Chapter 2
The Algebraic Bethe Ansatz

The algebraic Bethe Ansatz method for quantum integrable models was proposed by the Leningrad Group [1–7] in the late 1970s, based on YBE. This method was then generalized to open boundary integrable systems by Sklyanin [8] in 1988, through developing an equation accounting for the integrable boundaries. In the past several decades, the algebraic Bethe Ansatz method has become the most popular one for solving quantum integrable models. Particularly, the development of the nested algebraic Bethe Ansatz [9–19] makes it possible to diagonalize multi-component integrable models in a systematic way.

This chapter is devoted to a detailed description of the algebraic Bethe Ansatz method and its nested version, with the isotropic spin-chain models as examples. These approaches are applicable for all the integrable models under the condition that a proper reference state exists, though different tricks may be used to find proper generating operators for the eigenstates according to the properties of R-matrices defined under different algebras [20–24]. Based on the Bethe Ansatz solutions, the methods to construct low-lying elementary excitations and thermodynamics for the spin-$\frac{1}{2}$ chain are introduced. In addition, the fusion procedure and a quantity that is important throughout this book, the quantum determinant, are also introduced. The last section is devoted to a brief introduction of Sklyanin's separation of variables method [25–27].

2.1 The Periodic Heisenberg Spin Chain

2.1.1 The Algebraic Bethe Ansatz

To show the algebraic Bethe Ansatz procedure clearly, let us consider again the spin-$\frac{1}{2}$ Heisenberg chain model. For convenience, we define the homogeneous monodromy matrix and the corresponding transfer matrix as

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$$T_0(u) = R_{0,N}(u) \cdots R_{0,1}(u), \tag{2.1.1}$$

and

$$t(u) = tr_0 T_0(u), \tag{2.1.2}$$

respectively with the R-matrix defined in (1.5.2). With the properties of the permutation operator we have

$$\begin{aligned} \left.\frac{\partial \ln t(u)}{\partial u}\right|_{u=0} &= t^{-1}(0) \left.\frac{\partial t(u)}{\partial u}\right|_{u=0} \\ &= \sum_{j=1}^{N} P_{1,2} \cdots P_{1,N} tr_0 [P_{0,N} \cdots P_{0,j+1} P_{0,j-1} \cdots P_{0,1}] = \sum_{j=1}^{N} P_{j,j-1}. \end{aligned} \tag{2.1.3}$$

The Hamiltonian of the Heisenberg spin chain is thus expressed as

$$H = \frac{1}{2} \sum_{j=1}^{N} \sigma_j \cdot \sigma_{j+1} = \left.\frac{\partial \ln t(u)}{\partial u}\right|_{u=0} - \frac{1}{2} N, \tag{2.1.4}$$

with the periodic boundary condition $\sigma_{N+1} \equiv \sigma_1$.

To calculate the commutation relations among the elements of the monodromy matrix, let us write out the explicit forms of the R-matrix and the monodromy matrix in the auxiliary tensor space:

$$R_{0,\bar{0}}(u-v) = \begin{pmatrix} u-v+1 & 0 & 0 & 0 \\ 0 & u-v & 1 & 0 \\ 0 & 1 & u-v & 0 \\ 0 & 0 & 0 & u-v+1 \end{pmatrix}, \tag{2.1.5}$$

$$T_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \otimes I_{\bar{0}} = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}, \tag{2.1.6}$$

$$T_{\bar{0}}(v) = I_0 \otimes \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}, \tag{2.1.7}$$

where I_0 and $I_{\bar{0}}$ are the identity operators in the 0th space and $\bar{0}$th space, respectively. With the help of the Yang-Baxter relation (1.2.5), we can easily deduce the following commutation relations:

$$
\begin{aligned}
&[A(u), A(v)] = [D(u), D(v)] = 0,\\
&A(u)B(v) = \frac{u-v-1}{u-v}B(v)A(u) + \frac{1}{u-v}B(u)A(v),\\
&D(u)B(v) = \frac{u-v+1}{u-v}B(v)D(u) - \frac{1}{u-v}B(u)D(v),\\
&[B(u), B(v)] = [C(u), C(v)] = 0,\\
&[B(u), C(v)] = \frac{1}{u-v}[D(v)A(u) - D(u)A(v)].
\end{aligned} \tag{2.1.8}
$$

Based on the above relations, the following useful formulae can be derived:

$$
\begin{aligned}
A(u)B(\mu_1)\cdots B(\mu_M) = &\prod_{j=1}^{M}\frac{u-\mu_j-1}{u-\mu_j}B(\mu_1)\cdots B(\mu_M)A(u)\\
&+\sum_{j=1}^{M}\frac{1}{u-\mu_j}\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l-1}{\mu_j-\mu_l}B(\mu_1)\cdots B(\mu_{j-1})\\
&\times B(u)B(\mu_{j+1})\cdots B(\mu_M)A(\mu_j),
\end{aligned} \tag{2.1.9}
$$

$$
\begin{aligned}
D(u)B(\mu_1)\cdots B(\mu_M) = &\prod_{j=1}^{M}\frac{u-\mu_j+1}{u-\mu_j}B(\mu_1)\cdots B(\mu_M)D(u)\\
&-\sum_{j=1}^{M}\frac{1}{u-\mu_j}\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l+1}{\mu_j-\mu_l}B(\mu_1)\cdots B(\mu_{j-1})\\
&\times B(u)B(\mu_{j+1})\cdots B(\mu_M)D(\mu_j).
\end{aligned} \tag{2.1.10}
$$

Proof From the commutation relations (2.1.8) we know that Eq. (2.1.9) is satisfied for $M = 1$. Assuming that Eq. (2.1.9) is also satisfied for an arbitrary M, we have

$$
\begin{aligned}
A(u)B(\mu_{M+1})B(\mu_1)\cdots B(\mu_M) &= \frac{u-\mu_{M+1}-1}{u-\mu_{M+1}}B(\mu_{M+1})A(u)B(\mu_1)\cdots B(\mu_M)\\
&\quad+\frac{1}{u-\mu_{M+1}}B(u)A(\mu_{M+1})B(\mu_1)\cdots B(\mu_M)\\
&=\prod_{j=1}^{M+1}\frac{u-\mu_j-1}{u-\mu_j}B(\mu_1)\cdots B(\mu_M)B(\mu_{M+1})A(u)\\
&\quad+\frac{u-\mu_{M+1}-1}{u-\mu_{M+1}}\sum_{j=1}^{M}\frac{1}{u-\mu_j}\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l-1}{\mu_j-\mu_l}\\
&\quad\times B(\mu_1)\cdots B(\mu_{j-1})B(u)B(\mu_{j+1})\cdots B(\mu_{M+1})A(\mu_j)\\
&\quad+\frac{1}{u-\mu_{M+1}}\prod_{j=1}^{M}\frac{\mu_{M+1}-\mu_j-1}{\mu_{M+1}-\mu_j}B(\mu_1)\cdots B(\mu_M)B(u)A(\mu_{M+1})
\end{aligned}
$$

$$+\frac{1}{u-\mu_{M+1}}\sum_{j=1}^{M}\frac{1}{\mu_{M+1}-\mu_j}\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l-1}{\mu_j-\mu_l}$$
$$\times B(\mu_1)\cdots B(\mu_{j-1})B(u)B(\mu_{j+1})\cdots B(\mu_{M+1})A(\mu_j). \tag{2.1.11}$$

Combining the second and the fourth terms in the above equation, we obtain

$$A(u)B(\mu_1)\cdots B(\mu_{M+1}) = \prod_{j=1}^{M+1}\frac{u-\mu_j-1}{u-\mu_j}B(\mu_1)\cdots B(\mu_{M+1})A(u)$$
$$+\sum_{j=1}^{M+1}\frac{1}{u-\mu_j}\prod_{l\neq j}^{M+1}\frac{\mu_j-\mu_l-1}{\mu_j-\mu_l}B(\mu_1)\cdots B(\mu_{j-1})$$
$$\times B(u)B(\mu_{j+1})\cdots B(\mu_{M+1})A(\mu_j). \tag{2.1.12}$$

Therefore, Eq. (2.1.9) is also satisfied for $M+1$. Equation (2.1.10) can be proven similarly. □

Let us define the vacuum state of the system as

$$|0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_N, \tag{2.1.13}$$

where $|\uparrow\rangle_n$ is the local spin-up state of site n (Accordingly, the local spin-down state is denoted as $|\downarrow\rangle_n$.). For convenience, we introduce the notations

$$\sigma_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y). \tag{2.1.14}$$

The Pauli matrices applying on the states $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$ thus behave as

$$\begin{aligned}
&\sigma_j^-|\uparrow\rangle_j = |\downarrow\rangle_j, \quad \sigma_j^+|\downarrow\rangle_j = |\uparrow\rangle_j,\\
&\sigma_j^-|\downarrow\rangle_j = \sigma_j^+|\uparrow\rangle_j = 0,\\
&\sigma_j^z|\uparrow\rangle_j = |\uparrow\rangle_j, \quad \sigma_j^z|\downarrow\rangle_j = -|\downarrow\rangle_j.
\end{aligned} \tag{2.1.15}$$

From the definition of the R-matrix we have

$$\begin{aligned}
R_{0,n}(u)|0\rangle &= \begin{pmatrix} u+\frac{1}{2}(1+\sigma_n^z) & \sigma_n^- \\ \sigma_n^+ & u+\frac{1}{2}(1-\sigma_n^z)\end{pmatrix}|0\rangle\\
&= \begin{pmatrix} u+1 & \sigma_n^- \\ 0 & u\end{pmatrix}|0\rangle.
\end{aligned} \tag{2.1.16}$$

This directly induces

$$\begin{aligned} A(u)|0\rangle &= a(u)|0\rangle = (u+1)^N|0\rangle, \\ D(u)|0\rangle &= d(u)|0\rangle = u^N|0\rangle, \\ C(u)|0\rangle &= 0. \end{aligned} \tag{2.1.17}$$

The operator $B(u)$ can be treated as the spin flipping operator and used to construct the Bethe states

$$|\mu_1, \ldots, \mu_M\rangle = \prod_{j=1}^{M} B(\mu_j)|0\rangle, \tag{2.1.18}$$

where M is the number of flipped spins and $\{\mu_j\}$ is a set of parameters. Note that

$$t(u) = A(u) + D(u). \tag{2.1.19}$$

Applying $t(u)$ to the Bethe state, with the help of the commutation relations (2.1.9) and (2.1.10), we have

$$\begin{aligned} t(u)|\mu_1, \ldots, \mu_M\rangle &= \Lambda(u)|\mu_1, \ldots, \mu_M\rangle \\ &\quad + \sum_{j=1}^{M} \Lambda_j(u) B(\mu_1) \cdots B(\mu_{j-1}) B(u) B(\mu_{j+1}) \cdots B(\mu_M)|0\rangle, \end{aligned} \tag{2.1.20}$$

where

$$\Lambda(u) = a(u) \prod_{j=1}^{M} \frac{u - \mu_j - 1}{u - \mu_j} + d(u) \prod_{j=1}^{M} \frac{u - \mu_j + 1}{u - \mu_j}, \tag{2.1.21}$$

$$\Lambda_j(u) = \frac{1}{u - \mu_j} \left\{ a(\mu_j) \prod_{l \neq j}^{M} \frac{\mu_j - \mu_l - 1}{\mu_j - \mu_l} - d(\mu_j) \prod_{l \neq j}^{M} \frac{\mu_j - \mu_l + 1}{\mu_j - \mu_l} \right\}. \tag{2.1.22}$$

To ensure the Bethe state to be an eigenstate of the transfer matrix, the unwanted terms must vanish, i.e., $\Lambda_j(u) = 0$. This induces the Bethe Ansatz equations (BAEs)

$$\left(1 + \frac{1}{\mu_j}\right)^N = \prod_{l \neq j}^{M} \frac{\mu_j - \mu_l + 1}{\mu_j - \mu_l - 1}, \quad j = 1, \ldots, M. \tag{2.1.23}$$

The solutions $\{\mu_j | j = 1, \ldots, M\}$ of the above equations are the Bethe roots.

For convenience, we put $\mu_j = i\lambda_j - \frac{1}{2}$. The BAEs can be rewritten as

$$\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right)^N = -\prod_{l=1}^{M} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i}, \quad j = 1, \ldots, M. \tag{2.1.24}$$

From Eq. (2.1.4) we obtain the eigenvalue of the Hamiltonian in terms of the Bethe roots as

$$E(\lambda_1, \ldots, \lambda_M) = \left.\frac{\partial \ln \Lambda(u)}{\partial u}\right|_{u=0} - \frac{1}{2}N = -\sum_{j=1}^{M} \frac{1}{\lambda_j^2 + \frac{1}{4}} + \frac{1}{2}N. \tag{2.1.25}$$

Obviously, the $T-Q$ relation (1.4.1) holds for this model with the parametrization

$$Q(u) = \prod_{j=1}^{M} (u - \mu_j). \tag{2.1.26}$$

In addition, the unwanted terms $\Lambda_j(u)$ can be expressed in terms of the residue of $\Lambda(u)$ at the point $u = \mu_j$

$$\Lambda_j(u) = \frac{1}{\mu_j - u} \mathrm{res}\, \Lambda(u)|_{u=\mu_j}, \tag{2.1.27}$$

which indicates that the regularity of $\Lambda(u)$ already ensures the "unwanted" terms in Eq. (2.1.20) to be zero [28].

2.1.2 Selection Rules of the Bethe Roots

As we mentioned in Chap. 1, in order to get a self consistent set of BAEs, the poles μ_j must be simple. Indeed by carefully examining the Bethe states we can deduce the Pauli principle for the Bethe roots [29], i.e., the eigenvector is zero as long as $\mu_j = \mu_l$ for $j \neq l$. Such a selection rule can easily be verified by the coordinate Bethe Ansatz. In fact, to preserve the regularity of $\Lambda(u)$, doubly degenerate μ_j (if they exist) must satisfy the condition

$$\mathrm{res}\{(u - \mu_j)\Lambda(u)\}|_{u=\mu_j} = 0, \tag{2.1.28}$$

which gives rise to an additional equation apart from the $M-1$ Eq. (2.1.23) and makes the $M-1$ Bethe roots overdetermined.

Moreover, one may find that a pair $\mu_1 = 0$ and $\mu_2 = -1$ satisfy the BAEs (2.1.23). However, this solution also induces a zero Bethe vector. A simple proof is as follows: From the definition we know that

$$T_0(0) = P_{0,N} \cdots P_{0,1} = P_{0,1} P_{1,N} \cdots P_{1,2}. \tag{2.1.29}$$

Therefore,

$$B(0) = \sigma_1^- P_{1,N} \cdots P_{1,2}. \tag{2.1.30}$$

From the crossing symmetry property (1.5.6) we know that

$$\begin{aligned} T_0(-1) &= R_{0,N}(-1) \cdots R_{0,1}(-1) \\ &= (-1)^N \sigma_0^y P_{0,N}^{t_0} \cdots P_{0,1}^{t_0} \sigma_0^y, \end{aligned} \tag{2.1.31}$$

which gives

$$B(-1) = (-1)^{N-1} P_{1,2} \cdots P_{1,N} \sigma_1^-. \tag{2.1.32}$$

Equations (2.1.30) and (2.1.32) imply that

$$B(-1)B(0) = B(0)B(-1) = 0. \tag{2.1.33}$$

2.1.3 Ground State

A remarkable fact is that based on the BAEs, the physical quantities can be derived. Particularly, computing physical quantities becomes simpler in the thermodynamic limit.

Let us first consider the case of all the Bethe roots $\{\lambda_j | j = 1, \ldots, M\}$ being real. Taking the logarithm of (2.1.24), we have

$$\theta_1(\lambda_j) = \frac{2\pi I_j}{N} + \frac{1}{N} \sum_{l=1}^{M} \theta_2(\lambda_j - \lambda_l), \tag{2.1.34}$$

where $\theta_n(x) = 2 \arctan(2x/n)$, and $\{I_j\}$ are certain integers (half odd integers) for $N - M$ odd ($N - M$ even). For convenience, we define the counting function

$$Z(\lambda) = \frac{1}{2\pi} \left[\theta_1(\lambda) - \frac{1}{N} \sum_{l=1}^{M} \theta_2(\lambda - \lambda_l) \right]. \tag{2.1.35}$$

Obviously, $Z(\lambda_j) = I_j/N$ corresponds to the Eq. (2.1.34). In principle, each possible I_j may correspond to a λ_j solution of the BAEs. However, those solutions may not be occupied. We treat the occupied solutions as "particles" and the unoccupied solutions as "holes". For any consecutive I_j and $I_{j+1} = I_j + 1$, the following relation holds:

$$\frac{Z(\lambda_{j+1}) - Z(\lambda_j)}{\lambda_{j+1} - \lambda_j} = \frac{1}{N\delta\lambda_j}, \tag{2.1.36}$$

with $\delta\lambda_j = \lambda_{j+1} - \lambda_j$. In the thermodynamic limit $N \to \infty$, Eq. (2.1.36) becomes the density of states $\rho(\lambda) + \rho^h(\lambda)$ in the λ space, where $\rho(\lambda)$ and $\rho^h(\lambda)$ are the densities of the particles and holes, respectively. Taking the derivative of Eq. (2.1.35) with respect to λ, we obtain

$$\rho(\lambda) + \rho^h(\lambda) = \frac{dZ(\lambda)}{d\lambda} = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\rho(\mu)d\mu, \tag{2.1.37}$$

where

$$a_n(\lambda) = \frac{1}{2\pi}\frac{n}{\lambda^2 + n^2/4}. \tag{2.1.38}$$

From Eq. (2.1.25), we know that each real Bethe root λ_j contributes negative energy. In the ground state, the Bethe roots should fill the whole real axis and leave no hole for an even N, i.e., $\rho^h(\lambda) = 0$. This means that the density of particles in the ground state $\rho_g(\lambda)$ satisfies

$$\rho_g(\lambda) = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\rho_g(\mu)d\mu. \tag{2.1.39}$$

Equation (2.1.39) can be solved by the Fourier transformation defined for an arbitrary function $F(\lambda)$ as

$$\begin{aligned} \tilde{F}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega\lambda} F(\lambda)d\lambda, \\ F(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\lambda}\tilde{F}(\omega)d\omega. \end{aligned} \tag{2.1.40}$$

Taking the Fourier transform of $a_n(\lambda)$, we have

$$\tilde{a}_n(\omega) = e^{-\frac{n|\omega|}{2}}. \tag{2.1.41}$$

Taking the Fourier transform of Eq. (2.1.39), we obtain

$$\tilde{\rho}_g(\omega) = \frac{1}{2\cosh\frac{\omega}{2}}. \tag{2.1.42}$$

Thus the solution of Eq. (2.1.39) is

$$\rho_g(\lambda) = \frac{1}{2\cosh(\pi\lambda)}. \tag{2.1.43}$$

The density of flipped spins relative to the reference state is

$$\frac{M}{N} = \int_{-\infty}^{\infty} \rho_g(\lambda)d\lambda = \frac{1}{2}, \tag{2.1.44}$$

which means that the magnetization of the ground state is zero. The energy density of the ground state reads

$$e_g = -2\pi \int_{-\infty}^{\infty} a_1(\lambda)\rho_g(\lambda)d\lambda + \frac{1}{2} = \frac{1}{2} - 2\ln 2. \tag{2.1.45}$$

For an odd N, there is a hole at $\lambda^h = \pm\infty$ in the real axis which carries zero energy. The energy density of the ground state still takes the form of (2.1.45) but the state is doubly degenerate.

2.1.4 Spinon Excitations

Now let us consider the elementary excitations of the system. We focus on the even N case. The simplest excitation is the case of one less spin flipped, i.e., $M = N/2 - 1$. Such a configuration is described with two holes put at λ_1^h and λ_2^h in the λ sea. In this case, the density of holes is

$$\rho^h(\lambda) = \frac{1}{N}\delta(\lambda - \lambda_1^h) + \frac{1}{N}\delta(\lambda - \lambda_2^h). \tag{2.1.46}$$

The density $\rho(\lambda)$ will deviate from $\rho_g(\lambda)$ by $\delta\rho(\lambda)$ because of the presence of the two holes. From Eqs. (2.1.39) and (2.1.46) we obtain that

$$\delta\rho(\lambda) + \rho^h(\lambda) = -\int_{-\infty}^{\infty} a_2(\lambda - \mu)\delta\rho(\mu)d\mu, \tag{2.1.47}$$

which can be solved by Fourier transformation. After some calculations, we obtain the total spin S_e of this excitation as

$$S_e = -N\int_{-\infty}^{\infty} \delta\rho(\lambda)d\lambda = 1, \tag{2.1.48}$$

and the excitation energy as

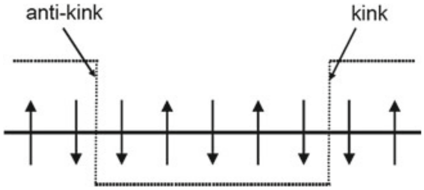

Fig. 2.1 Classical picture of the spin-triplet elementary excitations. Relative to the Neel state, the net spin carried by the flipped domain is one. Each domain boundary (kink or anti-kink) carries a spin of $\frac{1}{2}$

$$\Delta E = -2\pi N \int_{-\infty}^{\infty} a_1(\lambda)\delta\rho(\lambda)d\lambda = \varepsilon(\lambda_1^h) + \varepsilon(\lambda_2^h), \tag{2.1.49}$$

where $\varepsilon(\lambda)$ is the dressed energy with the definition

$$\varepsilon(\lambda) = 2\pi a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\varepsilon(\mu)d\mu. \tag{2.1.50}$$

From Eq. (2.1.49), we see that the energy of such excitations is the summation of the energies of two quasi-holes. Solving Eq. (2.1.50) by Fourier transformation, we obtain

$$\varepsilon(\lambda) = \frac{\pi}{\cosh(\pi\lambda)} = 2\pi\rho_g(\lambda). \tag{2.1.51}$$

Here the two holes together carry spin of one, and each of them may only carry a spin $\frac{1}{2}$. Note that such elementary excitations are unusual compared to those in the higher dimensional magnetic systems where one magnon carries a total spin of one. The classical picture of the spin-triplet excitation is shown in Fig. 2.1. Those excitations are usually called spinons [30], a typical fractional excitation in the one-dimensional quantum systems.

2.1.5 String Solutions

In the above we only considered the real Bethe roots. In fact, the BAEs may have complex solutions. For a complex λ_j with a positive imaginary part, we have

$$\left|\lambda_j - \frac{i}{2}\right| \leq \left|\lambda_j + \frac{i}{2}\right|. \tag{2.1.52}$$

This indicates that the left hand side of BAEs (2.1.24) tends to zero when $N \to \infty$. To keep the equality, the numerator of the right hand side of Eq. (2.1.24) must also tend to zero in this limit, which means that there exists another solution $\lambda_j^* \sim \lambda_j - i$ in the set of solutions. The general complex solutions of the Bethe roots read

$$\lambda_{j,\alpha}^{(n)} = \lambda_\alpha^{(n)} - \frac{i}{2}(n + 1 - 2j) + o(e^{-\delta N}), \quad j = 1, 2, \ldots, n. \qquad (2.1.53)$$

This is just the string hypothesis [31]. Here $\lambda_\alpha^{(n)}$ indicates the position of the αth n-string in the real axis, and δ is a small positive number to account for the finite size deviations.

Substituting the string solutions into the BAEs and taking the product for all j in the string, we readily obtain

$$\prod_{j=1}^{n}\left(\frac{\lambda_{j,\alpha}^{(n)} - \frac{i}{2}}{\lambda_{j,\alpha}^{(n)} + \frac{i}{2}}\right)^N = \prod_{j=1}^{n}\prod_{m=1}^{\infty}\prod_{m,l,\beta\neq n,j,\alpha}\frac{\lambda_{j,\alpha}^{(n)} - \lambda_{l,\beta}^{(m)} - i}{\lambda_{j,\alpha}^{(n)} - \lambda_{l,\beta}^{(m)} + i}. \qquad (2.1.54)$$

Considering the large N limit and omitting the exponentially small corrections, we reduce the above equation to

$$\begin{aligned}\left(\frac{\lambda_\alpha^{(n)} - \frac{i}{2}n}{\lambda_\alpha^{(n)} + \frac{i}{2}n}\right)^N = &-\prod_{m=1}^{\infty}\prod_{\beta}\frac{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} - \frac{i}{2}(m+n)}{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} + \frac{i}{2}(m+n)}\\ &\times\left[\frac{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} - \frac{i}{2}(m+n-2)}{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} + \frac{i}{2}(m+n-2)}\right]^2 \times \cdots\\ &\times\left[\frac{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} - \frac{i}{2}(|m-n|+2)}{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} + \frac{i}{2}(|m-n|+2)}\right]^2 \frac{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} - \frac{i}{2}|m-n|}{\lambda_\alpha^{(n)} - \lambda_\beta^{(m)} + \frac{i}{2}|m-n|}. \qquad (2.1.55)\end{aligned}$$

Taking the logarithm of the above equation we readily have

$$\theta_n(\lambda_\alpha^{(n)}) = \frac{2\pi I_\alpha^{(n)}}{N} + \frac{1}{N}\sum_{m,\beta}\theta'_{m,n}(\lambda_\alpha^{(n)} - \lambda_\beta^{(m)}), \qquad (2.1.56)$$

where $I_\alpha^{(n)}$ are integers or half odd integers depending on the parity of $N - \sum_{n=1}^{\infty} nM_n$ with M_n being the number of n-strings and

$$\begin{aligned}\theta'_{m,n}(\lambda) = &\theta_{m+n}(\lambda) + 2\theta_{m+n-2}(\lambda) + \cdots\\ &+ 2\theta_{|m-n|+2}(\lambda) + (1 - \delta_{m,n})\theta_{|m-n|}(\lambda). \qquad (2.1.57)\end{aligned}$$

As for the real solution case, we define the counting functions

$$Z_n(\lambda) = \frac{1}{2\pi}\left[\theta_n(\lambda) - \frac{1}{N}\sum_{m,\beta}\theta'_{m,n}(\lambda - \lambda_\beta^{(m)})\right]. \tag{2.1.58}$$

Obviously, $Z_n(\lambda_\alpha^{(n)}) = I_\alpha^{(n)}/N$ corresponds to Eq. (2.1.56). In the thermodynamic limit, we have

$$\frac{dZ_n(\lambda)}{d\lambda} = \rho_n(\lambda) + \rho_n^h(\lambda), \tag{2.1.59}$$

where $\rho_n(\lambda)$ and $\rho_n^h(\lambda)$ are the densities of n-strings and n-string holes, respectively. The density of flipped spins is

$$\frac{M}{N} = \sum_{n=1}^{\infty} n\int_{-\infty}^{\infty}\rho_n(\lambda)d\lambda. \tag{2.1.60}$$

Taking the derivative of (2.1.58), we obtain the relation between $\rho_n^h(\lambda)$ and $\rho_m(\lambda)$ as

$$\rho_n^h(\lambda) = a_n(\lambda) - \sum_{m=1}^{\infty}\int_{-\infty}^{\infty} A_{m,n}(\lambda-\mu)\rho_m(\mu)d\mu, \tag{2.1.61}$$

where

$$A_{m,n}(\lambda) = a_{m+n}(\lambda) + 2a_{m+n-2}(\lambda) + \cdots + 2a_{|m-n|+2}(\lambda) + a_{|m-n|}(\lambda),$$
$$a_0(\lambda) \equiv \delta(\lambda). \tag{2.1.62}$$

Equation (2.1.61) is significant for studying the elementary excitations and thermodynamics.

In order to give the complete picture of the elementary excitations in the system, let us consider another simple type of elementary excitation, i.e., a 2-string at λ_s plus two holes in the real axis. In this case, the corresponding density functions are

$$\rho_1^h(\lambda) = \frac{1}{N}\left[\delta(\lambda-\lambda_1^h) + \delta(\lambda-\lambda_2^h)\right], \tag{2.1.63}$$

$$\rho_2(\lambda) = \frac{1}{N}\delta(\lambda-\lambda_s). \tag{2.1.64}$$

For $n = 1$ in Eq. (2.1.61), we obtain

$$\rho_1(\lambda) + \rho_1^h(\lambda) = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda-\mu)\rho_1(\mu)d\mu$$
$$- \int_{-\infty}^{\infty}[a_1(\lambda-\mu) + a_3(\lambda-\mu)]\rho_2(\mu)d\mu. \tag{2.1.65}$$

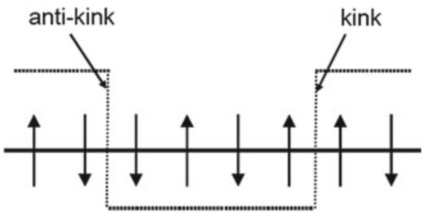

Fig. 2.2 Classical picture of the spin-singlet elementary excitations. Relative to the Neel state, the net spin carried by the flipped domain is zero. One domain boundary carries a spin of $\frac{1}{2}$ and the other carries a spin of -$\frac{1}{2}$

The deviation of the particle density from that of the ground state reads

$$\begin{aligned}\delta\rho_1(\lambda) = -\rho_1^h(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda-\mu)\delta\rho_1(\mu)d\mu \\ - \int_{-\infty}^{\infty}[a_1(\lambda-\mu)+a_3(\lambda-\mu)]\rho_2(\mu)d\mu. \qquad (2.1.66)\end{aligned}$$

This allows us to derive the excitation energy as

$$\begin{aligned}\Delta E &= -2\pi N\int_{-\infty}^{\infty} a_1(\lambda)\delta\rho_1(\lambda)d\lambda - 2\pi\left[a_1\left(\lambda_s+\frac{i}{2}\right)+a_1\left(\lambda_s-\frac{i}{2}\right)\right] \\ &= \varepsilon(\lambda_1^h)+\varepsilon(\lambda_2^h). \qquad (2.1.67)\end{aligned}$$

It is easy to check that $M = N/2$ in this case, indicating a spin singlet excitation as shown in Fig. 2.2. Interestingly, the excitation energy takes the same form as that of Eq. (2.1.49). This means that the contribution of the 2-string is completely canceled by that of the $\rho_1(\lambda)$ redistribution induced by the presence of the string. It can be proven that this statement is also valid for the more-holes cases with the presence of arbitrary strings: the n-strings contribute nothing to the energy, and the excitation energy only depends on the positions of the holes. However, the strings do affect the scattering matrix among the holes [32].

2.1.6 Thermodynamics

The thermodynamic Bethe Ansatz was first proposed by Yang and Yang [33] for the Lieb-Liniger model [34] and subsequently generalized to other integrable models by Gaudin [35], Takahashi [36–38] and Johnson and McCoy [39]. The central point lies in deriving the entropy from the distribution of the Bethe roots.

For the present model, the energy of an n-string in the external magnetic field h is

$$\varepsilon_n^0(\lambda) = \sum_{j=1}^{n}\left[\frac{\lambda+\frac{i}{2}(n+1-2j)-\frac{i}{2}}{\lambda+\frac{i}{2}(n+1-2j)+\frac{i}{2}}+\frac{\lambda+\frac{i}{2}(n+1-2j)+\frac{i}{2}}{\lambda+\frac{i}{2}(n+1-2j)-\frac{i}{2}}-2\right]+nh$$
$$= -2\pi a_n(\lambda)+nh. \tag{2.1.68}$$

The density of energy can be calculated by

$$E/N = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\varepsilon_n^0(\lambda)\rho_n(\lambda)d\lambda+\frac{1}{2}(1-h). \tag{2.1.69}$$

Let us consider an infinitely small interval $[\lambda, \lambda+d\lambda]$ in the λ space. The number of states allowed to be occupied by an n-string in this interval is

$$N[\rho_n(\lambda)+\rho_n^h(\lambda)]d\lambda.$$

Then the number of the possible physical states in this interval is

$$d\Omega(\lambda) = \prod_{n=1}^{\infty}\frac{[N(\rho_n(\lambda)+\rho_n^h(\lambda))d\lambda]!}{[N\rho_n(\lambda)d\lambda]![N\rho_n^h(\lambda)d\lambda]!}. \tag{2.1.70}$$

With the help of Sterling's formula $\ln N! \approx N\ln N$, we obtain the entropy in the interval

$$dS(\lambda) = \ln d\Omega(\lambda) \approx N\sum_{n=1}^{\infty}\Big\{[\rho_n(\lambda)+\rho_n^h(\lambda)]\ln[\rho_n(\lambda)+\rho_n^h(\lambda)]$$
$$-\rho_n(\lambda)\ln\rho_n(\lambda)-\rho_n^h(\lambda)\ln\rho_n^h(\lambda)\Big\}d\lambda. \tag{2.1.71}$$

We define the relative density of the free energy as

$$f = \frac{F}{N}-\frac{1}{2}(1-h), \tag{2.1.72}$$

where $F = E - TS$ is the usual free energy, T is the temperature and S is the entropy. Substituting Eqs. (2.1.69) and (2.1.71) into Eq. (2.1.72), we have

$$f = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\varepsilon_n^0(\lambda)\rho_n(\lambda)d\lambda - T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\Big\{[\rho_n(\lambda)+\rho_n^h(\lambda)]\ln[\rho_n(\lambda)+\rho_n^h(\lambda)]$$
$$-\rho_n(\lambda)\ln\rho_n(\lambda)-\rho_n^h(\lambda)\ln\rho_n^h(\lambda)\Big\}d\lambda. \tag{2.1.73}$$

For a thermal equilibrium state, the free energy should be minimized with the variation taken with respect to $\rho_n(\lambda)$, i.e.,

$$\frac{\delta f}{\delta \rho_n(\lambda)} = 0, \tag{2.1.74}$$

which leads to

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \Big\{ \varepsilon_n^0(\lambda)\delta\rho_n(\lambda) - T\ln[1+\eta_n(\lambda)]\delta\rho_n(\lambda) \\ -T\ln[1+\eta_n^{-1}(\lambda)]\delta\rho_n^h(\lambda) \Big\} d\lambda = 0, \tag{2.1.75}$$

where

$$\eta_n(\lambda) = \frac{\rho_n^h(\lambda)}{\rho_n(\lambda)}. \tag{2.1.76}$$

Note that $\delta\rho_n(\lambda)$ and $\delta\rho_m^h(\lambda)$ are not independent but are related through the following equation derived from Eq. (2.1.61):

$$\delta\rho_n^h(\lambda) = -\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} A_{m,n}(\lambda-\mu)\delta\rho_m(\mu)d\mu. \tag{2.1.77}$$

Substituting Eq. (2.1.77) into Eq. (2.1.75) and putting the coefficient of $\delta\rho_n(\lambda)$ to zero, we have

$$\ln[1+\eta_n(\lambda)] = \frac{\varepsilon_n^0(\lambda)}{T} + \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} A_{m,n}(\lambda-\mu)\ln[1+\eta_m^{-1}(\mu)]d\mu. \tag{2.1.78}$$

For convenience, we introduce the integral operators $[n]$ as

$$[n]F(\lambda) = \int_{-\infty}^{\infty} a_n(\lambda-\mu)F(\mu)d\mu. \tag{2.1.79}$$

Note that under Fourier transformation, $[n]$ becomes a multiplier $\exp(-n|\omega|/2)$. It can be easily demonstrated that the following relation holds:

$$[m][n] = [m+n]. \tag{2.1.80}$$

Further, we define

$$\hat{A}_{m,n} = [m+n] + 2[m+n-2] + \cdots + 2[|m-n|+2] + [|m-n|],$$
$$\hat{G} = \frac{[1]}{[0]+[2]}, \tag{2.1.81}$$

where the kernel of the operator $\hat{G}$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda\omega}\frac{e^{-\frac{1}{2}|\omega|}}{1+e^{-|\omega|}}d\omega = \frac{1}{2\cosh(\pi\lambda)} = \rho_g(\lambda). \tag{2.1.82}$$

It can be proven that the following operator identities hold:

$$\hat{G}[\hat{A}_{m,n+1} + \hat{A}_{m,n-1}] = -\delta_{m,n} + \hat{A}_{m,n}, \quad n > 1, \tag{2.1.83}$$
$$\hat{G}\hat{A}_{m,2} = -\delta_{1,m} + \hat{A}_{1,m}. \tag{2.1.84}$$

With the help of the above relations, we rewrite Eq. (2.1.78) as

$$\ln(1+\eta_n(\lambda)) = \frac{\varepsilon_n^0(\lambda)}{T} + \sum_{m=1}^{\infty}\hat{A}_{n,m}\ln(1+\eta_m^{-1}(\lambda)). \tag{2.1.85}$$

Applying the integral operator $\hat{G}$ to the summation of Eq. (2.1.85) with $n+1$ and $n-1$, we obtain

$$\begin{aligned}
&\hat{G}[\ln(1+\eta_{n+1}(\lambda)) + \ln(1+\eta_{n-1}(\lambda))] \\
&= \frac{1}{T}\hat{G}(\varepsilon_{n+1}^0(\lambda) + \varepsilon_{n-1}^0(\lambda)) + \sum_{m=1}^{\infty}\hat{G}(\hat{A}_{n+1,m} + \hat{A}_{n-1,m})\ln(1+\eta_m^{-1}(\lambda)) \\
&= \frac{\varepsilon_n^0(\lambda)}{T} - \ln(1+\eta_n^{-1}(\lambda)) + \sum_{m=1}^{\infty}\hat{A}_{n,m}\ln(1+\eta_m^{-1}(\lambda)).
\end{aligned} \tag{2.1.86}$$

Combining Eqs. (2.1.85) and (2.1.86), we arrive at

$$\ln\eta_n(\lambda) = \hat{G}[\ln(1+\eta_{n+1}(\lambda)) + \ln(1+\eta_{n-1}(\lambda))]. \tag{2.1.87}$$

Again, applying the integral operator $\hat{G}$ on Eq. (2.1.85) with $n=2$, we obtain

$$\ln\eta_1(\lambda) = -\frac{2\pi\rho_g(\lambda)}{T} + \hat{G}\ln(1+\eta_2(\lambda)). \tag{2.1.88}$$

For the case of $n \to \infty$, from Eq. (2.1.85), we learn that

$$\lim_{n\to\infty} \frac{\ln \eta_n}{n} = \frac{h}{T}. \tag{2.1.89}$$

Equations (2.1.87)–(2.1.89) form a closed set of equations for the thermodynamic quantity η_n.

Substituting Eq. (2.1.61) into Eq. (2.1.73) and using Eq. (2.1.75), we obtain

$$f = -T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} a_n(\lambda) \ln[1 + \eta_n^{-1}(\lambda)] d\lambda. \tag{2.1.90}$$

From (2.1.85), we know that

$$\hat{G} \ln(1 + \eta_1(\lambda)) = \frac{1}{T} \hat{G} \varepsilon_1^0(\lambda) + \sum_{m=1}^{\infty} [m] \ln(1 + \eta_m^{-1}(\lambda)). \tag{2.1.91}$$

Putting $\lambda = 0$ in the above equation we have

$$\begin{aligned} &\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} a_m(\lambda) \ln[1 + \eta_m^{-1}(\lambda)] d\lambda \\ &\quad = \frac{2 \ln 2 - \frac{1}{2h}}{T} + \int_{-\infty}^{\infty} \rho_g(\lambda) \ln[1 + \eta_1(\lambda)] d\lambda. \end{aligned} \tag{2.1.92}$$

Substituting (2.1.92) into (2.1.90), we finally get the expression for the free energy

$$F/N = e_g - T \int_{-\infty}^{\infty} \rho_g(\lambda) \ln[1 + \eta_1(\lambda)] d\lambda. \tag{2.1.93}$$

We remark that though initially the string hypothesis is used, the final formula for the free energy is only related to the real root distribution.

Generally, the thermodynamic BAEs cannot be solved exactly. Below, let us consider the low energy limit $T \to 0$ and $h \to 0$. Because $\varepsilon(\lambda) > 0$, the driving term of Eq. (2.1.88) tends to $-\infty$ and $\eta_1(\lambda) \to 0$. This indicates that $\rho_1^h(\lambda) = 0$ at zero temperature, which coincides with the density configuration of the ground state previously derived. In this case, all the $\eta_n(\lambda)$ become constants and the integral Eqs. (2.1.87)–(2.1.89) are reduced to

$$\eta_n^2 = (1 + \eta_{n+1})(1 + \eta_{n-1}), \quad n > 1. \tag{2.1.94}$$

The general solution of the above equations is [31]

$$\eta_n = \left(\frac{bz^n - b^{-1}z^{-n}}{z - z^{-1}}\right)^2 - 1, \tag{2.1.95}$$

where b and z are determined by $\eta_1 = 0$ and (2.1.89), i.e., $b = 1, z = e^{\frac{h}{2T}}$. Therefore, the solution of thermodynamic BAEs in the limit $T \to 0$ is

$$\eta_n = \frac{\sinh^2 \frac{nh}{2T}}{\sinh^2 \frac{h}{2T}} - 1. \tag{2.1.96}$$

In the case of $h = 0$, we have $\eta_n(h = 0) = n^2 - 1$. Putting $\eta_n = \exp[-\varepsilon_n/T]$, then $\varepsilon_1 \sim T^0$. Comparing the T^{-1} terms of Eq. (2.1.85) we have

$$\varepsilon_1(\lambda) = -\varepsilon_1^0(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - \mu)\varepsilon_1(\mu)d\mu. \tag{2.1.97}$$

From Eqs. (2.1.50) and (2.1.97), we obtain that $\varepsilon_1(\lambda)$ is the dressed energy $\varepsilon(\lambda)$.

Since the quasi momentum is $p(\lambda) = 2\pi Z(\lambda)$, under the Fermi liquid framework [40] we define the density of states as

$$N(\lambda) = \frac{1}{\pi}\left|\frac{dp(\lambda)}{d\varepsilon(\lambda)}\right| = 2\left|\frac{\rho_g(\lambda)}{\varepsilon'(\lambda)}\right|. \tag{2.1.98}$$

In the ground state, the density of states reads $N(\lambda) = \pi^{-2}|\coth(\pi\lambda)|$ and at the Fermi surface it is $N(\infty) = \pi^{-2}$. Up to leading order, the density of free energy reads

$$\begin{aligned} f &= -T\int_{-\infty}^{\infty} N(\lambda) \ln\left[1 + e^{-\frac{|\varepsilon(\lambda)|}{T}}\right] d\varepsilon(\lambda) \\ &\approx -\frac{T^2}{\pi^2}\int_{-\infty}^{\infty} \ln(1 + e^{-|x|})dx = -\frac{1}{6}T^2. \end{aligned} \tag{2.1.99}$$

2.2 The Open Heisenberg Spin Chain

2.2.1 The Algebraic Bethe Ansatz

The algebraic Bethe Ansatz for open integrable models can be performed through the combination of YBE and RE. As an example, let us consider the isotropic Heisenberg spin chain with two boundary magnetic fields, a model first exactly solved by Alcaraz et al. via coordinate Bethe Ansatz [41]. The model Hamiltonian is

$$H = \sum_{j=1}^{N-1} \sigma_j \cdot \sigma_{j+1} + h_1 \sigma_1^z + h_N \sigma_N^z, \tag{2.2.1}$$

where h_1 and h_N are the boundary fields.

For the open boundary models, rather than the one-row monodromy matrix, we need to introduce the double-row monodromy matrix

$$\mathscr{U}_0(u) = T_0(u) K_0^-(u) \hat{T}_0(u), \tag{2.2.2}$$

where $K_0^-(u)$ is the solution of RE and

$$\hat{T}_0(u) \equiv (1-u^2)^N T_0^{-1}(-u) = R_{1,0}(u) \cdots R_{N,0}(u). \tag{2.2.3}$$

Here, the R-matrix is defined by (1.5.2) and RE reads

$$\begin{aligned} &R_{1,2}(u-v) K_1^-(u) R_{2,1}(u+v) K_2^-(v) \\ &\quad = K_2^-(v) R_{1,2}(u+v) K_1^-(u) R_{2,1}(u-v). \end{aligned} \tag{2.2.4}$$

It can be demonstrated that $\mathscr{U}_0(u)$ also satisfies the RE

$$\begin{aligned} &R_{1,2}(u-v)\mathscr{U}_1(u)R_{1,2}(u+v)\mathscr{U}_2(v) \\ &\quad = R_{1,2}(u-v)T_1(u)K_1^-(u)\hat{T}_1(u)R_{1,2}(u+v)T_2(v)K_2^-(v)\hat{T}_2(v) \\ &\quad = R_{1,2}(u-v)T_1(u)K_1^-(u)T_2(v)R_{1,2}(u+v)\hat{T}_1(u)K_2^-(v)\hat{T}_2(v) \\ &\quad = R_{1,2}(u-v)T_1(u)T_2(v)K_1^-(u)R_{1,2}(u+v)\hat{T}_1(u)K_2^-(v)\hat{T}_2(v) \\ &\quad = T_2(v)T_1(u)R_{1,2}(u-v)K_1^-(u)R_{1,2}(u+v)K_2^-(v)\hat{T}_1(u)\hat{T}_2(v) \\ &\quad = T_2(v)T_1(u)K_2^-(v)R_{1,2}(u+v)K_1^-(u)R_{1,2}(u-v)\hat{T}_1(u)\hat{T}_2(v) \\ &\quad = T_2(v)K_2^-(v)T_1(u)R_{1,2}(u+v)K_1^-(u)\hat{T}_2(v)\hat{T}_1(u)R_{1,2}(u-v) \\ &\quad = T_2(v)K_2^-(v)T_1(u)R_{1,2}(u+v)\hat{T}_2(v)K_1^-(u)\hat{T}_1(u)R_{12}(u-v) \\ &\quad = T_2(v)K_2^-(v)\hat{T}_2(v)R_{1,2}(u+v)T_1(u)K_1^-(u)\hat{T}_1(u)R_{1,2}(u-v) \\ &\quad = \mathscr{U}_2(v)R_{1,2}(u+v)\mathscr{U}_1(u)R_{1,2}(u-v). \end{aligned} \tag{2.2.5}$$

Note that the following relations are used in deriving the above relation:

$$R_{1,2}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{1,2}(u-v), \tag{2.2.6}$$

$$R_{1,2}^{-1}(u-v) = \frac{1}{1-(u-v)^2} R_{1,2}(-u+v), \tag{2.2.7}$$

$$\hat{T}_1(u)R_{1,2}(u+v)T_2(v) = T_2(v)R_{1,2}(u+v)\hat{T}_1(u). \tag{2.2.8}$$

The transfer matrix of the model with open boundary conditions is constructed by the double-row monodromy matrix as

$$t(u) = tr_0 \left\{ K_0^+(u) \mathscr{U}_0(u) \right\}, \tag{2.2.9}$$

where $K_0^+(u)$ is a solution of the dual RE (1.2.16) which now reads as follows due to $\mathscr{M} = \mathrm{id}$ (see (1.5.10))

$$\begin{aligned} &R_{1,2}(v-u)\, K_1^+(u)\, R_{2,1}(-u-v-2)\, K_2^+(v) \\ &\quad = K_2^+(v) R_{1,2}(-u-v-2)\, K_1^+(u)\, R_{2,1}(v-u). \end{aligned} \tag{2.2.10}$$

With the help of (2.2.5) and the properties (1.5.4)–(1.5.10), we can derive that

$$\begin{aligned}
t(u)\, t(v) &= tr_1 \left\{ K_1^+(u) \mathscr{U}_1(u) \right\} tr_2 \left\{ K_2^+(v) \mathscr{U}_2(v) \right\} \\
&= tr_1 \left\{ K_1^{+t_1}(u) \mathscr{U}_1^{t_1}(u) \right\} tr_2 \left\{ K_2^+(v) \mathscr{U}_2(v) \right\} \\
&= tr_{1,2} \left\{ K_1^{+t_1}(u) \mathscr{U}_1^{t_1}(u)\, K_2^+(v) \mathscr{U}_2(v) \right\} \\
&= tr_{1,2} \left\{ K_1^{+t_1}(u) K_2^+(v)\, \mathscr{U}_1^{t_1}(u) \mathscr{U}_2(v) \right\} \\
&= tr_{1,2} \left\{ K_1^{+t_1}(u) K_2^+(v)\, R_{2,1}^{t_1,-1}(v+u) R_{2,1}^{t_1}(v+u)\, \mathscr{U}_1^{t_1}(u) \mathscr{U}_2(v) \right\} \\
&= tr_{1,2} \left\{ [K_2^+(v) K_1^{+t_1}(u) R_{2,1}^{t_1,-1}(v+u)]^{t_1}\, [R_{2,1}^{t_1}(v+u) \mathscr{U}_1^{t_1}(u) \mathscr{U}_2(v)]^{t_1} \right\} \\
&= tr_{1,2} \left\{ [K_2^+(v) R_{2,1}^{t_1,-1,t_1}(v+u) K_1^+(u)]\, [\mathscr{U}_1(u) R_{2,1}(v+u) \mathscr{U}_2(v)] \right\} \\
&= tr_{1,2} \left\{ [K_2^+(v) R_{2,1}^{t_1,-1,t_1}(v+u) K_1^+(u)]\, [R_{1,2}^{-1}(u-v) R_{1,2}(u-v)] \right. \\
&\qquad \left. \times\, [\mathscr{U}_1(u) R_{2,1}(v+u) \mathscr{U}_2(v)] \right\} \\
&= tr_{1,2} \left\{ [K_2^+(v) R_{2,1}^{t_1,-1,t_1}(v+u) K_1^+(u) R_{1,2}^{-1}(u-v)] \right. \\
&\qquad \left. \times\, [R_{1,2}(u-v) \mathscr{U}_1(u) R_{2,1}(v+u) \mathscr{U}_2(v)] \right\} \\
&\overset{(2.2.5)}{=} tr_{1,2} \left\{ [K_2^+(v) R_{2,1}^{t_1,-1,t_1}(v+u) K_1^+(u) R_{1,2}^{-1}(u-v)] \right. \\
&\qquad \left. \times [\mathscr{U}_2(v) R_{1,2}(u+v) \mathscr{U}_1(u) R_{2,1}(-v+u)] \right\} \\
&\overset{(1.2.17)}{=} tr_{1,2} \left\{ [R_{2,1}^{-1}(-v+u) K_1^+(u) R_{1,2}^{t_2,-1,t_2}(u+v) K_2^+(v)] \right. \\
&\qquad \left. \times [\mathscr{U}_2(v) R_{1,2}(u+v) \mathscr{U}_1(u) R_{2,1}(-v+u)] \right\} \\
&= tr_{1,2} \left\{ R_{2,1}^{-1}(-v+u)\, [K_1^+(u) R_{1,2}^{t_2,-1,t_2}(u+v) K_2^+(v) \right. \\
&\qquad \left. \times\, \mathscr{U}_2(v) R_{1,2}(u+v) \mathscr{U}_1(u)]\, R_{2,1}(-v+u) \right\} \\
&= tr_{1,2} \left\{ K_1^+(u) R_{1,2}^{t_2,-1,t_2}(u+v) K_2^+(v) \mathscr{U}_2(v) R_{1,2}(u+v) \mathscr{U}_1(u) \right\} \\
&= tr_{1,2} \left\{ [K_1^+(u) R_{1,2}^{t_2,-1,t_2}(u+v) K_2^+(v)]^{t_2}\, [\mathscr{U}_2(v) R_{1,2}(u+v) \mathscr{U}_1(u)]^{t_2} \right\} \\
&= tr_{1,2} \left\{ [K_1^+(u) K_2^{+t_2}(v) R_{1,2}^{t_2,-1}(u+v)]\, [R_{1,2}^{t_2}(u+v) \mathscr{U}_2^{t_2}(v) \mathscr{U}_1(u)] \right\}
\end{aligned}$$

$$
\begin{aligned}
&= tr_{1,2}\left\{[K_1^+(u)K_2^{+t_2}(v)\ \mathscr{U}_2^{t_2}(v)\mathscr{U}_1(u)]\right\} \\
&= tr_{1,2}\left\{[K_2^{+t_2}(v)\mathscr{U}_2^{t_2}(v)]\ [K_1^+(u)\mathscr{U}_1(u)]\right\} \\
&= tr_2\left\{K_2^+(v)\mathscr{U}_2(v)\right\}\ tr_1\left\{K_1^+(u)\mathscr{U}_1(u)\right\} = t(v)\,t(u).
\end{aligned} \tag{2.2.11}
$$

Therefore, the transfer matrices with different spectral parameters are mutually commutative,

$$
[t(u),\ t(v)] = 0. \tag{2.2.12}
$$

The general solutions of $K_0^{\pm}(u)$ were given in [42–45]. Here we choose the diagonal ones

$$
K_0^-(u) = p + u\sigma_0^z, \quad K_0^+(u) = q + (u+1)\sigma_0^z, \tag{2.2.13}
$$

which allow us to perform the algebraic Bethe Ansatz, where p and q are two boundary parameters. Taking the derivative of the logarithm of the transfer matrix, we obtain

$$
\left.\frac{\partial t(u)}{\partial u}\right|_{u=0} = 2pK_N^+(0) + 4pq\sum_{j=1}^{N-1} P_{j,j+1} + 2q\sigma_1^z. \tag{2.2.14}
$$

Therefore, the Hamiltonian (2.2.1) can be constructed by the transfer matrix as

$$
H = \frac{1}{2pq}\left.\frac{\partial t(u)}{\partial u}\right|_{u=0} - N, \tag{2.2.15}
$$

with the boundary parameters p and q determined by h_1 and h_N as

$$
p = \frac{1}{h_1}, \quad q = \frac{1}{h_N}. \tag{2.2.16}
$$

Denote the double-row monodromy matrix as

$$
\mathscr{U}(u) \equiv \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) \end{pmatrix}. \tag{2.2.17}
$$

By using RE (2.2.5), we can derive the following useful commutation relations:

$$
[\mathscr{B}(u), \mathscr{B}(v)] = [\mathscr{C}(u), \mathscr{C}(v)] = 0, \tag{2.2.18}
$$

$$
\begin{aligned}
\mathscr{A}(u)\mathscr{B}(v) &= \frac{(u+v)(u-v-1)}{(u-v)(u+v+1)}\mathscr{B}(v)\mathscr{A}(u) + \frac{u+v}{(u-v)(u+v+1)}\mathscr{B}(u)\mathscr{A}(v) \\
&- \frac{1}{u+v+1}\mathscr{B}(u)\mathscr{D}(v),
\end{aligned} \tag{2.2.19}
$$

$$\mathscr{D}(u)\mathscr{B}(v) = \frac{(u-v+1)(u+v+2)}{(u-v)(u+v+1)}\mathscr{B}(v)\mathscr{D}(u) - \frac{(u+v+2)}{(u-v)(u+v+1)}\mathscr{B}(u)\mathscr{D}(v)$$
$$- \frac{2}{(u-v)(u+v+1)}\mathscr{B}(v)\mathscr{A}(u) + \frac{(u-v+2)}{(u-v)(u+v+1)}\mathscr{B}(u)\mathscr{A}(v). \quad (2.2.20)$$

For convenience, let us introduce

$$\overline{\mathscr{D}}(u) = (2u+1)\mathscr{D}(u) - \mathscr{A}(u). \quad (2.2.21)$$

The transfer matrix thus reads

$$\begin{aligned} t(u) &= (q+u+1)\mathscr{A}(u) + (q-u-1)\mathscr{D}(u) \\ &= \frac{q-u-1}{2u+1}\overline{\mathscr{D}}(u) + \left(\frac{q-u-1}{2u+1} + q + u + 1\right)\mathscr{A}(u), \end{aligned} \quad (2.2.22)$$

and

$$\overline{\mathscr{D}}(u)\mathscr{B}(v) = \frac{(u-v+1)(u+v+2)}{(u-v)(u+v+1)}\mathscr{B}(v)\overline{\mathscr{D}}(u) - \frac{2(u+1)}{(u-v)(2v+1)}\mathscr{B}(u)\overline{\mathscr{D}}(v)$$
$$+ \frac{4(u+1)v}{(2v+1)(u+v+1)}\mathscr{B}(u)\mathscr{A}(v), \quad (2.2.23)$$

$$\mathscr{A}(u)\mathscr{B}(v) = \frac{(u+v)(u-v-1)}{(u-v)(u+v+1)}\mathscr{B}(v)\mathscr{A}(u) - \frac{1}{(u+v+1)(2v+1)}\mathscr{B}(u)\overline{\mathscr{D}}(v)$$
$$+ \frac{2v}{(u-v)(2v+1)}\mathscr{B}(u)\mathscr{A}(v). \quad (2.2.24)$$

Let us introduce further the notations

$$\begin{aligned} \mathscr{B}_M &= \mathscr{B}(\lambda_1)\cdots\mathscr{B}(\lambda_M), \\ \mathscr{B}_M^j &= \mathscr{B}(\lambda_1)\cdots\mathscr{B}(\lambda_{j-1})\mathscr{B}(u)\mathscr{B}(\lambda_{j+1})\cdots\mathscr{B}(\lambda_M). \end{aligned} \quad (2.2.25)$$

By using the commutation relations (2.2.23) and (2.2.24), we can prove that

$$\mathscr{A}(u)\mathscr{B}_M = \prod_{j=1}^{M}\frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)}\mathscr{B}_M\mathscr{A}(u)$$
$$- \sum_{j=1}^{M}\frac{1}{(u+\lambda_j+1)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\overline{\mathscr{D}}(\lambda_j)$$

$$+\sum_{j=1}^{M}\frac{2\lambda_j}{(u-\lambda_j)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\mathscr{A}(\lambda_j), \tag{2.2.26}$$

$$\begin{aligned}\overline{\mathscr{D}}(u)\mathscr{B}_M &= \mathscr{B}_M\overline{\mathscr{D}}(u)\prod_{j=1}^{M}\frac{(u+\lambda_j+2)(u-\lambda_j+1)}{(u+\lambda_j+1)(u-\lambda_j)}\\ &-\sum_{j=1}^{M}\frac{2(u+1)}{(u-\lambda_j)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\overline{\mathscr{D}}(\lambda_j)\\ &+\sum_{j=1}^{M}\frac{4\lambda_j(u+1)}{(u+\lambda_j+1)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\mathscr{A}(\lambda_j).\end{aligned} \tag{2.2.27}$$

Proof Obviously, Eq. (2.2.26) is satisfied for $M=1$. Suppose it is also satisfied for an arbitrary M. We have

$$\begin{aligned}&\mathscr{A}(u)\mathscr{B}_{M+1}=\mathscr{A}(u)\mathscr{B}_M\mathscr{B}(\lambda_{M+1})\\ &=\prod_{j=1}^{M}\frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)}\mathscr{B}_M\mathscr{A}(u)\mathscr{B}(\lambda_{M+1})\\ &-\sum_{j=1}^{M}\frac{1}{(u+\lambda_j+1)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\overline{\mathscr{D}}(\lambda_j)\mathscr{B}(\lambda_{M+1})\\ &+\sum_{j=1}^{M}\frac{2\lambda_j}{(u-\lambda_j)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\mathscr{B}_M^j\mathscr{A}(\lambda_j)\mathscr{B}(\lambda_{M+1}).\end{aligned} \tag{2.2.28}$$

The first term can be calculated as

$$\begin{aligned}&\mathscr{B}_{M+1}\mathscr{A}(u)\prod_{j=1}^{M+1}\frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)}\\ &+\prod_{j=1}^{M}\frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)}\frac{2\lambda_{M+1}}{(u-\lambda_{M+1})(2\lambda_{M+1}+1)}\mathscr{B}_{M+1}^{M+1}\mathscr{A}(\lambda_{M+1})\\ &-\prod_{j=1}^{M}\frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)}\frac{\mathscr{B}_{M+1}^{M+1}\overline{\mathscr{D}}(\lambda_{M+1})}{(u-\lambda_{M+1}+1)(2\lambda_{M+1}+1)}.\end{aligned} \tag{2.2.29}$$

The second term can be calculated as

$$-\sum_{j=1}^{M}\frac{1}{(u+\lambda_j+1)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}$$

$$
\begin{aligned}
&\times\left[\frac{(\lambda_j-\lambda_{M+1}+1)(\lambda_j+\lambda_{M+1}+2)}{(\lambda_j-\lambda_{M+1})(\lambda_j+\lambda_{M+1}+1)}\mathscr{B}^j_{M+1}\overline{\mathscr{D}}(\lambda_j)\right.\\
&-\frac{2(\lambda_j+1)}{(\lambda_j-\lambda_{M+1})(2\lambda_{M+1}+1)}\mathscr{B}^j_M\mathscr{B}(\lambda_j)\overline{\mathscr{D}}(\lambda_{M+1})\\
&\left.+\frac{4(\lambda_j+1)\lambda_{M+1}}{(\lambda_j+\lambda_{M+1}+1)(2\lambda_{M+1}+1)}\mathscr{B}^j_M\mathscr{B}(\lambda_j)\mathscr{A}(\lambda_{M+1})\right].
\end{aligned}
\tag{2.2.30}
$$

The third term can be calculated as

$$
\begin{aligned}
&\sum_{j=1}^{M}\frac{2\lambda_j}{(u-\lambda_j)(2\lambda_j+1)}\prod_{l\neq j}^{M}\frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}\\
&\quad\times\left[\frac{(\lambda_j+\lambda_{M+1})(\lambda_j-\lambda_{M+1}-1)}{(\lambda_j-\lambda_{M+1})(\lambda_j+\lambda_{M+1}+1)}\mathscr{B}^j_{M+1}\mathscr{A}(\lambda_j)\right.\\
&\quad-\frac{1}{(\lambda_j+\lambda_{M+1}+1)(2\lambda_{M+1}+1)}\mathscr{B}^j_M\mathscr{B}(\lambda_j)\overline{\mathscr{D}}(\lambda_{M+1})\\
&\quad\left.+\frac{2\lambda_{M+1}}{(\lambda_j-\lambda_{M+1})(2\lambda_{M+1}+1)}\mathscr{B}^j_M\mathscr{B}(\lambda_j)\mathscr{A}(\lambda_{M+1})\right].
\end{aligned}
\tag{2.2.31}
$$

Comparing the coefficients of the terms including $\mathscr{A}(u)$, $\mathscr{A}(\lambda_j)$, $\overline{\mathscr{D}}(\lambda_j)$, $\mathscr{A}(\lambda_{M+1})$, $\overline{\mathscr{D}}(\lambda_{M+1})$ and using the properties

$$
\mathscr{B}^{M+1}_{M+1}=\mathscr{B}(u)\mathscr{B}_M,
\tag{2.2.32}
$$

$$
\mathscr{B}_{M+1}=\mathscr{B}(\lambda_{M+1})\mathscr{B}_M,
\tag{2.2.33}
$$

we arrive at Eq. (2.2.26). Equation (2.2.27) can be proven similarly. □

With the crossing property (1.5.6), we obtain the duality relation between $\hat{T}(u)$ and $T(u)$

$$
\begin{aligned}
\sigma_0^y[\hat{T}_0(u)]^{t_0}\sigma_0^y &= \sigma_0^y[R_{0,1}(u)\cdots R_{0,N}(u)]^{t_0}\sigma_0^y\\
&=\sigma_0^y R_{0,N}^{t_0}(u)\sigma_0^y\sigma_0^y R_{0,N-1}^{t_0}(u)\sigma_0^y\cdots\sigma_0^y R_{0,1}^{t_0}(u)\sigma_0^y\\
&=(-1)^N R_{0,N}(-u-1)R_{0,N-1}(-u-1)\cdots R_{0,1}(-u-1)\\
&=(-1)^N T_0(-u-1).
\end{aligned}
\tag{2.2.34}
$$

Thus the matrix elements of $\hat{T}(u)$ can be expressed by those of $T(u)$ with a different spectral parameter

$$
\hat{T}_0(u)=(-1)^N\begin{pmatrix} D(-u-1) & -B(-u-1)\\ -C(-u-1) & A(-u-1)\end{pmatrix}.
\tag{2.2.35}
$$

The double-row monodromy matrix is thus factorized as

$$\begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) \end{pmatrix} = (-1)^N \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \times \begin{pmatrix} p+u & 0 \\ 0 & p-u \end{pmatrix} \begin{pmatrix} D(-u-1) & -B(-u-1) \\ -C(-u-1) & A(-u-1) \end{pmatrix}. \tag{2.2.36}$$

With the help of Eq. (2.1.8), we obtain

$$\begin{aligned} \mathscr{A}(u)|0\rangle &= (p+u)(u+1)^{2N}|0\rangle, \\ \overline{\mathscr{D}}(u)|0\rangle &= 2(p-u-1)u^{2N+1}|0\rangle, \\ \mathscr{C}(u)|0\rangle &= 0. \end{aligned} \tag{2.2.37}$$

Therefore, $|0\rangle$ is an eigenstate of $\mathscr{A}(u)$ and $\overline{\mathscr{D}}(u)$. $\mathscr{B}(u)$ can be used as the generating operator for the eigenstates.

Assume that the eigenstates of the transfer matrix take the following form

$$|\, \lambda_1, \ldots, \lambda_M\rangle = \prod_{j=1}^{M} \mathscr{B}(\lambda_j)|0\rangle. \tag{2.2.38}$$

The transfer matrix applied on the state (2.2.38) gives

$$\begin{aligned} t(u)\,|\, \lambda_1, \ldots, \lambda_M\rangle &= \Lambda(u)\,|\, \lambda_1, \ldots, \lambda_M\rangle \\ &+ \sum_{j=1}^{M} \Lambda_j(u)\mathscr{B}(\lambda_1)\cdots\mathscr{B}(\lambda_{j-1})\mathscr{B}(u)\mathscr{B}(\lambda_{j+1})\cdots\mathscr{B}(\lambda_M)\,|\,0\rangle, \end{aligned} \tag{2.2.39}$$

where $\Lambda(u)$ is the eigenvalue term

$$\begin{aligned} \Lambda(u) = &\left(\frac{q-u-1}{2u+1} + q + u + 1\right) \\ &\times (p+u)(u+1)^{2N} \prod_{j=1}^{M} \frac{(u+\lambda_j)(u-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)} \\ &+ 2\frac{q-u-1}{2u+1}(p-u-1)u^{2N+1} \prod_{j=1}^{M} \frac{(u-\lambda_j+1)(u+\lambda_j+2)}{(u-\lambda_j)(u+\lambda_j+1)}, \end{aligned} \tag{2.2.40}$$

and the unwanted coefficients $\Lambda_j(u)$ read

$$\Lambda_j(u) = \frac{4(u+1)(q+\lambda_j)(p+\lambda_j)\lambda_j}{(u-\lambda_j)(u+\lambda_j+1)(2\lambda_j+1)}(\lambda_j+1)^{2N} \prod_{l \neq j}^{M} \frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}$$

$$-\frac{4(u+1)(q-\lambda_j-1)(p-\lambda_j-1)}{(u-\lambda_j)(u+\lambda_j+1)(2\lambda_j+1)}\lambda_j^{2N+1}\prod_{l\neq j}^{M}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l)(\lambda_j+\lambda_l+1)}.$$

Putting $\Lambda_j(u)=0$, we obtain the BAEs

$$\frac{(\lambda_j+q)(\lambda_j+p)}{(\lambda_j+1-q)(\lambda_j+1-p)}\left(1+\frac{1}{\lambda_j}\right)^{2N} \times\prod_{l\neq j}^{M}\frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}=1,\quad j=1,\ldots,M. \tag{2.2.41}$$

Similarly, the selection rules $\lambda_j\neq\lambda_l$ and $\lambda_j\neq-\lambda_l-1$ are required as those for the periodic boundary case discussed in Sect. 2.1.2.

Assume that $\lambda_j=i\mu_j-\frac{1}{2}$. Equation (2.2.41) can be rewritten as

$$\frac{(\mu_j-i\bar{q})(\mu_j-i\bar{p})}{(\mu_j+i\bar{q})(\mu_j+i\bar{p})}\left(\frac{\mu_j-\frac{i}{2}}{\mu_j+\frac{i}{2}}\right)^{2N}=\prod_{l\neq j}^{M}\frac{(\mu_j-\mu_l-i)(\mu_j+\mu_l-i)}{(\mu_j-\mu_l+i)(\mu_j+\mu_l+i)}, \tag{2.2.42}$$

with $\bar{p}=p-\frac{1}{2},\bar{q}=q-\frac{1}{2}$ and $j=1,\ldots,M$. The eigenvalue of the Hamiltonian is

$$\begin{aligned}E&=\frac{1}{2pq}\frac{\partial\Lambda(u)}{\partial u}\Big|_{u=0}-N\\&=-\sum_{j=1}^{M}\frac{2}{\mu_j^2+\frac{1}{4}}+N-1+\frac{1}{p}+\frac{1}{q}.\end{aligned} \tag{2.2.43}$$

Note that the unwanted terms can also be expressed as

$$\Lambda_j(u)=\frac{(u+1)(2\lambda_j+1)}{(\lambda_j-u)(u+\lambda_j+1)(\lambda_j+1)}\mathrm{res}\Lambda(u)|_{u=\lambda_j}, \tag{2.2.44}$$

which indicates that Baxter's $T-Q$ relation (1.4.1) also holds for this model with

$$\begin{aligned}a(u)&=\frac{2u+2}{2u+1}(u+p)(u+q)(u+1)^{2N},\\d(u)&=\frac{2u}{2u+1}(u-p+1)(u-q+1)u^{2N},\\Q(u)&=\prod_{j=1}^{M}(u-\lambda_j)(u+\lambda_j+1).\end{aligned} \tag{2.2.45}$$

2.2.2 Surface Energy and Boundary Bound States

Both the boundary fields and the open boundary itself contribute finite values to physical quantities such as the ground state energy and the free energy. The surface energy is a typical quantity to describe the boundary effects. It was first studied by Gaudin [46] and subsequently studied by a number of authors [41, 47–54].

Let us consider the case of $\bar{p},\ \bar{q} \geq 0$. In the ground state, all the Bethe roots should take real values. By taking the logarithm of Eq. (2.2.42), we obtain

$$\begin{aligned}&\theta_{2\bar{p}}(\mu_j) + \theta_{2\bar{q}}(\mu_j) + 2N\theta_1(\mu_j)\\ &\quad = 2\pi I_j + \sum_{l=1}^{M}[\theta_2(\mu_j - \mu_l) + \theta_2(\mu_j + \mu_l)] - \theta_1(\mu_j),\end{aligned} \tag{2.2.46}$$

where the θ_n-functions are defined below (2.1.34) and I_j are integers. Similar to the periodic case, we define

$$\begin{aligned}Z(u) = \frac{1}{2\pi}\Bigg\{&\theta_1(u) + \frac{1}{2N}[\theta_{2\bar{p}}(u) + \theta_{2\bar{q}}(u) + \theta_1(u)\\ &- \sum_{l=1}^{M}(\theta_2(u - \mu_l) + \theta_2(u + \mu_l))]\Bigg\}.\end{aligned} \tag{2.2.47}$$

It is obvious that $Z(\mu_j) = I_j/(2N)$. In the thermodynamic limit, the density distributions are determined by

$$\rho(u) + \rho^h(u) = \frac{dZ(u)}{du}. \tag{2.2.48}$$

Taking the derivative of $Z(u)$, we obtain

$$\begin{aligned}\rho(u) = a_1(u) &+ \frac{1}{2N}\left[a_{2\bar{p}}(u) + a_{2\bar{q}}(u) + a_1(u) - \delta(u)\right]\\ &- \int_{-\infty}^{\infty} a_2(u - v)\rho(v)dv,\end{aligned} \tag{2.2.49}$$

where we put $\rho^h(u) = \frac{1}{2N}\delta(u)$. The existence of $\delta(u)$ in the above equation is due to the hole at $I_j = 0$, which is a solution of the BAEs but leaves a zero wave function.

The density deviation from that of the periodic case satisfies

$$\begin{aligned}\delta\rho(u) = &\frac{1}{2N}\left[a_{2\bar{p}}(u) + a_{2\bar{q}}(u) + a_1(u) - \delta(u)\right]\\ &- \int_{-\infty}^{\infty} a_2(u - v)\delta\rho(v)dv.\end{aligned} \tag{2.2.50}$$

With the help of Fourier transformation, we obtain

$$\delta\tilde{\rho}(\omega) = \frac{1}{2N}\frac{e^{-\bar{p}|\omega|} + e^{-\bar{q}|\omega|} + e^{-\frac{|\omega|}{2}} - 1}{1 + e^{-|\omega|}}. \tag{2.2.51}$$

The surface energy can be calculated as

$$\begin{aligned}\varepsilon_b &= h_1 + h_N - 1 - 4\pi N\int_{-\infty}^{\infty} a_1(u)\delta\rho(u)du \\ &= h_1 + h_N - 1 - 2N\int_{-\infty}^{\infty} \tilde{a}_1(\omega)\delta\tilde{\rho}(\omega)d\omega \\ &= h_1 + h_N - 1 - 2\int_0^{\infty} \frac{e^{-p\omega} + e^{-q\omega} + e^{-\omega} - e^{-\frac{\omega}{2}}}{1 + e^{-\omega}}d\omega \\ &= h_1 + h_N - 1 + \pi - 2\ln 2 - 2\int_0^{\infty} \frac{e^{-\frac{\omega}{h_1}} + e^{-\frac{\omega}{h_N}}}{1 + e^{-\omega}}d\omega. \end{aligned} \tag{2.2.52}$$

Note that

$$N\int_{-\infty}^{\infty} \delta\rho(u)du = N\delta\tilde{\rho}(0) = \frac{1}{2}, \tag{2.2.53}$$

indicating that there is a boundary spin in the system. In fact, this boundary spin is carried by a boundary hole at $\lambda^h \to \infty$. If we replace $\rho^h(u)$ with

$$\rho^h(u) = \frac{1}{2N}\left[\delta(u) + \delta(u - \lambda^h) + \delta(u + \lambda^h)\right], \tag{2.2.54}$$

we obtain that $N\int_{-\infty}^{\infty}\delta\rho(u)du = 0$. This boundary hole carries zero energy and corresponds to the Majorana modes at the two boundaries [55].

For $\bar{p} < 0$, an imaginary mode $\mu_j = i\bar{p}$ may exist. This mode contributes a negative bare energy for $-\frac{1}{2} < \bar{p} < 0$ and a positive bare energy for $\bar{p} < -\frac{1}{2}$, indicating that the bound state is only stable in the former case.

In addition to the above solutions, the following boundary string

$$\mu_{b,l}^{m,n} = i\bar{p} + il, \quad l = -n, \ldots, 0, \ldots, m, \tag{2.2.55}$$

may exist, where $n > \bar{p} > 0$ or $m > -\bar{p} > 0$ is needed to preserve the equality of the BAEs. The bare energy of this boundary string reads

$$-2\pi\left[a_{2(m+p)}(0) + a_{2(n-p+1)}(0)\right].$$

The deviation of $\rho(u)$ implied by the boundary string satisfies

$$\delta\rho(u) = -[2]\delta\rho(u) - \frac{1}{2N}\left[a_{2(m+1+\bar{p})}(u) + a_{2(m+\bar{p})}(u) + a_{2(n+1-\bar{p})}(u) + a_{2(n-\bar{p})}(u)\right]. \tag{2.2.56}$$

With the help of Fourier transformation we have

$$\delta\tilde{\rho}(\omega) = -\frac{1}{2N}\left[e^{-(m+\bar{p})|\omega|} + e^{-(n-\bar{p})|\omega|}\right]. \tag{2.2.57}$$

Therefore, the contribution of the boundary string to the energy is

$$\begin{aligned}\varepsilon_{bs} &= \int_{-\infty}^{\infty}\left[e^{-(m+\bar{p})|\omega|} + e^{-(n-\bar{p})|\omega|}\right]e^{-\frac{1}{2}|\omega|}d\omega \\ &\quad -2\pi\left[a_{2(m+p)}(0) + a_{2(n-p+1)}(0)\right] \\ &= 0.\end{aligned} \tag{2.2.58}$$

The effect of the boundary string is similar to that of the bulk strings, i.e., contributing nothing to the energy.

2.3 Nested Algebraic Bethe Ansatz for $SU(n)$-Invariant Spin Chain

The integrability of the multi-component models was first studied by Sutherland [56] on the basis of Yang's work [57]. Subsequently, Sutherland realized that the corresponding $SU(n)$ spin chain is also exactly solvable [58]. In this section, we introduce the nested algebraic Bethe Ansatz method with the $SU(n)$-invariant quantum spin chain as an example.

The model Hamiltonian reads

$$H = \sum_{j=1}^{N} P_{j,j+1}, \tag{2.3.1}$$

where the permutation operator is defined in the tensor space of n-dimensional linear spaces

$$P_{j,j+1} = \sum_{\mu,\nu=1}^{n} E_j^{\mu,\nu} E_{j+1}^{\nu,\mu}, \tag{2.3.2}$$

and $\mu, \nu = 1, \ldots, n$, $E_j^{\mu,\nu}$ is the Weyl matrix (or the Hubbard operator)

$$E^{\mu,\nu} = |\mu\rangle\langle\nu|. \tag{2.3.3}$$

The R-matrix of the system is

$$R_{i,j}(u) = \alpha(u) + \beta(u) P_{i,j}, \tag{2.3.4}$$

where

$$\alpha(u) = \frac{u}{u+\eta}, \quad \beta(u) = \frac{\eta}{u+\eta}. \tag{2.3.5}$$

One can easily check that the above R-matrix satisfies YBE.

The monodromy matrix of the system is constructed by the R-matrices as

$$T_0(u) = R_{0,N}(u) R_{0,N-1}(u) \cdots R_{0,1}(u). \tag{2.3.6}$$

We can easily deduce the following Yang-Baxter relation:

$$\check{R}_{1,2}(u-v)[T(u) \otimes T(v)] = [T(v) \otimes T(u)] \check{R}_{1,2}(u-v), \tag{2.3.7}$$

where $\check{R}$ is the braided R-matrix with the definition

$$\check{R}_{1,2}(u) = P_{1,2} R_{1,2}(u). \tag{2.3.8}$$

The braided R-matrices satisfy the braided YBE

$$\check{R}_{1,2}(u-v) \check{R}_{2,3}(u) \check{R}_{1,2}(v) = \check{R}_{2,3}(v) \check{R}_{1,2}(u) \check{R}_{2,3}(u-v). \tag{2.3.9}$$

We write out the explicit form of the monodromy matrix in the auxiliary space:

$$T(u) = \begin{pmatrix} A_{1,1}(u) & \cdots & A_{1,n-1}(u) & B_1(u) \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1,1}(u) & \cdots & A_{n-1,n-1}(u) & B_{n-1}(u) \\ C_1(u) & \cdots & C_{n-1}(u) & D(u) \end{pmatrix}. \tag{2.3.10}$$

The transfer matrix of the system is the trace of the monodromy matrix in the auxiliary space

$$t(u) = tr_0 T_0(u) = A_{1,1}(u) + A_{2,2}(u) + \cdots + D(u). \tag{2.3.11}$$

From the Yang-Baxter relation (2.3.7) one can easily check that the transfer matrices with different spectral parameters are mutually commutative,

$$[t(u),\ t(v)] = 0. \tag{2.3.12}$$

Thus the system is integrable and the Hamiltonian (2.3.1) can be derived from the transfer matrix $t(u)$ as

$$H = \eta \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0} + N. \tag{2.3.13}$$

From (2.3.7), we can derive the useful commutation relations:

$$D(u)C_{b_1}(\lambda) = \frac{1}{\alpha(\lambda - u)} C_{b_1}(\lambda)D(u) - \frac{\beta(\lambda - u)}{\alpha(\lambda - u)} C_{b_1}(u)D(\lambda), \tag{2.3.14}$$

$$\begin{aligned} A_{b_1,b_2}(u)C_{b_3}(\lambda) = & \frac{\check{R}^{(1)}(u-\lambda)^{b_3,b_2}_{b_4,b_5}}{\alpha(u-\lambda)} C_{b_5}(\lambda)A_{b_1,b_4}(u) \\ & - \frac{\beta(u-\lambda)}{\alpha(u-\lambda)} C_{b_2}(u)A_{b_1,b_3}(\lambda), \end{aligned} \tag{2.3.15}$$

$$C_{b_1}(u)C_{b_2}(\lambda) = \check{R}^{(1)}(u-\lambda)^{b_2,b_1}_{c_1,c_2} C_{c_2}(\lambda)C_{c_1}(u), \tag{2.3.16}$$

where all the subscripts and superscripts take values of $1, \ldots, n-1$, repeated indices mean summation and

$$\begin{aligned} \check{R}^{(1)}_{i,j}(u) = & \sum_{b_1,b_2=1}^{n-1} \beta(u) E_i^{b_1,b_1} \otimes E_j^{b_2,b_2} + \sum_{b_1,b_2=1}^{n-1} \alpha(u) E_i^{b_1,b_2} \otimes E_j^{b_2,b_1} \\ \equiv & \beta(u) + \alpha(u) P^{(1)}_{i,j}, \end{aligned} \tag{2.3.17}$$

where $P^{(1)}_{i,j}$ is the permutation operator defined in the $SU(n-1)$ algebra. The $\check{R}^{(1)}(u)$ is the braided R-matrix of the $SU(n-1)$-invariant spin chain.

To construct the eigenstate, we choose the local vacuum as $|0\rangle_j = (0, 0, \ldots, 1)^t$, where t means transposition. The global vacuum state is the direct product of the local vacuum, $|0\rangle = \otimes_{j=1}^N |0\rangle_j$. Obviously,

$$R_{0,j}(u)|0\rangle_j = \begin{pmatrix} \alpha(u) & 0 & \cdots & 0 \\ 0 & \alpha(u) & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ \beta(u)E_j^{1,n} & \beta(u)E_j^{2,n} & \cdots & 1 \end{pmatrix} |0\rangle_j. \tag{2.3.18}$$

The above relation allows us to arrive at

$$T(u)|0\rangle = \begin{pmatrix} \alpha^N(u) & 0 & \cdots & 0 \\ 0 & \alpha^N(u) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ C_1(u) & C_2(u) & \cdots & 1 \end{pmatrix} |0\rangle. \tag{2.3.19}$$

Suppose that the Bethe states take the form:

$$|\lambda_1^{(1)}, \ldots, \lambda_{L_1}^{(1)}|F_1\rangle = C_{a_1}(\lambda_1^{(1)}) \cdots C_{a_{L_1}}(\lambda_{L_1}^{(1)})|0\rangle F_1^{a_{L_1} \cdots a_1}, \tag{2.3.20}$$

where $F_1^{a_{L_1}\cdots a_1}$ is a function of the Bethe roots $\lambda_j^{(1)}$ and L_1 is the number of the first set of Bethe roots. Applying the transfer matrix to the Bethe state (2.3.20) and using the commutation relations (2.3.14)–(2.3.16) repeatedly, we readily have

$$\begin{aligned}
&t(u)C_{a_1}(\lambda_1^{(1)})\cdots C_{a_{L_1}}(\lambda_{L_1}^{(1)})|0\rangle F_1^{a_{L_1}\cdots a_1}\\
&\quad=\left\{\alpha^N(u)\Lambda^{(1)}(u)\prod_{j=1}^{L_1}\frac{1}{\alpha(u-\lambda_j^{(1)})}+\prod_{j=1}^{L_1}\frac{1}{\alpha(\lambda_j^{(1)}-u)}\right\}|\lambda_1^{(1)},\ldots,\lambda_{L_1}^{(1)}|F_1\rangle\\
&\qquad-\sum_{j=1}^{L_1}\frac{\beta(u-\lambda_j^{(1)})}{\alpha(u-\lambda_j^{(1)})}\left\{\prod_{k=1,\neq j}^{L_1}\frac{1}{\alpha(\lambda_j^{(1)}-\lambda_k^{(1)})}\alpha^N(\lambda_j^{(1)})t^{(1)}(\lambda_j^{(1)})\right.\\
&\qquad\left.-\prod_{k=1,\neq j}^{L_1}\frac{1}{\alpha(\lambda_k^{(1)}-\lambda_j^{(1)})}\right\}|\ldots,\lambda_{j-1}^{(1)},u,\lambda_{j+1}^{(1)},\ldots|F_1\rangle,
\end{aligned}\tag{2.3.21}$$

where $\Lambda^{(1)}(u)$ is the eigenvalue of the next transfer matrix $t^{(1)}(u)$ defined below and

$$|\ldots,\lambda_{j-1}^{(1)},u,\lambda_{j+1}^{(1)},\ldots|F_1\rangle\equiv\cdots C_{a_{j-1}}(\lambda_{j-1}^{(1)})C_{a_j}(u)C_{a_{j+1}}(\lambda_{j+1}^{(1)})\cdots|0\rangle F_1^{a_{L_1}\cdots a_1},$$

indicates the unwanted terms. If the Bethe state is an eigenstate of the transfer matrix, the unwanted terms must be canceled. This leads to the first set of BAEs:

$$\begin{aligned}
&\prod_{k\neq j}^{L_1}\frac{\alpha(\lambda_j^{(1)}-\lambda_k^{(1)})}{\alpha(\lambda_k^{(1)}-\lambda_j^{(1)})}\frac{1}{\alpha^N(\lambda_j^{(1)})}F_1^{b_{L_1}\cdots b_1}=t^{(1)}(\lambda_j^{(1)})_{a_1\cdots a_{L_1}}^{b_1\cdots b_{N_1}}F_1^{a_{L_1}\cdots a_1},\\
&\qquad j=1,2,\ldots,L_1.
\end{aligned}\tag{2.3.22}$$

The corresponding eigenvalue of the transfer matrix thus reads

$$\Lambda(u)=\prod_{j=1}^{L_1}\frac{1}{\alpha(u-\lambda_j^{(1)})}\alpha^N(u)\Lambda^{(1)}(u)+\prod_{j=1}^{L_1}\frac{1}{\alpha(\lambda_j^{(1)}-u)}.\tag{2.3.23}$$

We note that the first set of BAEs is in fact a new eigenvalue problem.

Let us define the nested monodromy matrix as

$$T_0^{(1)}(u)=R_{0,L_1}^{(1)}(u-\lambda_{L_1}^{(1)})R_{0,L_1-1}^{(1)}(u-\lambda_{L_1-1}^{(1)})\cdots R_{0,1}^{(1)}(u-\lambda_1^{(1)}),\tag{2.3.24}$$

where $R_{0,j}^{(1)}(u)=P_{0,j}^{(1)}\check{R}_{0,j}^{(1)}(u)$. Then the nested transfer matrix $t^{(1)}(u)$ is

$$t^{(1)}(u)=tr_0T_0^{(1)}(u).\tag{2.3.25}$$

It can be checked that the following Yang-Baxter relation holds

$$\check{R}^{(1)}_{1,2}(u-v)[T^{(1)}(u)\otimes T^{(1)}(v)] = [T^{(1)}(v)\otimes T^{(1)}(u)]\check{R}^{(1)}_{1,2}(u-v). \quad (2.3.26)$$

Note that now some inhomogeneous parameters $\{\ldots,\lambda_j^{(1)},\ldots\}$ enter the nested monodromy matrix and the nested transfer matrix. The above process reduces the eigenvalue problem to the $SU(n-1)$ level. Repeating the process, we obtain

$$\begin{aligned}\Lambda^{(r)}(u) = &\prod_{j=1}^{L_{r+1}} \frac{1}{\alpha(u-\lambda_j^{(r+1)})} \prod_{l=1}^{L_r} \alpha(u-\lambda_l^{(r)})\Lambda^{(r+1)}(u)\\ &+\prod_{j=1}^{L_{r+1}} \frac{1}{\alpha(\lambda_j^{(r+1)}-u)}, \quad r=1,\ldots,n-1, \end{aligned}\quad (2.3.27)$$

with the boundary condition $\Lambda^{(n)}(u)=1$, where L_r is the number of the rth set of Bethe roots. The BAEs are given by

$$\begin{aligned}\Lambda^{(r)}(\lambda_j^{(r)}) &= \prod_{k\neq j}^{L_r} \frac{\alpha(\lambda_j^{(r)}-\lambda_k^{(r)})}{\alpha(\lambda_k^{(r)}-\lambda_j^{(r)})}\frac{1}{\alpha^N(\lambda_j^{(r)})},\\ &\quad j=1,\ldots,L_r, \quad r=1,\ldots,n-1. \end{aligned}\quad (2.3.28)$$

Substituting Eq. (2.3.27) into the above equation we readily have

$$\begin{aligned}&\prod_{k\neq j}^{L_r} \frac{\lambda_j^{(r)}-\lambda_k^{(r)}-1}{\lambda_j^{(r)}-\lambda_k^{(r)}+1} = \prod_{l=1}^{L_{r-1}} \frac{\lambda_j^{(r)}-\lambda_l^{(r-1)}}{\lambda_j^{(r)}-\lambda_l^{(r-1)}+1} \prod_{m=1}^{L_{r+1}} \frac{\lambda_j^{(r)}-\lambda_m^{(r+1)}-1}{\lambda_j^{(r)}-\lambda_m^{(r+1)}},\\ &\quad j=1,\ldots,L_r, \quad r=1,2,\ldots,n-1. \end{aligned}\quad (2.3.29)$$

For convenience, we put $\lambda_j^{(r)}\to i\mu_j^{(r)}-r/2$. The BAEs are thus transformed to more symmetric form

$$\begin{aligned}&\prod_{k\neq j}^{L_r} \frac{\mu_j^{(r)}-\mu_k^{(r)}-i}{\mu_j^{(r)}-\mu_k^{(r)}+i} = \prod_{l=1}^{L_{r-1}} \frac{\mu_j^{(r)}-\mu_l^{(r-1)}-\frac{i}{2}}{\mu_j^{(r)}-\mu_l^{(r-1)}+\frac{i}{2}} \prod_{m=1}^{L_{r+1}} \frac{\mu_j^{(r)}-\mu_m^{(r+1)}-\frac{i}{2}}{\mu_j^{(r)}-\mu_m^{(r+1)}+\frac{i}{2}},\\ &\quad j=1,\ldots,L_r, \quad r=1,2,\ldots,n-1. \end{aligned}\quad (2.3.30)$$

Note that $L_0=N$, $L_N=0$ and $\lambda_j^{(0)}=0$ are assumed. The eigenvalue of the transfer matrix is

$$\Lambda(u) = \alpha^N(u)\sum_{r=1}^{n-1}\prod_{j=1}^{L_r}\frac{1}{\alpha(u-\lambda_j^{(r)})}\prod_{l=1}^{L_{r+1}}\frac{1}{\alpha(\lambda_j^{(r+1)}-u)}$$

$$+\prod_{j=1}^{L_1}\frac{1}{\alpha(\lambda_j^{(1)}-u)}, \tag{2.3.31}$$

which allows us to derive the eigenvalue of the Hamiltonian in terms of the Bethe roots:

$$E(\mu_1^{(1)},\ldots,\mu_{L_1}^{(1)})=\frac{\partial\ln\Lambda(u)}{\partial u}\bigg|_{u=0}=-\sum_{j=1}^{L_1}\frac{1}{\mu_j^{(1)^2}+\frac{1}{4}}+N. \tag{2.3.32}$$

The physical properties including the ground state energy, the elementary excitations and the thermodynamics of this model can also be derived by a similar process for the spin-$\frac{1}{2}$ model. For these topics we direct the readers' attention to some excellent reviews [59–62]. Finally, we note that for the $SU(n)$-invariant quantum spin chain (2.3.1), the nested $T-Q$ relation can also be constructed. Details will be given in Chap. 7.

2.4 Quantum Determinant, Projectors and Fusion

An important quantity throughout this book is the quantum determinant, which is related to the inverse monodromy matrix. To show how the quantum determinant is defined, let us first check the inverse R-matrix for the XXX spin-$\frac{1}{2}$ chain. The crossing relation (1.5.6) indicates that

$$R_{0,j}(-u)=-\sigma_0^y R_{0,j}^{t_0}(u-\eta)\sigma_0^y. \tag{2.4.1}$$

The unitary relation (1.5.5) indicates that

$$R_{0,j}^{-1}(u)=\varphi^{-1}(u)\sigma_0^y R_{0,j}^{t_0}(u-\eta)\sigma_0^y. \tag{2.4.2}$$

This allows us to define the inverse monodromy matrix

$$\begin{aligned}T_0^{-1}(u)&=[R_{0,N}(u)\cdots R_{0,1}(u)]^{-1}=R_{0,1}^{-1}(u)\cdots R_{0,N}^{-1}(u)\\&=a^{-1}(u)d^{-1}(u-\eta)\sigma_0^y R_{0,1}^{t_0}(u-\eta)\cdots R_{0,N}^{t_0}(u-\eta)\sigma_0^y\\&=a^{-1}(u)d^{-1}(u-\eta)\sigma_0^y T_0^{t_0}(u-\eta)\sigma_0^y,\end{aligned} \tag{2.4.3}$$

where $a(u)$ and $d(u)$ are two R-matrix dependent functions given in (2.1.17) for the R-matrix (1.5.2). The quantum determinant is thus defined as

$$\mathrm{Det}_q\{T(u)\}=T_0(u)\sigma_0^y T_0^{t_0}(u-\eta)\sigma_0^y=a(u)d(u-\eta). \tag{2.4.4}$$

Since

$$\sigma_0^y T_0^{t_0}(u-\eta)\sigma_0^y = \begin{pmatrix} D(u-\eta) & -B(u-\eta) \\ -C(u-\eta) & A(u-\eta) \end{pmatrix}, \tag{2.4.5}$$

we have

$$\begin{aligned} \mathrm{Det}_q\{T(u)\} &= A(u)D(u-\eta) - B(u)C(u-\eta) \\ &= D(u)A(u-\eta) - C(u)B(u-\eta). \end{aligned} \tag{2.4.6}$$

In addition, the following operator identities also hold:

$$\begin{aligned} &A(u)B(u-\eta) = B(u)A(u-\eta), \quad C(u)D(u-\eta) = D(u)C(u-\eta), \\ &D(u-\eta)B(u) = B(u-\eta)D(u), \quad A(u-\eta)C(u) = C(u-\eta)A(u). \end{aligned} \tag{2.4.7}$$

The exact definition of the quantum determinant is given by the fusion procedure [7, 63–66]. Given a tensor space $\mathbf{V} \otimes \mathbf{V}$ spanned by an orthonormal basis $\{|\Phi_{j,\alpha}\rangle | j, \alpha = 0, 1, \ldots\}$, a one-dimensional projection operator, which projects all vectors onto a one-dimensional subspace $\mathbf{V}^{(j,\alpha)}$, is defined as

$$P_{1,2}^{(j,\alpha)} = |\Phi_{j,\alpha}\rangle\langle\Phi_{j,\alpha}|, \tag{2.4.8}$$

which possesses the properties

$$P_{1,2}^{(j,\alpha)} P_{1,2}^{(l,\beta)} = \delta_{j,l}\delta_{\alpha,\beta} P_{1,2}^{(j,\alpha)}. \tag{2.4.9}$$

Since all operators defined in the tensor space can be expressed as linear combinations of $|\Phi_{j,\alpha}\rangle\langle\Phi_{k,\beta}|$, for a given operator $A_{1,2}(u)$, the following relation holds:

$$P_{1,2}^{(j,\alpha)} A_{1,2}(u) P_{1,2}^{(j,\alpha)} = \mathbb{A}_{(j,\alpha)}(u) P_{1,2}^{(j,\alpha)}, \tag{2.4.10}$$

with $\mathbb{A}_{(j,\alpha)}(u)$ being a scalar function.

For any R-matrix possessing the properties (1.5.4)–(1.5.9), we define the quantum determinant of the one-row monodromy matrices as

$$\begin{aligned} \mathrm{Det}_q\{T(u)\} &= tr_{1,2}\left\{P_{1,2}^{(-)} T_1(u-\eta) T_2(u) P_{1,2}^{(-)}\right\}, \\ \mathrm{Det}_q\{\hat{T}(u)\} &= tr_{1,2}\left\{P_{1,2}^{(-)} \hat{T}_1(u-\eta) \hat{T}_2(u) P_{1,2}^{(-)}\right\}. \end{aligned} \tag{2.4.11}$$

Since $tr_{1,2} P_{1,2}^{(-)} = 1$ and $P_{1,2}^{(-)}$ is a one-dimensional projector, $\mathrm{Det}_q\{T(u)\}$ must be a scalar function. With YBE and the fusion condition (1.5.9) we have

$$P_{1,2}^{(-)} R_{1,j}(u-\eta) R_{2,j}(u) = R_{2,j}(u) R_{1,j}(u-\eta) P_{1,2}^{(-)}$$

$$= P^{(-)}_{1,2} R_{1,j}(u-\eta) R_{2,j}(u) P^{(-)}_{1,2} = \mathrm{Det}_q\{R(u)\} P^{(-)}_{1,2}, \tag{2.4.12}$$

with

$$\mathrm{Det}_q\{R(u)\} = tr_{1,2}\left\{P^{(-)}_{1,2} R_{1,j}(u-\eta) R_{2,j}(u) P^{(-)}_{1,2}\right\}. \tag{2.4.13}$$

The above relation leads to

$$\mathrm{Det}_q\{T(u)\} = \prod_{j=1}^{N} \mathrm{Det}_q\{R(u)\}. \tag{2.4.14}$$

Accordingly, the quantum determinants of the reflection matrices, which are useful to compute the quantum determinant of the double-row monodromy matrix, are defined as

$$\mathrm{Det}_q\{K^-(u)\} = tr_{1,2}\left\{P^{(-)}_{1,2} K^-_1(u-\eta) R_{1,2}(2u-\eta) K^-_2(u)\right\}, \tag{2.4.15}$$

$$\mathrm{Det}_q\{K^+(u)\} = tr_{1,2}\left\{P^{(-)}_{1,2} K^+_2(u) R_{1,2}(-2u-\eta) K^+_1(u-\eta)\right\}. \tag{2.4.16}$$

We note that the quantities $\mathrm{Det}_q\{K^{\pm}(u)\}$ and $\mathrm{Det}_q\{R(u)\}$ can easily be derived with the explicit expressions for the R-matrix, K-matrices and $P^{(-)}_{1,2}$. The quantum determinant for high-rank R-matrices can be defined similarly with singlet projectors. Details will be given in Chap. 7.

In fact, the quantum determinant is only a special case of fusion with a singlet projector. The fusion procedure can be generalized to cases with high-dimensional projectors in the associated algebras. For a given j, the states $\{|\Phi_{j,\alpha}\rangle | \alpha = 1, \ldots, n_j\}$ span an n_j-dimensional subspace. The corresponding projection operator onto this subspace is thus defined as

$$P^{(j)}_{1,2} = \sum_{\alpha=1}^{n_j} P^{(j,\alpha)}_{1,2}, \tag{2.4.17}$$

which possesses the property

$$P^{(j)}_{1,2} P^{(l)}_{1,2} = \delta_{j,l} P^{(j)}_{1,2}. \tag{2.4.18}$$

Given an R-matrix defined in the tensor space $\mathbf{V}_1 \otimes \mathbf{V}_2$ (the dimensions of the two vector spaces $\mathbf{V}_1$ and $\mathbf{V}_2$ may be different), at some special point $u = u_0$ (e.g., $u_0 = \pm\eta$ in (1.5.9)) the corresponding R-matrix becomes decorated projector, namely,

$$R_{1,2}(u_0) = P^{(j)}_{1,2} \times \gamma^{(j)}_{1,2} = P^{(j)}_{1,2} R_{1,2}(u_0), \tag{2.4.19}$$

where $P^{(j)}_{1,2}$ is a projector with the same rank of $R_{1,2}(u_0)$ and $\gamma^{(j)}_{1,2}$ is some non-degenerate matrix. For some particular R-matrices, the corresponding value of u_0, $P^{(j)}_{1,2}$ and $\gamma^{(j)}_{1,2}$ will be seen in the following chapters. From YBE we obtain

$$\begin{aligned} R_{2,m}(u)R_{1,m}(u+u_0)R_{1,2}(u_0) &= P^{(j)}_{1,2}R_{1,2}(u_0)R_{1,m}(u+u_0)R_{2,m}(u) \\ &= P^{(j)}_{1,2}R_{2,m}(u)R_{1,m}(u+u_0)R_{1,2}(u_0). \end{aligned} \tag{2.4.20}$$

Multiplying (2.4.20) by the inversion of $\gamma^{(j)}_{1,2}$ from the right side, we have

$$P^{(j)}_{1,2}R_{2,m}(u)R_{1,m}(u+u_0)P^{(j)}_{1,2} = R_{2,m}(u)R_{1,m}(u+u_0)P^{(j)}_{1,2}. \tag{2.4.21}$$

Similarly, we can derive

$$P^{(j)}_{1,2}T_2(u)T_1(u+u_0)P^{(j)}_{1,2} = T_2(u)T_1(u+u_0)P^{(j)}_{1,2}. \tag{2.4.22}$$

The relations (2.4.19)–(2.4.22) are useful to construct operator product identities of high-rank and high-spin integrable models. Details will be given in Chaps. 7–9.

2.5 Sklyanin's Separation of Variables

According to Liouville's theorem, a remarkable feature of classical integrable systems is that their variables are completely separable. Sklyanin realized that quantum integrable models also possess such a feature and the separation of variables of quantum integrable models can be performed in the framework of the algebraic Bethe Ansatz [25–27]. We use a simple example, i.e., the periodic spin-$\frac{1}{2}$ Heisenberg chain to explain the main idea of the quantum SoV method.

2.5.1 SoV Basis

Let us start from the monodromy matrix like (1.5.11) denoted by

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.5.1}$$

From the definition of the R-matrix we know that $D(u)$ is an operator valued polynomial of u with a degree N and can be expressed as

$$D(u) = (u-\mathbf{d}_1)(u-\mathbf{d}_2)\cdots(u-\mathbf{d}_N), \tag{2.5.2}$$

where $\{\mathbf{d}_j | j = 1, \ldots, N\}$ are certain u-independent operators. From the commutative property $[D(u), D(v)] = 0$ we readily have

$$[\mathbf{d}_j,\ \mathbf{d}_l] = 0, \quad j, l = 1, \ldots, N, \tag{2.5.3}$$

which indicate that $\{\mathbf{d}_j | j = 1, \ldots, N\}$ form a mutually commutative operator family and have common eigenstates. These operators thus serve as the quantum counterpart of action variables or the canonical momenta in the Liouville theory.

Given a common eigenstate $|\Omega\rangle$ of $\{\mathbf{d}_j | j = 1, \ldots, N\}$, let us assume

$$\mathbf{d}_j |\Omega\rangle = d_j |\Omega\rangle, \quad j = 1, \ldots, N, \tag{2.5.4}$$

where $\{d_j | j = 1, \ldots, N\}$ are the corresponding eigenvalues. Let us also assume that the operator $D(u)$ is simple, i.e., there should be 2^N (which is equal to the dimension of the Hilbert space) possible sets of $\{d_j | j = 1, \ldots, N\}$. Such a condition can always be realized with generic inhomogeneity included in the monodromy matrix, which allows us to choose one set of them and assume that $d_j \neq d_l \neq d_j \pm \eta$ for $j \neq l$.

Obviously,

$$D(d_j)|\Omega\rangle = 0, \quad j = 1, \ldots, N. \tag{2.5.5}$$

This relation allows us to define other non-null eigenstates of $D(u)$ such as

$$|d_{p_1}, \ldots, d_{p_n}\rangle = \prod_{j=1}^{n} B(d_{p_j})|\Omega\rangle, \tag{2.5.6}$$

with $p_j \in \{1, \ldots, N\}$, $p_1 < p_2 < \cdots < p_n$ and $0 \leq n \leq N$ and

$$D(u)|d_{p_1}, \ldots, d_{p_n}\rangle = \prod_{j=1}^{n}(u - d_{p_j} + \eta) \prod_{l \neq \{p_1, \ldots, p_n\}}^{N} (u - d_l)|d_{p_1}, \ldots, d_{p_n}\rangle. \tag{2.5.7}$$

With the inhomogeneous parameters $\{\theta_j | j = 1, \ldots, N\}$, a natural choice of the initial state is $|\Omega\rangle = |0\rangle$. In this case, $d_j = \theta_j$ and the eigenstates of $D(u)$ read

$$|\theta_{p_1}, \cdots, \theta_{p_n}\rangle = \prod_{j=1}^{n} B(\theta_{p_j})|0\rangle, \quad n = 0, \cdots, N, \tag{2.5.8}$$

$$\langle\theta_{p_1}, \cdots, \theta_{p_n}| = \prod_{j=1}^{n} \langle 0|C(\theta_{p_j}), \quad n = 0, \cdots, N. \tag{2.5.9}$$

Let us consider the quantity $\langle\theta_{q_1},\ldots,\theta_{q_m}|D(u)|\theta_{p_1},\ldots,\theta_{p_n}\rangle$. Acting $D(u)$ to the left and to the right alternatively, we readily have

$$\langle\theta_{q_1},\ldots,\theta_{q_m}|\theta_{p_1},\ldots,\theta_{p_n}\rangle = f_n(\theta_{p_1},\ldots,\theta_{p_n})\delta_{m,n}\prod_{j=1}^{n}\delta_{p_j,q_j}, \quad (2.5.10)$$

with

$$f_n(\theta_{p_1},\ldots,\theta_{p_n}) = \langle\theta_{p_1},\ldots,\theta_{p_n}|\theta_{p_1},\ldots,\theta_{p_n}\rangle. \quad (2.5.11)$$

The total number of states defined in (2.5.8) from $n = 0$ to $n = N$ is exactly 2^N by a simple counting. Therefore, the left eigenstates (2.5.9) and the right eigenstates (2.5.8) are orthogonal and respectively form a left basis and a right basis of the Hilbert space [67].

2.5.2 Functional Relations

Let $\langle\Psi|$ denote a left eigenstate of the transfer matrix $t(u) = A(u) + D(u)$ with the eigenvalue $\Lambda(u)$. In addition, we define the scalar product $F_n(u_1,\cdots,u_n)$ as

$$F_n(u_1,\ldots,u_n) = \langle\Psi|\prod_{j=1}^{n}B(u_j)|0\rangle, \quad n = 0,\ldots,N. \quad (2.5.12)$$

Note the fact that $B(\theta_j)B(\theta_j-\eta) = 0$, which can be proven with a similar procedure in Sect. 2.1.2. With the help of the commutation relations (2.1.9)–(2.1.10) and by computing the quantities

$$\langle\Psi|t(\theta_j-\eta)|\theta_1,\ldots,\theta_n\rangle, \quad \langle\Psi|t(\theta_j)|\theta_1,\ldots,\theta_j-\eta,\ldots,\theta_n\rangle,$$

we obtain

$$\Lambda(\theta_j-\eta)F_n(\theta_1,\ldots,\theta_n) = -\prod_{l\neq j}^{n}\frac{\theta_j-\theta_l-\eta}{\theta_j-\theta_l}a(\theta_j)F_n(\ldots,\theta_j-\eta,\ldots), \quad (2.5.13)$$

$$\Lambda(\theta_j)F_n(\ldots,\theta_j-\eta,\ldots) = -\prod_{l\neq j}^{n}\frac{\theta_j-\theta_l}{\theta_j-\theta_l-\eta}d(\theta_j-\eta)F_n(\theta_1,\ldots,\theta_n),$$
$$j = 1,\ldots,n, \quad (2.5.14)$$

which readily give the functional relations (1.5.19), where $a(u) = d(u+\eta) = \prod_{j=1}^{N}(u-\theta_j+\eta)$. Provided that $\Lambda(u)$ is parameterized by the homogeneous $T-Q$

relation (1.5.22), the associated eigenstates are the usual Bethe states and the solution of the above Eqs. (2.5.13) and (2.5.14) can be given in terms of a certain determinant such as (4.6.18). Detailed derivation of $F_n(\theta_{p_1}, \ldots, \theta_{p_n})$ for antiperiodic and open boundary conditions will be introduced in Chaps. 4 and 5 respectively. The derivation of this quantity for the periodic boundary case can be found in [29, 67].

2.5.3 Operator Decompositions

From the definition of the monodromy matrix we know that $B(u)$ and $C(u)$ are operator valued polynomials of u with a degree $N-1$. From the commutation relations (2.1.8), we can see that the coefficients of $B(u)$ [or $C(u)$] are mutually commutative. Accordingly, we can make the following useful operator decompositions

$$B(u) = \sum_{j=1}^{N} \prod_{l \neq j}^{N} \frac{u - b_l}{b_j - b_l} B(b_j), \quad C(u) = \sum_{j=1}^{N} \prod_{l \neq j}^{N} \frac{u - b_l}{b_j - b_l} C(b_j), \tag{2.5.15}$$

where $\{b_j | j = 1, \ldots, N\}$ are arbitrary complex numbers with $b_j \neq b_l \neq b_j \pm \eta$.

The above operator decompositions are convenient to compute inner products and scalar products. Put $b_j = \theta_j - \eta$ for $j = 1, \ldots, n$ and $b_j = \theta_j$ for $j = n+1, \ldots, N$. We have

$$\begin{aligned} B(\theta_n) = &\sum_{l=1}^{n} \prod_{k \neq l}^{n} \frac{\theta_n - \theta_k + \eta}{\theta_l - \theta_k} \prod_{k=n+1}^{N} \frac{\theta_n - \theta_k}{\theta_l - \theta_k - \eta} B(\theta_l - \eta) \\ &+ \sum_{l=n+1}^{N} \prod_{k=1}^{n} \frac{\theta_n - \theta_k + \eta}{\theta_l - \theta_k + \eta} \prod_{k=n+1, \neq l}^{N} \frac{\theta_n - \theta_k}{\theta_l - \theta_k} B(\theta_l). \end{aligned} \tag{2.5.16}$$

With the above relation and the fact $B(\theta_j)B(\theta_j - \eta) = 0$, we readily obtain

$$\begin{aligned} f_n(\theta_1, \ldots, \theta_n) = &\prod_{k=1}^{n-1} \frac{\theta_n - \theta_k + \eta}{\theta_n - \theta_k} \prod_{k=n+1}^{N} \frac{\theta_n - \theta_k}{\theta_n - \theta_k - \eta} \\ &\times \langle \theta_1, \ldots, \theta_{n-1} | C(\theta_n) B(\theta_n - \eta) | \theta_1, \ldots, \theta_{n-1} \rangle. \end{aligned} \tag{2.5.17}$$

The expression (2.4.6) of the quantum determinant implies

$$\begin{aligned} f_n(\theta_1, \ldots, \theta_n) = &-a(\theta_n) d(\theta_n - \eta) \prod_{k=1}^{n-1} \frac{\theta_n - \theta_k + \eta}{\theta_n - \theta_k} \\ &\times \prod_{k=n+1}^{N} \frac{\theta_n - \theta_k}{\theta_n - \theta_k - \eta} f_{n-1}(\theta_1, \ldots, \theta_{n-1}), \end{aligned} \tag{2.5.18}$$

which directly gives the solution

$$f_n(\theta_1, \ldots, \theta_n) = \prod_{j=1}^{n} \left\{ a(\theta_j) d_j(\theta_j) \prod_{k \neq j}^{n} \frac{\theta_j - \theta_k + \eta}{\theta_j - \theta_k} \right\}, \tag{2.5.19}$$

with $d_j(\theta_j) = \eta \prod_{l \neq j}^{N} (\theta_j - \theta_l)$.

The eigenstate $\langle \Psi |$ can be expressed as

$$\langle \Psi | = \sum_{\{p_j\}} \frac{F_n(\theta_{p_1}, \ldots, \theta_{p_n})}{f_n(\theta_{p_1}, \ldots, \theta_{p_n})} \langle \theta_{p_1}, \ldots, \theta_{p_n} |. \tag{2.5.20}$$

Similarly, the right eigenstate $| \Psi \rangle$ can be derived as

$$| \Psi \rangle = \sum_{\{p_j\}} \frac{F_n(\theta_{p_1}, \ldots, \theta_{p_n})}{f_n(\theta_{p_1}, \ldots, \theta_{p_n})} | \theta_{p_1}, \ldots, \theta_{p_n} \rangle. \tag{2.5.21}$$

We remark that for each matrix element of a monodromy matrix satisfying the Yang-Baxter relation or Sklyanin's reflection relation, its eigenstates can be constructed in a similar way, as long as the elements with different spectral parameters are mutually commutative. In such a sense, the SoV scheme gives a precise definition of quantum integrability or Yang-Baxter integrability. As we shall show in Chaps. 4 and 5, depending on the boundary conditions, either diagonal or off-diagonal elements of the monodromy matrices can be used to construct a convenient basis. The key point is that the number of independent eigenstates must be the same as the dimension of the Hilbert space. Obviously, the eigenstates of $C(u)$ [or $B(u)$] for the periodic spin chain can not form a complete basis.

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