Chapter 2
Some Preliminaries

This chapter will present some basic facts from complex analysis, operator theory and von Neumann algebras. These results will be needed in the sequel.

2.1 Some Preliminaries in Complex Analysis

In this section, we will present some standard results in complex analysis. First, some basic notations will be provided. Let $\mathbb{C}$ denote the complex plane, and $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. By *the punctured disk* we refer to $\mathbb{D} - \{0\}$. For each positive integer d, $\mathbb{C}^d$ denotes the d-th Cartesian product of $\mathbb{C}$. We use $\mathbb{B}_d$ to denote *the unit ball* in $\mathbb{C}^d$, i.e.

$$\mathbb{B}_d = \{(z_1, \cdots, z_d) \in \mathbb{C}^d : \sum_{k=1}^{d} |z_k|^2 < 1\}.$$

Write

$$\mathbb{D}^d = \{(z_1, \cdots, z_d) \in \mathbb{C}^d : |z_k| < 1, k = 1, \cdots, d\},$$

called *the unit polydisk*.

A function f on some domain $\Omega (\Omega \subseteq \mathbb{C}^d)$ is called *holomorphic* if for each $z_0 \in \Omega$, there is some neighborhood V of z_0 on which f can be expressed as a uniformly convergent power series

$$f(z) = \sum_{\alpha} c_\alpha (z - z_0)^\alpha,$$

K. Guo, H. Huang, *Multiplication Operators on the Bergman Space*,
Lecture Notes in Mathematics 2145, DOI 10.1007/978-3-662-46845-6_2

where α are multi-indices and each term $(z - z_0)^\alpha$ represents a monomial in $z_1 - z_{01}, \cdots, z_d - z_{0d}$. It is known [Hor, Kran] that if Ω is a domain in $\mathbb{C}^d (d \geq 2)$, then a function f on Ω is holomorphic if and only if f is holomorphic in each variable. Denote by $Hol(\Omega)$ the set of all holomorphic functions on Ω. All bounded functions in $Hol(\Omega)$ consist of a class, denoted by $H^\infty(\Omega)$, which is a Banach space with the sup-norm defined by

$$\|f\|_\infty = \sup_{z\in\Omega} |f(z)|, f \in H^\infty(\Omega).$$

In particular, $H^\infty(\mathbb{D})$ denotes the set of bounded holomorphic functions over $\mathbb{D}$. The uniform closure of polynomials on Ω is written by $A(\Omega)$. For example, one can refer to [Ru1, Ru2, Ru3] and [Ga, Hof1] for the study of $A(\mathbb{D})$, $A(\mathbb{D}^d)$, and $A(\mathbb{B}_d)$.

Let Ω be a domain on the complex plane $\mathbb{C}$. A holomorphic function f on Ω is called *univalent* if it is injective. If in addition f is onto, then f is called *biholomorphic*.

It is well-known that the terms "holomorphic" and "analytic" are synonym. In this book, we always take the term "holomorphic" except for the following cases: analytic continuation, analytic curve, and analytic Toeplitz operator. Sometimes we also mention the terms "conformal", "conformal map" and "conformal isomorphism", by which we mean the corresponding map is biholomorphic.

Rouche's theorem is a well-known result in complex analysis.

Theorem 2.1.1 (Rouche) *Suppose both f and g are holomorphic functions over a domain G on $\mathbb{C}$, and Ω is a sub-domain of G whose boundary $\partial\Omega$ consists of finitely many Jordan curves and $\partial\Omega \subseteq G$. If*

$$|f(z) - g(z)| < |f(z)|, z \in \partial\Omega,$$

then f and g have the same number of zeros in Ω, counting multiplicity.

Let Ω_0 and Ω be two domains in $\mathbb{C}^d$. A holomorphic map $\Phi : \Omega_0 \to \Omega$ is called *proper* if $\Phi^{-1}(E)$ is compact for every compact subset E of Ω. Equivalently, for any sequence $\{z_n\}$ ($z_n \in \Omega_0$) without limit point in Ω_0, $\{\Phi(z_n)\}$ has no limit point in Ω. In particular, when $d = 1$, it is always an *n-folds map* [Mi, Appendix E]; that is, for every $z \in \Omega_0$, $\Phi - \Phi(z)$ has exactly n zeros in Ω_0, counting multiplicity. For example, a finite Blaschke product B with order n is a proper map from $\mathbb{D}$ onto $\mathbb{D}$. For each $w \in \mathbb{D}$, $B - w$ has n zeros in $\mathbb{D}$, counting multiplicity. In general, by applying Rouche's theorem one can give the following.

Proposition 2.1.2 *Suppose $\phi : \Omega_0 \to \Omega$ is a proper map with $\Omega_0, \Omega \subseteq \mathbb{C}$, then there is a constant integer n such that for any $w \in \Omega$, $\phi - w$ has exactly n zeros in Ω_0, counting multiplicity.*

Proof For two distinct points w_0 and w_1 in Ω, let γ be a curve in Ω connecting w_0 with w_1. Since the curve γ is compact and ϕ is proper, $\phi^{-1}(\gamma)$ is also compact. Then it is not difficult to construct a sub-domain Ω_1 of Ω_0 such that $\phi^{-1}(\gamma) \subseteq \Omega_1$

and $\partial\Omega_1$ is a subset of Ω_0 consisting of several smooth Jordan curves. By the compactness of $\phi^{-1}(\gamma)$, one can apply Rouche's theorem to show that for all $w \in \gamma$, $\phi - w$ has the same number of zeros in Ω_1. Since $\Omega_1 \supseteq \phi^{-1}(\gamma)$, $\phi - w$ has no zero on $\Omega_0 - \Omega_1$ for any $w \in \gamma$. Thus, $\phi - w$ has the same number of zeros in Ω_0. In particular, $\phi - w_0$ and $\phi - w_1$ has the same number of zeros in Ω_0, counting multiplicity. □

The following theorem characterizes all proper maps on $\mathbb{D}^n$ and $\mathbb{B}_n$, see [Ru1, Ru2].

Theorem 2.1.3

(I) By omitting a permutation, every proper holomorphic map Φ of $\mathbb{D}^n$ into $\mathbb{D}^n$ must have the following form:

$$\Phi(z) = (\varphi_1(z_1), \cdots, \varphi_n(z_n)), z \in \mathbb{D}^n$$

where all $\varphi_i (1 \leq i \leq n)$ are finite Blaschke products, see [Ru1]. In particular, every proper holomorphic map of $\mathbb{D}$ into $\mathbb{D}$ is a finite Blaschke product.

(II) For $n > 1$, every proper holomorphic map Φ of $\mathbb{B}_n$ into $\mathbb{B}_n$ is a holomorphic automorphism of the unit ball $\mathbb{B}_n$, see [Ru2].

For more about proper maps, one can refer to [Ru2, Chap. 15]. The remaining part of this section concerns with Blaschke products. In this book, we always write

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z},$$

with $\lambda, z \in \mathbb{D}$. By definition, a *Blaschke product* is a unimodular constant tuple of the following:

$$B(z) \triangleq z^m \prod_n \frac{a_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z} \equiv z^m \prod_n \frac{a_n}{|a_n|} \varphi_{a_n}(z),\ z \in \mathbb{D}, m \in \mathbb{Z}_+$$

where $\{a_n\}$ is a nonzero sequence in $\mathbb{D}$ satisfying $\sum_n (1 - |a_n|^2) < \infty$. If the sequence $\{a_n\}$ contains $N - m$ elements, i.e. B has N zeros counting multiplicity, then B is called a *finite Blaschke product* with order N. A Blaschke product of order one is called a Blaschke factor. It is well-known that when $\{a_n\}$ is an infinite sequence, the product $B(z)$ converges and is holomorphic in $\mathbb{D}$ [Hof1]; in this case, B is called an *infinite Blaschke product*.

The class of inner functions enlarges the class of Blaschke products. Recall that if ϕ is a function in $H^\infty(\mathbb{D})$ admitting unimodular radial limits almost everywhere on $\mathbb{T}$ according to the arc length measure, then ϕ is called an *inner function*. Frostman's theorem states a relationship between inner functions and Blaschke products, for example, see [Ga, p. 75, Theorem 6.4].

Theorem 2.1.4 (Frostman) *Let f be a nonconstant inner function on $\mathbb{D}$. Then for all λ in $\mathbb{D}$ except possibly for a set of capacity zero, the function*

$$\varphi_\lambda(f) = \frac{\lambda - f}{1 - \bar{\lambda} f}$$

is a Blaschke product.

The definition of capacity will be illustrated in the next section.

The following gives a generalization of Frostman's theorem due to Rudin [Ru4].

Theorem 2.1.5 (Rudin) *For each nonconstant function $f \in H^\infty(\mathbb{D})$, the inner part of $f - f(\lambda)(\lambda \in \mathbb{D})$ is a Blaschke product with distinct zeros, except for λ in a set of capacity zero.*

By Frostman's theorem, the study of reducing subspaces for M_f acting on function spaces where f is an arbitrary inner function is equivalent to studying those for the multiplication operator defined by a Blaschke product $\varphi_\lambda(f)$; and this context will be considered in the next chapters.

It is well-known [Hof1] that a sequence $\{z_n\}$ in $\mathbb{D}$ is the zero sequence of a function $f \in H^\infty(\mathbb{D})$ if and only if

$$\sum_n (1 - |z_n|^2) < \infty.$$

This fact can be imported from the unit disk to the half plane.

Lemma 2.1.6 *Let f be a bounded holomorphic function defined on the half plane $\{z|\, Re z > \lambda\}$ with $\lambda \in \mathbb{R}$, and let $\lambda_1, \lambda_2, \cdots$ be all zeros of f on the half real axis $(\lambda, +\infty)$ (repeated according to multiplicity). Then for any $\delta > 0$,*

$$\sum_{0 \neq \lambda_i > \lambda + \delta} |\lambda_i|^{-1} < +\infty.$$

In particular, if $k = 1, 2, \cdots$ are the zeros of a bounded holomorphic function f, then f is necessarily zero [Hof1, p. 74, Exercise 3].

Proof of Lemma 2.1.6 The proof is from [Huang1]. Given $\delta > 0$, the mapping

$$w = \frac{z - \lambda - \delta}{z - \lambda + \delta}$$

is a bi-holomorphic map from $\{z|\, Re\, z > \lambda\}$ onto the unit disk. Let $z = z(w)$ be its inverse, and then $f\big(z(w)\big)$ is in $H^\infty(\mathbb{D})$. Since $w(\lambda_1), w(\lambda_2), \cdots$ are the zeros of $f\big(z(w)\big)$, they must satisfy the Blaschke condition:

$$\sum_i (1 - |z(\lambda_i)|^2) < \infty.$$

Therefore, $\sum_{\lambda_i > \lambda+\delta}(1-|z(\lambda_i)|) < \infty$. That is,

$$2\delta \sum_{\lambda_i > \lambda+\delta} (\lambda_i - \lambda + \delta)^{-1} < \infty,$$

which easily leads to the desired conclusion. The proof of Lemma 2.1.6 is complete. $\square$

We shall adopt the notations from [Ga]. For a holomorphic function f on $\mathbb{D}$ and $\xi \in \mathbb{T}$, *the cluster set* of f at ξ is

$$\mathrm{Cl}(f,\xi) = \bigcap_{r>0} \overline{f(\mathbb{D} \cap O(\xi, r))},$$

where $O(\xi, r)) = \{z : |z - \xi| < r\}$. Clearly, $w \in \mathrm{Cl}(f, \xi)$ if and only if there is a sequence $\{z_n\}$ in $\mathbb{D}$ such that $z_n \to \xi$ and $f(z_n) \to w$ as n tends to infinity. The range set of f at ξ is defined to be

$$\mathcal{R}(f,\xi) = \bigcap_{r>0} f\big(\mathbb{D} \cap O(\xi, r)\big).$$

The following theorem gives a characterization for the singularities of inner functions, see [Ga, p. 80, Theorem 6.6].

Theorem 2.1.7 *Let f be an inner function on $\mathbb{D}$, and ξ is a singular point of f (i.e. a point at which f does not extend across the unit circle analytically). Then* $\mathrm{Cl}(f,\xi) = \overline{\mathbb{D}}$ *and* $\mathcal{R}(f,\xi) = \mathbb{D} - L$*, where L is a set of capacity zero.*

Proof The proof is from [Ga].

Since $\mathcal{R}(f,\xi) \subseteq \mathrm{Cl}(f,\xi) \subseteq \overline{\mathbb{D}}$, it suffices to show that $\mathcal{R}(f,\xi)$ equals $\mathbb{D}$ minus some set of capacity zero. By Theorem 2.1.4, there is a set L' of capacity zero such that $\varphi_\lambda(f)$ is always a Blaschke product whenever $\lambda \in \mathbb{D} - L'$. Now fix a point $\lambda \in \mathbb{D} - L'$. Since f is singular at ξ, so is the Blaschke product $\varphi_\lambda(f)$. Recall that a Blaschke product B with zero sequence $\{z_n\}$ can be extended analytically to $\mathbb{T} - E$, where E is the set of accumulation points of $\{z_n\}$ [Hof1]. Then it follows that there is a zero subsequence $\{z_n'\}$ of $\varphi_\lambda(f)$ tending to ξ. Thus, $\varphi_\lambda(f)(z_n') = 0$, i.e. $f(z_n') = \lambda$ for each n. This shows that $\lambda \in \mathcal{R}(f,\xi)$, and hence $\mathbb{D} - L' \subseteq \mathcal{R}(f,\xi) \subseteq \mathbb{D}$, completing the proof. $\square$

Concerning Blaschke products, two important classes will be introduced: interpolating and thin Blaschke products. On the unit disk, *the pseudohyperbolic distance* between two points z and λ is defined by

$$d(z,\lambda) = |\varphi_\lambda(z)|, \; z, \lambda \in \mathbb{D}.$$

An infinite Blaschke product B is called *an interpolating Blaschke product* if for any $(a_n) \in l^\infty$, there exists a function f in $H^\infty(\mathbb{D})$ such that

$$f(z_n) = a_n, \ \ n = 1, 2, \cdots,$$

where $\{z_n\}$ denotes its zero sequence. This is equivalent to the following:

$$\inf_k \prod_{j\geq 1, j\neq k} d(z_j, z_k) > 0,$$

see [Ga].

A Blaschke product B is called *a thin Blaschke product* [GM2] if the zero sequence $\{z_n\}$ of B satisfies

$$\lim_{k\to\infty} \prod_{j\geq 1, j\neq k} d(z_j, z_k) = 1.$$

Since

$$\prod_{j\geq 1, j\neq k} d(z_j, z_k) = (1 - |z_k|^2)|B'(z_k)|,$$

the above condition is equivalent to

$$\lim_{k\to\infty} (1 - |z_k|^2)|B'(z_k)| = 1.$$

A thin Blaschke product is never a covering map. This follows easily from the fact that if a Blaschke product B is a covering map whose zero sequence is $\{z_n\}$, then $(1 - |z_k|^2)|B'(z_k)|$ is a constant not depending on k [Cow1, Theorem 6]. Also, a thin Blaschke product has many good properties, see Sect. 5.2 in Chap. 5. Here, we just mention one of them as below, for example see [GM2, Lemma 3.2(3)], [Hof2, pp. 86, 106], or [Ga, pp. 404, 310].

Proposition 2.1.8 *Let B be a thin Blaschke product. Then the following hold:*

(1) *Each value in $\mathbb{D}$ can be achieved for infinitely many times by B.*
(2) *For every point w in $\mathbb{D}$, $\varphi_w \circ B$ is a thin Blaschke product.*

Note that in Proposition 2.1.8(1) is a direct consequence of (2). Besides, Proposition 2.1.8 also implies that a thin Blaschke product is never a covering map because its image is exactly the unit disk.

Another result on thin Blaschke products will also be used in the sequel, see [La, Ga] and [GM3, Lemma 4.2].

Proposition 2.1.9 *Suppose B is a thin Blaschke product with the zero sequence $\{z_k\}$. Then for each r with $0 < r < 1$, there exists an $\varepsilon \in (0,1)$ such that $|B(z)| \geq r$ whenever $d(z, z_n) \geq \varepsilon$ for all n.*

Proof The proof is from [Ga, pp. 395–397].

Suppose B is a thin Blaschke product with the zero sequence $\{z_k\}$. For each $z \in \mathbb{D}$ and $\varepsilon \in (0,1)$, put

$$\Delta(z, \varepsilon) \triangleq \{w \in \mathbb{D} : d(z, w) < \varepsilon\}.$$

To prove Proposition 2.1.9, it suffices to show for enough large $r(0 < r < 1)$, there is a constant $\varepsilon \in (0,1)$ satisfying

$$B^{-1}(\Delta(0, r)) \subseteq \bigcup_{n \geq 0} \Delta(z_n, \varepsilon). \tag{2.1}$$

First, assume that there is a constant δ such that

$$\prod_{j; j \neq k} d(z_j, z_k) \geq \delta.$$

In this case, we will prove that for $r = r(\delta) \in (0,1)$, $|B(z)| \geq r$ whenever $d(z, z_n) \geq \varepsilon$ for all n and $r(\delta) \to 1$ as $\delta \to 1^-$.

For each $a \in \mathbb{D}$, put

$$L_a(z) \triangleq -\varphi_a(-z) = \frac{z + a}{1 + \bar{a}z},$$

and set

$$h_n(\zeta) = B\left(\frac{\zeta + z_n}{1 + \overline{z_n}\zeta}\right) = B \circ L_{z_n}(\zeta).$$

By using the maximum value theorem, one can show that for each holomorphic function $f : \mathbb{D} \to \mathbb{D}$,

$$d(f(z), f(a)) \equiv \frac{|f(z) - f(a)|}{|1 - \overline{f(a)}f(z)|} \leq \frac{|z - a|}{|1 - \bar{a}z|}, \quad z \neq a \quad \text{and} \quad z, a \in \mathbb{D}. \tag{2.2}$$

Since $\|h_n\|_\infty = 1$ and $h_n(0) = 0$, then $\frac{h_n(\zeta)}{\zeta}$ is a holomorphic function from $\mathbb{D}$ into $\mathbb{D}$. Taking $f(\zeta) = \frac{h_n(\zeta)}{\zeta}$ and $a = 0$ in (2.2), we get

$$d\left(\frac{h_n(\zeta)}{\zeta}, h_n'(0)\right) \leq |\zeta|. \tag{2.3}$$

Write $\lambda = 1 - \sqrt{1-\delta} < \delta$. By (2.3), when $|\zeta| = \lambda$,

$$|\frac{h_n(\zeta)}{\zeta}| \geq \frac{|h_n'(0)| - |\zeta|}{1 - |\zeta||h_n'(0)|} \geq \frac{\delta - \lambda}{1 - \lambda\delta}.$$

For $|\zeta| = \lambda$,

$$|h_n(\zeta)| \geq \frac{\delta - \lambda}{1 - \lambda\delta}\lambda \triangleq r.$$

By the argument principle, for each w with $|w| < r$, $h_n - w$ and h_n has the same number of zeros in $\{\zeta : |\zeta| < \lambda\}$, counting multiplicity. Thus, $h_n - w$ has exactly one zero in $\{\zeta : |\zeta| < \lambda\}$, counting multiplicity. Put

$$V_n \triangleq L_{z_n}\big(h_n^{-1}(\Delta(0,r)) \cap \lambda\mathbb{D}\big),$$

and then B maps V_n biholomorphically onto $\Delta(0,r)$. Since

$$V_n \subseteq \Delta(z_n, \lambda),$$

(2.1) reduces to the following

$$B^{-1}(\Delta(0,r)) \subseteq \cup_n V_n. \tag{2.4}$$

Below, we will prove (2.4). For each $w(|w| < r)$ define

$$B_w(z) = \frac{B(z) - w}{1 - \overline{w}B(z)},$$

which has exactly one zero $z_n(w)$ in $V_n (n \geq 0)$ and $z_n(w)$ is a holomorphic function of $w(|w| < r)$ satisfying

$$z_n(0) = z_n.$$

To prove (2.4), we must show that B_w has no zero outside $\cup_n V_n$. Now let $H_w(z)$ denote the Blaschke product with the zero sequence $\{z_n(w)\}$, and then

$$B_w = H_w g_w,$$

where g_w is a bounded holomorphic function satisfying $\|g_w\|_\infty \leq 1$. Note that

$$|B_w(0)| = |g_w(0)| \prod_{n=1}^{\infty} |z_n(w)|.$$

Write

$$G(w) = \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|} z_n(w),$$

This product converges on $\{w : |w| < r\}$ because its partial products are bounded by 1 and it converges at $w = 0$. Since $\|g_w\|_\infty \leq 1$,

$$|G(w)| = \prod_{n=1}^{\infty} |z_n(w)| \geq |B_w(0)| = |\frac{B(0) - w}{1 - \overline{B(0)}w}|.$$

Write

$$F(w) = \frac{B(0) - w}{1 - \overline{B(0)}w} G(w)^{-1}, \ |w| < r,$$

and clearly

$$|F(w)| \leq 1 \quad \text{and} \quad |F(0)| = 1,$$

forcing $|F(w)| = 1$, $|w| < r$. This immediately shows that $|g_w(0)| = 1$, which shows that g_w is a constant because $\|g_w\|_\infty \leq 1$. Thus, B_w has the same zeros as H_w. Since all zeros $\{z_n(w)\}$ of H_w lie in $\cup_n V_n$ for each $w(|w| < r)$, then

$$B^{-1}(\{w : |w| < r\}) = \cup_n V_n \subseteq \cup_n \Delta(z_n, \delta),$$

as desired. The proof of (2.4) is complete. That is, with $r = r(\delta)$ we get (2.1).

In general, since B is a thin Blaschke product,

$$\prod_{j; j \neq k} d(z_j, z_k) \to 1, \ (k \to \infty).$$

For a fixed number $\delta \in (0, 1)$, there is natural number N_0 such that

$$\prod_{j; j \neq N_0} d(z_j, z_k) > \delta.$$

Let B_1 denote the finite Blaschke product with zero sequence $\{z_k : 1 \leq k < N_0\}$, and B_2 denotes the Blaschke product with zero sequence $\{z_k : k \geq N_0\}$. Clearly, $B = B_1 B_2$. By the observations that

$$\{|B| < r^2(\delta)\} \subseteq \{|B_1| < r(\delta)\} \cup \{|B_2| < r(\delta)\},$$

and that $\lim\limits_{\delta \to 1^-} r(\delta) = 1$, the remaining is an easy exercise. $\square$

To establish another result on thin Blaschke products, we first present the following lemma, see [Mas, Theorem 3.6] for example.

Lemma 2.1.10 *If ψ is a finite Blaschke product and $\lambda \in \mathbb{T}$, then*

$$|\psi'(\lambda)| = \lim_{z\to\lambda} \frac{1-|\psi(z)|^2}{1-|z|^2}.$$

Proof Assume ψ is a finite Blaschke product. We must show that for each $\lambda \in \mathbb{T}$,

$$|\psi'(\lambda)| = \lim_{z\to\lambda} \frac{1-|\psi(z)|^2}{1-|z|^2}.$$

The proof is divided into two part.

Step I. First, it will be shown that $\lim\limits_{z\to\lambda} \frac{1-|\psi(z)|^2}{1-|z|^2}$ always exists for $\lambda \in \mathbb{T}$.

This will be handled by induction. In more detail, suppose ψ is a finite Blashcke product. If order $\psi = 1$, then one may write $\psi(z) = \varphi_a(z)$ with $a \in \mathbb{D}$. By computations, we have

$$1-|\psi(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

Then

$$\frac{1-|\psi(z)|^2}{1-|z|^2} = \frac{1-|a|^2}{|1-\bar{a}z|^2} = \left|-\frac{1-|a|^2}{(1-\bar{a}z)^2}\right| = |\psi'(z)|.$$

By induction, assume that $\lim\limits_{w\to\lambda} \frac{1-|\psi(z)|^2}{1-|z|^2}$ exists if order $\psi = k$. Now write that $\psi(z) = B(z)\varphi_a(z)$, where $a \in \mathbb{D}$ and B is a finite Blashcke product of order k. Note that

$$\frac{1-|\psi(z)|^2}{1-|z|^2} = \frac{1-|B(z)|^2}{1-|z|^2} + |B|^2 \frac{1-|\varphi_a(z)|^2}{1-|z|^2},$$

from which it follows that $\lim\limits_{z\to\lambda} \frac{1-|\psi(z)|^2}{1-|z|^2}$ exists.

Step II. To complete the proof, it suffices to show that $|\psi'(\lambda)| = \lim\limits_{k\to\infty} \frac{1-|\psi(w_k)|^2}{1-|w_k|^2}$ holds for some sequence $\{w_k\}$ tending to λ.

Put $z^* = \psi(\lambda) \in \mathbb{T}$, define

$$\gamma(t) = tz^*,\ 0 \le t \le 1,$$

and write $z_k = \gamma(1-\frac{1}{k+1})$. It is known that the derivative of a finite Blaschke product never vanishes on $\mathbb{T}$. Then there is an $r_0 \in [0,1)$ such that $\psi^{-1}(\gamma[r_0,1])$ consists of

d ($d =$ order ψ) disjoint arcs,

$$\gamma_1, \cdots, \gamma_d.$$

Without loss of generality, put $r_0 = 0$. For each z_k, $\psi^{-1}(z_k)$ consists of d different complex numbers, denoted by

$$w_k^1, \cdots, w_k^d,$$

where $w_k^j \in \gamma_j (1 \leq j \leq d)$ for each k. Without loss of generality, assume that $\lambda \in \gamma_1$. Rewrite w_k for w_k^1 and $w^* \equiv \lambda$. Note that z_k tends to z^* as k tends to infinity, and then w_k tends to w^* as k tends to infinity. Since $\psi'(w^*) \neq 0$, ψ is conformal at w^*. Then from

$$z_k = z^* - \frac{1}{k+1} z^*, k \to \infty,$$

it follows that

$$w_k = w^* - \delta_k w^* + o(1)\delta_k, k \to \infty,$$

where δ_k is a positive infinitesimal, and $o(1)$ denotes an infinitesimal. After some computations, we have

$$\frac{1 - |z_k|^2}{1 - |w_k|^2} \Big/ \frac{\frac{1}{k+1}}{\delta_k} \to 1, k \to \infty. \tag{2.5}$$

Since

$$\left|\frac{\psi(w_k) - \psi(w^*)}{w_k - w^*}\right| = \left|\frac{z_k - z^*}{w_k - w^*}\right| = \left|\frac{1 - z_k \overline{z^*}}{1 - w_k \overline{w^*}}\right|,$$

it is easy to verify that

$$\left|\frac{\psi(w_k) - \psi(w^*)}{w_k - w^*}\right| \Big/ \frac{\frac{1}{k+1}}{\delta_k} \to 1, k \to \infty.$$

Therefore, $|\psi'(w^*)| / \frac{\frac{1}{k+1}}{\delta_k} \to 1, k \to \infty$. Then by (2.5),

$$\lim_{k \to \infty} |\psi'(w^*)| \frac{1 - |w_k|^2}{1 - |z_k|^2} = 1,$$

and thus $|\psi'(\lambda)| = \lim_{k \to \infty} \frac{1 - |\psi(w_k)|^2}{1 - |w_k|^2}$. The proof is complete. □

Now we are ready to state the following.

Proposition 2.1.11 *Given an infinite Blashcke product B and a finite Blaschke product ψ, B is a thin Blaschke product if and only if $B \circ \psi$ is a thin Blaschke product.*

Proof First, suppose B is an infinite Blaschke product with zero sequence $\{a_k\}_{k=1}^{\infty}$, ψ is a finite Blaschke product with order n, and $B \circ \psi$ is a thin Blaschke product. Put $\phi = B \circ \psi$, and write $\psi^{-1}(a_k)$ as a finite sequence

$$b_k^1, \cdots, b_k^n.$$

Since $B \circ \psi$ is a thin Blaschke product, we have

$$|\phi'(b_k^j)|(1 - |b_k^j|^2) \to 1,\ k \to \infty (j = 1, \cdots, n).$$

That is,

$$|B'(a_k)|(1 - |b_k^j|^2)|\psi'(b_k^j)| \to 1,\ k \to \infty (j = 1, \cdots, n).$$

Also, by Lemma 2.1.10 we have

$$\frac{(1 - |b_k^j|^2)|\psi'(b_k^j)|}{1 - |a_k|^2} \to 1,\ k \to \infty (j = 1, \cdots, n),$$

forcing

$$|B'(a_k)(1 - |a_k|^2)| \to 1,\ k \to \infty,$$

which shows that B is a thin Blaschke product.

The inverse direction follows from similar discussion as above. □

Remark 2.1.12 With a bit more effort, one can show that if B is a thin Blashcke product and $h \in H^{\infty}(\mathbb{D})$ satisfying $B = h \circ \psi$ with ψ a finite Blaschke product, then h is a thin Blashcke product.

In Chap. 6, geometric properties of thin Blaschke products are investigated in considerable detail. For that aim, we present Böttcher's theorem which studies the behavior of a holomorphic function over a neighborhood of 0. It is of independent interest, see [Mi, Theorem 9.1] or [CaG, p. 33, Theorem 4.1].

Theorem 2.1.13 (Böttcher) *Suppose*

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

where $a_n \neq 0$ for $n \geq 2$. Then there exists a local holomorphic change of coordinate $w = \varphi(z)$ such that $\varphi(0) = 0$ and $\varphi \circ f \circ \varphi^{-1}(w) = w^n$ holds on a neighborhood of 0. Furthermore, φ is unique up to multiplication by an $(n-1)$-th root of unity.

Proof The proof is from [CaG].

First, we prove the existence of φ. By the Fourier expansion of f at 0, there are two constants δ and C satisfying $0 < \delta < 1 < C$, $C\delta < 1$, and

$$|f(z)| \leq C|z|^n, \ |z| < \delta.$$

By induction, write $f^{[1]} = f$ and $f^{[k+1]} = f^{[k]} \circ f (k \geq 2)$. Also, we have

$$|f^{[k]}(z)| \leq (C|z|)^{n^k} \leq 1, \ |z| < \delta, \ k = 1, 2, \cdots . \tag{2.6}$$

If we change variables by setting $\zeta = cz$ where $c^{n-1} = \frac{1}{a_n}$, then we have conjugated f to the form $f(\zeta) = \zeta^n + \cdots$, where suspension points "$\cdots$" denote terms of higher degrees here and below. Thus, we may assume that $a_n = 1$. In this case, we will prove that there is a map $\varphi(z) = z + \cdots$ satisfying

$$\varphi(f(z)) = \varphi(z)^n.$$

Put

$$\varphi_k(z) \triangleq f^{[k]}(z)^{n^{-k}} = \sqrt[n^k]{(z^{n^k} + \cdots)} = z \sqrt[n^k]{1 + \cdots},$$

which is well-defined in a neighborhood of 0. These φ_k's satisfy

$$\varphi_{k-1} \circ f = (f^{[k-1]} \circ f)^{n^{-k+1}} = \varphi_k^n.$$

If one can show that on some small disk Δ centered at 0, $\{\varphi_k\}$ converges uniformly to a function φ, then φ is necessarily holomorphic and satisfies that

$$\varphi(f(z)) = \varphi(z)^n, \ z \in \Delta.$$

Also,

$$\varphi'(0) = \lim_{k \to \infty} \varphi_k'(0) = 1,$$

which will finish the proof. Now it remains to show that $\{\varphi_k\}$ converges. To see this, recall that $f^{[k+1]} = f^{[k]} \circ f$. Note that by (2.6)

$$\begin{aligned}
\frac{\varphi_{k+1}(z)}{\varphi_k(z)} &= \Big(\frac{\varphi_1 \circ f^{[k]}(z)}{f^{[k]}(z)}\Big)^{n^{-k}} \\
&= \big(1 + O(|f^{[k]}(z)|)\big)^{n^{-k}} \\
&= 1 + O(n^{-k})O(|f^{[k]}(z)|) \\
&= 1 + O(n^{-k})O(1) \\
&= 1 + O(n^{-k}), \quad |z| < \delta.
\end{aligned}$$

Thus, there is an enough large number N_0, such that the product $\prod_{k=N_0}^{\infty} \frac{\varphi_{k+1}}{\varphi_k}$ converges uniformly on the disk $\{z : |z| < \delta\}$, which implies that $\{\varphi_k\}$ converges uniformly on the disk $\{z : |z| < \delta\}$, to a function φ as desired. The proof for the existence of φ is complete.

Next, it remains to deal with the uniqueness of φ. If ψ is a biholomorphic map on a neighborhood of 0 such that $\psi(0) = 0$ and $\psi \circ f \circ \psi^{-1}(w) = w^n$ holds on a neighborhood of 0. Since

$$\varphi \circ f \circ \varphi^{-1}(w) = w^n,$$

then

$$\varphi^{-1} \circ T \circ \varphi(w) = f(w),$$

where $T(w) = w^n$. Thus,

$$\psi \circ \varphi^{-1} \circ T \circ \varphi \circ \psi^{-1}(w) = T(w)$$

holds on a neighborhood V of 0. Write $h = \varphi \circ \psi^{-1}$, and then

$$T(h(w)) = h(T(w)), \ w \in V. \tag{2.7}$$

Write $h(z) = c_1 z + c_2 z^2 + \cdots$, and by (2.7) one can show $c_1^{n-1} = 1$ and

$$c_k = 0, \ k = 2, 3, \cdots .$$

This shows that φ is unique up to multiplication by an $(n-1)$-th root of unity. □

Roughly speaking, Theorem 2.1.13 tells us that if f is a non-constant function that is holomorphic at z_0 and $f'(z_0) = 0$, then on an enough small neighborhood of z_0, it behaves no more complicated than $z \mapsto z^n$ at $z = 0$.

Below, we present the definitions of covering maps and branched covering maps.

Let Ω be a bounded planar domain. A mapping $\phi : \mathbb{D} \to \Omega$ is called *a holomorphic covering map* if for each point of Ω there exists a connected open neighborhood U in Ω such that ϕ maps each component of $\phi^{-1}(U)$ conformally onto U. That is, the holomorphic function ϕ is topologically a covering map. It is well-known that such a map ϕ always exists, and ϕ is unique up to a conformal automorphism of the unit disk [Gol, Mi, V], see as follows.

Theorem 2.1.14 (The Koebe Uniformization Theorem) *Given a point z_0 in $\mathbb{D}$ and w_0 in Ω with the cardinality $\sharp\partial\Omega \geq 2$, there is a unique holomorphic covering map ϕ of $\mathbb{D}$ onto Ω with $\phi(z_0) = w_0$ and $\phi'(z_0) > 0$.*

Thus, covering maps over $\mathbb{D}$ exist abundantly. For example, let E be a discrete subset of $\mathbb{D}$, and ϕ is a holomorphic covering map from $\mathbb{D}$ onto $\mathbb{D} - E$. Then one can show that ϕ is an interpolating Blaschke product if $0 \notin E$, see Example 6.3.9 for details.

It is notable that Riemann mapping theorem can be regarded as a special case of Theorem 2.1.14. It appears as a classical result in standard textbooks of complex analysis, see [Ne, p. 175] and [BG, p. 174, Theorem 2.6.21] for example.

Theorem 2.1.15 (Riemann Mapping Theorem) *Let Ω be a simple connected domain whose boundary consists of more than one point. Then for each fixed point $w_0 \in \Omega$ and $z_0 \in \mathbb{D}$, there is a unique conformal map ϕ from $\mathbb{D}$ onto Ω satisfying $\phi(z_0) = w_0$ and $\phi'(z_0) > 0$.*

As follows, we introduce the notion of branched covering map [Mi, Appendix E], which is a natural generalization of covering map. Given a holomorphic map ϕ : $\mathbb{D} \to \Omega$, write

$$G(\phi) = \{\gamma; \gamma \text{ is a conformal automorphism of } \mathbb{D} \text{ satisfying } \phi \circ \gamma = \phi\},$$

which is called *the deck transformation group* of ϕ. For this map $\phi : \mathbb{D} \to \Omega$, we pose the following conditions:

(1) every point of Ω has a connected open neighborhood U such that each connected component V of $\phi^{-1}(U)$ maps onto U by a proper map $\phi|_V$;
(2) for any $z_1, z_2 \in \mathbb{D}$, $\phi(z_1) = \phi(z_2)$ implies that there is a member $\gamma \in G(\phi)$ such that $\gamma(z_1) = z_2$.

A holomorphic map $\phi : \mathbb{D} \to \Omega$ is called *a branched covering map* if it satisfies (1); and it is called *regular* if (2) is satisfied. For each $w \in \Omega$, $\phi^{-1}(w)$ is called a *fiber* over w. Note that condition (2) tells us that the deck transformation group $G(\phi)$ acts transitively on each fiber.

Regular branched covering maps share several good properties. Let Ω be a bounded planar domain, and let $\phi : \mathbb{D} \to \Omega$ be such a map. Then it is not difficult to check that for each $w \in \Omega$, the multiplicities at $z(z \in \phi^{-1}(w))$ for zeros of $\phi - w$ only depend on w [Mi, Appendix E]. Also, ϕ has the following property [Mi, BG].

Proposition 2.1.16 *For a planar domain Ω, let $\phi : \mathbb{D} \to \Omega$ be a holomorphic regular branched covering map, and put*

$$\mathcal{E}_\phi = \{\phi(z) : z \in \mathbb{D} \text{ and } \phi'(z) = 0\}.$$

Then both $\mathcal{E}_\phi$ and $\phi^{-1}(\mathcal{E}_\phi)$ are discrete in Ω and $\mathbb{D}$, respectively.

Proof Assume that $\phi : \mathbb{D} \to \Omega$ is a holomorphic regular branched covering map. We first show that $\mathcal{E}_\phi$ is discrete in Ω. For this, it suffices to show that for each $w \in \Omega$, there is a neighborhood V of w such that $V \subseteq \Omega$ and $\mathcal{E}_\phi \cap V$ is finite. Now for a given $w \in \Omega$, there exists an enough small neighborhood V $(V \subseteq \Omega)$ of w such that

$$\phi^{-1}(V) = \bigsqcup_{i \geq 0} V_i,$$

where V_i are components of $\phi^{-1}(V)$; and for each i, $\phi|_{V_i} : V_i \to V$ is a proper map. In particular, $\phi(V_i) = V$ for each i. Also, we may assume that at least one V_i satisfies that $\overline{V_i} \subseteq \mathbb{D}$, say,

$$\overline{V_0} \subseteq \mathbb{D}.$$

Put

$$F_0 \triangleq \{\phi(z) : z \in V_0 \text{ and } \phi'(z) = 0\}.$$

Since $Z(\phi') \cap V_0$ is a finite set, so is F_0.

In addition, by regularity of ϕ, for any $z_1, z_2 \in \mathbb{D}$, $\phi(z_1) = \phi(z_2)$ implies that there is a $\gamma \in G(\phi)$ such that $\gamma(z_1) = z_2$. This implies that if $\phi'(z_0) = 0$ and $\phi(z_0) \in V$ for some $z_0 \in \mathbb{D}$, then there is a member ρ in $G(\phi)$ such that $\rho(z_0) \in V_0$, and thus $\phi'(\rho(z_0)) = 0$ since $\phi \circ \rho = \phi$. By arbitrariness of z_0,

$$\mathcal{E}_\phi \cap V = F_0,$$

which is finite, as desired. Thus, $\mathcal{E}_\phi$ is discrete in Ω.

It remains to show that $\phi^{-1}(\mathcal{E}_\phi)$ is discrete in $\mathbb{D}$; equivalently, $\phi^{-1}(\mathcal{E}_\phi) \cap r\mathbb{D}$ is finite for any $r \in (0, 1)$. With r fixed, the discrete property of $\mathcal{E}_\phi$ shows that $\phi(r\mathbb{D}) \cap \mathcal{E}_\phi$ is finite, and hence $\phi^{-1}(\mathcal{E}_\phi) \cap r\mathbb{D}$ is finite. By arbitrariness of r, $\phi^{-1}(\mathcal{E}_\phi)$ is discrete in $\mathbb{D}$, as desired. The proof is complete. □

For example, a finite Blaschke product B is a branched covering map over $\mathbb{D}$, but it is rarely regular. If B is also regular, then there is a $\lambda \in \mathbb{D}$ such that $B = c\varphi_\lambda^n$ for some unimodular constant c. In general, a finite Blaschke product B with order n is always an n-folds map. When restricted on $\mathbb{D} - B^{-1}(\mathcal{E}_B)$, B becomes a covering map.

The following result comes from the proof of [Cow1, Theorem 6].

Proposition 2.1.17 *Suppose $\phi : \mathbb{D} \to \Omega$ is a bounded holomorphic regular branched covering map. For each $a_0 \in \mathbb{D} - \phi^{-1}(\mathcal{E}_\phi)$, let B denote the Blaschke product whose zero set is $\phi^{-1}\big(\phi(a_0)\big)$. Then both $\dfrac{\phi - \phi(a_0)}{B}$ and $\dfrac{B}{\phi - \phi(a_0)}$ are in $H^\infty(\mathbb{D})$.*

Proof The proof is from [Cow1].

It is enough to deal with the case of $\phi : \mathbb{D} \to \Omega$ being a bounded holomorphic covering map. As will be seen later, the proof for general case is similar. Now fix $a_0 \in \mathbb{D} - \phi^{-1}(\mathcal{E}_\phi)$. Without loss of generality, assume that $\phi(a_0) = 0$, and the deck transformation group $G(\phi)$ of ϕ is infinite. Then there is an enough small $r > 0$ such that $r\mathbb{D} \subseteq \Omega$ and

$$\phi^{-1}(r\mathbb{D}) = \bigsqcup_{n=0}^{\infty} V_n,$$

where $\phi|_{V_n} : V_n \to r\mathbb{D}$ are biholomorphic for all n. Write $\{a_n\}_{n=0}^{\infty} = \phi^{-1}\big(\phi(a_0)\big)$, where $a_n \in V_n$ for each n, and let B denote the Blaschke product whose zero sequence is $\{a_n\}_{n=0}^{\infty}$. Clearly, $\frac{\phi}{B} \in H^{\infty}(\mathbb{D})$, see [Hof1].

To see $\frac{B}{\phi} \in H^{\infty}(\mathbb{D})$, it suffices to show that $|\frac{\phi}{B}|$ is bounded below. For this, note that if $z \in \mathbb{D} - \bigsqcup_{n=0}^{\infty} V_n$, then

$$\left|\frac{\phi(z)}{B(z)}\right| \geq |\phi(z)| \geq r.$$

Now consider $\frac{\phi}{B}$ on $\overline{V_0}$. Since a_0 is the simple zero of B and ϕ, then by theory of complex analysis there is a constant $c > 0$ satisfying

$$\left|\frac{\phi(z)}{B(z)}\right| \geq c,\ z \in \overline{V_0}.$$

In general, let ρ be a member of $G(\phi)$ satisfying $\rho(a_k) = a_0$, and then ρ maps V_k onto V_0. Since $B \circ \rho$ is also a Blaschke product, whose zero sequence equals $\{a_k\}_{k=0}^{\infty}$, we have

$$|B(\rho(z))| = |B(z)|,\ z \in \mathbb{D}.$$

This, combined with $\phi \circ \rho = \phi$, yields that

$$\left|\frac{\phi(z)}{B(z)}\right| = \left|\frac{\phi(\rho(z))}{B(\rho(z))}\right|,\ z \in V_k,$$

forcing $|\frac{\phi}{B}| \geq c$ on V_k. Therefore,

$$\left|\frac{\phi(z)}{B(z)}\right| \geq c,\ z \in \bigsqcup_{n=0}^{\infty} V_n.$$

This leads to the conclusion that $|\frac{\phi}{B}|$ is bounded below, forcing $\frac{B}{\phi} \in H^{\infty}(\mathbb{D})$, as desired. The proof is complete. □

Next, we would like to mention Runge's theorem, see [Hor, Theorem 1.3.1].

Theorem 2.1.18 (Runge) *Let Ω be an open set in $\mathbb{C}$ and K a compact subset of Ω. The following are equivalent.*

(1) Every function which is holomorphic on a neighborhood of K can be approximated uniformly on K by functions in $A(\Omega)$;

(2) The open set $\Omega - K$ *has no component which is relatively compact in* Ω*;*
(3) For every $z \in \Omega - K$*, there is a function* $f \in A(\Omega)$ *such that*

$$|f(z)| > \sup_{w \in K} |f(w)|.$$

The following is obtained by taking $\Omega = \mathbb{C}$, also see [Hor].

Corollary 2.1.19 *Every function which is holomorphic on a neighborhood of the compact set* K *can be approximated by polynomials uniformly on* K *if and only if* $\mathbb{C} - K$ *is connected, if and only if for every* $z \notin K$ *there is a polynomial* f *such that*

$$|f(z)| > \sup_{w \in K} |f(w)|.$$

The multi-variable version of Runge's theorem is Oka-Weil theorem [Oka, Weil]. Before stating it, we recall the notion of convex hull, or in short, hull. Let K be a compact set in some domain G in $\mathbb{C}^d$, and $\mathcal{F}$ be a family of holomorphic functions over G. The *hull* $\hat{K}$ of K with respect to $\mathcal{F}$ is defined to be

$$\{z \in G : |f(z)| \leq \|f\|_{K,\infty}, f \in \mathcal{F}\},$$

where $\|f\|_{K,\infty} = \sup\{|f(z)| : z \in K\}$.

Theorem 2.1.20 (Oka-Weil) *Let* G *be a domain in* $\mathbb{C}^d$*, and suppose the compact set* $K(K \subseteq G)$ *coincides with its hull with respect to the algebra* $Hol(G)$ *of all holomorphic functions on* G*. Then for any function* f *holomorphic in a neighborhood of* K*, and for any* $\varepsilon > 0$*, there is a function* $F \in Hol(G)$ *such that*

$$\max_{z \in K} |f(z) - F(z)| < \varepsilon.$$

Finally, some results are presented from real analysis, which will be used in the sequel.

For a topological space X, $C(X)$ always denotes the algebra of all continuous complex-valued functions over X. If X is a compact Hausdorff space, then $C(X)$ is equipped with the maximal-norm:

$$\|h\| \triangleq \max_{x \in X} |h(x)|, \ h \in C(X).$$

Theorem 2.1.21 (Stone-Weierstrass Theorem) *Let* X *be a compact Hausdorff space and let* $\mathcal{S}$ *be a subset of* $C(X)$ *which separates points in* X*. That is, for any* $x, y \in X$ *with* $x \neq y$*, there is a function* $f \in \mathcal{S}$ *such that* $f(x) \neq f(y)$*. Then the complex unital* $*$*-algebra generated by* $\mathcal{S}$ *is dense in* $C(X)$*.*

Given a function $h \in L^1(\mathbb{T})$, expand its Fourier series as follows:

$$h \sim \sum_{n=-\infty}^{\infty} a_n e^{i\theta},$$

where the right hand side only represents a formal series. Write $s_k = \sum_{n=-k}^{k} a_n e^{i\theta}$, and put

$$\sigma_k = \frac{s_0 + \cdots + s_{k-1}}{k},$$

called the *k-th Cesaro mean* of h, see [Hof1]. It is well known that Cesaro means have good property in approximation. For example, if in addition $h \in L^p(\mathbb{T})$ for some $p \in [1, +\infty)$, then $\{\sigma_k\}$ converges to h in $L^p(\mathbb{T})$-norm. Besides, if $h \in L^\infty(\mathbb{T})$, then

$$\|\sigma_k\|_\infty \leq \|h\|_\infty, \ \ k = 1, 2, \cdots.$$

In this case, $\{\sigma_k\}$ converges to h almost everywhere with respect to the arc-length measure, and thus $\{\sigma_k\}$ converges to h in the weak*-topology of $L^\infty(\mathbb{T})$.

Theorem 2.1.22 (Lebesgue's Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions on a complete measure space (X, Σ, μ). Suppose there is a non-negative $g \in L^1(X, \mu)$ such that $|f_n| \leq g\,(n \geq 1)$ hold almost everywhere and one of the following holds:*

(1) $\{f_n\}$ converges to f almost everywhere;
(2) $\{f_n\}$ converges to f in measure.

Then f is integrable and

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0.$$

In particular, $\lim\limits_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Also presented is Lusin's theorem, which is well known in real analysis. One can refer to [Hal1, p. 242] or [Ru3, Theorem 2.24].

Theorem 2.1.23 (Lusin's Theorem) *Suppose X is a locally compact Hausdorff space, and $\mathcal{B}$ is the class of all Borel subsets of X. Let $(X, \mathcal{B}, \mu)$ be a Radon measure space, and let f be a Borel measurable function on $E (E \in \mathcal{B})$ with $\mu E < \infty$, then for each $\varepsilon > 0$, there is a compact subset F of E such that $f|_F$ is continuous, and $\mu(E - F) < \varepsilon$.*

For a Radon measure space $(X, \mathcal{B}, \mu)$, by definition μ is inner regular and for each $x \in X$, there exists a open neighborhood O_x of x such that $\mu(O_x) < \infty$. It is known that for a measure space $(X, \mathcal{B}, \mu)$, if X is a locally compact Hausdorff space such that every open set in X is a countable union of compact sets and $\mu(K)$ is finite for every compact set K, then μ is always regular and locally finite [Ru3, Theorem 2.18]. Recall that a measure μ is called *regular* if for any measurable set E, we have

$$\mu E = \inf\{\mu U : U \text{ is an open set containing } E\} \qquad \text{(outer regularity)}$$

and

$$\mu E = \sup\{\mu F : F \text{ is a compact subset of } E\} \qquad \text{(inner regularity).}$$

For example, if μ is the Lebesgue measure m on $\mathbb{R}^n$, then μ is a regular measure. In this case, it is well-known that for any Lebesgue measurable function f, there is a Borel measurable function $\hat{f}$ satisfying $f = \hat{f}$ a.e. with respect to m. Thus, Lusin's theorem also holds for Lebesgue measurable functions.

In Lusin's theorem, since X is a locally compact Hausdorff space, then applying Tietze extension theorem shows that there is a continuous function g on X such that $g|_F = f$. Also, in Theorem 2.1.23 one can require $\|g\|_\infty \le \|f\|_\infty$ provided that $f \in L^\infty(X, \mu)$.

Below, we present Tietze Extension Theorem, see [Arm, Ke]. Given a topological space X, if every pair (K_1, K_2) of disjoint closed sets in X can be separated by two disjoint open sets U_1 and U_2, satisfying $U_1 \supseteq K_1$ and $U_2 \supseteq K_2$, then X is called *a normal space*.

Theorem 2.1.24 *Let K be a closed subset of X, a normal topological space, and f is a continuous function from K into $[a, b]$ with $a, b \in \mathbb{R}$. Then there exists a continuous function g from X into $[a, b]$ such that*

$$g(x) = f(x), \ \ x \in K.$$

2.2 The Notion of Capacity

In this section, we will give the notion of capacity. This section mainly consults [Ga, Ru1] and [CL].

Let K be a compact set in the complex plane, and μ a positive measure supported on K with $\mu \neq 0$. Define

$$U_\mu(z) = \int_K \ln \frac{1}{|\zeta - z|} d\mu(\zeta), \ z \in \mathbb{C},$$

called *the logarithmic potential* of μ. The following displays equivalent conditions for a compact set in $\mathbb{C}$ to have positive (logarithmic) capacity, refer to [Ga, pp. 78, 79] and [Ru1, pp. 56, 67].

Proposition 2.2.1 *Suppose K is a compact subset of $\mathbb{C}$. Then the following are equivalent:*

(1) there is a nonzero positive measure μ on K whose logarithmic potential is bounded on some neighborhood of K.
(2) K carries a nonzero positive measure μ whose logarithmic potential is continuous in $\mathbb{C}$.

If either (1) or (2) holds, then K is called to have positive capacity. In particular, if $K \subseteq \mathbb{D}$, then K has positive capacity if and only if K supports a nonzero positive measure ν for which the function

$$G_\nu(z) = \int_K \ln \left|\frac{1-\overline{\zeta}z}{\zeta - z}\right| d\nu(\zeta)$$

is bounded on $\mathbb{D}$.

In general, a set $E(E \subset \mathbb{C})$ is called to *have positive capacity* if there is some compact subset K of E having positive capacity. If E does not have positive capacity, then we say *E has capacity zero*.

It is known that a capacity-zero subset K of $\mathbb{R}$ (regarded as a subset of $\mathbb{C}$) must have null Lebesgue measure [Ru1, p. 57]. In general, a set of capacity zero is of linear measure zero [CL]. Here, a subset E of $\mathbb{C}$ has linear measure zero if and only if for any $\varepsilon > 0$, there is a sequence of disks $O(z_n, r_n)$ whose union covers E and $\sum_n r_n < \varepsilon$; that is, the one-dimensional Hausdorff measure of E is zero. However, the converse does not hold. As mentioned in [CL, p. 10], the standard Cantor's ternary set P is of positive capacity [Nev, Fr]. However, P is a perfect set with null Lebesgue measure. Recall that a set is called *perfect* if it is a closed set with no isolated point.

Let Ω be a domain in $\mathbb{C}$ and E is a compact subset of Ω. Then E is called *H^∞-removable* if every bounded holomorphic function on $\Omega - E$ can be analytically extended to Ω. It should be pointed out that the property of H^∞-removable does not depend on the choice of Ω. One may refer to [Ma, Du] for an account on H^∞-removable property.

The following theorem is known in the theory of complex analysis, which was shown by Painlevé, and later by Besicovitch [Bes].

Theorem 2.2.2 *If E is a compact subset in $\mathbb{C}$ whose one-dimensional Hausdorff measure is zero, then E is H^∞-removable. In particular, if E is a compact set of capacity zero, then E is H^∞-removable.*

The above paragraphs have provided some descriptions for E having positive capacity. Below, to each subset E of $\mathbb{C}$ one can assign a precise value $\operatorname{cap} E$, which is exactly the logarithmic capacity of E. The following content is from [CL]. Denote by $\mathcal{M}$ the set of all Borel probability measures μ supported on E, i.e. all Borel measures μ satisfying

$$\int_E d\mu(\zeta) = 1.$$

Let F be a compact subset of $\mathbb{C}$. If $\mathbb{C}-F$ were not connected, then the linear measure of F would never be zero, which means that in some sense F were "large". Now, assume that F is a compact subset of $\mathbb{C}$ such that $\mathbb{C}-F$ is connected. Let μ be a non-negative Borel measure over $\mathbb{C}$ and $\mu \in \mathcal{M}$, and set

$$u(z) = \int_F \ln \frac{1}{|z-\zeta|} d\mu(\zeta), z \notin F,$$

which is harmonic. Define

$$V_F = \inf_{\mu \in \mathcal{M}} \big(\sup_{z \notin F} u(z)\big),$$

which is called the *equilibrium potential* of F, and the *capacity of the set F* is defined to be

$$\operatorname{cap} F \triangleq \exp(-V_F).$$

Here, $\exp(-\infty)$ is assigned to be 0. In general, the capacity of a Borel set E is defined to be the supremum of capacities of all compact subsets of E. From the definition, it follows that $\operatorname{cap} A \leq \operatorname{cap} B$ whenever $A \subseteq B$. Also, it is clear that the union of two capacity-zero sets is of capacity zero. Furthermore, the union of a countable family of capacity-zero sets is of capacity zero [Nev, Fr]. In particular, any countable subset of the complex plane has capacity zero. Concerning with capacity, there are many interesting results arising from function theory, see [CL, Theorem 1.7] and [Ru2, 3.6.2].

The following property of closed capacity-zero sets proves useful.

Lemma 2.2.3 ([CL, Theorem 1.7]) *If E is a closed, bounded set of capacity zero, then there exists a probability measure μ on E such that the potential*

$$u(z) = \int_E \ln \frac{1}{|z-\zeta|} d\mu(\zeta),\ z \notin E$$

tends to $+\infty$ as z tends to an arbitrary point of E.

The following result will be needed in the sequel, which collects two known results from [Ga] and [CL]. The reader can also consult [GM2, Theorem 1.1].

Proposition 2.2.4 ([Ga, CL]) *If $f : \mathbb{D} \to \Omega$ is a holomorphic covering map, then f is an inner function if and only if $\Omega = \mathbb{D} - E$, where E is a relatively closed subset of $\mathbb{D}$ with capacity zero.*

Proof Since the image of a non-constant holomorphic map is open, the "only if" part follows directly from [Ga, p. 80, Theorem 6.6] or Theorem 2.1.7. Thus it remains to deal with the "if" part, whose proof comes from [CL, pp. 37, 38]. Here, we include the proof for completeness. Now assume that $f : \mathbb{D} \to \Omega$ is a holomorphic covering map, where $\Omega = \mathbb{D} - E$, with E being a relatively closed, capacity-zero subset of $\mathbb{D}$. We will show that f is an inner function. For this, we first make the following claim:

Claim If, on a set $F(F \subseteq \mathbb{T})$ of positive measure, the radial limit values of f lie in a set E of capacity zero, then f is identically constant.

The proof reduces to the case of E being a compact set of capacity zero. For this, assume that on a set $F(F \subseteq \mathbb{T})$ of positive measure, f admits radial limits in E, a set of capacity zero. By standard analysis, the function $f|_F$(taking radial limits) is Lebesgue measurable over F. By Lusin's theorem, there is a compact subset F_0 of F with positive measure such that $f|_{F_0}$ is continuous, and thus $f(F_0)$ is a compact subset of E with capacity zero. Then one can replace E with $f(F_0)$.

Now we may assume that E is compact. By Lemma 2.2.3, there is a probability measure μ on E such that the potential

$$u(w) = \int_E \ln \frac{1}{|w-\zeta|} d\mu(\zeta) \quad (w \notin E)$$

tends to $+\infty$ as w tends to any given point of E. Set

$$u_1(w) = \int_E \ln \frac{2}{|w-\zeta|} d\mu(\zeta),$$

which is a non-negative harmonic function in $\mathbb{D} - E$. Also, $u_1(w)$ tends to $+\infty$ as w tends to any point of E. Define $U(z) = u_1 \circ f$, a non-negative harmonic function in $\mathbb{D}$. By our assumption on f, there is a subset J of $\mathbb{T}$ with positive measure such that

$$\{f(\zeta) : \zeta \in J\} \subseteq E,$$

where $f(\zeta)$ is defined to be the radial limit of f at ζ, i.e. $\lim_{r\to 1^-} f(r\zeta)$. Thus for each $\zeta \in J$, $\lim_{r\to 1^-} U(r\zeta) = +\infty$.

Let V denote a harmonic conjugate of U, and $U + iV$ is holomorphic in $\mathbb{D}$. Define

$$F = \exp(-U - iV),$$

a bounded holomorphic function on $\mathbb{D}$. Since for each $\zeta \in J$, $\lim_{r\to 1^-} U(r\zeta) = +\infty$, $\lim_{r\to 1^-} F(r\zeta) = 0$. By a uniqueness theorem of Riesz [Hof1], F is identically zero, which is a contradiction. Therefore, f is identically constant. The proof of the claim is complete.

To show that f is an inner function, it suffices to prove that for almost everywhere $\zeta \in \mathbb{T}$, the radial limit $f(\zeta)$ at ζ exists and lies in $\{z; |z| = 1\}$. Assume conversely that f possesses radial limits w with $|w| < 1$ on a set of positive measure on $\mathbb{T}$. Then we would see that *those radial limits w lie in E*. If so, then applying the above claim yields that f is a constant, which is a contradiction. Therefore, f is an inner function, as desired.

To complete the proof, it remains to show that if the radial limit w lies in the unit disk, then $w \in E$. Otherwise, there would be some $t \in \mathbb{R}$ such that

$$\lim_{r\to 1^-} f(re^{it}) = w_0 \in \Omega = \mathbb{D} - E.$$

Since f is a covering map from $\mathbb{D}$ onto Ω, there is a disk O containing w_0 such that $O \subseteq \Omega$ and

$$f^{-1}(O) = \bigsqcup_n U_n$$

where U_n are all connected components, and $f : U_n \to O$ is a biholomorphic map for each n. Since $\lim_{r\to 1^-} f(re^{it}) = w_0$, there is an enough small $\delta > 0$ satisfying

$$f(re^{it}) \in O, \ r \in (1-\delta, 1).$$

Then there is some integer n_0 such that

$$re^{it} \in U_{n_0}, \ r \in (1-\delta, 1).$$

Since $f : U_{n_0} \to O$ is a bijection, there is a point $a \in U_{n_0}$ such that $f(a) = w_0$. Then there is a $r_0 > 0$ and $\varepsilon > 0$ such that

$$\overline{O(a, r_0)} \subseteq U_{n_0} \quad \text{and} \quad O(w_0, \varepsilon) \subseteq f(O(a, r_0)) \subseteq O.$$

Noting $\lim_{r\to 1^-} f(re^{it}) = w_0$, we deduce that there is an $r_1 \in (1-\delta, 1)$ such that

$$f(r_1 e^{it}) \in O(w_0, \varepsilon) \quad \text{and} \quad r_1 e^{it} \notin O(a, r_0).$$

Since $f(O(a, r_0)) \supseteq O(w_0, \varepsilon)$, it follows that there is a point $a' \in O(a, r_0)$ satisfying $f(a') = f(r_1 e^{it})$ and $a' \neq r_1 e^{it}$, which is a contradiction to the bijectivity of the map $f : U_{n_0} \to O$, completing the proof. $\square$

Remark 2.2.5 By similar discussion as above, one can prove a similar version of Proposition 2.2.4 for regular branched covering maps; that is, if $\phi : \mathbb{D} \to \Omega$ is a regular branched covering map, then ϕ is an inner function if and only if $\Omega = \mathbb{D} - E$, where E is a relatively closed subset of $\mathbb{D}$ with capacity zero.

For more results on singularities and sets of capacity zero, we call the reader's attention to [CL, pp. 10–13].

2.3 Local Inverse and Analytic Continuation

This section mainly introduces the notations of local inverse and analytic continuation, which proves useful in the analysis of geometric property of holomorphic functions, see Chaps. 3–5.

Let Ω_0 be a domain of the complex plane and f be a holomorphic function on Ω_0. If ρ is a map defined on some sub-domain V of Ω_0 such that $\rho(V) \subseteq \Omega_0$ and $f(\rho(z)) = f(z), z \in V$, then ρ is called *a local inverse* of f on V [T1]. In this book, we take $\Omega_0 = \mathbb{D}$ in most cases.

For example, put $f(z) = z^n$ $(z \in \mathbb{D})$ and $\rho(z) = \xi z$ $(z \in \mathbb{D})$, where ξ is one of the n-th root of unit. Then ρ is a local inverse of f on $\mathbb{D}$. To see more examples, assume that ϕ is a covering map over $\mathbb{D}$. Then each local inverse of ϕ can be analytically extended to an automorphism of $\mathbb{D}$, precisely, a member in the deck transformation group of ϕ.

We need some definitions from [Ru3, Chap. 16]. *A function element* is an ordered pair (f, D), where D is a simply-connected open set and f is a holomorphic function on D. Two function elements (f_0, D_0) and (f_1, D_1) are called *direct continuations* if $D_0 \cap D_1$ is not empty and $f_0 = f_1$ holds on $D_0 \cap D_1$. By a *curve* or a *path*, we mean a continuous map from [0, 1] into $\mathbb{C}$. A *loop* is a path σ satisfying $\sigma(0) = \sigma(1)$. Given a function element (f_0, D_0) and a curve γ with $\gamma(0) \in D_0$, if there is a partition of [0, 1]:

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

and function elements $(f_j, D_j)(0 \le j \le n)$ such that

1. (f_j, D_j) and (f_{j+1}, D_{j+1}) are direct continuation for all j with $0 \le j \le n-1$;
2. $\gamma[s_j, s_{j+1}] \subseteq D_j(0 \le j \le n-1)$ and $\gamma(1) \in D_n$,

then (f_n, D_n) is called *an analytic continuation of* (f_0, D_0) *along* γ; and (f_0, D_0) is called to *admit* an analytic continuation along γ. In this case, we write $f_0 \sim f_n$. Clearly, this is an equivalence and we write $[f]$ for *the equivalent class* of f.

In Chaps. 4 and 5, we consider the case $f = B$, a thin or finite Blaschke product. In such a situation, put

$$E = \mathbb{D} - B^{-1}(\mathcal{E}_B),$$

where $\mathcal{E}_B$ denotes the critical value set; that is

$$\mathcal{E}_B = \{B(z) : z \in \mathbb{D} \text{ and } B'(z) = 0\}.$$

All functions mentioned in the last paragraph (such as f_j, ρ and σ, γ) are well-defined on some subsets of E.

The following theorem is well-known, which states that the analytic continuation along a curve must be unique, see [Ru3, Theorem 16.11].

Theorem 2.3.1 *If (f, D) is a function element and γ is a curve which starts at the center of D, then (f, D) admits at most one analytic continuation along γ.*

As follows we present some examples, which come from standard textbooks of complex analysis.

Example 2.3.2 Let D_0 be the unit disk $\mathbb{D}$, and D_1 be the upper half plane $\{z : Imz > 0\}$. Write

$$f_0(z) = \sum_{n=0}^{\infty} z^n,\ z \in D_0,$$

and put

$$f_1(z) = \frac{1}{1-z},\ z \in D_1.$$

Then by direct computation one sees that (f_0, D_0) and (f_1, D_1) are direct continuation.

Set $V_0 = \{z : |z-1| < 1\}$, define

$$g_0(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n},\ z \in V_0.$$

That is, $g_0(z) = \ln z$, where $\ln 1 = 0$. With $V_1 = D_1$, the upper half plane, put

$$g_1(z) = \int_1^z \frac{1}{\zeta} d\zeta,\ z \in V_1,$$

where the integral is along any curve in V_1. Then it is not difficult to verify that for any $x \in (0,1)$, $g_0(x) = g_1(x)$, and hence by the uniqueness theorem $g_0 = g_1$ on $V_0 \cap V_1$. Therefore, (g_0, V_0) and (g_1, V_1) are direct continuation.

Example 2.3.3 Let V_2 and V_3 denote the left and lower half plane, respectively. That is,

$$V_2 = \{z : Rez < 0\} \quad \text{and} \quad V_3 = \{z : Imz < 0\}.$$

Similarly, define

$$g_2(z) = \int_{-1}^{z} \frac{1}{\zeta} d\zeta + \pi i, \; z \in V_2,$$

where the integral is along any curve in $\overline{V_2} - \{0\}$. Define

$$g_3(z) = \int_{1}^{z} \frac{1}{\zeta} d\zeta + 2\pi i, \; z \in V_3,$$

where the integral is along any curve in $\overline{V_3} - \{0\}$. Then one can check that (g_i, V_i) and (g_{i+1}, V_{i+1}) are direct continuation for $i = 0, 1, 2$. However, (g_3, V_3) and (g_1, V_1) are not direct continuation since $g_1(1) \neq g_3(1)$.

Furthermore, let γ be a curve in $\mathbb{C} - \{0\}$ with $\gamma(0) \in V_1$. Then it is not difficult to show that there is a unique analytic continuation of g_0. Inspired by this fact, one can present more examples. For instance, define

$$z^{\alpha} = \exp(\alpha \ln z),$$

where α is an irrational real number. Along γ there always exists an analytic continuation of $(\exp(\alpha g_1), V_1)$.

To enclose this section, an important result on analytic continuations will be presented, known as the monodromy theorem, see [Ru3] for example.

Theorem 2.3.4 (The Monodromy Theorem) *Suppose Ω is a simply connected domain, (f, D) is a function element with $D \subseteq \Omega$, and (f, D) can be analytically continued along every curve in Ω that starts at the center of D. Then there exists $g \in Hol(\Omega)$ such that $g(z) = f(z)$, $z \in D$.*

For example, let L be a simple curve connecting 0 and ∞, say $L = [0, +\infty)$. Then set $\Omega = \mathbb{C} - L$, a simply connected domain. It is easy to see that the function element (g_2, V_2) in Example 2.3.3 satisfies the assumptions in Theorem 2.3.4. Therefore, there is a holomorphic function g in Ω satisfying $g|_{V_2} = g_2$. Precisely,

$$g(z) = \int_{-1}^{z} \frac{1}{\zeta} d\zeta + \pi i, \; z \in \Omega,$$

where the integral is along any smooth curve in Ω.

2.4 Uniformly Separated Sequence

As usual, let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let dA be the normalized area measure on $\mathbb{D}$. Denote by $L^2_a(\mathbb{D})$ the *Bergman space* over $\mathbb{D}$ consisting of all holomorphic functions over $\mathbb{D}$, which are square integrable with respect to dA. In general, for a domain Ω in $\mathbb{C}^d$, denote by $L_a^2(\Omega)$ the *Bergman space over* Ω, which consists of all holomorphic functions over Ω that are square integrable with respect to the volume measure dV on Ω. We also introduce the weighted Bergman space. Precisely, for each $\alpha > -1$, denote by $L_{a,\alpha}^2(\mathbb{D})$ the *weighted Bergman space*, which consists of all holomorphic functions over $\mathbb{D}$ that are square integrable with respect to the normalized measure $(\alpha+1)(1-|z|^2)^\alpha dA(z)$. When $\alpha = 0$, $L_{a,0}^2(\mathbb{D})$ is exactly the usual Bergman space over $\mathbb{D}$. Denote by $L_{a,\alpha}^p(\mathbb{D})(0 < p < \infty)$ the space of all holomorphic functions f over $\mathbb{D}$ satisfying

$$\|f\|_p = \Big(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)\Big)^{\frac{1}{p}} < \infty.$$

When $p \geq 1$, it is well-known that $L_{a,\alpha}^p(\mathbb{D})$ is a Banach space.

Another classical model of reproducing kernel Hilbert space is the Hardy space $H^2(\mathbb{D})$. In general, for $0 < p < +\infty$, $H^p(\mathbb{D})$ consists of all holomorphic functions f on $\mathbb{D}$ which satisfies

$$\|f\|_p = \sup_{0<r<1} \Big(\int_0^{2\pi} |f_r(\theta)|^p d\theta\Big)^{\frac{1}{p}} < \infty.$$

For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is a Banach space. The Hardy space $H^2(\mathbb{D}^n)$ over the polydisk $\mathbb{D}^n$ is defined to be the class of all holomorphic functions f on $\mathbb{D}^n$ which satisfies

$$\sup_{0<r<1} \Big(\int_{\mathbb{T}^n} |f(r\zeta)|^2 dm_n(\zeta)\Big)^{\frac{1}{2}} < \infty,$$

where dm_n denotes the normalized Lebesgue measure on $\mathbb{T}^n$. Note that each holomorphic function f on $\mathbb{D}^n$ has the following expansion:

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha,$$

and it is well-known that $f \in H^2(\mathbb{D}^n)$ if and only if the coefficients c_α are square-summable.

A sequence $\{z_k\}$ in $\mathbb{D}$ is called *uniformly separated* if there is a numerical constant $\delta > 0$ such that

$$\delta_k = \prod_{j \neq k} d(z_j, z_k) \equiv \prod_{j \neq k} \Big|\frac{z_j - z_k}{1 - \overline{z_j} z_k}\Big| \geq \delta, \quad k = 1, 2, \cdots.$$

A sequence $\{z_k\}$ is uniformly separated if and only if it is an interpolating sequence for $H^\infty(\mathbb{D})$; namely, for each bounded complex sequence $\{w_k\}$ there is an f in $H^\infty(\mathbb{D})$ satisfying $f(z_k) = w_k$ [Ga, Hof1].

A sequence $\{z_k\}$ of distinct points in $\mathbb{D}$ is called *an interpolating sequence for* $L_a^p(\mathbb{D})(0 < p < \infty)$ if the equations $f(z_k) = w_k$ for $k = 1, 2, \cdots$ have a common solution f in $L_a^p(\mathbb{D})$ whenever

$$\sum_{k=1}^{\infty}(1 - |z_k|^2)^2 |w_k|^p < \infty.$$

A sequence $\{z_k\}$ is called *an interpolating sequence for* $H^p(\mathbb{D})(0 < p < \infty)$ if the equations $f(z_k) = w_k$ for $k = 1, 2, \cdots$ has a solution f in $H^p(\mathbb{D})$ whenever

$$\sum_{k=1}^{\infty}(1 - |z_k|^2)|w_k|^p < \infty.$$

The following result is known[DS, pp. 157, 175].

Theorem 2.4.1 *A uniformly separated sequence is an interpolating sequence for both* $L_a^p(\mathbb{D})$ *and* $H^p(\mathbb{D})$*, where* $0 < p < \infty$.

In addition, if a sequence $\{z_k\}$ is interpolating for some $H^p(\mathbb{D})$, then it is uniformly separated, and hence interpolating for any $H^p(\mathbb{D})$. For details, see [DS, Sect. 6.2]. However, a similar version for $L_a^p(\mathbb{D})$ fails.

A sequence $\{z_k\}$ in $\mathbb{D}$ is called a $L_{a,\alpha}^2(\mathbb{D})$-*interpolating sequence* if for any sequence $\{w_k\}$ of complex numbers satisfying

$$\sum_{k=0}^{\infty}(1 - |z_k|^2)^{2+\alpha}|w_k|^2 < +\infty,$$

there exists an $h \in L_{a,\alpha}^2(\mathbb{D})$ satisfying

$$h(z_k) = w_k, k = 1, 2 \cdots .$$

In the case of $p = 2$, Theorem 2.4.1 has a generalization, which essentially comes from Seip [Se1].

Proposition 2.4.2 *A uniformly separated sequence* $\{z_k\}$ *is interpolating for all* $L_{a,\alpha}^2(\mathbb{D})(\alpha > -1)$.

Proof Assume that $\{z_k\}$ is a uniformly separated sequence. By the method in [Se1], a sequence $\{\lambda_k\}$ is interpolating for $L_{a,\alpha}^2(\mathbb{D})$ if and only if $\{\lambda_k\}$ is separated and $D^+(\{\lambda_k\}) < \frac{\alpha+1}{2}$, where $D^+(\{\lambda_k\})$ denotes the upper density of $\{\lambda_k\}$ (for details, refer to [DS] and [HKZ, Theorem 5.22]). When $\{\lambda_k\}$ is uniformly separated,

$$D^+(\{\lambda_k\}) = 0,$$

see [DS, pp. 174, 175]. Thus, a uniformly separated sequence $\{z_k\}$ is interpolating for all $L^2_{a,\alpha}(\mathbb{D})(\alpha > -1)$, as desired. □

By a careful verification, one can show that all results on [DS, pp. 103–109] hold for $L^p_{a,\alpha}(\mathbb{D})$. In particular, one gets the following, refer to [Ho, McS, Sh] and [DS].

Theorem 2.4.3 *Let B denote the Blaschke product whose zero sequence is $\{z_k\}$. Then the following statements are equivalent:*

(1) The Blaschke sequence $\{z_k\}$ is a finite union of uniformly separated sequences;
(2) $\sum_{k=1}^{\infty}(1-|z_k|^2)|f(z_k)|^p < \infty$ for all f in $H^p(\mathbb{D})$ for some $p \in (0,\infty)$;
(3) $\sup_{\lambda\in\mathbb{D}} \sum_{k=1}^{\infty}(1-|\varphi_\lambda(z_k)|) < \infty$;
(4) The Blaschke product B is a universal divisor of $L^p_a(\mathbb{D})$ for some $p \in (0,\infty)$; that is, $\frac{f}{B} \in L^p_a(\mathbb{D})$ for every function f in $L^p_a(\mathbb{D})$ which vanishes on $\{z_k\}$;
(5) The Blaschke product B is a universal divisor of $L^p_{a,\alpha}(\mathbb{D})(0 < p < \infty)$;
(6) The multiplication operator M_B is bounded below on some $L^p_a(\mathbb{D})$ with $p > 0$; that is, there is a positive constant c such that $\|Bf\|_p \geq c\|f\|_p$ for all $f \in L^p_a(\mathbb{D})$;
(7) The multiplication operator M_B is bounded below on some $L^p_{a,\alpha}(\mathbb{D})$ $(0 < p < \infty)$.

Let B be such a Blaschke product as in Theorem 2.4.3, and denote by $\mathcal{N}$ the subspace of all functions in $L^p_{a,\alpha}(\mathbb{D})(0 < p < \infty, \alpha > -1)$ that vanish on the zero set of B, counting multiplicity. Then by Theorem 2.4.3(7),

$$\mathcal{N} = BL^p_{a,\alpha}(\mathbb{D}).$$

A sequence $\{z_k\}$ of distinct points in $\mathbb{D}$ is called *uniformly discrete* [DS] if

$$\inf_{j\neq k} d(z_j, z_k) > 0.$$

It is worthwhile to mention that in [HKZ] an equivalent definition of uniformly discrete is formulated.

The following result comes essentially from [Has, Lu], also see [Zhu3, Theorem 2.25].

Theorem 2.4.4 *Given $p > 0$ and $\alpha > -1$, if $\{z_k\}$ is uniformly discrete sequence in $\mathbb{D}$, then there is a constant $C > 0$ such that*

$$\sum_{k=1}^{\infty}(1-|z_k|^2)^{2+\alpha}|f(z_k)|^p \leq C\|f\|_p^p,\ f \in L^p_{a,\alpha}(\mathbb{D}).$$

In particular, if $\{z_k\}$ is uniformly discrete, then

$$\sum_{k=1}^{\infty}(1-|z_k|^2)^{2+\alpha}|f(z_k)|^2 \leq C\|f\|_2^2,\ f \in L^2_{a,\alpha}(\mathbb{D}).$$

In the case of $\alpha = 0$, Theorem 2.4.4 is a partial result of [DS, p. 70, Theorem 15].

To establish Theorem 2.4.4, we first present an estimate from [Zhu3, Lemma 2.20] as follows.

Fix $r \in (0, 1)$. Given any $z \in \mathbb{D}$, $\Delta(z, r)$ denotes the pseudohyperbolic disk centered at z with radius r. For $w \in \Delta(z, r)$, write $\lambda = \varphi_z(w)$, and then $w = \varphi_z(\lambda)$. Since the pseudohyperbolic metric d is invariant under Möbius map,

$$d(0, \lambda) = d(z, w) < r, \ i.e. \ |\lambda| < r.$$

By computations,

$$1 - |w|^2 = \frac{(1 - |z|^2)(1 - |\lambda|^2)}{|1 - \overline{\lambda} z|^2};$$

that is,

$$\frac{1 - |w|^2}{1 - |z|^2} = \frac{1 - |\lambda|^2}{|1 - \overline{\lambda} z|^2}.$$

Since $|z| \leq 1$ and $|\lambda| \leq r$, then there is a constant $C_0 > 0$ such that

$$\frac{1}{C_0} \leq \frac{1 - |\lambda|^2}{|1 - \overline{\lambda} z|^2} \leq C_0,$$

and thus

$$\frac{1}{C_0} \leq \frac{1 - |w|^2}{1 - |z|^2} \leq C_0, \quad w \in \Delta(z, r). \tag{2.8}$$

Since

$$\begin{aligned}
(1 - |w|^2)(1 - |z|^2) &= 1 - (|z|^2 + |w|^2) + |z|^2|w|^2 \\
&\leq 1 - 2|z||w| + |z|^2|w|^2 \\
&= (1 - |z||w|)^2 \\
&\leq |1 - \overline{w} z|^2,
\end{aligned}$$

then by (2.8), there is a constant C_1 such that

$$|1 - \overline{w} z| \geq C_1(1 - |z|^2), \quad w \in \Delta(z, r). \tag{2.9}$$

Proof of Theorem 2.4.4 The proof of Theorem 2.4.4 comes from [Zhu3, pp. 57–61].

Let f be a holomorphic function in $\mathbb{D}$. Since $|f|^p$ is subharmonic, then for all $a \in \mathbb{D}$ and $r \in (0, 1-|a|)$, we have

$$|f|^p(a) \le \int_0^{2\pi} |f|^p(a + r\rho e^{i\theta})\frac{d\theta}{2\pi}, \ \ 0 \le \rho \le 1.$$

By integration in polar coordinates, one gets

$$|f|^p(a) \le \int_{\mathbb{D}} |f|^p(a + rz) dv_\alpha(z),$$

where

$$dv_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z).$$

In particular, for any $f \in L^p_{a,\alpha}(\mathbb{D})$,

$$|f|^p(0) \le \frac{1}{v_\alpha(\Delta(0,r))} \int_{\Delta(0,r)} |f(w)|^p dv_\alpha(w).$$

Replacing f with $f \circ \varphi_z$ gives

$$|f|^p(z) \le \frac{1}{v_\alpha(\Delta(0,r))} \int_{\Delta(0,r)} |(f \circ \varphi_z)(w)|^p dv_\alpha(w).$$

With the change of variable $w = \varphi_z$, one gets

$$|f|^p(z) \le \frac{1}{v_\alpha(\Delta(0,r))} \int_{\Delta(z,r)} |f(w)|^p \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{2(2+\alpha)}} dv_\alpha(w).$$

From (2.9) it follows that when $r(0 < r < 1)$ is fixed, there is a constant C_2 depending on both p and α, satisfying

$$|f|^p(z) \le \frac{C_2}{(1-|z|^2)^{2+\alpha}} \int_{\Delta(z,r)} |f(w)|^p dv_\alpha(w).$$

That is,

$$(1-|z|^2)^{2+\alpha}|f|^p(z) \le C_2 \int_{\Delta(z,r)} |f(w)|^p dv_\alpha(w). \tag{2.10}$$

Since $\{z_k\}$ is uniformly discrete,

$$\delta \triangleq \inf_{j \ne k} d(z_j, z_k) > 0.$$

Write $r = \frac{\delta}{2}$. We have $\Delta(z_j, r) \cap \Delta(z_k, r) = \emptyset$ for $j \neq k$. By (2.10),

$$\sum_{k=1}^{\infty}(1-|z_k|^2)^{2+\alpha}|f(z_k)|^p \leq C_2 \int_{\bigsqcup_k \Delta(z_k,r)} |f(w)|^p dv_\alpha(w) \leq C_2\|f\|_p^p,\ f \in L^p_{a,\alpha}(\mathbb{D}),$$

with $C = C_2$, the proof of Theorem 2.4.4 is complete. □

Combining Proposition 2.4.2 with Theorem 2.4.4 yields the following.

Proposition 2.4.5 *For a uniformly separated sequence* $\{z_j\}$*, let* B *be the Blaschke product for* $\{z_j\}$*. Then for* $\alpha > -1$*,*

$$h \mapsto \{(1-|z_j|^2)^{\frac{2+\alpha}{2}} h(z_j)\} \tag{2.11}$$

is a bounded invertible linear map from $L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})$ *onto* l^2*.*

Proof The proof comes from [Huang2].

To prove Proposition 2.4.5, it suffices to show that (2.11) defines a bounded linear bijection. Assume that $\{z_j\}$ is a uniformly separated sequence. Let B denote the Blaschke product for $\{z_j\}$, and $\mathcal{N}$ denotes the closed subspace of those functions in $L^2_{a,\alpha}(\mathbb{D})$ which vanish on $\{z_j : j \in \mathbb{Z}_+\}$. By the comments below Theorem 2.4.3, $\mathcal{N} = BL^2_{a,\alpha}(\mathbb{D})$, and hence

$$L^2_{a,\alpha}(\mathbb{D}) = BL^2_{a,\alpha}(\mathbb{D}) \oplus \Big(L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})\Big). \tag{2.12}$$

By Theorem 2.4.4, (2.11) is a map into l^2. By Proposition 2.4.2, $\{z_j\}$ is interpolating for $L^2_{a,\alpha}(\mathbb{D})(\alpha > -1)$, which shows that (2.11) is a surjective linear map from $L^2_{a,\alpha}(\mathbb{D})$ onto l^2, and by (2.12) it is also a surjective map from $L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})$ onto l^2. Its injectivity follows directly from the identity $\mathcal{N} = BL^2_{a,\alpha}(\mathbb{D})$. Thus (2.11) is a bijection.

We claim that the linear map (2.11)

$$h \mapsto \{(1-|z_j|^2)^{\frac{2+\alpha}{2}} h(z_j)\},\ h \in L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})$$

is bounded. For this, write A for this map. Suppose that (h_n, Ah_n) is a sequence tending to $(h, \mathbf{d})$, where all h_n and h are in $L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})$, and $\mathbf{d}$ is in l^2. Soon one will see that $Ah = \mathbf{d}$. Since $\{h_n\}$ converges to h in norm, it follows that $\{h_n\}$ converges to h at each point in $\mathbb{D}$, and hence $(Ah_n)_j$ tends to $(Ah)_j$ for each j. Since Ah_n converges to $\mathbf{d}$, then for each j we have $(Ah)_j = \mathbf{d}_j$. Then $Ah = \mathbf{d}$, as desired. By applying the closed graph theorem, the map A is bounded. Therefore

$$h \mapsto \{(1-|z_j|^2)^{\frac{2+\alpha}{2}} h(z_j)\}$$

is a bounded linear bijection from $L^2_{a,\alpha}(\mathbb{D}) \ominus BL^2_{a,\alpha}(\mathbb{D})$ to l^2, as desired. □

2.5 Some Results in von Neumann Algebras

This section provides some preliminaries from the theory of von Neumann algebras.

We first review some common terminology. As mentioned in the introduction, *a von Neumann algebra* $\mathcal{A}$ is a unital C^*-algebra on a Hilbert space $\mathcal{H}$, which is closed in the weak operator topology [Con1]. By a *projection* P in $\mathcal{A}$, we mean a self-adjoint operator P satisfying $P^2 = P$. If in addition, P commutes with each member in $\mathcal{A}$, then P is called *a central projection*. A projection P is called *minimal* if the only nonzero projection majorized by P is itself. By *the rank of a projection*, we mean the dimension of its range. For each closed subspace M, P_M always denote the orthogonal projection onto M.

The following results are basic and useful, see [Con1, Proposition 13.3].

Proposition 2.5.1 *Let $\mathcal{A}$ be a von Neumann algebra in $B(H)$ and let $A \in \mathcal{A}$.*

(a) If A is normal and ϕ is a bounded Borel function on the spectrum of A, then $\phi(A) \in \mathcal{A}$.
(b) The operator A is the linear combination of four unitary operators that belong to $\mathcal{A}$.
(c) If E and F are the projections onto the closure of Range A and $\ker A$*, respectively, then $E, F \in \mathcal{A}$. Here,* $\ker A \triangleq \{x \in H : Ax = 0\}$.
(d) If $A = W|A|$ is the polar decomposition of A, then both W and $|A|$ belong to $\mathcal{A}$.
(e) A von Neumann algebra is the norm closed linear span of its projections.

Two projections $P, Q \in \mathcal{A}$ are called *equivalent*, if there is an operator V in $\mathcal{A}$ such that $V^*V = P$ and $VV^* = Q$. This operator V must be a partial isometry, and in this case we write $P \sim Q$. A projection $P \in \mathcal{A}$ is called finite if there exists no projection $Q \in \mathcal{A}$ such that $Q < P$ and $Q \sim P$. Otherwise P is called infinite. A von Neumann algebra $\mathcal{A}$ is called *finite* if its identity is finite; otherwise, $\mathcal{A}$ is called infinite. By a simply reasoning one can prove that a von Neumann algebra $\mathcal{A}$ is finite if and only if $\mathcal{A}$ contains no non-unitary isometry. A projection P in $\mathcal{A}$ is called *abelian* if $P\mathcal{A}P$ is an abelian algebra.

As we mention the dimension of a von Neumann algebra (or a C*-algebra) $\mathcal{A}$, we refer to the algebraic dimension of $\mathcal{A}$.

Recall that the center $\mathrm{Z}(\mathcal{A})$ of a von Neumann algebra $\mathcal{A}$ is the set consisting of all members of $\mathcal{A}$ that commute with each operator in $\mathcal{A}$; that is,

$$\mathrm{Z}(\mathcal{A}) = \{A \in \mathcal{A} : AB = BA,\ \forall B \in \mathcal{A}\}.$$

A projection in $\mathrm{Z}(\mathcal{A})$ is called a *cental projection* in $\mathcal{A}$.

A von Neumann algebra is called *homogeneous* if there is a family of orthogonal abelian projections that are mutually equivalent and whose sum is the identity, see [Con1, p. 285]. The following characterizes the structure of homogenous von Neumann algebras, see [Con1, Proposition 50.15] and its corollary.

Theorem 2.5.2 *If $\mathcal{A}$ is a homogeneous von Neumann algebra in $B(\mathcal{H})$, and let $\{E_i\}$ be a collection of pairwise orthogonal, mutually equivalent projections in $\mathcal{A}$ with $\sum_i E_i = I$. If $\{E_i\}$ has cardinality n, then $\mathcal{A}$ is unitarily isomorphic to $M_n(\mathcal{B})$, where $\mathcal{B} = \mathcal{A}|_{E_1\mathcal{H}}$ is $*$-isomorphic to $\mathrm{Z}(\mathcal{A})$.*

A linear map $\tau : \mathcal{A} \to \mathrm{Z}(\mathcal{A})$ is called a *faithful, center-valued trace* on $\mathcal{A}$ if τ satisfies the following:

(1) $\tau(I) = I$;
(2) for any $A \in \mathcal{A}$ with $A \geq 0$, we have $\tau(A) \geq 0$; and $\tau(A) = 0$ if and only if $A = 0$;
(3) $\tau(AB) = \tau(BA)$ for all A and B in $\mathcal{A}$.

The following result is well-known. The reader may refer to Corollaries 50.13 and 55.9 in [Con1] for example.

Theorem 2.5.3 *A von Neumann algebra is finite if and only if it has a faithful, center-valued trace.*

If $\mathrm{Z}(\mathcal{A}) = \mathbb{C}I$, then the von Neumann algebra $\mathcal{A}$ is called *a factor*.

(1) type I factor– if there is a minimal projection $E \neq 0$, i.e. a projection E such that there is no nonzero projection F satisfying $F < E$;
(2) type II factor–if there is no minimal projection but there is a non-zero finite projection. By a II_1 factor we mean that it is a type II factor and its identity is finite, otherwise, II_∞.
(3) type III factor–if it does not contain any nonzero finite projection at all.

The following result tell us that for any factor $\mathcal{A}$, both $\mathcal{A}$ and $\mathcal{A}'$ must share the same type, see [Con1, Corollary 48.17] and [Bla].

Proposition 2.5.4 *A factor $\mathcal{A}$ is of type* I, II *or* III *if and only if $\mathcal{A}'$ is of type* I, II *or* III*, respectively. Moreover, any factor is exactly one of the types* I_n, I_∞, II_1, II_∞ *or* III.

For the definition of type I_n factor, see the paragraph before Theorem 2.5.10.

It is well-known that to a great extent the study of von Neumann algebras reduces to the study of factors [Di, Con1, Jon]. A type I factor is always unitarily equivalent to the tensor product of the operator algebra $B(\mathcal{H})$ and I, see Theorem 2.5.10. By Theorem 2.5.3, one can give an equivalent characterization for type II_1 factors: a factor is a type II_1 factor if and only if $\mathcal{A}$ is an infinite dimensional factor and $\mathcal{A}$ has a faithful, finite, complex-valued trace. This definition can be reformulated as follows [Jon, Definition 6.1.10]: $\mathcal{A}$ is called a *type II_1 factor* if $\mathcal{A}$ is an infinite dimensional factor and there is a nonzero linear map $tr : \mathcal{A} \to \mathbb{C}$ such that for $A, B \in \mathcal{A}$,

(i) $tr(AB) = tr(BA)$;
(ii) $tr(A^*A) \geq 0$;
(iii) tr is ultraweakly continuous; that is, tr is continuous under the weak* topology, where the weak* topology is induced by semi-norm family

$\{\tau_A : A$ is in the trace class $\}$, and $\tau_A(B) = |Tr(BA)|$, $B \in B(\mathcal{H})$, and Tr is the classical trace on trace class.

The above map tr turns out to be faithful, and it is unique if we require that $tr(I) = 1$.

Familiar type II_1 factors arise from group von Neumann algebras, $\mathcal{L}(G)$ and $\mathcal{R}(G)$, where G is a countable, discrete group. For this context, see [Con1] or Sect. 6.5 in Chap. 6 of this book. In this book, we would encounter concrete factors of type I and II arising from multiplication operators defined on classical reproducing kernel Hilbert spaces, the Bergman space and the Hardy space; and our attention is mainly focused on the Bergman space.

Now we turn back to some technical results in the theory of von Neumann algebra. The following seems likely to be known.

Lemma 2.5.5 *Suppose P is a minimal projection in a von Neumann algebra $\mathcal{A}$. Then for each projection Q, either $PQ = 0$ or there is some projection Q_0 such that $Q_0 \leq Q$ and $Q_0 \sim P$.*

Proof Suppose P is a minimal projection in $\mathcal{A}$, whose range is denoted by M. Now assume that Q is a projection onto N satisfying $PQ \neq 0$, and set $W = PQ$. Let $W = U|W|$ be the polar decomposition. Since P is minimal, the partial isometry U in $\mathcal{A}$ satisfies

$$UU^* = P \quad \text{and} \quad U^*U = Q_0,$$

where Q_0 denotes the orthogonal projection onto the range of U^*(and hence, $Q_0 \leq Q$). That is, $Q_0 \sim P$ and $Q_0 \leq Q$. The proof is complete. □

The following result is a direct consequence of Lemma 2.5.5.

Corollary 2.5.6 *For two minimal projections E and F in a von Neumann algebra $\mathcal{A}$, either $E \perp F$ or $E \sim F$.*

Also, we require the following. For a collection $\mathcal{E}$ of projections, let $\bigvee_{P\in\mathcal{E}} P$ denote the supremum of $\mathcal{E}$ [Con1, pp. 242, 243], which proves to be the orthogonal projection onto the closed space spanned by the ranges of P, where $P \in \mathcal{E}$.

Corollary 2.5.7 *Suppose P is a minimal projection in a von Neumann algebra $\mathcal{A}$. Then the projection $\bigvee_{Q\sim P} Q$ (where Q run over all projections in $\mathcal{A}$) is in the center of $\mathcal{A}$.*

Proof Suppose P is a minimal projection in a von Neumann algebra $\mathcal{A}$. By [Con1, Proposition 43.3], $\bigvee_{Q\sim P} Q$ is in $\mathcal{A}$. By Proposition 2.5.1, a von Neumann algebra is the norm-closed span of its projections. Then it suffices to show that $\bigvee_{Q\sim P} Q$ commutes with each projection P_0 in $\mathcal{A}$.

To see this, rewrite $\hat{P}$ for $\bigvee_{Q\sim P} Q$. First, we assume that there is no projection Q_0 satisfying $Q_0 \leq P_0$ and $Q_0 \sim P$. In this case, by Lemma 2.5.5 we get $P_0 \perp P$; similarly, by the minimality of P we have $P_0 \perp Q$ whenever $Q \sim P$. This implies that $P_0 \perp \hat{P}$, and we are done.

Otherwise, there is a projection Q_0 in $\mathcal{A}$ satisfying $Q_0 \leq P_0$ and $Q_0 \sim P$. In this case, consider $P_0 - Q_0$. By the same reasoning as above, either $P_0 - Q_0 \perp \hat{P}$ or there is a projection Q_1 in $\mathcal{A}$ satisfying $Q_1 \leq P_0 - Q_0$ and $Q_1 \sim P$. Applying Zorn's Lemma shows that there is a maximal family of mutually orthogonal projections Q_i such that $\bigvee_i Q_i \leq P_0$ with $Q_i \sim P$ for each i. Then there is no projection Q satisfying $Q \leq P_0 - \bigvee_i Q_i$ and $Q \sim P$, and hence by Lemma 2.5.5 $(P_0 - \bigvee_i Q_i) \perp P$. By the same reasoning, if P is replaced with any $\tilde{P}$ satisfying $\tilde{P} \sim P$, then $(P_0 - \bigvee_i Q_i) \perp \tilde{P}$, and thus

$$(P_0 - \bigvee_i Q_i) \perp \bigvee_{\tilde{P} \sim P} \tilde{P}.$$

That is,

$$(P_0 - \bigvee_i Q_i) \perp \hat{P}.$$

Also, we have $\bigvee_i Q_i \leq \hat{P}$, which implies that $\hat{P}$ commutes with P_0 since

$$P_0 = (P_0 - \bigvee_i Q_i) + \bigvee_i Q_i.$$

The proof is complete. □

By the proof of Corollary 2.5.7, we have the following consequence.

Corollary 2.5.8 *Suppose P is a minimal projection in a von Neumann algebra $\mathcal{A}$ and put $\hat{P} = \bigvee_{Q \sim P} Q$ (where Q run over all projections in $\mathcal{A}$). Then there is a family of mutually orthogonal projections $\{Q_i\}$ such that $Q_i \sim P$ and*

$$\hat{P} = \bigvee_i Q_i.$$

The following will be concerned with the structure of type I factors. It is clear that the full matrix algebra $M_n(\mathbb{C})$ is a factor of type I. Here, we allow n to be ∞, and $M_\infty(\mathbb{C})$ represents the algebra of all bounded operators on an ∞-dimensional separable Hilbert space, say l^2.

To investigate the structure of type I factors, we need the following result [Jon], which shows that all minimal projections in a type I factor has the same rank.

Proposition 2.5.9 *Let P and Q be nonzero projections in a factor $\mathcal{A}$. Then there exists a unitary operator $U \in \mathcal{A}$ such that $PUQ \neq 0$. Furthermore, if P, Q are minimal, then $P \sim Q$.*

Proof Suppose conversely that $PUQ = 0$ for any unitary operator U in $\mathcal{A}$. Then $U^*PUQ = 0$, and hence

$$(\bigvee_{U \in \mathcal{A}} U^*PU)Q = 0, \tag{2.13}$$

where U run over all unitary operators in $\mathcal{A}$. Since $\bigvee_{U \in \mathcal{A}} U^*PU$ commutes with any unitary operator in $\mathcal{A}$, $\bigvee_{U \in \mathcal{A}} U^*PU$ lies in $\mathcal{A}' \bigcap \mathcal{A}$. Since $\mathcal{A}$ is a factor, $\bigvee_{U \in \mathcal{A}} U^*PU = I$, which is a contradiction to (2.13). Thus, there exists a unitary operator $U \in \mathcal{A}$ such that $PUQ \neq 0$.

Assume that P and Q are minimal projections. Pick a unitary operator U such that $PUQ \neq 0$, and let V be the partial isometry in the polar decomposition of PUQ. Then

$$VV^* \leq P \quad \text{and} \quad V^*V \leq Q.$$

By the minimality of P and Q,

$$VV^* = P \quad \text{and} \quad V^*V = Q.$$

That is, $P \sim Q$. □

Now assume $\mathcal{A}$ is a type I factor, and so is $\mathcal{A}'$. By Proposition 2.5.9, the rank of a minimal projection in $\mathcal{A}$ is a constant integer, which does not depend on the choice of the minimal projection. Then set n_1 = rank of a minimal projection in $\mathcal{A}$, and n_2 = rank of a minimal projection in $\mathcal{A}'$. It is easy to verify that n_1 is equal to the cardinality of a maximal family of mutually orthogonal minimal projections in $\mathcal{A}$, and similar is true for n_2. A *type I_n factor* is by definition one for which $n = n_2$, and the integer n_1 is called *the multiplicity of the factor*. The following theorem can be found in most books on operator algebra, see [Jon] for instance. It describes how a type I factor looks like.

Theorem 2.5.10 *Assume $\mathcal{A}$ is a type I factor on a Hilbert space H. Then there exist Hilbert spaces H_1, H_2 with* $\dim H_1 = n_2$, $\dim H_2 = n_1$ *and a unitary operator $U : H \to H_1 \otimes H_2$ such that $U\mathcal{A}U^* = B(H_1) \otimes I_{H_2}$.*

For two operators A and B defined on Hilbert spaces H and K, respectively, A is called *unitarily equivalent* to B if there is a unitary operator $U : H \to K$ such that $B = UAU^*$. Write $Ad_U : B(H) \to B(K), A \to UAU^*$. Given two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, if there is a unitary operator U such that $\mathcal{B} = Ad_U(\mathcal{A})$, then $\mathcal{A}$ is called *unitarily isomorphic* to $\mathcal{B}$, or spatially isomorphic to $\mathcal{B}$.

The following is an immediate consequence of Theorem 2.5.10, which determines the structure of all finite dimensional von Neumann algebras, see [Da, Theorem III.1.2] or [Jon].

Theorem 2.5.11 *Assume that $\mathcal{A}$ is a finite dimensional von Neuamm algebra on a Hilbert space H. Then $\mathcal{A}$ is $*$-isomorphic to $\bigoplus_{k=1}^{r} M_{n_k}(\mathbb{C})$, where r equals the dimension of the center* $\mathrm{Z}(\mathcal{A})$. *Precisely, $\mathcal{A}$ is unitarily isomorphic to the direct sum*

$$\bigoplus_{k=1}^{r}\Big(M_{n_k}(\mathbb{C}) \otimes I_{H_k}\Big),$$

where H_k are subspaces of H.

Proof Note that $\mathrm{Z}(\mathcal{A})$ is a finite dimensional abelian von Neumann algebra and that if P is a minimal projection in $\mathrm{Z}(\mathcal{A})$, then $P\mathcal{A}P$ is a type I factor on PH. Let $\{P_1, \cdots, P_r\}$ be a maximal family of mutually orthogonal minimal projections in $\mathrm{Z}(\mathcal{A})$. Then we have the decomposition:

$$\mathcal{A} = \bigoplus_{k=1}^{r} P_k \mathcal{A} P_k,$$

and hence the conclusion follows directly from Theorem 2.5.10. □

In von Neumann algebras, an important result is *the von Neumann Bicommutant Theorem*. This theorem relates the closure of a set of bounded operators on a Hilbert space in certain topologies to the bicommutant of that set. In essence, it is a connection between the algebraic and topological sides of operator theory.

Let $\mathcal{A}$ be a subset of $B(H)$, and the commutant of $\mathcal{A}$ be defined as

$$\mathcal{A}' = \{S \in B(H) : AS = SA,\ \forall A \in \mathcal{A}\}.$$

Then $\mathcal{A}'$ is a WOT-closed subalgebra of $B(H)$. The set $\mathcal{A}$ is called self-adjoint if $A^* \in \mathcal{A}$ for all A in $\mathcal{A}$. Let $\mathcal{A}_s$ and $\mathcal{A}_w$ be the SOT-closure and the WOT-closure of $\mathcal{A}$, respectively.

The formal statement of the bicommutant theorem is as follows:

Theorem 2.5.12 (von Neumann Bicommutant Theorem) *Let $\mathcal{A}$ be a self-adjoint subalgebra of $B(H)$ and $I \in \mathcal{A}$, then*

$$\mathcal{A}'' = \mathcal{A}_w = \mathcal{A}_s.$$

Proof The proof comes from [Ar2]. By the fact that $\mathcal{A}_s \subseteq \mathcal{A}_w \subseteq \mathcal{A}''$, we only need to prove that for each $B \in \mathcal{A}''$, and any $\varepsilon > 0$, and $h_1, \cdots, h_n \in H$, there exists $A \in \mathcal{A}$, such that

$$\sum_{k=1}^{n} \|(B - A)h_k\|^2 < \varepsilon^2.$$

Given any $h \in H$, it is easy to see that the closure $\overline{\mathcal{A}h}$ of $\mathcal{A}h$ is a reducing subspace of $\mathcal{A}$, and hence the orthogonal projection P onto $\overline{\mathcal{A}h}$ belongs to $\mathcal{A}'$. Therefore, $PB = BP$. Since $1 \in \mathcal{A}$, and $Ph = h$, this implies that $Bh \in \overline{\mathcal{A}h}$. It follows that for any $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\|(B - A)h\| < \varepsilon$.

Next we will use the above reasoning to complete the proof. Write $H_n = H \oplus \cdots \oplus H$, and let $\mathcal{A}_n = \{S \oplus \cdots \oplus S : S \in \mathcal{A}\}$, then $\mathcal{A}_n$ is a unital self-adjoint subalgebra on H_n. It is easy to show that

$$\mathcal{A}_n' = \{[T_{ij}]_{n\times n} : T_{ij} \in \mathcal{A}'\},$$

and $[S_{ij}]$ commutes with each element in $\mathcal{A}_n'$ if and only if there exists $S \in \mathcal{A}''$ such that

$$[S_{ij}] = S \oplus \cdots \oplus S.$$

This implies that

$$\mathcal{A}_n'' = \{S \oplus \cdots \oplus S : S \in \mathcal{A}''\}.$$

It follows that $B_n = B \oplus \cdots \oplus B \in \mathcal{A}_n''$ if $B \in \mathcal{A}''$. Set $\mathbf{h} = h_1 \oplus \cdots \oplus h_n$, the above reasoning shows that $B_n\mathbf{h} \in \overline{\mathcal{A}_n\mathbf{h}}$, and hence there exists $A \in \mathcal{A}$ such that

$$\|B_n\mathbf{h} - A_n\mathbf{h}\| < \varepsilon,$$

where $A_n = A \oplus \cdots \oplus A$, that is, $\sum_{k=1}^{n} \|(B - A)h_k\|^2 < \varepsilon^2$. □

Corollary 2.5.13 *Let $\mathcal{A}$ be a self-adjoint subalgebra of $B(H)$ and $I \in \mathcal{A}$, then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*

2.6 Some Results in Operator Theory

In this section, we present some operator-theoretic results.

For any bounded holomorphic function ϕ over $\mathbb{D}$, let M_ϕ be the multiplication operator defined on the Bergman space $L_a^2(\mathbb{D})$ with the symbol ϕ. As done before, let $\mathcal{W}^*(\phi)$ denote the von Neumann algebra generated by M_ϕ and put $\mathcal{V}^*(\phi) \triangleq \mathcal{W}^*(\phi)'$, the commutant algebra of $\mathcal{W}^*(\phi)$. It is well-known that $\mathcal{V}^*(\phi)$ equals the von Neumann algebra generated by the orthogonal projections onto M, where M run over all reducing subspaces of M_ϕ. If M_ϕ has no nontrivial reducing subspace, then M_ϕ is called *irreducible*; in this case, $\mathcal{V}^*(\phi) = \mathbb{C}I$, and by von Neumann bi-commutant theorem $\mathcal{W}^*(\phi) = \mathcal{V}^*(\phi)' = B(L_a^2(\mathbb{D}))$, all bounded linear operators on $L_a^2(\mathbb{D})$.

Now, let us have a look at the reducing subspaces from the view of von Neumann algebra. Given two reducing subspaces M and N of M_ϕ, if there exists a unitary operator U from M onto N and U commutes with M_ϕ, then M is called to be *unitarily equivalent* to N. In this case we can extend U to $\tilde{U}$ such that $\tilde{U}|_M = U$ and $\tilde{U}|_{M^\perp} = 0$. It follows that $\tilde{U}$ commutes with both M_ϕ and M_ϕ^*. Write P and Q for the orthogonal projections from $L_a^2(\mathbb{D})$ onto M and N, respectively. Observe that $P = \tilde{U}^*\tilde{U}$ and $Q = \tilde{U}\tilde{U}^*$. That is, two projections P and Q are equivalent in $\mathcal{V}^*(\phi)$. In this way, *the unitary equivalence between reducing subspaces can be identified with the equivalence between projections in* $\mathcal{V}^*(\phi)$.

A more general setting is presented as follows. Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a commuting operator tuple acting on a separable Hilbert space H. Write $\mathcal{W}^*(\mathbf{T})$ for von Neumann algebra generated by $T_1, \cdots, T_d$, and $\mathcal{V}^*(\mathbf{T})$ for the commutant algebra of $\mathcal{W}^*(\mathbf{T})$, i.e. $\mathcal{V}^*(\mathbf{T}) = (\mathcal{W}^*(\mathbf{T}))'$, which is also a von Neumann algebra. A closed subspace M of H is called *a reducing subspace* for the tuple $\mathbf{T}$ if M is invariant for both T_i and T_i^*, $i = 1, \cdots, d$; equivalently, both M and $M^\perp$ are invariant for all T_i. A reducing subspace M is called *minimal* if there is no nonzero reducing subspace N satisfying $N \subsetneqq M$. To put it in another way, P_M is a minimal projection in $\mathcal{V}^*(\mathbf{T})$. Two reducing subspaces M_1 and M_2 are called *unitarily equivalent* if there exists a unitary operator U from M onto N and U commutes with $T_i (1 \le i \le d)$. In this case, we write

$$M_1 \overset{U}{\cong} M_2.$$

One can show that $M_1 \overset{U}{\cong} M_2$ if and only if P_{M_1} and P_{M_2} are equivalent in $\mathcal{V}^*(\mathbf{T})$. Later in Chap. 8, we take $\mathbf{T}$ to be a tuple of multiplication operators acting on a function space, such as the Hardy space, the Bergman space, and etc.

Now set $H_0 = H \ominus (T_1 H + \cdots + T_d H)$, and put $Q = P_{H_0}$, the orthogonal projection onto H_0. We claim that $Q \in \mathcal{W}^*(\mathbf{T})$. To see this, note that each $T_i \in \mathcal{W}^*(\mathbf{T})$, and thus the range projection of $T_i T_i^*$ is in $\mathcal{W}^*(\mathbf{T})$, i.e.

$$P_{\overline{T_i H}} \in \mathcal{W}^*(\mathbf{T}).$$

Then

$$\bigvee_{1 \le i \le d} P_{\overline{T_i H}} \in \mathcal{W}^*(\mathbf{T}).$$

That is, $I - P_{H_0} \in \mathcal{W}^*(\mathbf{T})$, forcing $Q \in \mathcal{W}^*(\mathbf{T})$.

Set $\mathcal{R}(\mathbf{T}) = Q\mathcal{W}^*(\mathbf{T})Q$. By the theory of von Neumann algebras, $\mathcal{R}(\mathbf{T})$ is a von Neumann algebra on H_0, and

$$\mathcal{R}'(\mathbf{T}) = Q\big(\mathcal{W}^*(\mathbf{T})\big)'Q = Q\mathcal{V}^*(\mathbf{T})Q.$$

For the tuple $\mathbf{T} = (T_1, \cdots, T_d)$ acting on a Hilbert space H, write $\mathcal{H}_n$ for all homogeneous polynomials in $\mathbf{T}$ with degree n, $n = 0, 1, \cdots$. Now define a map $\tau : \mathcal{V}^*(\mathbf{T}) \to Q\mathcal{V}^*(\mathbf{T})Q$ by setting

$$\tau(A) = QA \equiv QAQ, \quad A \in \mathcal{V}^*(\mathbf{T}).$$

Since $Q \in \mathcal{W}^*(\mathbf{T})$, it is easy to see that τ is a $*$-homomorphism. Furthermore, we have the following result, due to Guo [Guo5].

Theorem 2.6.1 *If* $\cap_n \overline{\mathcal{H}_n H} = 0$, *then the map* τ *is a* $*$*-isomorphism.*

Proof It suffices to show that τ is injective. Assume that there exists an operator $A \in \mathcal{V}^*(\mathbf{T})$ such that $QAQ = 0$. This implies that $A|_{H_0} = 0$. Since

$$H = H_0 \oplus \overline{(T_1 H + \cdots + T_d H)}$$

and $AT_i = T_i A$ for $i = 1, \cdots, d$, then we have

$$\begin{aligned} AH &\subseteq \overline{A(T_1 H + \cdots + T_d H)} \\ &= \overline{T_1 AH + \cdots + T_d AH} \\ &\subseteq \overline{\sum_{i,j} T_i T_j AH} \\ &\subseteq \overline{\sum_{i,j,k} T_i T_j T_k AH} \\ &\subseteq \cdots . \end{aligned}$$

This immediately gives that $AH \subseteq \cap_n \overline{\mathcal{H}_n H}$, and hence $AH = 0$, i.e. $A = 0$. □

The following result, due to Guo [Guo5], characterizes when $\mathcal{V}^*(\mathbf{T})$ is abelian.

Proposition 2.6.2 *Suppose* $\cap_n \overline{\mathcal{H}_n H} = 0$, *and*

$$k \triangleq \dim H \ominus (T_1 H + \cdots + T_d H) < \infty.$$

Then the following are equivalent:

(1) $\mathcal{V}^*(\boldsymbol{T})$ *is abelian.*
(2) any two distinct minimal projections in $\mathcal{V}^*(\boldsymbol{T})$ *are orthogonal;*
(3) there exist at most k *minimal reducing subspaces for the tuple* $\boldsymbol{T}$*;*
(4) there exist at most finitely many minimal reducing subspaces for $\boldsymbol{T}$.

Proof By Theorem 2.6.1,

$$\dim \mathcal{V}^*(\mathbf{T}) = \dim Q\mathcal{V}^*(\mathbf{T})Q \leq (\dim QH)^2 = k^2 < \infty.$$

That is, $\mathcal{V}^*(\mathbf{T})$ is a finite dimensional von Neumann algebra. Then the equivalence between (1)–(4) in Proposition 2.6.2 follows directly from Theorem 2.5.11, which states that any finite dimensional von Neumann algebra is unitarily isomorphic to the direct sum

$$\bigoplus_{k=1}^{r}\Big(M_{n_k}(\mathbb{C})\otimes I_{H_k}\Big),$$

where H_k are subspaces of H. The proof is complete. □

The following is an immediate consequence [Guo5].

Corollary 2.6.3 *If* $\bigcap_n \overline{\mathcal{H}_n H} = 0$, *and* $k = \dim H \ominus (T_1 H + \cdots + T_d H) < \infty$, *then* $\mathcal{V}^*(\boldsymbol{T})$ *is finite dimensional, and* $\dim \mathcal{V}^*(\boldsymbol{T}) \le k^2$.

By Corollary 2.5.6, we have the following.

Proposition 2.6.4 *Let* M_1 *and* M_2 *be two minimal reducing subspaces for* $\boldsymbol{T}$. *If* M_1 *is not orthogonal to* M_2, *then* $M_1 \overset{U}{\cong} M_2$. *Equivalently, if two minimal projections* P, Q *in* $\mathcal{V}^*(\boldsymbol{T})$ *satisfy* $PQ \ne 0$, *then* $P \sim Q$.

Proof Here, we provide a different proof.

By assumption, $P_{M_1}P_{M_2} \ne 0$, and hence by spectral decomposition, there are positive constants λ_1 and λ_2 such that

$$(P_{M_1}P_{M_2})(P_{M_1}P_{M_2})^* = P_{M_1}P_{M_2}P_{M_1} = \lambda_1 P_{M_1},$$

and

$$(P_{M_1}P_{M_2})^*(P_{M_1}P_{M_2}) = P_{M_2}P_{M_1}P_{M_2} = \lambda_2 P_{M_2}.$$

This leads to the identity

$$\lambda_1^2 P_{M_1} = P_{M_1}(P_{M_2}P_{M_1}P_{M_2})P_{M_1} = \lambda_2 P_{M_1}P_{M_2}P_{M_1} = \lambda_1\lambda_2 P_{M_1},$$

forcing $\lambda_1 = \lambda_2$. Write $V = \frac{1}{\sqrt{\lambda_1}}P_{M_1}P_{M_2}$, and then

$$P_{M_1} = VV^*, \quad P_{M_2} = V^*V.$$

This is $P_{M_1} \sim P_{M_2}$. Equivalently, $M_1 \overset{U}{\cong} M_2$. □

By applying Theorem 2.5.11, one can show that if $\dim \mathcal{V}^*(\mathbf{T}) < \infty$, then $\mathcal{V}^*(\mathbf{T})$ is abelian if and only if for any distinct projections P and Q, P is never equivalent to Q. The following example is from [Guo5, Example 3], which shows that $\mathcal{V}^*(\mathbf{T})$ is not necessarily abelian in general.

Example 2.6.5 Given $p_1, \cdots, p_n \in C[z_1, z_2]$, let $M_{p_1}, \cdots, M_{p_n}$ be multiplication operators on the Hardy space $H^2(\mathbb{D}^2)$. Assume that the common zeros $\bigcap_{k=1}^n Z(p_k) \bigcap \mathbb{D}^2$ is finite and nonempty, then by [CG, Theorem 2.2.15, Coroallry 2.26],

$$l = \dim H^2(D^2) \ominus (p_1 H^2(D^2) + \cdots + p_n H^2(D^2)) < \infty,$$

and l is equal to the cardinality of the common zeros (counting multiplicities). By Corollary 2.6.3,

$$\dim \mathcal{V}^*(M_{p_1}, \cdots, M_{p_n}) \leq l^2.$$

In general, the von Neumann algebra $\mathcal{V}^*(M_{p_1}, \cdots, M_{p_n})$ is not abelian. For example, put $p_1 = z^2,\ p_2 = w^3$, and then $\mathcal{V}^*(M_{p_1}, M_{p_2})$ is $*$-isomorphic to

$$\mathcal{V}^*(M_{p_1}) \otimes \mathcal{V}^*(M_{p_2}),$$

where $\mathcal{V}^*(M_{p_i})$ are defined over $H^2(\mathbb{D})$ for $i = 1, 2$. Note that none of $\mathcal{V}^*(M_{p_i})$ is abelian.

However, if $\mathcal{V}^*(M_{z^2}, M_{w^3})$ is defined over $L_a^2(\mathbb{D}^2)$, then by the same reasoning shows that $\mathcal{V}^*(M_{z^2}, M_{w^3})$ is abelian. This is because on the Bergman space $L_a^2(\mathbb{D})$, $\mathcal{V}^*(M_{z^n})$ is abelian for all positive integer n.

The following describes the structure of $\mathcal{V}^*(S)$ in the case of S being a pure isometry.

Example 2.6.6 Let S be a pure isometry, i.e., $\bigcap_n S^n H = 0$. Set $H_0 = H \ominus SH$, and hence $Q = I - SS^*$. We have $Q\mathcal{V}^*(S)Q = B(H_0)$, and hence by Theorem 2.6.1, $\mathcal{V}^*(S)$ is $*$-isomorphic to $B(H_0)$.

To see this, since

$$S^*Q = 0,\ QS = 0,$$

and

$$S^{*m}S^n = S^{n-m} \text{ if } n \geq m; \quad S^{*m}S^n = S^{(n-m)*} \text{ if } n < m,$$

we see

$$Q\mathcal{W}^*(S)Q = \mathbb{C}\, Q,$$

and hence

$$Q\mathcal{V}^*(S)Q = (Q\mathcal{W}^*(S)Q)' = B(H_0).$$

This example says that there exists a one-to-one and onto correspondence between reducing subspaces of S and closed subspaces of H_0.

The next example is presented by Guo.

Example 2.6.7 Let $f \in H^\infty(\mathbb{D})$, and let M_f be multiplication operator on $L_a^2(\mathbb{D})$ defined by f. If f has zero points in $\mathbb{D}$, then $\bigcap_n \overline{f^n L_a^2(\mathbb{D})} = 0$. We assume $k = \dim L_a^2(\mathbb{D}) \ominus f L_a^2(\mathbb{D}) < \infty$. Corollary 2.6.3 implies that $\mathcal{V}^*(M_f)$ is finite dimensional, and $\dim \mathcal{V}^*(M_f) \leq k^2$. In particular, if $k = 1$, then M_f has no nontrivial reducing subspace.

Now let us turn back to Proposition 2.6.2. Consider $\mathbf{T} = M_B$, the multiplication operator defined by a finite Blaschke product B on the Bergman space $L_a^2(\mathbb{D})$. Then one has the following conclusion, also see [GH1].

Proposition 2.6.8 *Let B be a finite Blaschke product of order n. Then the following are equivalent.*

(1) $\mathcal{V}^*(B)$ has at most n distinct minimal projections;
(2) $\mathcal{V}^*(B)$ is abelian;
(3) All minimal projections in $\mathcal{V}^*(B)$ are mutually orthogonal.

Zhu conjectured that for a finite Blaschke product B of order n, there are exactly n distinct minimal reducing subspaces [Zhu1]. This is equivalent to the fact that $\mathcal{V}^*(B)$ has exactly n minimal projections. In fact, by applying [SZZ2, Theorem 3.1] Zhu's conjecture holds only if $B(z) = \phi^n$ for some Möbius transform ϕ. Therefore, the conjecture is modified as follows: M_B has at most n distinct minimal reducing subspaces [DSZ]. Therefore the modified conjecture is equivalent to assertion that $\mathcal{V}^*(B)$ is abelian. In the case of order $B = 3, 4, 5, 6$, the modified conjecture is demonstrated in [GSZZ, SZZ1, GH1]. By using the techniques of local inverse and group-theoretic methods, it was proved that $\mathcal{V}^*(B)$ is abelian if order $B = 7, 8$ in [DSZ]. The latest progress is an affirmative answer to the modified conjecture due to Douglas et al. [DPW], see Chap. 4.

Springer

http://www.springer.com/978-3-662-46844-9

Multiplication Operators on the Bergman Space
Guo, K.; Huang, H.
2015, VIII, 322 p. 11 illus., 1 illus. in color., Softcover
ISBN: 978-3-662-46844-9