Chapter 2
Linear Partial Differential Equations

Several important heat and fluid flow processes in technical applications and in nature can approximately be described by linear partial differential equations. As stated in the previous chapter, linear partial differential equations are normally much simpler to solve than nonlinear partial differential equations. In addition, a large body of literature exists on how to solve linear PDEs.

The following four chapters focus on the solution of linear partial differential equations. This chapter is concerned with the classification of second order partial differential equations and presents a short introduction into the method of separation of variables. Chapter 3 focuses on convective heat transfer in laminar and turbulent pipe and channel flows. Here parabolic problems are considered and the general eigenvalue problems, associated with these equations, are explained. Chapter 4 discusses some specific methods for the analytical solution of eigenvalue problems, in the case of large eigenvalues. Finally, Chap. 5 deals with convective heat transfer problems in laminar or turbulent pipe and channel flows for low Peclet numbers (liquid metals). For this type of applications, the axial heat conduction within the flow can no longer be ignored and the resulting energy equation remains elliptic in nature. This has strong implications on the solution method for the energy equation.

2.1 Classification of Second-Order Partial Differential Equations

In the following, we are concerned with a linear second order partial differential equation which depends on the two independent variables x and y. The most general form of the homogeneous equation is given by

$$\begin{aligned} & A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} \\ & + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = 0 \end{aligned} \tag{2.1}$$

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where $A,\ldots, F$ are constants or functions of x and y and are sufficiently differentiable in the domain of interest.

The form of Eq. (2.1) resembles the quadratic equation of a conic sections in analytical geometry. The equation

$$a\,x^2 + 2b\,xy + c\,y^2 + d\,x + e\,y + f = 0 \tag{2.2}$$

represents an ellipse, parabola or hyperbola depending on whether $b^2 - ac <, =, > 0$, respectively.

The classification of the second-order partial differential equation is based on the fact that Eq. (2.1) can be transformed into a standard form. This is very similar to the treatment of the quadratic Eq. (2.2) of conic sections in analytical geometry. We distinguish the following different cases (for the point x_0, y_0 under consideration):

$$1.\ B^2(x_0, y_0) - A(x_0, y_0)\ C(x_0, y_0) > 0 \tag{2.3}$$

is of hyperbolic type. There exist two real characteristic curves.

$$2.\ B^2(x_0, y_0) - A(x_0, y_0)\ C(x_0, y_0) = 0 \tag{2.4}$$

is of parabolic type. There exists one real characteristic curve.

$$3.\ B^2(x_0, y_0) - A(x_0, y_0)\ C(x_0, y_0) < 0 \tag{2.5}$$

is of elliptic type. The two characteristic curves are conjugate complex.

Each of these equations can be transformed into its standard form, if we introduce the following new coordinates into Eq. (2.1)

$$\begin{aligned} \xi &= \xi(x, y) \\ \eta &= \eta(x, y) \end{aligned} \tag{2.6}$$

Is the equation of hyperbolic type, the standard form is given by

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = f_1\left(\xi, \eta,\ u,\ \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) \tag{2.7}$$

If we introduce the new coordinates

$$\begin{aligned} \bar{\xi} &= \xi + \eta \\ \bar{\eta} &= \xi - \eta \end{aligned} \tag{2.8}$$

we obtain an alternative standard form of the hyperbolic equation

$$\frac{\partial^2 u}{\partial\bar{\xi}^2} - \frac{\partial^2 u}{\partial\bar{\eta}^2} = f_2\left(\bar{\xi}, \bar{\eta},\ u,\ \frac{\partial u}{\partial\bar{\xi}}, \frac{\partial u}{\partial\bar{\eta}}\right) \tag{2.9}$$

If the equation is of parabolic type, then the standard form is given by

$$\frac{\partial^2 u}{\partial \xi^2} = f_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) \quad \text{or} \quad \frac{\partial^2 u}{\partial \eta^2} = \bar{f}_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) \tag{2.10}$$

Finally, if the equation is of elliptic type, the standard form of the equation is given by

$$\frac{\partial^2 u}{\partial \tilde{\xi}^2} + \frac{\partial^2 u}{\partial \tilde{\eta}^2} = f_4\left(\tilde{\xi}, \tilde{\eta}, u, \frac{\partial u}{\partial \tilde{\xi}}, \frac{\partial u}{\partial \tilde{\eta}}\right) \tag{2.11}$$

where the new coordinates $\tilde{\xi}$, $\tilde{\eta}$ are defined by

$$\tilde{\xi} = \frac{1}{2}(\xi + \eta), \quad \tilde{\eta} = \frac{1}{2i}(\xi - \eta) \tag{2.12}$$

In order to obtain one of the above standard or canonical forms of the equation, we need to perform the coordinate transformation given by Eq. (2.6). Since we want the transformed equation to be equivalent to the original equation, we assume that ξ and η are twice continuously differentiable and we insist that the Jacobian $J \neq 0$

$$J = \left| \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \right| = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \neq 0 \tag{2.13}$$

in the region under consideration. By assuming that Eq. (2.13) holds, we have always a unique transformation between x, y and ξ, η.

Use of the chain rule gives:

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_y &= \left(\frac{\partial u}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial x}\right)_y + \left(\frac{\partial u}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial x}\right)_y \\ \left(\frac{\partial u}{\partial y}\right)_x &= \left(\frac{\partial u}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial y}\right)_x + \left(\frac{\partial u}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial y}\right)_x \end{aligned} \tag{2.14}$$

and also

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) \\ &\quad + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} \end{aligned} \tag{2.15}$$

After inserting the above expressions into Eq. (2.1), one obtains

$$\bar{A}(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} + 2\bar{B}(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C}(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} + \bar{D}(\xi,\eta)\frac{\partial u}{\partial \xi} + \bar{E}(\xi,\eta)\frac{\partial u}{\partial \eta} + Fu = 0 \tag{2.16}$$

with the coefficients

$$\begin{aligned}
\bar{A} &= A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 \\
\bar{B} &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
\bar{C} &= A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \\
\bar{D} &= A\frac{\partial^2 \xi}{\partial x^2} + 2B\frac{\partial^2 \xi}{\partial x \partial y} + C\frac{\partial^2 \xi}{\partial y^2} + D\frac{\partial \xi}{\partial x} + E\frac{\partial \xi}{\partial y} \\
\bar{E} &= A\frac{\partial^2 \eta}{\partial x^2} + 2B\frac{\partial^2 \eta}{\partial x \partial y} + C\frac{\partial^2 \eta}{\partial y^2} + D\frac{\partial \eta}{\partial x} + E\frac{\partial \eta}{\partial y}
\end{aligned} \tag{2.17}$$

Now we need to specify our change of variables, expressed by Eq. (2.6), in order to obtain one of the previously given standard or canonical forms. For example, if we want to obtain the hyperbolic equation in the form of Eq. (2.7) we have to assume that $\bar{A}$ and $\bar{C}$ are equal to zero. Because these two expressions are identical if we exchange ξ and η, we can achieve the condition $\bar{A} = \bar{C} = 0$, only if ξ and η are both solutions of the following equation

$$A\left(\frac{\partial \Omega}{\partial x}\right)^2 + 2B\left(\frac{\partial \Omega}{\partial x}\right)\left(\frac{\partial \Omega}{\partial y}\right) + C\left(\frac{\partial \Omega}{\partial y}\right)^2 = 0; \quad \Omega = \xi \text{ or } \eta \tag{2.18}$$

Along a curve Ω = const. one has

$$d\Omega = \frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{\partial \Omega}{\partial x}\Big/\frac{\partial \Omega}{\partial y} \tag{2.19}$$

we obtain from Eqs. (2.18) and (2.19) the following ordinary differential equation

$$A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = 0 \tag{2.20}$$

This differential equation can be solved for *dy*/*dx* and one obtains the following two cases:

$$\frac{dy}{dx} = \left(B + \sqrt{B^2 - AC}\right)\Big/A \qquad (2.21)$$
$$\frac{dy}{dx} = \left(B - \sqrt{B^2 - AC}\right)\Big/A$$

These two equations are known as the characteristic equations. They prescribe the functional relationship between the families of curves in the *xy*-plane for which ξ = const. and η = const. This means that a change of variables according to

$$\xi = f_1(x, y) \qquad (2.22)$$
$$\eta = f_2(x, y)$$

will transform Eq. (2.1) into its standard form. From Eq. (2.21) it is apparent that there are three cases to be considered:

Case 1 Hyperbolic equation $(B^2 - AC) > 0$
The preceding analysis results in a canonical form for the hyperbolic equation. For this case we have two real characteristics, which can be obtained from the ordinary differential Eq. (2.21).

Case 2 Elliptic equation $(B^2 - AC) < 0$
For this case, one obtains from Eq. (2.21) no real, but two conjugate complex solutions. Therefore, the elliptic equation has two conjugate complex characteristics. The elliptic case needs not to be recalculated again, because it can be deduced from the calculation of the hyperbolic case. This can be shown as follows: let us consider the canonical form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = f_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) \qquad (2.7)$$

Now, we have two conjugate complex characteristics. Therefore, we can introduce into the above equation the coordinate transform given by Eq. (2.12)

$$\tilde{\xi} = \frac{1}{2}(\xi + \eta) \qquad (2.12)$$
$$\tilde{\eta} = \frac{1}{2i}(\xi - \eta)$$

Use of the chain rule according to Eq. (2.15) gives

$$\begin{aligned}\frac{\partial^2 u}{\partial\xi\partial\eta} &= \frac{\partial^2 u}{\partial\tilde{\xi}^2}\frac{\partial\tilde{\xi}}{\partial\xi}\frac{\partial\tilde{\xi}}{\partial\eta} + \frac{\partial^2 u}{\partial\tilde{\xi}\partial\tilde{\eta}}\left(\frac{\partial\tilde{\xi}}{\partial\xi}\frac{\partial\tilde{\eta}}{\partial\eta} + \frac{\partial\tilde{\xi}}{\partial\eta}\frac{\partial\tilde{\eta}}{\partial\xi}\right)\\ &\quad + \frac{\partial^2 u}{\partial\tilde{\eta}^2}\frac{\partial\tilde{\eta}}{\partial\xi}\frac{\partial\tilde{\eta}}{\partial\eta} + \frac{\partial u}{\partial\tilde{\xi}}\frac{\partial^2\tilde{\xi}}{\partial\xi\partial\eta} + \frac{\partial u}{\partial\tilde{\eta}}\frac{\partial^2\tilde{\eta}}{\partial\xi\partial\eta}\\ &= \frac{\partial^2 u}{\partial\tilde{\xi}^2}\frac{1}{4} + \frac{\partial^2 u}{\partial\tilde{\xi}\partial\tilde{\eta}}\left(-\frac{1}{4i} + \frac{1}{4i}\right) + \frac{\partial^2 u}{\partial\tilde{\eta}^2}\frac{1}{4} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial\tilde{\xi}^2} + \frac{\partial^2 u}{\partial\tilde{\eta}^2}\right)\end{aligned}$$

and Eq. (2.7) is transformed into the standard form for the elliptic type, given by Eq. (2.11)

$$\frac{\partial^2 u}{\partial\tilde{\xi}^2} + \frac{\partial^2 u}{\partial\tilde{\eta}^2} = f_4\left(\tilde{\xi}, \tilde{\eta}, u, \frac{\partial u}{\partial\tilde{\xi}}, \frac{\partial u}{\partial\tilde{\eta}}\right) \tag{2.11}$$

Case 3 Parabolic equation $(B^2 - AC) = 0$
For this case, it can be seen from Eq. (2.21) that only one family of real characteristics exists. Because of this fact, we can set in Eq. (2.17) for example $\eta = x$. Note that this is only possible if ξ depends on y, so that the Jacobian, defined by Eq. (2.13), is not zero. Then we obtain immediately from Eq. (2.17) that $\bar{C} = A$ while $\bar{B}$ is equal to

$$\bar{B} = A\,\frac{\partial\xi}{\partial x} + B\,\frac{\partial\xi}{\partial y} \tag{2.23}$$

This expression is identical to zero, as it can be deduced from Eq. (2.18) rewritten as follows—for the case $(B^2 - AC) = 0$:

$$A\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\xi}{\partial y}\right) + \frac{B^2}{A}\left(\frac{\partial\xi}{\partial y}\right)^2 = \frac{1}{A}\left(A\frac{\partial\xi}{\partial x} + B\frac{\partial\xi}{\partial y}\right)^2 = 0 \tag{2.24}$$

Finally, the standard or canonical form of the parabolic equation, given by Eq. (2.10), is obtained if we divide Eq. (2.16) by A.

In order to illustrate the above shown classification, we will investigate some simple examples:

Example 1 The equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.25}$$

is parabolic, because the expression $B^2 - AC = 1 - 1 = 0$. The characteristic Eq. (2.21) reduces to

$$\frac{dy}{dx} = 1 \tag{2.26}$$

which has the solution

$$y - x = C_1 \tag{2.27}$$

If we change the coordinates according to

$$\begin{aligned} \xi &= y - x \\ \eta &= x \end{aligned} \tag{2.28}$$

where η has been selected arbitrarily, although always respecting the condition that the Jacobian, defined in Eq. (2.13), is not equal to zero. Introducing the new coordinates into Eq. (2.25), we obtain

$$\frac{\partial^2 u}{\partial \eta^2} = 0 \tag{2.29}$$

This equation has the general solution

$$u = \eta F(\xi) + H(\xi) = xF(y - x) + H(y - x) \tag{2.30}$$

Example 2 Consider the equation

$$y^4 \frac{\partial^2 u}{\partial x^2} - x^4 \frac{\partial^2 u}{\partial y^2} - 2yx^2 \frac{\partial u}{\partial x} = 0 \tag{2.31}$$

From this equation, we obtain $B^2 - AC = x^4 y^4 > 0$. This shows that Eq. (2.31) is hyperbolic everywhere except along the coordinate axis ($x = 0$ or $y = 0$).

From Eq. (2.21) we obtain the following two ordinary differential equations for the characteristics

$$\frac{dy}{dx} = \left(B + \sqrt{B^2 - AC}\right) \Big/ A = \frac{x^2 y^2}{y^4} = \left(\frac{x}{y}\right)^2 \tag{2.32}$$

and

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^2 \tag{2.33}$$

From the two Eqs. (2.32) and (2.33) we calculate the characteristic curves to be:

$$\frac{1}{3}\left(y^3 - x^3\right) = C_1; \quad \frac{1}{3}\left(y^3 + x^3\right) = C_2 \tag{2.34}$$

In order to transform Eq. (2.31) into its standard form, we have to perform the following coordinate transformation

$$\begin{aligned} \xi &= \frac{1}{3}\left(y^3 - x^3\right) \\ \eta &= \frac{1}{3}\left(y^3 + x^3\right) \end{aligned} \tag{2.35}$$

Using Eqs. (2.14) and (2.15), one obtains

$$\begin{aligned} -4x^4y^4\frac{\partial^2 u}{\partial\xi\partial\eta} - 2\left(xy^4 + yx^4\right)\frac{\partial u}{\partial\xi} + 2\left(xy^4 - yx^4\right)\frac{\partial u}{\partial\eta} \\ - 2yx^2\left(-x^2\frac{\partial u}{\partial\xi} + x^2\frac{\partial u}{\partial\eta}\right) = 0 \end{aligned} \tag{2.36}$$

and after simplifying and replacing x *and* y by ξ and η we finally get

$$\frac{\partial^2 u}{\partial\xi\partial\eta} - \frac{1}{3}\frac{1}{\xi - \eta}\frac{\partial u}{\partial\xi} - \frac{1}{3}\frac{3\xi - \eta}{\eta^2 - \xi^2}\frac{\partial u}{\partial\eta} = 0 \tag{2.37}$$

Example 3 We consider the wave equation

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\frac{\partial^2\Phi}{\partial x^2} = 0 \tag{2.38}$$

This equation describes for example the one-dimensional propagation of sound in a pipe. Because $B^2 - AC > 0$, the equation is hyperbolic in the region of interest.

The characteristic equations are given by

$$\frac{dt}{dx} = \left(B + \sqrt{B^2 - AC}\right)\Big/A = \frac{c}{-c^2} = -\frac{1}{c} \tag{2.39}$$

and

$$\frac{dt}{dx} = \frac{1}{c} \tag{2.40}$$

From these two equations we obtain $ct + x = C_1$ and $-ct + x = C_2$ and we get the following new coordinates:

$$\xi = x + ct \qquad (2.41)$$
$$\eta = x - ct$$

If we introduce this new coordinates into Eq. (2.38), we get the following simple partial differential equation

$$\frac{\partial^2 \Phi}{\partial \xi \, \partial \eta} = 0 \qquad (2.42)$$

which has the general solution

$$\Phi(\xi, \eta) = \Psi_1(\eta) + \Psi_2(\xi) \qquad (2.43)$$

or rewritten in x, t coordinates

$$\Phi(x, t) = \Psi_1(x - ct) + \Psi_2(x + ct) \qquad (2.44)$$

This shows that the solution of Eq. (2.38) can be expressed as the superposition of two waves, which travel with constant velocity c into different directions of the solution domain. This shows also nicely how the information in the problem is transported by the two real characteristics. The solution obtained here is known in literature as the d'Alembert solution.

2.2 Character of the Solutions for the Partial Differential Equations

In the preceding section we have concentrated on the classification of the different second-order partial differential equations. However, for solving actual physical problems, it is of great importance to discuss also the character of their solutions as well as the associated boundary conditions.

2.2.1 Parabolic Second-Order Equations

Let us start our discussion with the parabolic partial differential equation. As stated before, this equation has one real characteristic. As an example, we look at the heat conduction equation for a one-dimensional unsteady conduction problem in a slab (see Fig. 2.1). The slab has the thickness l and the spatial coordinate ranges from $0 < x < l$.

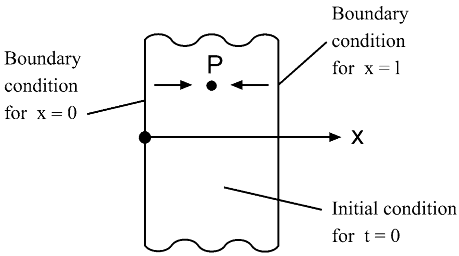

Fig. 2.1 Transient heat conduction in a slab

Assuming that the material properties of the slab are constant, the temperature distribution in the slab can be obtained from the solution of the following equation

$$\rho c \, \frac{\partial T}{\partial t} = k \, \frac{\partial^2 T}{\partial x^2} \tag{2.45}$$

The temperature distribution of the slab at the beginning of the process ($t = 0$) is given. This is the initial condition of the problem

$$T(0, x) = f_1(x) \tag{2.46}$$

In addition to this initial condition, boundary conditions have to be prescribed at the surface of the slab for $x = 0$ and $x = l$. Here the following different types of boundary conditions are possible:

- Boundary conditions of the first kind (Dirichlet wall boundary conditions). Here the temperature at the boundary is specified, for example

$$T(t, 0) = f_2(t), \quad T(t, l) = f_3(t) \tag{2.47}$$

- Boundary conditions of the second kind (Neumann conditions). For this type of boundary conditions the gradient is specified at the boundaries, for example

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = f_4(t), \quad \left(\frac{\partial T}{\partial x}\right)_{x=l} = f_5(t) \tag{2.48}$$

- Boundary conditions of the third kind. Here a combination of temperature and temperature gradient is prescribed at the surface. Such boundary conditions are relatively common in technical systems, since they describe, for example, the case of a slab which is heated or cooled by a fluid at temperature T_1 or T_2 flowing over the boundaries of the slab. A typical example is

$$\begin{aligned} k\left(\frac{\partial T}{\partial x}\right)_{x=0} &= h_1(T(t, 0) - T_1), \\ k\left(\frac{\partial T}{\partial x}\right)_{x=l} &= h_2(T(t, l) - T_2) \end{aligned} \tag{2.49}$$

where h_1 and h_2 are heat transfer coefficients.

Of course, all the above mentioned boundary conditions can be present in any possible combination, for example: at $x = 0$, a constant wall temperature is prescribed; whereas, at $x = l$, a boundary condition of the third kind is applied.

Summarising the above discussion, it can be seen that for the parabolic second-order partial differential equation an initial condition together with boundary conditions at the surface need to be specified. This means that the temperature at an arbitrary point P in the domain (see Fig. 2.1) is always influenced by the wall boundary conditions at $x = 0$ and $x = l$. In addition, all disturbances, which are specified for $t = 0$, will propagate into the solution domain for all subsequent times. On the other hand disturbances, which are introduced at a later time t_1, can not influence the solution for $t < t_1$. This shows nicely the character of the solution, which depends only on one real characteristic.

2.2.2 *Elliptic Second-Order Equations*

For the elliptic equation, the characteristic curves are families of conjugate complex functions. In order to investigate the character of the solutions and the boundary conditions needed for this type of equations, we select as an example the steady-state heat conduction in a square plate $(0 \le x \le a,\ 0 \le y \le a)$, as shown in Fig. 2.2.

The steady-state temperature distribution is obtained from the solution of the following equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.50}$$

For this type of equation, boundary conditions have to be prescribed along each point of the boundary. We might illustrate this for the here given example. Here the following different types of boundary conditions can be assigned:

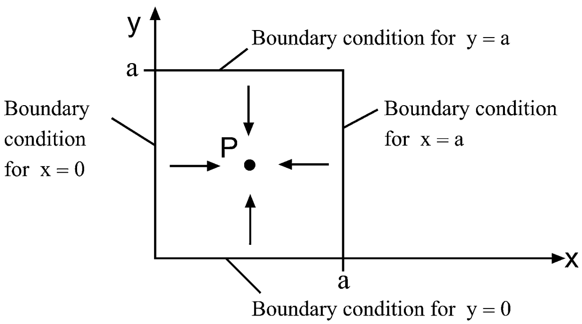

Fig. 2.2 Steady-state heat conduction in a square plate

- Boundary conditions of the first kind (Dirichlet conditions). The temperature is specified at each point of the boundary

$$\begin{aligned} T(x,0) &= T_1(x), \quad T(x,a) = T_2(x) \\ T(0,y) &= T_3(y), \quad T(a,y) = T_4(y) \end{aligned} \tag{2.51}$$

- Boundary conditions of the second kind (Neumann conditions). The heat flux normal to the wall is specified along the boundary.

$$\begin{aligned} \left(\frac{\partial T}{\partial x}\right)_{x=0} &= f_1(y), \quad \left(\frac{\partial T}{\partial x}\right)_{x=a} = f_2(y) \\ \left(\frac{\partial T}{\partial y}\right)_{y=0} &= f_3(x), \quad \left(\frac{\partial T}{\partial y}\right)_{y=a} = f_4(x) \end{aligned} \tag{2.52}$$

- Boundary conditions of the third kind. This might be again a combination of the normal gradient at the surface and the temperature. One example is:

$$\begin{aligned} k\left(\frac{\partial T}{\partial x}\right)_{x=0} &= f_1(T(0,y) - T_1), \quad k\left(\frac{\partial T}{\partial x}\right)_{x=a} = f_2(T(a,y) - T_2) \\ k\left(\frac{\partial T}{\partial y}\right)_{y=0} &= f_3(T(x,0) - T_3), \quad k\left(\frac{\partial T}{\partial y}\right)_{y=a} = f_4(T(x,a) - T_4) \end{aligned} \tag{2.53}$$

Again, the boundary conditions can be applied as mixed boundary conditions.

From the above examples, it can be seen that for an elliptic equation we have to deal with a boundary-value problem, whereas for the parabolic equation, we had to solve a combined initial-boundary value problem. This means that for the elliptic problem any disturbance, which is brought into the region of interest (for example by slightly changing one boundary condition), will immediately influence the solution of the problem at a given point in the domain (see point P in Fig. 2.2).

2.2.3 Hyperbolic Second-Order Equations

Hyperbolic partial differential equations mainly appear in vibration and wave problems. These equations have two real characteristics. The one-dimensional wave equation for a perfectly flexible string serves here as an example to explain the character of the solution and the associated boundary conditions (see Fig. 2.3).

The differential equation is given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{b^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{2.54}$$

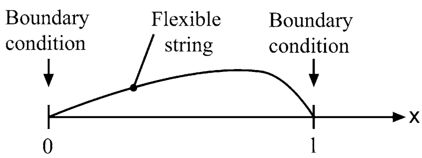

Fig. 2.3 Flexible string

where b is a constant. As shown in the previous section (see Eq. (2.44)), the general solution of this equation is given by

$$u(x,t) = \Psi_1(x+bt) + \Psi_2(x-bt) \tag{2.55}$$

Therefore, it would be easiest to prescribe boundary conditions for Eq. (2.54) along two parts of the characteristics intersecting at one point. This would be a complete initial value problem. However, for most physical problems, this description of the boundary conditions is not typical. Instead, the following boundary conditions might be normally applied:

- Boundary conditions of the first kind. Here the deflection of the string at the location $x = 0$ and $x = l$ might be prescribed.

$$u(0,t) = f_1(t), \quad u(l,t) = f_2(t) \tag{2.56}$$

  If the string is fixed at the locations $x = 0$ and $x = l$, f_1 and f_2 will be zero.
- Boundary conditions of the second kind. Here we will prescribe $\partial u/\partial x$ for $x = 0$ and $x = l$.
- For the boundary conditions of the third kind, we will specify a combination of u and $\partial u/\partial x$ for $x = 0$ and $x = l$.

As for all other types of equations, the boundary conditions can also be applied in a mixed form.

In addition to the two boundary conditions at $x = 0$ and $x = l$, initial conditions for $t = 0$ have to be specified for the problem. These initial conditions for the finite string could be that $u(x,0) = f_3(x), \partial u/\partial t(x,0) = f_4(x)$.

2.3 Separation of Variables

This section gives an introduction into the method of separation of variables. This method is one of the most commonly used methods for solving linear partial differential equations. The method is explained in the following sections by two basic examples. In the next chapters, more advanced problems are considered.

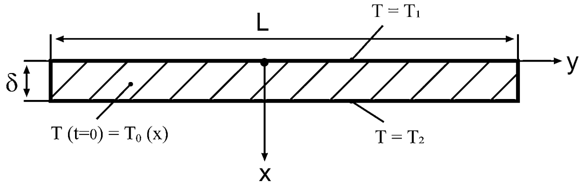

Fig. 2.4 Transient heat conduction in a slab

2.3.1 One-Dimensional Transient Heat Conduction in a Slab

We consider the energy equation for heat conduction in a slab. The problem is depicted in Fig. 2.4. The slab has the thickness δ and the length L. At the two surfaces $x = 0$ and $x = \delta$ the slab is subjected to constant temperatures.

Under the assumption that the material properties of the plate are constant, the energy equation takes the following form

$$\rho c \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{2.57}$$

We now assume that the dimension L is much larger than δ, so that the heat conduction in the y-direction is negligible compared to the heat conduction in the x-direction. Therefore, the problem simplifies to

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} \tag{2.58}$$

where $a = k/(\rho c)$ is the thermal diffusivity of the material. Equation (2.57) has to be solved together with the following boundary conditions

$$\begin{aligned} x &= 0 : T = T_1 \\ x &= \delta : T = T_2 \end{aligned} \tag{2.59}$$

and the initial condition

$$t = 0 : T = T_0(x) \tag{2.60}$$

This problem is described by a parabolic equation. This means that one real characteristic exists for the solution. Before solving the above given problem, we first introduce the dimensionless quantities

$$\tilde{x} = \frac{x}{\delta}, \quad \tilde{t} = \frac{at}{\delta^2}, \quad \Theta = \frac{T - T_1}{T_2 - T_1} \tag{2.61}$$

This results in the following problem

$$\frac{\partial \Theta}{\partial \tilde{t}} = \frac{\partial^2 \Theta}{\partial \tilde{x}^2} \tag{2.62}$$

with the boundary conditions

$$\begin{aligned} &\tilde{x} = 0 : \Theta = 0 \\ &\tilde{x} = 1 : \Theta = 1 \\ &\tilde{t} = 0 : \Theta = (T_0(\tilde{x}) - T_1)/(T_2 - T_1) = \Theta_0(\tilde{x}) \end{aligned} \tag{2.63}$$

Before applying the method of separation of variables to the Eqs. (2.62) and (2.63), we want to investigate the solution domain, shown in Fig. 2.5. Here the parabolic nature of the problem is clearly visible. The initial condition for $\tilde{t} = 0$ is propagated into the solution domain for larger times. Any disturbance introduced into the problem at $\tilde{t} = \tilde{t}_1$ will therefore only influence the solution at subsequent times. The solution for $\tilde{t} < \tilde{t}_1$ remains unchanged.

Let us assume that the solution of the problem can be expressed in the form

$$\Theta = H(\tilde{t})G(\tilde{x}) \tag{2.64}$$

Introducing Eq. (2.64) into the boundary conditions, results in:

$$\begin{aligned} &\tilde{x} = 0 : H(\tilde{t})G(0) = 0 \\ &\tilde{x} = 1 : H(\tilde{t})G(1) = 1 \end{aligned} \tag{2.65}$$

From this equation we notice immediately that using the expression (2.64) can not result in a solution to the present problem, because the boundary condition for $\tilde{x} = 1$ can not be satisfied if $H(\tilde{t})$ is an arbitrary function of $\tilde{t}$. Therefore, we conclude that we first have to make the boundary conditions homogenous, in order to find a solution with the help of Eq. (2.64). This can be done by splitting the solution into two parts

$$\Theta = \Theta_S(\tilde{x}) + \Theta_T(\tilde{x}, \tilde{t}) \tag{2.66}$$

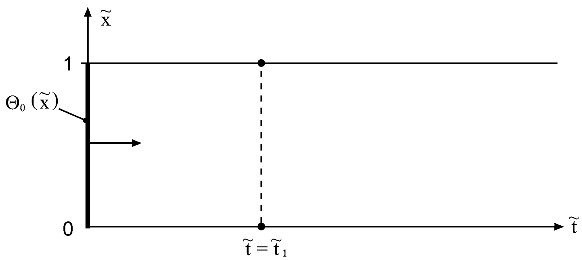

Fig. 2.5 Solution domain for the transient conduction in the slab

The steady-state solution Θ_S is simply a linear distribution and is given by

$$\Theta_S(\tilde{x}) = \tilde{x} \tag{2.67}$$

Introducing Eq. (2.66) into the Eqs. (2.62) and (2.63) results in the following problem for Θ_T

$$\frac{\partial \Theta_T}{\partial \tilde{t}} = \frac{\partial^2 \Theta_T}{\partial \tilde{x}^2} \tag{2.68}$$

with the boundary conditions

$$\begin{aligned} \tilde{x} = 0 &: \Theta_T = 0 \\ \tilde{x} = 1 &: \Theta_T = 0 \\ \tilde{t} = 0 &: \Theta_T = \Theta_0(\tilde{x}) - \Theta_S(\tilde{t} = 0) = \Theta_0(\tilde{x}) - \tilde{x} \end{aligned} \tag{2.69}$$

Introducing now again the product of functions, given by Eq. (2.64), we obtain the boundary conditions

$$\begin{aligned} \tilde{x} = 0 &: H(\tilde{t})G(0) = 0 \quad \Rightarrow G(0) = 0 \\ \tilde{x} = 1 &: H(\tilde{t})G(1) = 0 \quad \Rightarrow G(1) = 0 \end{aligned} \tag{2.70}$$

This shows that the expression given by Eq. (2.64) is able to satisfy the two boundary conditions of the problem. Therefore, this approach promises to be successful. Introducing Eq. (2.64) into the partial differential Eq. (2.68), one obtains

$$H'(\tilde{t})G(\tilde{x}) = G''(\tilde{x})H(\tilde{t}) \tag{2.71}$$

By separating the variables, this equation can be rewritten as

$$\frac{H'(\tilde{t})}{H(\tilde{t})} = \frac{G''(\tilde{x})}{G(\tilde{x})} \tag{2.72}$$

The left hand side of this equation is only a function of $\tilde{t}$, whereas the right hand side is only a function of $\tilde{x}$. Therefore, both sides of the equation must be constant. This constant is set to C_1.

$$\frac{H'(\tilde{t})}{H(\tilde{t})} = \frac{G''(\tilde{x})}{G(\tilde{x})} = C_1 \tag{2.73}$$

Let us first investigate the differential equation for the function H. This equation takes the form

$$\frac{H'(\tilde{t})}{H(\tilde{t})} = C_1 \tag{2.74}$$

and can easily been integrated to give

$$H(\tilde{t}) = C_2 \exp(C_1 \tilde{t}) \tag{2.75}$$

If we have a closer look at this equation, it can be seen that the function $H(\tilde{t})$ tends to infinity for $\tilde{t} \to \infty$ if $C_1 > 0$. However, this would not lead to a physically meaningful solution for the problem, because the temperature would tend to infinity for large times. For $C_1 = 0$, Eq. (2.75) results in a constant for $H(\tilde{t})$ and the time dependence of the solution would be lost. Therefore, we can conclude that the constant C_1 must always be smaller than zero for the problem under consideration. This can be expressed by replacing the constant by $C_1 = -\lambda^2$. Then we obtain for the function H

$$H(\tilde{t}) = C_2 \exp\left(-\lambda^2 \tilde{t}\right) \tag{2.76}$$

For the function $G(\tilde{x})$, one obtains the following ordinary differential equation from Eq. (2.73)

$$\frac{G''(\tilde{x})}{G(\tilde{x})} = -\lambda^2 \quad \Rightarrow G'' + \lambda^2 G = 0 \tag{2.77}$$

This equation has to be solved together with the homogeneous boundary conditions given by Eq. (2.70). It has the trivial solution $G = 0$ and will have further solutions for selected values of λ. These selected values of λ are called the eigenvalues of Eq. (2.77). The problem given by Eq. (2.77) and associated boundary conditions, see Eq. (2.70), is called an eigenvalue problem. This sort of problem is discussed in more detail in Chap. 3. The general solution of Eq. (2.77) is given by

$$G(\tilde{x}) = C_3 \sin(\lambda \tilde{x}) + C_4 \cos(\lambda \tilde{x}) \tag{2.78}$$

where the solution must satisfy the two boundary conditions

$$G(0) = 0, \quad G(1) = 0 \tag{2.79}$$

From the boundary condition $G(0) = 0$, it follows that C_4 is zero and one obtains

$$G = C_3 \sin(\lambda \tilde{x}) \tag{2.80}$$

Now, if we apply the second boundary condition $G(1) = 0$, we find

$$0 = C_3 \sin(\lambda \cdot 1) \tag{2.81}$$

Equation (2.81) shows that either C_3 has to be zero (which would be the trivial solution of the problem, where $G = 0$) or that $\sin(\lambda)$ has to be zero. The latter is only possible if

$$\lambda = n\pi \quad \text{with} \quad n = 1, 2, 3, \ldots \tag{2.82}$$

These special values of λ are the eigenvalues of Eq. (2.77) and are shown in Fig. 2.6.

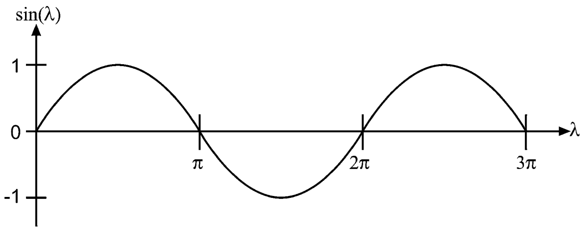

Fig. 2.6 Eigenvalues of the eigenfunction $\sin(\lambda)$

The solution for Θ_T is obtained from the Eqs. (2.75) and (2.80) as

$$\Theta_T = C_2 \exp\left(-\lambda^2 \tilde{t}\right)\, C_3 \sin(\lambda \tilde{x}) \tag{2.83}$$

For simplicity, we combine the constants C_2 and C_3 and have

$$\Theta_T = C\ \sin(\lambda \tilde{x}) \exp\left(-\lambda^2 \tilde{t}\right) \tag{2.84}$$

Now we try to fulfill the initial condition using Eq. (2.84). Inserting Eq. (2.84) into Eq. (2.69) results in

$$\begin{aligned} \tilde{t} = 0 : \Theta_T = \Theta_0(\tilde{x}) - \tilde{x} = C \sin(\lambda \tilde{x}) \exp\left(-\lambda^2 0\right) \\ \Rightarrow \Theta_0(\tilde{x}) - \tilde{x} = C \sin(\lambda \tilde{x}) \end{aligned} \tag{2.85}$$

From the equation above, one can see that Eq. (2.84) is automatically a solution of the problem if

$$\Theta_0(\tilde{x}) = \tilde{x} + C \sin(n\pi x), \quad n = 1, 2, 3, \ldots \tag{2.86}$$

However, from Eq. (2.82) it is clear that there is an infinite number of eigenvalues. Because the partial differential equation is linear, we use the principle of superposition to construct the final solution of the problem. This means that the solution will be given by

$$\Theta_T = \sum_{n=1}^{\infty} C_n \sin(\lambda_n \tilde{x}) \exp\left(-\lambda_n^2 \tilde{t}\right) \tag{2.87}$$

This solution has to fulfil the initial condition. Inserting Eq. (2.87) into Eq. (2.69) results in

$$\Theta_0(\tilde{x}) - \tilde{x} = \sum_{n=1}^{\infty} C_n \sin(\lambda_n \tilde{x}) \tag{2.88}$$

which means that we have to represent the function $\Theta_0(\tilde{x}) - \tilde{x}$ by a Fourier series (see Stephenson 1986, Myint-U and Debnath 1987, Zauderer 1989, Sommerfeld 1978).

In order to obtain the unknown coefficients C_n, from Eq. (2.88), we multiply both sides of the equation by $\sin(\lambda_m \tilde{x})$ and integrate the resulting expressions across the region of interest for $\tilde{x}$ between zero and one. This results in

$$\int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(\lambda_m \tilde{x}) d\tilde{x} = \int_0^1 \sum_{n=1}^{\infty} C_n \sin(\lambda_n \tilde{x}) \sin(\lambda_m \tilde{x}) d\tilde{x} \tag{2.89}$$

Exchanging the summation and integration signs on the right hand side of this equation leads to

$$\int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(\lambda_m \tilde{x}) d\tilde{x} = \sum_{n=1}^{\infty} C_n \int_0^1 \sin(\lambda_m \tilde{x}) \sin(\lambda_n \tilde{x}) d\tilde{x} \tag{2.90}$$

If we now evaluate the integrals on the right side of Eq. (2.90), we find that

$$\begin{aligned} \int_0^1 \sin(\lambda_m \tilde{x}) \sin(\lambda_n \tilde{x}) d\tilde{x} = 0 \quad \text{for} \ \ n \neq m \\ = \int_0^1 \sin^2(\lambda_n \tilde{x}) d\tilde{x} = \frac{1}{2} \quad \text{for} \ \ n = m \end{aligned} \tag{2.91}$$

Writing Eq. (2.90) in detail gives

$$\begin{aligned} \int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(m\pi \tilde{x}) d\tilde{x} = C_1 \int_0^1 \sin(m\pi \tilde{x}) \sin(\pi \tilde{x}) d\tilde{x} \\ + C_2 \int_0^1 \sin(m\pi \tilde{x}) \sin(2\pi \tilde{x}) d\tilde{x} \\ + C_n \int_0^1 \sin^2(n\pi \tilde{x}) d\tilde{x} \quad (\text{for}\, n = m) \\ + C_\alpha \int_0^1 \sin(m\pi \tilde{x}) \sin(\alpha\pi \tilde{x}) d\tilde{x} + \cdots \end{aligned} \tag{2.90}$$

From Eq. (2.91) one can see that in the sum on the right hand side of this equation only the term containing C_n will be non-zero. Therefore, Eq. (2.90) reduces to

$$\int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(n\pi \tilde{x}) d\tilde{x} = C_n \int_0^1 \sin^2(n\pi \tilde{x}) d\tilde{x} \tag{2.92}$$

From Eq. (2.92), the unknown constants C_n can be evaluated to be

$$C_n = \frac{\int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(n\pi\tilde{x}) d\tilde{x}}{\int_0^1 \sin^2(n\pi\tilde{x}) d\tilde{x}} = 2 \int_0^1 (\Theta_0(\tilde{x}) - \tilde{x}) \sin(n\pi\tilde{x}) d\tilde{x} \tag{2.93}$$

Thus the solution of the problem is given by Eq. (2.66) with the steady-state solution according to Eq. (2.67) and the transient solution according to Eq. (2.87). Thus,

$$\Theta = \tilde{x} + \sum_{n=1}^{\infty} C_n \sin(n\pi\tilde{x}) \exp\left(-(n\pi)^2 \tilde{t}\right) \tag{2.94}$$

In order to show the transient evolution of the temperature field, we select $\Theta_0(\tilde{x}) = 1$ for the above example. Using this initial condition, Eq. (2.93) becomes

$$C_n = 2 \int_0^1 (1 - \tilde{x}) \sin(n\pi\tilde{x}) d\tilde{x} = \frac{2}{n\pi} \tag{2.95}$$

and the temperature distribution in the solid is given by

$$\Theta = \tilde{x} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi\tilde{x}) \exp\left(-(n\pi)^2 \tilde{t}\right) \tag{2.96}$$

This temperature distribution is shown in Fig. 2.7 for different times. One can see nicely that the temperature distribution in the solid changes from the prescribed

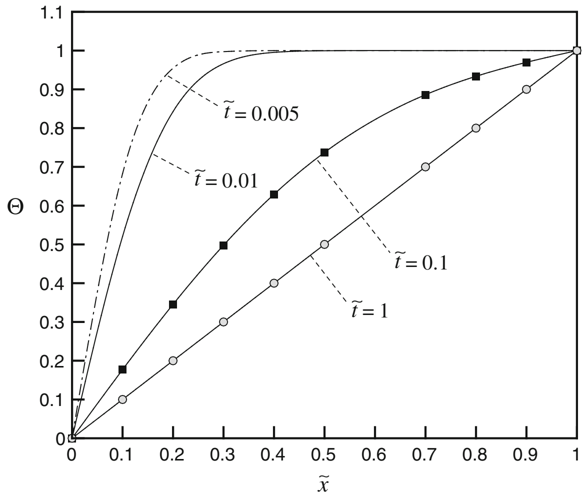

Fig. 2.7 Transient temperature distribution in the slab for selected times

constant initial temperature distribution to the linear temperature shape for the steady-state temperature distribution for $\tilde{t} \to \infty$.

In addition, one can see from Eq. (2.96) that the individual parts of the sum in this equation are decaying rapidly with increasing time (notice that the argument of the exponential function contains $(n\pi)^2$ as a multiplier).

2.3.2 Steady-State Heat Conduction in a Rectangular Plate

As a second example to explain the method of separation of variables, we investigate the heat conduction in a rectangular plate, with the height c and width b (see Fig. 2.8).

All four sides of the plate are set to a constant temperature T_0. Inside the rectangular area, a sink is located with constant sink intensity K. Assuming constant physical properties, the energy equation for this steady-state heat conduction problem is given by

$$0 = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + K \tag{2.97}$$

with the boundary conditions

$$\begin{aligned} T(b/2, y) &= T_0, \quad T(-b/2, y) = T_0 \\ T(x, c/2) &= T_0, \quad T(x, -c/2) = T_0 \end{aligned} \tag{2.98}$$

As in the first example we first introduce dimensionless quantities, before proceeding with the solution of the problem. Suitable dimensionless quantities are given by

$$\Theta = \frac{T - T_0}{T_0}, \quad \tilde{x} = \frac{x}{c/2}, \quad \tilde{y} = \frac{y}{c/2} \tag{2.99}$$

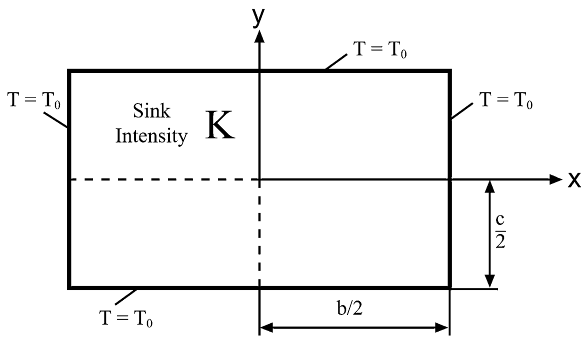

Fig. 2.8 Geometrical configuration and boundary conditions for the heat conduction problem in a flat plate

Introducing these quantities into Eqs. (2.97) and (2.98) results in

$$0 = \frac{\partial^2 \Theta}{\partial \tilde{x}^2} + \frac{\partial^2 \Theta}{\partial \tilde{y}^2} + \bar{K} \tag{2.100}$$

$$\begin{aligned} \Theta(A, \tilde{y}) = 0, \quad \Theta(-A, \tilde{y}) = 0 \\ \Theta(\tilde{x}, 1) = 0, \quad \Theta(\tilde{x}, -1) = 0 \end{aligned} \tag{2.101}$$

where the following abbreviations have been used:

$$\bar{K} = \frac{Kc^2}{4kT_0}, \quad A = \frac{b}{c} \tag{2.102}$$

This problem is described by an elliptic second-order partial differential equation. The boundary conditions, expressed by Eq. (2.101), are homogeneous, but the differential Eq. (2.100) is not.

In order to find a solution of the problem, we make once again use of the method of superposition, since the partial differential equation is linear, and split the solution into two parts

$$\Theta = \Theta_h + \Theta_p \tag{2.103}$$

Θ_h represents the solution of the homogeneous differential equation (without a sink, $K = 0$) and Θ_p is one particular solution of the problem. Let us first focus on the particular solution of the problem. In order to find this solution, we assume that $\Theta_p = f(\tilde{y})$. Alternatively, we could also assume $\Theta_p = f(\tilde{x})$ and obtain the same final solution of the problem. The analysis, however, would be altered.

If we substitute $\Theta_p = f(\tilde{y})$, into Eq. (2.100), the following relation for the function $f(\tilde{y})$ is obtained

$$f''(\tilde{y}) = -\bar{K} \quad \Rightarrow f(\tilde{y}) = -\frac{1}{2}\bar{K}\tilde{y}^2 + C_1\tilde{y} + C_2 \tag{2.104}$$

Since we need only one particular solution of the problem, we could set C_1 and C_2 equal to zero. However, a better choice is to select the constants C_1 and C_2 in such a way that the two boundary conditions $\Theta_p(\tilde{x}, 1) = 0,\ \Theta_p(\tilde{x}, -1) = 0$ are satisfied. If we do so, we obtain the following solution of the problem

$$\Theta_p = \frac{\bar{K}}{2}\left(1 - \tilde{y}^2\right) \tag{2.105}$$

After having obtained the solution for Θ_p, one has to solve the following problem for Θ_h which is deduced from the Eqs. (2.100) and (2.101)

$$0 = \frac{\partial^2 \Theta_h}{\partial \tilde{x}^2} + \frac{\partial^2 \Theta_h}{\partial \tilde{y}^2} \tag{2.106}$$

$$\begin{aligned} &\Theta_h(A, \tilde{y}) = -\bar{K}/2\left(1 - \tilde{y}^2\right), \quad &&\Theta_h(-A, \tilde{y}) = -\bar{K}/2\left(1 - \tilde{y}^2\right) \\ &\Theta_h(\tilde{x}, 1) = 0, &&\Theta_h(\tilde{x}, -1) = 0 \end{aligned} \tag{2.107}$$

Note that the differential equation for Θ_h is now homogeneous, whereas two of the boundary conditions are non-homogeneous. Furthermore, note that the two boundary conditions, corresponding to a constant value of $\tilde{y}$, are still homogeneous. This is of importance for the subsequent analysis of the problem.

We now assume that Eq. (2.106) has a solution, which can be obtained by the method of separation of variables. Thus

$$\Theta_h = F(\tilde{x})G(\tilde{y}) \tag{2.108}$$

Introducing Eq. (2.108) into the Eqs. (2.106) and (2.107) results in

$$F''(\tilde{x})G(\tilde{y}) + G''(\tilde{y})F(\tilde{x}) = 0 \tag{2.109}$$

from which we obtain

$$\frac{F''(\tilde{x})}{F(\tilde{x})} = -\frac{G''(\tilde{y})}{G(\tilde{y})} = \pm\lambda^2 \tag{2.110}$$

From Eq. (2.110), one notices that a physically plausible solution occurs for both $+\lambda^2$ and $-\lambda^2$. In order to investigate this problem further, we analyse in more detail the solutions for the function $G(\tilde{y})$. From Eq. (2.110) we get

$$G_1''(\tilde{y}) + \lambda^2 G_1(\tilde{y}) = 0 \quad \text{for } +\lambda^2 \tag{2.111}$$

$$G_2''(\tilde{y}) - \lambda^2 G_2(\tilde{y}) = 0 \quad \text{for } -\lambda^2 \tag{2.112}$$

which gives rise to the following two possible solutions

$$G_1(\tilde{y}) = C_3 \cos(\lambda\tilde{y}) + C_4 \sin(\lambda\tilde{y}) \tag{2.113}$$

$$G_2(\tilde{y}) = C_3 \cosh(\lambda\tilde{y}) + C_4 \sinh(\lambda\tilde{y}) \tag{2.114}$$

If we now reconsider the problem to be solved (Eqs. (2.106) and (2.107)), one can see that $\Theta_h = F(\tilde{x})\,G(\tilde{y})$ has to be zero for $\tilde{y} = \pm 1$. If we satisfy these two boundary conditions by Eq. (2.114), we obtain only the trivial solution, because the functions $\cosh(\lambda\tilde{y})$ and $\sinh(\lambda\tilde{y})$ have only one zero point. Instead, if we satisfy the two boundary conditions by Eq. (2.113), we obtain an equation, which determines the eigenvalues. Therefore, Eq. (2.113) is the desired solution and thus, we have to select $+\lambda^2$ in Eq. (2.110).

For the function $F(\tilde{x})$, one obtains from Eq. (2.110)

$$F''(\tilde{x}) - \lambda^2 F(\tilde{x}) = 0 \tag{2.115}$$

which has the solution

$$F(\tilde{x}) = C_5 \cosh(\lambda\tilde{x}) + C_6 \sinh(\lambda\tilde{x}) \tag{2.116}$$

Combining the solutions for F and G leads to the following expression for Θ_h

$$\Theta_h = (C_3 \cos(\lambda\tilde{y}) + C_4 \sin(\lambda\tilde{y}))(C_5 \cosh(\lambda\tilde{x}) + C_6 \sinh(\lambda\tilde{x})) \tag{2.117}$$

This expression has to satisfy the boundary conditions given by Eq. (2.107)

$$\begin{aligned} &\Theta_h(A,\tilde{y}) = -\bar{K}/2\left(1-\tilde{y}^2\right), \quad &&\Theta_h(-A,\tilde{y}) = -\bar{K}/2\left(1-\tilde{y}^2\right) \\ &\Theta_h(\tilde{x},1) = 0, &&\Theta_h(\tilde{x},-1) = 0 \end{aligned} \tag{2.107}$$

Applying the two boundary conditions for fixed values of $\tilde{y}$, the following two equations are obtained

$$\begin{gathered} C_3 \cos(\lambda) + C_4 \sin(\lambda) = 0 \\ C_3 \cos(\lambda(-1)) + C_4 \sin(\lambda(-1)) = 0 \end{gathered} \tag{2.118}$$

Because $\cos(\lambda) = \cos(-\lambda)$ and $\sin(\lambda) = -\sin(-\lambda)$, one obtains from the above equations that $C_4 = 0$ and that

$$C_3 \cos(\lambda) = 0 \tag{2.119}$$

From this equation it follows that

$$\lambda = \frac{2n-1}{2}\pi, \quad n = 1, 2, 3, \ldots \tag{2.120}$$

and the following solution for Θ_h is obtained

$$\Theta_h = C_3 \cos(\lambda\tilde{y})(C_5 \cosh(\lambda\tilde{x}) + C_6 \sinh(\lambda\tilde{x})) \tag{2.121}$$

From the two boundary conditions, corresponding to a fixed value of $\tilde{x}$ in Eq. (2.107), it can be seen that $\Theta_h(A,\tilde{y}) = \Theta_h(-A,\tilde{y})$, which indicates Θ_h is an even function in $\tilde{x}$. Therefore, it follows that $C_6 = 0$. Thus

$$\Theta_h = C \cos(\lambda\tilde{y}) \cosh(\lambda\tilde{x}), \quad C = C_3 C_5 \tag{2.122}$$

Since an infinite number of eigenvalues has been found from Eq. (2.120), the solution for Θ_h can be constructed by superimposing all these individual solutions.

This results in

$$\Theta_h = \sum_{n=1}^{\infty} C_n \cos(\lambda_n \tilde{y}) \cosh(\lambda_n \tilde{x}) \tag{2.123}$$

The unknown coefficients C_n can be obtained by matching the boundary condition $\Theta_h(A, \tilde{y}) = -\bar{K}/2\,(1-\tilde{y}^2)$ by Eq. (2.122). This results in

$$-\frac{\bar{K}}{2}(1-\tilde{y}^2) = \sum_{n=1}^{\infty} C_n \cos(\lambda_n \tilde{y}) \cosh(\lambda_n A) \tag{2.124}$$

If we multiply both sides of the above equation by $\cos(\lambda_m \tilde{y})$ and integrate the resulting expressions between –1 and 1 we obtain:

$$-\frac{\bar{K}}{2}\int_{-1}^{1} (1-\tilde{y}^2)\cos(\lambda_m \tilde{y})d\tilde{y} = \int_{-1}^{1}\left(\sum_{n=1}^{\infty} C_n \cos(\lambda_n \tilde{y})\cos(\lambda_m \tilde{y})\cosh(\lambda_n A)\right)d\tilde{y} \tag{2.125}$$

Again, the summation and integration signs on the right hand side of Eq. (2.125) can be interchanged. Then, it is obvious that from the sum only one term will not be equal to zero, because the integral

$$\begin{aligned} \int_{-1}^{1} \cos(\lambda_n \tilde{y})\cos(\lambda_m \tilde{y})d\tilde{y} &= 0 \quad \text{for} \quad \lambda_n \neq \lambda_m \\ &= 1 \quad \text{for} \quad \lambda_n = \lambda_m \end{aligned} \tag{2.126}$$

Finally, the following equation is obtained for the determination of the unknown coefficients

$$C_n = \frac{-\frac{\bar{K}}{2}\int_{-1}^{1}(1-\tilde{y}^2)\cos(\lambda_n \tilde{y})d\tilde{y}}{\cosh(\lambda_n A)} = \frac{2\bar{K}(-1)^n}{\lambda_n^3 \cosh(\lambda_n A)} \tag{2.127}$$

The solution of the problem, given by Eqs. (2.100) and (2.101), is obtained by combining the two parts of the solution Θ_h and Θ_p. This gives finally

$$\Theta = \frac{\bar{K}}{2}\left(1 - \tilde{y}^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\lambda_n^3 \cosh(\lambda_n A)} \cos(\lambda_n \tilde{y}) \cosh(\lambda_n \tilde{x})\right) \tag{2.128}$$

The solution obtained here shall serve as an example that solutions, which are obtained for heat conduction problems, can be very useful for other applications.

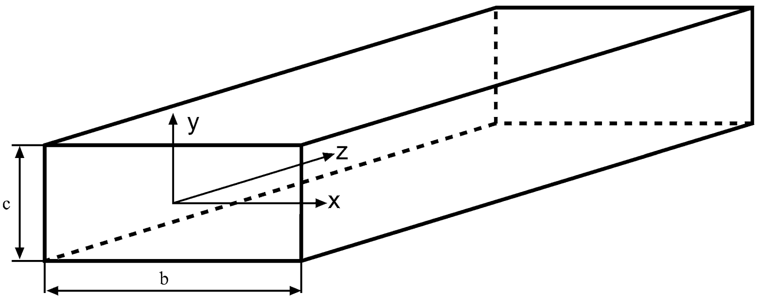

Fig. 2.9 Geometrical configuration and coordinate system for the flow in a rectangular channel

To that aim, let us investigate the flow in a rectangular channel. The channel has the width b and height c. The geometry under consideration is shown in Fig. 2.9.

For this problem, u, v, w are the flow velocities in the x-, y- and z-direction. Under the assumption of a steady, incompressible, laminar flow with constant fluid properties, the Navier-Stokes equations reduce to

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=F_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{1.1}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=F_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{1.2}$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=F_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{1.3}$$

and the mass continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.6}$$

If we now further assume that the flow is hydrodynamically fully developed, i.e. the velocity profile does not change along the axial direction, only the w component of the flow velocity is present. The u and v components are identically zero. Additionally, the w component of the flow can only be a function of the x and y coordinate for the hydrodynamically fully developed flow. Then the problem simplifies to (see for example Spurk 1987)

$$0=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{2.129}$$

where the pressure gradient in the axial direction is constant for a hydrodynamically fully developed flow. The boundary conditions are the no slip conditions at all boundaries of the channel. Thus

$$\begin{aligned} w(b/2, y) &= 0, \quad w(-b/2, y) = 0 \\ w(x, c/2) &= 0, \quad w(x, -c/2) = 0 \end{aligned} \tag{2.130}$$

If we introduce into Eq. (2.129) the abbreviation $\bar{K} = -1/\mu \, \partial p/\partial z$, it can be seen that the problem for determining the fully developed velocity profile is identical to the heat conduction problem in a plate containing a heat sink with constant sink intensity (the derivation of the fully developed velocity field in a rectangular channel is given for example in Spurk (1987)). This shows nicely the similarity of the equations describing problems in Fluid Mechanics and Heat Transfer.

2.3.3 Separation of Variables for the General Case of a Linear Second-Order Partial Differential Equation

At the beginning of this chapter, we have been concerned with a linear second order partial differential equation, which depended on the two variables x and y. The most general form of this homogeneous equation is given by

$$\begin{aligned} A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} \\ + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = 0 \end{aligned} \tag{2.1}$$

where $A, \ldots, F$ are constants or functions of x and y, which are sufficiently differentiable in the domain of interest.

In the previous two examples, we used the method of separation of variables to derive a solution of the linear partial differential equation as an infinite sum. However, we only addressed very special cases of Eq. (2.1). It is now interesting to evaluate, under which conditions a separation of variables is possible for Eq. (2.1). In order to answer this question, we consider the transformed Eq. (2.16)

$$\begin{aligned} \bar{A}(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} + 2\bar{B}(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C}(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} \\ + \bar{D}(\xi,\eta)\frac{\partial u}{\partial \xi} + \bar{E}(\xi,\eta)\frac{\partial u}{\partial \eta} + F(\xi,\eta)u = 0 \end{aligned} \tag{2.16}$$

where the new coordinates ξ and η, defined by Eq. (2.6), have been used. Let us substitute

$$u = H(\xi)\, G(\eta) \tag{2.131}$$

into Eq. (2.16). From this we obtain

$$\begin{aligned} &\bar{A}(\xi,\eta)H''(\xi)G(\eta) + 2\bar{B}(\xi,\eta)H'(\xi)G'(\eta) + \bar{C}(\xi,\eta)H(\xi)G''(\eta) \\ &\quad + \bar{D}(\xi,\eta)H'(\xi)G(\eta) + \bar{E}(\xi,\eta)H(\xi)G'(\eta) + FH(\xi)G(\eta) = 0 \end{aligned} \tag{2.132}$$

where the prime indicates the differentiation of the functions $H(\xi)$ and $G(\eta)$ with respect to the independent variable. Dividing Eq. (2.132) by $H(\xi)\, G(\eta)$ results in

$$\begin{aligned} &\bar{A}(\xi,\eta)\frac{H''(\xi)}{H(\xi)} + 2\bar{B}(\xi,\eta)\frac{H'(\xi)}{H(\xi)}\frac{G'(\eta)}{G(\eta)} + \bar{C}(\xi,\eta)\frac{G''(\eta)}{G(\eta)} \\ &\quad + \bar{D}(\xi,\eta)\frac{H'(\xi)}{H(\xi)} + \bar{E}(\xi,\eta)\frac{G'(\eta)}{G(\eta)} + F(\xi,\eta) = 0 \end{aligned} \tag{2.133}$$

From the above equation it can be seen that the variables can only be separated if $\bar{B}(\xi,\eta) = 0$. According to Eq. (2.17), this requires that the new coordinates are chosen in a way to ensure that

$$\bar{B} = A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = 0 \tag{2.134}$$

After setting $\bar{B}(\xi,\eta) = 0$, Eq. (2.132) can be rewritten in the following way

$$\begin{aligned} &\frac{\bar{A}(\xi,\eta)}{N(\xi,\eta)}\frac{H''(\xi)}{H(\xi)} + \frac{\bar{D}(\xi,\eta)}{N(\xi,\eta)}\frac{H'(\xi)}{H(\xi)} + \frac{\bar{C}(\xi,\eta)}{N(\xi,\eta)}\frac{G''(\eta)}{G(\eta)} \\ &\quad + \frac{\bar{E}(\xi,\eta)}{N(\xi,\eta)}\frac{G'(\eta)}{G(\eta)} + \frac{F(\xi,\eta)}{N(\xi,\eta)} = 0 \end{aligned} \tag{2.135}$$

where the whole equation has been divided by the function $N(\xi,\eta)$. If we further assume that

$$\frac{F(\xi,\eta)}{N(\xi,\eta)} = f_1(\xi) + f_2(\eta) \tag{2.136}$$

Equation (2.135) can be written as

$$\begin{aligned} &\frac{\bar{A}(\xi,\eta)}{N(\xi,\eta)}\frac{H''(\xi)}{H(\xi)} + \frac{\bar{D}(\xi,\eta)}{N(\xi,\eta)}\frac{H'(\xi)}{H(\xi)} + f_1(\xi) \\ &= -\frac{\bar{C}(\xi,\eta)}{N(\xi,\eta)}\frac{G''(\eta)}{G(\eta)} - \frac{\bar{E}(\xi,\eta)}{N(\xi,\eta)}\frac{G'(\eta)}{G(\eta)} - f_2(\eta) = \text{const.} \end{aligned} \tag{2.137}$$

The left side of this equation should now only be a function of ξ and the right side of the equation should only be a function of η. This is only possible, if the following restrictions are satisfied:

$$\frac{\bar{A}(\xi,\eta)}{N(\xi,\eta)}=f_3(\xi),\quad \frac{\bar{D}(\xi,\eta)}{N(\xi,\eta)}=f_4(\xi),$$
$$\frac{\bar{C}(\xi,\eta)}{N(\xi,\eta)}=f_5(\eta),\quad \frac{\bar{E}(\xi,\eta)}{N(\xi,\eta)}=f_6(\eta) \tag{2.138}$$

The present analysis might be very helpful in order to check in advance if the method of separation of variables can be applied to the problem under consideration. It would be incorrect, however, to assume that the method leads to a solution in all cases, where the separation of variables is possible.

Problems

2-1 Consider the partial differential equation

$$4\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2$$

(a) Determine the type of the differential equation.
(b) What are the characteristics of this equation?
(c) Transform the equation into its standard form.
(d) Determine the general solution of the above given differential equation (*hint*: use the substitution: $v = \partial u/\partial \eta$).

2-2 Consider the partial differential equation

$$-x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}-\frac{1}{2}\frac{\partial u}{\partial x}+A(x,y)\frac{\partial u}{\partial y}=0$$

Solve this equation using the method of separation of variables.

(a) Insert $u = F(x)G(y)$ into the equation. How should the function $A(x, y)$ look like, so that the method of separation of variables can lead to a solution of the above equation?
(b) Determine the type of the partial differential equation. Show the result in a x, y-diagram for $-\infty < x < +\infty$ and $-\infty < y < +\infty$.
(c) Calculate for $x > 0, y > 0$ the characteristics of the equation and transform the equation into its standard form (use for this $A(x, y) = -1/2$).

2-3 Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} + 10\frac{\partial u}{\partial y} = \sin x$$

(a) Determine the type of the differential equation.
(b) What are the characteristics of this equation?
(c) Transform the equation into its standard form.

2-4 A thin rectangular plate with side lengths a and b is subjected to a constant temperature T_1 at three sides $(T(x,0) = T(x,b) = T(0,y) = T_1)$, whereas the remaining side of this plate is subjected to the temperature distribution

$$T(a,y) = T_1\left\{\sin^3\left(\frac{2\pi y}{b}\right) + 1\right\}$$

We are interested in the steady-state temperature distribution in the plate. All material properties of the plate are constant. The steady-state temperature distribution can be calculated from the energy equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

with the above given boundary conditions.

(a) Make the energy equation and the boundary conditions dimensionless.
(b) Solve the problem using the method of separation of variables.

2-5 Consider a slab, which has extensions in the y- and z-direction much bigger than in the x-direction. The slab has constant initial temperature T_i. At $t = 0$, the slab is exposed at both sides ($x = 0$, $x = \delta$) to convective cooling. The temperatures of the surrounding fluid are given by T_C and T_G. The problem under consideration is described by the following, simplified energy equation

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

and the following boundary conditions

$$t = 0: \; T = T_i$$
$$x = 0: h_G(T_G - T(x=0)) + k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$
$$x = \delta: h_G(T_C - T(x=\delta)) - k\left.\frac{\partial T}{\partial x}\right|_{x=\delta} = 0$$

(a) Introduce dimensionless quantities into the above equations. Use

$$\Theta = \frac{T_G - T}{T_G - T_i}, \ \tilde{t} = \frac{at}{\delta^2}, \ a = \frac{k}{\rho c}, \ \tilde{x} = \frac{x}{\delta},$$
$$\mathrm{Bi}_G = \frac{h_G \delta}{k}, \ \mathrm{Bi}_C = \frac{h_C \delta}{k}$$

(b) Split the solution of the problem into the steady-state solution and into the solution of the transient part. Show that for the transient part of the solution, the two boundary conditions for $\tilde{x} = 0$ and $\tilde{x} = 1$ are homogeneous.
(c) Solve the first problem. What is the steady-state temperature distribution in the slab?
(d) Solve the transient problem. What is the complete solution of the problem?

2-6 Consider the transient heat conduction in a slab of length l. The slab has the initial temperature distribution

$$T(0, x) = \frac{x - l}{l}(T_2 - T_1) + T_2$$

At both sides of the slab, the following constant temperatures are applied

$$T(t, 0) = T_1, \quad T(t, l) = T_2$$

In addition, the slab contains a heat source. The above given problem can be described by the following partial differential equation (where a and B are constants)

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \frac{xBa(T_2 - T_1)}{l^3}$$

(a) Make the differential equation and the boundary conditions dimensionless by introducing suitable variables.
(b) Split the problem into different simpler problems.
(c) Solve the different problems and give the complete solution.

2-7 Consider a sphere with radius R. For $t = 0$ the sphere has constant temperature T_i. The surface of the sphere is set to the constant temperature T_0 for $t > 0$. The sphere contains also a heat source with constant source intensity $\dot{q}_{i0}$. The material properties of the sphere are considered to be constant. The temperature distribution in the sphere can be calculated from the energy equation in spherical coordinates

$$\rho c \frac{\partial T}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{k}{r^2 \sin \Psi} \frac{\partial}{\partial \Psi}\left(\sin \Psi \frac{\partial T}{\partial \Psi}\right)$$
$$+ \frac{k}{r^2 \sin \Psi} \frac{\partial}{\partial \varphi}\left(\frac{\partial T}{\partial \varphi}\right) + \dot{q}_{i0}$$

(a) Simplify the above given energy equation for the case of rotational symmetry. What are the needed boundary conditions?
(b) Transform the problem under consideration by using $T(r,t) = U(r,t)/r$. What is the resulting differential equation and what are the boundary conditions?
(c) Introduce dimensionless quantities, so that the spatial boundary conditions are homogeneous.
(d) Solve the transformed problem by using the method of separation of variables.
(e) Derive from the solution given under (d) the temperature distribution $T(r,t)$.

2-8 An older professor investigates at home a linear second order partial differential equation. The function u is dependent on x and y. After a while he obtains the two characteristics of the partial differential equation, given by

$$\xi = y - 2x$$
$$\eta = y + 2x$$

With this, he is able to reduce the original partial differential equation to

$$B^* \frac{\partial^2 u}{\partial \xi \partial \eta} = (\xi + \eta)(\xi - \eta)$$

After achieving this, he goes for lunch. When he returns, he notices that his dog has eating the manuscript. He can't remember how the original equation was looking like. The only thing he knows is that the coefficient in front of the term $\partial^2 u/\partial x^2$ was equal to "1".

(a) Please help the professor and try to reconstruct the original linear second order partial differential equation for u.
(b) Determine the coefficient B^* in the above equation.
(c) Determine the general solution of the above given partial differential equation.

2-9 Consider the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + m \frac{\partial^2 u}{\partial xy} + m \frac{\partial^2 u}{\partial y^2} = 2xy$$

where m is a rational number.

(a) Determine the type of the differential equation in dependence of m. When will the equation be elliptic, hyperbolic and parabolic? Please set now the parameter $m = 0$. The resulting equation should now be solved together with the boundary conditions

$$u(0, y) = y^2$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = y$$

(b) Determine first the general solution of the problem.
(c) Use the two above given boundary conditions in order to predict the solution of the problem.

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