_x, \mathbf{H}_y] \end{aligned} \tag{2.79}$$

where $\mathbf{H}_{xy}$, $\mathbf{H}_x$, $\mathbf{H}_y$, $\mathbf{A}$ to $\mathbf{E}$ and $\mathbf{a}$ to $\mathbf{j}$ are presented in Eqs. (2.60) and (2.61). Therefore, the displacement expressions of the plate can be rewritten in the vector form as:

$$\begin{aligned} u(x,y) &= \mathbf{HG}^u \\ v(x,y) &= \mathbf{HG}^v \\ w(x,y) &= \mathbf{HG}^w \\ \phi_x(x,y) &= \mathbf{HG}^x \\ \phi_y(x,y) &= \mathbf{HG}^y \end{aligned} \tag{2.80}$$

For free vibration analysis, the Lagrangian energy functional (L) of the plate can be simplified and written in terms of the strain energy and kinetic energy functions as:

$$L = T - U_s - U_{sp} \tag{2.81}$$

According to Eqs. (1.50), (1.51) and (1.54). The kinetic energy and strain energy functions of the laminated plate are:

$$T = \frac{\omega^2}{2}\int_x \int_y \left\{ \begin{array}{l} I_0 u^2 + 2I_1 u\phi_x + I_2\phi_x^2 \\ +I_0 v^2 + 2I_1 v\phi_y + I_2\phi_y^2 + I_0 w^2 \end{array} \right\} dxdy \tag{2.82}$$

$$\begin{aligned} U_{sp} = {} & \frac{1}{2}\int_0^b \left\{ \begin{array}{l} [k_{x0}^u u^2 + k_{x0}^v v^2 + k_{x0}^w w^2 + K_{x0}^x \phi_x^2 + K_{x0}^y \phi_y^2]_{x=0} \\ +[k_{x1}^u u^2 + k_{x1}^v v^2 + k_{x1}^w w^2 + K_{x1}^x \phi_x^2 + K_{x1}^y \phi_y^2]_{x=a} \end{array} \right\} dy \\ & + \frac{1}{2}\int_0^a \left\{ \begin{array}{l} [k_{y0}^u u^2 + k_{y0}^v v^2 + k_{y0}^w w^2 + K_{y0}^x \phi_x^2 + K_{y0}^y \phi_y^2]_{y=0} \\ +[k_{y1}^u u^2 + k_{y1}^v v^2 + k_{y1}^w w^2 + K_{y1}^x \phi_x^2 + K_{y1}^y \phi_y^2]_{y=b} \end{array} \right\} dx \end{aligned} \tag{2.83}$$

$$
\begin{aligned}
U_s = \frac{1}{2}\int_0^a \int_0^b &\left\{\begin{array}{l} A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + A_{22}\left(\frac{\partial v}{\partial y}\right)^2 + A_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \\ +2A_{12}\left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right) + 2A_{16}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right) \\ +2A_{26}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)^2 \\ +2A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right)\left(\phi_x + \frac{\partial w}{\partial x}\right) + A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 \end{array}\right\} dydx \\
+ \frac{1}{2}\int_0^a \int_0^b &\left\{\begin{array}{l} D_{11}\left(\frac{\partial \phi_x}{\partial x}\right)^2 + D_{22}\left(\frac{\partial \phi_y}{\partial y}\right)^2 + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2 \\ +2D_{12}\left(\frac{\partial \phi_y}{\partial y}\right)\left(\frac{\partial \phi_x}{\partial x}\right) + 2D_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ \times\left(\frac{\partial \phi_x}{\partial x}\right) + 2D_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\left(\frac{\partial \phi_y}{\partial y}\right) \end{array}\right\} dydx \\
+ \int_0^a \int_0^b &\left\{\begin{array}{l} B_{11}\left(\frac{\partial \phi_x}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + B_{12}\left(\frac{\partial \phi_y}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right) \\ +B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + B_{12}\left(\frac{\partial \phi_x}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) \\ +B_{22}\left(\frac{\partial \phi_y}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) \\ +B_{16}\left(\frac{\partial \phi_x}{\partial x}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + B_{26}\left(\frac{\partial \phi_y}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ +B_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{array}\right\} dydx
\end{aligned}
\tag{2.84}
$$

Substituting the displacement expressions of the plate (Eqs. 2.57 and 2.58) into the Lagrangian energy functional (Eq. 2.81) and minimizing the total expression of the Lagrangian energy functional by taking its derivatives with respect to each of the undetermined coefficients and making them equal to zero

$$
\begin{aligned}
&\frac{\partial L}{\partial \Xi_{mn}} = 0, \quad \text{and}\left\{\begin{array}{l} \Xi = A, B, C, D, E \\ m = 0, 1, \ldots M; \quad n = 0, 1, \ldots N \end{array}\right. \\
&\frac{\partial L}{\partial \Psi_{ln}} = 0, \quad \text{and}\left\{\begin{array}{l} \Psi = a, c, e, g, i \\ l = 1, 2; \quad n = 0, 1, \ldots N \end{array}\right. \\
&\frac{\partial L}{\partial \Upsilon_{lm}} = 0, \quad \text{and}\left\{\begin{array}{l} \Upsilon = b, d, f, h, j \\ l = 1, 2; \quad m = 0, 1, \ldots M \end{array}\right.
\end{aligned}
\tag{2.85}
$$

a total of $5*(M + 1)*(N + 1) + 10*(M + N + 2)$ equations can be obtained. They are summed up in a matrix form as:

$$
\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{G} = \mathbf{0} \tag{2.86}
$$

where **K** is the stiffness matrix of the plate, and **M** is the mass matrix. Both of them are symmetric matrices and they can be expressed as

$$
\mathbf{K} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uv} & \mathbf{K}_{uw} & \mathbf{K}_{ux} & \mathbf{K}_{uy} \\ \mathbf{K}_{uv}^T & \mathbf{K}_{vv} & \mathbf{K}_{vw} & \mathbf{K}_{vx} & \mathbf{K}_{vy} \\ \mathbf{K}_{uw}^T & \mathbf{K}_{vw}^T & \mathbf{K}_{ww} & \mathbf{K}_{wx} & \mathbf{K}_{wy} \\ \mathbf{K}_{ux}^T & \mathbf{K}_{vx}^T & \mathbf{K}_{wx}^T & \mathbf{K}_{xx} & \mathbf{K}_{xy} \\ \mathbf{K}_{uy}^T & \mathbf{K}_{vy}^T & \mathbf{K}_{wy}^T & \mathbf{K}_{xy}^T & \mathbf{K}_{yy} \end{bmatrix} \tag{2.87a}
$$

$$
\mathbf{M} = \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{ux} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{vv} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{vy} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{ww} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{ux}^{T} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{vy}^{T} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{yy} \end{bmatrix} \tag{2.87b}
$$

The superscript T represents the transposition operator. The explicit forms of submatrices in the stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$ are listed as follows

$$
\begin{aligned}
\mathbf{K}_{uu} &= \int_0^a \int_0^b \left\{ A_{11} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + A_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + A_{16} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + A_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} \right\} dydx \\
&\quad + \int_0^b \left\{ k_{x0}^u \mathbf{H}^T \mathbf{H}|_{x=0} + k_{x1}^u \mathbf{H}^T \mathbf{H}|_{x=a} \right\} dy + \int_0^a \left\{ k_{y0}^u \mathbf{H}^T \mathbf{H}\big|_{y=0} + k_{y1}^u \mathbf{H}^T \mathbf{H}\big|_{y=b} \right\} dx \\
\mathbf{K}_{uv} &= \int_0^a \int_0^b \left\{ A_{12} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + A_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + A_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + A_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} \right\} dydx \\
\mathbf{K}_{uw} &= \mathbf{0} \\
\mathbf{K}_{ux} &= \int_0^a \int_0^b \left\{ B_{11} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + B_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + B_{16} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + B_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} \right\} dydx \\
\mathbf{K}_{uy} &= \int_0^a \int_0^b \left\{ B_{12} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + B_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + B_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + B_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} \right\} dydx \\
\mathbf{K}_{vv} &= \int_0^a \int_0^b \left\{ A_{22} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + A_{26} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + A_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + A_{66} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} \right\} dydx \\
&\quad + \int_0^b \left\{ k_{x0}^v \mathbf{H}^T \mathbf{H}|_{x=0} + k_{x1}^v \mathbf{H}^T \mathbf{H}|_{x=a} \right\} dy + \int_0^a \left\{ k_{y0}^v \mathbf{H}^T \mathbf{H}\big|_{y=0} + k_{y1}^v \mathbf{H}^T \mathbf{H}\big|_{y=b} \right\} dx \\
\mathbf{K}_{vw} &= \mathbf{0} \\
\mathbf{K}_{vx} &= \int_0^a \int_0^b \left\{ B_{12} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + B_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + B_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + B_{66} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} \right\} dydx \\
\mathbf{K}_{vy} &= \int_0^a \int_0^b \left\{ B_{22} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + B_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + B_{26} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + B_{66} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} \right\} dydx \\
\mathbf{K}_{ww} &= \int_0^a \int_0^b \left\{ A_{44} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + A_{45} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + A_{45} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + A_{55} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} \right\} dydx \\
&\quad + \int_0^b \left\{ k_{x0}^w \mathbf{H}^T \mathbf{H}|_{x=0} + k_{x1}^w \mathbf{H}^T \mathbf{H}|_{x=a} \right\} dy + \int_0^a \left\{ k_{y0}^w \mathbf{H}^T \mathbf{H}\big|_{y=0} + k_{y1}^w \mathbf{H}^T \mathbf{H}\big|_{y=b} \right\} dx \\
\mathbf{K}_{wx} &= \int_0^a \int_0^b \left\{ A_{45} \frac{\partial \mathbf{H}^T}{\partial y} \mathbf{H} + A_{55} \frac{\partial \mathbf{H}^T}{\partial x} \mathbf{H} \right\} dydx \\
\mathbf{K}_{wy} &= \int_0^a \int_0^b \left\{ A_{44} \frac{\partial \mathbf{H}^T}{\partial y} \mathbf{H} + A_{45} \frac{\partial \mathbf{H}^T}{\partial x} \mathbf{H} \right\} dydx \\
\mathbf{K}_{xx} &= \int_0^a \int_0^b \left\{ \begin{array}{l} D_{11} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + D_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + D_{16} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} \\ + D_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + A_{55} \mathbf{H}^T \mathbf{H} \end{array} \right\} dydx \\
&\quad + \int_0^b \left\{ K_{x0}^x \mathbf{H}^T \mathbf{H}|_{x=0} + K_{x1}^x \mathbf{H}^T \mathbf{H}|_{x=a} \right\} dy + \int_0^a \left\{ K_{y0}^x \mathbf{H}^T \mathbf{H}\big|_{y=0} + K_{y1}^x \mathbf{H}^T \mathbf{H}\big|_{y=b} \right\} dx \\
\mathbf{K}_{xy} &= \int_0^a \int_0^b \left\{ \begin{array}{l} D_{12} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + D_{16} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + D_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} \\ + D_{66} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} + A_{45} \mathbf{H}^T \mathbf{H} \end{array} \right\} dydx \\
\mathbf{K}_{vv} &= \int_0^a \int_0^b \left\{ \begin{array}{l} D_{22} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial y} + D_{26} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial y} + D_{26} \frac{\partial \mathbf{H}^T}{\partial y} \frac{\partial \mathbf{H}}{\partial x} \\ + D_{66} \frac{\partial \mathbf{H}^T}{\partial x} \frac{\partial \mathbf{H}}{\partial x} + A_{44} \mathbf{H}^T \mathbf{H} \end{array} \right\} dydx \\
&\quad + \int_0^b \left\{ K_{x0}^y \mathbf{H}^T \mathbf{H}|_{x=0} + K_{x1}^y \mathbf{H}^T \mathbf{H}|_{x=a} \right\} dy + \int_0^a \left\{ K_{y0}^y \mathbf{H}^T \mathbf{H}\big|_{y=0} + K_{y1}^y \mathbf{H}^T \mathbf{H}\big|_{y=b} \right\} dx \\
\mathbf{M}_{uu} &= \mathbf{M}_{vv} = \mathbf{M}_{ww} = \int_0^a \int_0^b I_0 \mathbf{H}^T \mathbf{H} dydx \\
\mathbf{M}_{ux} &= \mathbf{M}_{vy} = \int_0^a \int_0^b I_1 \mathbf{H}^T \mathbf{H} dydx \\
\mathbf{M}_{xx} &= \mathbf{M}_{xy} = \int_0^a \int_0^b I_2 \mathbf{H}^T \mathbf{H} dydx
\end{aligned} \tag{2.88}
$$

G is a column vector which contains, in an appropriate order, the unknown expansion coefficients that appear in the series expansions, namely:

$$\mathbf{G} = [\mathbf{G}^u, \mathbf{G}^v, \mathbf{G}^w, \mathbf{G}^x, \mathbf{G}^y]^T \tag{2.89}$$

where $\mathbf{G}^u$, $\mathbf{G}^v$, $\mathbf{G}^w$, $\mathbf{G}^x$ and $\mathbf{G}^y$ are given in Eq. (2.79). Obviously, the natural frequencies and eigenvectors can be easily obtained by solving a standard matrix eigenproblem. Once the coefficient eigenvector **G** is determined for a given frequency, the displacements of the plate can be determined by substituting the coefficients into Eqs. (2.57) and (2.58).

The modified Fourier series solution procedure derived herein is focused on rectangular plates, it can readily be used for other laminated structures, such as beams, cylindrical shells, conical shells, spherical shells and shallow shell, see Chaps. 3–8.

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