Chapter 2
Mathematical Preliminaries

2.1 Introduction

In this chapter, the fundamental mathematical concepts and analysis tools in systems theory are summarized, which will be used in control design and stability analysis in the subsequent chapters. Much of the material is described in classical control theory textbooks and robotics books as standard form. Thus, some standard theorems, lemmas, and corollaries, which are available in references, are sometimes given without a proof. This chapter serves as a short review and as a convenient reference when necessary. In addition, for robotic control, stability analysis is the key core for all closed-loop systems, therefore, some metric or norms need to be defined such that system could be measured. Those norms that are defined to easily manipulate for control design and also, all norms that have some physical significance.

2.2 Linear Algebra

This chapter reviews some basic linear algebra facts that are essential in the study of this text. Most topics are developed intuitively for readers to grasp the ideas better. Detailed discussion can be found in the references listed at the end of the chapter.

2.2.1 Linear Subspaces

Denote $\mathbb{R}$ as the real scalar field, let $\mathbb{R}^n$ denote the vector space over $\mathbb{R}$. For any $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, an element of the form $\alpha_1 x_1 + \cdots + \alpha_k x_k$ with $\alpha_i \in \mathbb{R}$ is then called a linear combination over $\mathbb{R}$ of $x_1, \ldots, x_k$. The set of all linear combinations of $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ is a subspace called the span of $x_1, x_2, \ldots, x_k$, denoted by

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$$\text{span}\{x_1, x_2, \ldots, x_k\} := \{x = \alpha_1 x_1 + \cdots + \alpha_k x_k \colon \alpha_i \in \mathbb{R}\}. \tag{2.1}$$

The set of vectors $x_1, x_2, \ldots, x_k$ is said to be linearly dependent over $\mathbb{R}$ if there exists $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ not all zero such that

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0. \tag{2.2}$$

If the only set of α_i for which (2.2) holds is $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_k = 0$, then the set of vectors are said to be linearly independent.

Let $S \subset \mathbb{R}^n$, then a set of vectors $\{x_1, x_2, \ldots, x_k\} \in S$ is called a basis for S if $x_1, x_2, \ldots, x_k$ are linearly independent and $S = \text{span}\{x_1, x_2, \ldots, x_k\}$. Note that, the basis of a subspace S is not unique but all bases for S have the same number of elements, which is called the dimension of S, denoted by $\dim(S)$.

Vectors $\{x_1, x_2, \ldots, x_k\}$ are called mutually orthogonal if $x_i^T x_j = 0$ for all $i \neq j$ and orthonormal if $x_i^T x_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta function with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. More generally, a collection of subspaces $S_1, S_2, \ldots, S_k$ of $\mathbb{R}^n$ is said to be mutually orthogonal if $x^T y = 0$ with any $x \in S_i$ and $y \in S_j$ for $i \neq j$.

Let U and V be subspaces of S. If $U \cap V = \{0\}$, then $U + V$ is direct sum. Direct sum is denoted as $U \dot{+} V$. Let U, V be subspaces of S and $U \dot{+} V = S$, then $U \dot{+} V$ is referred to as a direct sum decomposition of S. Then U and V are a pair of complementary subspaces with respect to S. U is called the complementary subspace of V with respect to S. For any subspace U of S, there exists subspace $V \subset S$ such that

$$S = U \dot{+} V. \tag{2.3}$$

The orthogonal complement of a subspace S is defined as

$$S^\perp := \{y \in \mathbb{R}^n \colon y^T x = 0, \ \forall\, x \in S\}. \tag{2.4}$$

Then, the set of vectors $\{\mu_1, \mu_2, \ldots, \mu_k\}$ is said to be an orthonormal basis of S if the vectors form a basis of S and are orthonormal. It is always possible to extend such a basis to a full orthonormal basis $\{\mu_1, \mu_2, \ldots, \mu_n\}$ for $\mathbb{R}^n$. Note that in this case

$$S^\perp = \text{span}\{\mu_{k+1}, \ldots, \mu_n\}, \tag{2.5}$$

and $\{\mu_{k+1}, \ldots, \mu_n\}$ is called an orthonormal complement of $\{\mu_1, \mu_2, \ldots, \mu_k\}$.

Let $A \in \mathbb{R}^{m \times n}$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$; that is,

$$A \colon \mathbb{R}^n \mapsto \mathbb{R}^m. \tag{2.6}$$

Then the kernel or null space of the linear transformation A is defined by

$$\text{Ker } A = N(A) := \{x \in \mathbb{R}^n \colon Ax = 0\}, \tag{2.7}$$

and the image or range of A is

$$\text{Im } A = R(A) := \{y \in \mathbb{R}^m : y = Ax, x \in \mathbb{R}^n\}. \tag{2.8}$$

Let a_i $(i = 1, 2, \ldots, n)$ denotes the columns of a matrix A; then

$$\text{Im } A = \text{span}\{a_1, a_2, \ldots, a_n\}. \tag{2.9}$$

A square matrix $U \in \mathbb{R}^{n\times n}$ whose columns form an orthonormal basis for $\mathbb{R}^n$ is called an orthogonal matrix, and it satisfies $U^T U = UU^T = I$.

Now let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$; then the trace of A is defined as

$$\text{trace}\,(A) := \sum_{i=1}^{n} a_{ii}. \tag{2.10}$$

2.2.2 *Eigenvalues and Eigenvectors*

Let $A \in \mathbb{R}^{n\times n}$; then the eigenvalues of A are the n roots of its characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. The spectrum of A is the set of all λ that are eigenvalues of A. The spectral radius is defined as the maximal modulus of the eigenvalues, and is given by

$$\rho(A) := \max_{1\leq i\leq n} |\lambda_i| \tag{2.11}$$

if λ_i is a root of $p(\lambda)$, where, as usual, $|\cdot|$ denotes the magnitude. Nonzero vector x that satisfies

$$Ax = \lambda x \tag{2.12}$$

is referred to as a right eigenvector of A. Dually, a nonzero vector y is called a left eigenvector of A if

$$y^T A = \lambda y^T. \tag{2.13}$$

We just call the right eigenvector an eigenvector, if it seldom causes confusion. We also let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest eigenvalue and the smallest eigenvalue of A, respectively.

Lemma 2.1 *Consider the Sylvester equation*

$$AX + XB = C, \tag{2.14}$$

where A, B, and C are given matrices. There exists a unique solution X if and only if $\lambda_i(A) + \lambda_j(B) \neq 0$, $\forall i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

In particular, if $B = A^T$, Eq. (2.14) is called the Lyapunov equation.

2.2.3 Vector Norms and Matrix Norms

A real-valued function $\|\cdot\|$ defined on vector space X is called norm if for any $x \in X$ and $y \in Y$, it satisfies:

(i) $\|x\| \geq 0$ (positivity);
(ii) $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);
(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, for any scalar α (homogeneity) ;
(iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Let $x \in \mathbb{R}^n$. The vector p-norm ($1 \leq p < \infty$) is then defined by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

In particular, when $p = 1$, $p = 2$, and $p = \infty$, it has

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|;$$
$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2};$$
$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|.$$

Generally speaking, norm is an abstraction and extension of our usual concept of length in three-dimensional Euclidean space. A vector norm is a measure of the vector "length," for example, $\|x\|_2$ is the Euclidean distance of the vector x from the origin.

Similarly, we can introduce some kinds of measure for a matrix. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$; then the matrix norm induced by the vector p-norm is defined as

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

The matrix norms induced by vector p-norms are sometimes called induced p-norms. This is because $\|A\|_p$ is defined by or induced from a vector p-norm. In fact, A can be viewed as a mapping from a vector space $\mathbb{R}^n$ equipped with a vector norm $\|\cdot\|_p$. So from a system theoretical point of view, the induced norms have the interpretation of input/output amplification gains.

Lemma 2.2 *Let $\|A\|_{p1}$ and $\|A\|_{p2}$ be any two different norms, then there exist positive constants c_1, c_2, depending only on the choice of the norms, such that for all A,*

$$c_1 \|A\|_{p2} \leq \|A\|_{p1} \leq c_2 \|A\|_{p2}.$$

In particular, the induced matrix 2-norm can be computed as

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}.$$

We shall adopt the following convention throughout this book for the vector and matrix norms unless specified otherwise: Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$; then we shall denote the Euclidean 2-norm of x simply by

$$\|x\| := \|x\|_2$$

and the induced 2-norm of A by

$$\|A\| := \|A\|_2.$$

Another often used matrix norm is the so-called Frobenius norm. It is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

The symbol L_p^m for $1 \leq p < \infty$ is used in this book. It is defined as the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$ such that

$$\|u\|_{L_p} = \left(\int_0^\infty \|u(t)\|^p \, dt \right)^{1/p} < \infty$$

The subscript p in L_p^m refers to the type of p-norm used to define the space, while the superscript m is the dimension of the signal u.

2.2.4 Similarity Transformation

Consider an $n \times n$ matrix A. It maps $\mathbb{R}^n$ into itself. If we associate with $\mathbb{R}^n$ the orthonormal basis $\{i_1, i_2, \ldots, i_n\}$ with

$$i_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, i_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, i_{n-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, i_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

then the ith column of A is the representation of A_i with respect to the orthonormal basis. Now if we select a different set of basis $\{q_1, q_2, \ldots, q_n\}$, then the matrix A has a different representation $\overline{A}$. It turns out that the ith column of $\overline{A}$ is the representation of Aq_i with respect to the basis $\{q_1, q_2, \ldots, q_n\}$.

Consider the equation

$$Ax = y \tag{2.15}$$

The square matrix A maps x in $\mathbb{R}^n$ into y in $\mathbb{R}^n$. With respect to the basis $\{q_1, q_2, \ldots, q_n\}$, the equation becomes

$$\overline{A}\overline{x} = \overline{y} \tag{2.16}$$

where $\overline{x}$ and $\overline{y}$ are the representations of x and y with respect to the basis $\{q_1, q_2, \ldots, q_n\}$. They are related by

$$x = Q\overline{x} \qquad y = Q\overline{y}$$

with

$$Q = [q_1 \; q_2 \; \ldots \; q_n]$$

being an $n \times n$ nonsingular matrix. Substituting these into (2.15) yields

$$AQ\overline{x} = Q\overline{y} \quad \text{or} \quad Q^{-1}AQ\overline{x} = \overline{y}.$$

Comparing this with (2.16) yields

$$\overline{A} = Q^{-1}AQ \quad \text{or} \quad A = Q\overline{A}Q^{-1}. \tag{2.17}$$

This is called the similarity transformation and A and $\overline{A}$ are said to be similar. We write (2.17) as

$$AQ = Q\overline{A}$$

or

$$A[q_1 \; q_2 \; \ldots \; q_n] = [Aq_1 \; Aq_2 \; \ldots \; Aq_n] = [q_1 \; q_2 \; \ldots \; q_n]\overline{A}.$$

This shows that the ith column of $\overline{A}$ is indeed the representation of Aq_i with respect to the basis $\{q_1, q_2, \ldots, q_n\}$.

2.2.5 Singular Value Decomposition

Singular value decomposition (SVD) is a very useful tool in matrix analysis. It will be seen that singular values of a matrix are good measures of the "size" of the matrix and that the corresponding singular vectors are good indications of strong/weak input or output directions.

Lemma 2.3 *Let $A \in \mathbb{R}^{m \times n}$. There exist unitary matrices*

$$U = [\mu_1, \mu_2, \ldots, \mu_m]$$
$$V = [\nu_1, \nu_2, \ldots, \nu_n]$$

such that

$$A = U \Sigma V^*, \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_p \end{bmatrix}$$

and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0, \ \ p = \min\{m, n\}.$$

The σ_i is the ith singular value of A, and the vectors μ_i and ν_j are, respectively, the ith left singular vector and the jth right singular vector. It is easy to verify that

$$A\nu_i = \sigma_i \mu_i$$
$$A^T \mu_i = \sigma_i \nu_i.$$

The preceding equations can also be written as

$$A^T A \nu_i = \sigma_i^2 \nu_i$$
$$A A^T \mu_i = \sigma_i^2 \mu_i.$$

Hence σ_i^2 is an eigenvalue of AA^T or $A^T A$, μ_i is an eigenvector of AA^T, and ν_i is an eigenvector of $A^T A$.

The following notations are often adopted:

$$\overline{\sigma}(A) = \sigma_{max}(A) = \sigma_1 = \text{the largest singular value of } A;$$

and

$$\underline{\sigma}(A) = \sigma_{min}(A) = \sigma_p = \text{the smallest singular value of } A.$$

2.3 Controllability and Observability

2.3.1 Controllability

This section deals with the controllability properties of nonlinear systems described by linear time-varying state-space representations. In particular, consider a nonlinear system defined by the state-space representation:

$$\dot{x}(t) = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \ x \in \Omega_x \subset \mathbb{R}^n \tag{2.18}$$

where $u = [u_1, u_2, \ldots, u_n]^T \in \Omega_u \subset \mathbb{R}^m$ is the input vector. The system (2.18) is defined to be controllable if there exists an admissible input vector $u(t)$ such that the state $x(t)$ can converge from an initial point $x(t_0 = 0) = x_0 \in \Omega_x$ to the final point $x(t_f) \in \Omega_x$ within a finite time interval t_f. The controllability means that the control system is with a set of input channels through which the input can excite the states effectively to converge to the destination x_f. Then, the controllability of (2.18) should mainly depend on the function forms of all $f(x)$ and $g_i(x)$. The controllability of the nonlinear system (2.18) is based on a useful mathematical concept called Lie algebra, which is defined as follows:

Definition 2.4 A Lie algebra over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ is a vector space $\mathbb{G}$ for which a bilinear map $(X, \ Y) \to [X, \ Y]$ is defined from $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that

$$[X, \ Y] = -[Y, \ X] \tag{2.19}$$

$$[X, \ [Y, \ Z]] + [Y, [Z, X]] + [Z, [X, \ Y]] = 0 \tag{2.20}$$

for any $X, Y, Z \in \mathbb{G}$.

From the above definition, a Lie algebra is a vector space where an operator $[\cdot]$ is installed, which is called a Lie bracket, can be defined arbitrarily as long as it satisfies two conditions (2.19) and (2.20) simultaneously. The condition (2.19) is often called a skew symmetric relation and obviously implies that $[X, Y] = 0$. The condition (2.20) is called the Jacobi identity, which reveals a closed-loop cyclic relation among any three elements in a Lie algebra.

Define a special Lie algebra $\mathcal{E}$ that collects all n-dimensional differentiable vector fields in $\mathbb{R}^n$ along with a commutative derivative relation: For any two vector fields f and $g \in \mathbb{R}^n$, which are functions of $x \in \mathbb{R}^n$, we have

$$[f, \ g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{2.21}$$

It can be seen that the above equation satisfies the two conditions (2.19) and (2.20) of a Lie algebra.

It is easy to extend the above Lie bracket between two vector fields to higher order derivatives, a more compact notation may be defined based on *an adjoint operator*, that is, $[f, g] = \mathrm{ad}_f g$. This new notation treats the Lie bracket $[f,\ g]$ as vector field g operated on by an adjoint operator $\mathrm{ad}_f = [f, \cdot]$. Therefore, for an n-order Lie bracket ($n > 1$), one can simply write

$$[f,\ \cdots [f, g] \cdots] = \mathrm{ad}^n_f g \tag{2.22}$$

For a general control system given by (2.18), we define a control Lie algebra Δ, which is spanned by all up to order $(n-1)$ Lie brackets among f and g_1 through g_m as

$$\Delta = \mathrm{span}\left\{g_1, \ldots, g_m, \mathrm{ad}_f g_1, \ldots, \mathrm{ad}_f g_m, \ldots, \mathrm{ad}^{n-1}_f g_1, \ldots, \mathrm{ad}^{n-1}_f g_m\right\} \tag{2.23}$$

With the control Lie algebra concept, we can show that the following theorem is true and is also a general effective testing criterion for system controllability.

Theorem 2.5 *The control system (2.18) is controllable if and only if* $\dim(\Delta) = \dim(\Omega_x) = n$.

Note that because each element in Δ is a function of x, the dimension of Δ may be different from one point to another. Thus, if the preceding condition of dimension is valid only in a neighborhood of a point in $\Omega_x \subset \mathbb{R}^n$, we say that the system (2.18) is locally controllable. On the other hand, if the condition of dimension can cover all of the region Ω_x, then it is globally controllable.

2.3.2 Observability

Consider the observability for the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{2.24}$$

where $y \in \mathbb{R}^m$ is the output. This system is said to be observable if for each pair of distinct states x_1 and x_2, the corresponding outputs y_1 and y_2 are also distinguishable. Clearly, the observability can be interpreted as a testing criterion to check whether the entire system has sufficient output channels to measure (or observe) each internal state change. Intuitively, the observability should depend on the function forms of both $f(x)$ and $h(x)$.

We introduce a Lie derivative, which is virtually *a directional derivative* for a scalar field $\lambda(x)$, with $x \in \mathbb{R}^n$ along the direction of an n-dimensional vector field $f(x)$. The mathematical expression is given as

$$L_f \lambda(x) = \frac{\partial \lambda(x)}{\partial x} f(x) \tag{2.25}$$

Since $\frac{\partial \lambda(x)}{\partial x}$ is a $1 \times n$ gradient vector of the scalar $\lambda(x)$ and the norm of a gradient vector represents the maximum rate of function value changes, the product of the gradient and the vector field $f(x)$ in (2.24) becomes the directional derivative of $\lambda(x)$ along $f(x)$. Therefore, the Lie derivative of a scalar field defined by (2.25) is also a scalar field. If each component of a vector field $h(x) \in \mathbb{R}^m$ is considered to take a Lie derivative along $f(x) \in \mathbb{R}^n$, then all components can be acted on concurrently and the result is a vector field that has the same dimension as $h(x)$; its ith element is the Lie derivative of the ith component of $h(x)$. Namely, if $h(x) = [h_1(x), \ldots, h_m(x)]^T$ and each component $h_i(x)$, $i = 1, \ldots, m$ is a scalar field, then the Lie derivative of the vector field $h(x)$ is defined as

$$L_f h(x) = \begin{bmatrix} L_f h_1(x) \\ \vdots \\ L_f h_m(x) \end{bmatrix} \tag{2.26}$$

With the Lie derivative concept, we now define an observation space Ω_0 over $\mathbb{R}^n$ as

$$\Omega_0 = \text{span}\{h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\} \tag{2.27}$$

In other words, this space is spanned by all up to order $(n-1)$ Lie derivatives of the output function $h(x)$. Then, we further define an observability distribution, denoted by $d\Omega_0$, which collects the "gradient" vector of every component in Ω_0. Namely,

$$d\Omega_0 = \text{span}\{\frac{\partial \phi}{\partial x} | \phi \in \Omega_0\} \tag{2.28}$$

With these definitions, we can present the following theorem for testing the observability.

Theorem 2.6 *The system (2.24) is observable if and only if* $\dim(d\Omega_0) = n$.

Similar to the controllability case, this testing criterion also has locally observable and globally observable cases, depending on whether the condition of dimension in the theorem is valid only in a neighborhood of a point or over the entire state region.

2.4 Stability Theory

2.4.1 Definitions

Let us consider the following nonautonomous system:

$$\dot{x} = f(t, x) \tag{2.29}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, with $D \subseteq \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

Definition 2.7 The origin $x = 0$ is said to be an equilibrium point of system (2.29) if for all $t \geq 0$,

$$f(t, 0) = 0$$

Definition 2.8 A continuous function $\alpha\colon [0, a) \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition 2.9 A continuous function $\beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. It is said to belong to class $\mathcal{KL}_\infty$ if, in addition, for each fixed s the mapping $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r.

Definition 2.10 The equilibrium point $x = 0$ of system (2.29) is said to be

(1) stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0 \geq 0 \tag{2.30}$$

(2) uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ independent of t_0 such that (2.30) is satisfied;
(3) unstable if it is not stable;
(4) asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$;
(5) uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \forall t \geq t_0 + T(\eta), \forall \|x(t_0)\| < c. \tag{2.31}$$

(6) globally uniformly asymptotically stable (GUAS) if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \forall t \geq t_0 + T(\eta, c), \forall \|x(t_0)\| < c. \tag{2.32}$$

Definition 2.11 The system (2.29) is said to be exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k(\|x(t_0)\|)e^{-\lambda(t-t_0)}, \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c \tag{2.33}$$

and further is globally exponentially stable (GES) if (2.33) holds for any initial state $x(t_0)$.

Definition 2.12 The system (2.29) is $\mathcal{K}$-exponentially stable if there exist positive constants c and λ and a class $\mathcal{K}$ function α such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|)e^{-\lambda(t-t_0)}, \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c \tag{2.34}$$

and further is globally $\mathcal{K}$-exponentially stable if (2.34) holds for any initial state $x(t_0)$.

Definition 2.13 The solution of (2.29) is as follows:

(1) uniformly bounded if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \forall t \geq t_0. \tag{2.35}$$

(2) globally uniformly bounded if (2.35) holds for an arbitrarily large a;
(3) uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$ there is $T = T(a, b) \geq 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T. \tag{2.36}$$

(4) globally uniformly ultimately bounded if (2.36) holds for an arbitrarily large a.

2.4.2 Lemmas and Theorems

Lemma 2.14 *Assume that $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfies*

$$P\left[\frac{\partial d}{\partial x}\right] + \left[\frac{\partial d}{\partial x}\right]^T P \geq 0, \forall x \in \mathbb{R}^n, \tag{2.37}$$

when $P = P^T > 0$. Then

$$(x - y)^T P(d(x) - d(y)) \geq 0, \forall x, y \in \mathbb{R}^n. \tag{2.38}$$

Theorem 2.15 *Let $D = \{x \in \mathbb{R}^n \big| \|x\| < r\}$ with $x = 0$ being an equilibrium of (2.29). Let $V : D \times \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable function such that $\forall t \geq 0, \forall x \in D$,*

$$\begin{aligned} \gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\gamma_3(\|x\|). \end{aligned} \tag{2.39}$$

Then the system (2.29) is

(1) uniformly stable, if γ_1 and γ_2 are class $\mathcal{K}$ functions on $[0, r)$ and $\gamma_3 \geq 0$ on $[0, r)$;
(2) uniformly asymptotically stable, if γ_1, γ_2, and γ_3 are class $\mathcal{K}$ functions on $[0, r)$;
(3) exponentially stable if $\gamma_i(\rho) = k_i \rho^\alpha$ on $[0, r)$, $k_i > 0, \alpha > 0, i = 1, 2, 3$;
(4) globally uniformly stable if $D = \mathbb{R}^n$, γ_1, and γ_2 are class $\mathcal{K}_\infty$ functions, and $\gamma_3 \geq 0$ on $\mathbb{R}_+$;
(5) GUAS if $D = \mathbb{R}^n$, γ_1 and γ_2 are class K_∞ functions, and γ_3 is a class $\mathcal{K}$ function on $\mathbb{R}_+$;
(6) GES, if $D = \mathbb{R}^n$, $\gamma_i(\rho) = k_i \rho^\alpha$ on $\mathbb{R}_+$, $k_i > 0, \alpha > 0, i = 1, 2, 3$.

Theorem 2.16 *If there exists a continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\begin{aligned} \gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|), \\ \dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x) \leq 0. \end{aligned} \tag{2.40}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where γ_1 and γ_2 are class $\mathcal{K}_\infty$ functions, and W is a continuous function. Then all solutions of (2.29) are globally uniformly bounded and satisfy

$$\lim_{t \to \infty} W(x(t)) = 0. \tag{2.41}$$

In addition, if $W(x)$ is positive definite, then the equilibrium point $x = 0$ is GUAS.

Theorem 2.17 *Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function and $D \in \mathbb{R}^n$ be a domain that contains the origin, if*

$$\begin{aligned} \alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x), \forall \|x\| \geq \mu > 0 \end{aligned} \tag{2.42}$$

for all $t \geq 0$ and $x \in D$ where α_1 and α_2 are class $\mathcal{K}$ functions, and W is a continuously positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r)). \tag{2.43}$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, there is $T > 0$ (dependent on $x(t_0)$ and μ) such that the solutions of (2.29) satisfy

$$\begin{aligned} &\|x(t)\| \le \beta(\|x(t_o)\|, t - t_0), \forall t_0 \le t \le t_0 + T, \\ &\|x(t)\| < \alpha_1^{-1}(\alpha_2(\mu)), \forall t \ge t_0 + T. \end{aligned} \tag{2.44}$$

Moreover, if $D = \mathbb{R}^n$ and α_1 belongs to class K_∞, then (2.44) holds for any initial state $x(t_0)$ with no restriction on how large μ is.

2.4.3 Input-to-State Stability

Definition 2.18 We indicate the essential supremum norm of an essentially bounded function with the symbol $\|\cdot\|_\infty$. A function μ is said to be essentially bounded if $\text{esssup}_{t\ge 0}\|\mu(t)\| < \infty$. For given times $0 \le T_1 < T_2$, we indicate with $\mu_{[T_1,T_2)} : [0, +\infty) \to R_m$ the function given by $\mu_{[T_1,T_2)}(t) = \mu(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input μ is said to be locally essentially bounded if, for any $T > 0$, $\mu_{[0,T)}$ is essentially bounded. A function $w : [0, b) \to R$, $0 < b \le +\infty$, is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0, c]$, $0 < c < b$.

Definition 2.19 The system

$$\dot{x} = f(t, x, u), \tag{2.45}$$

where f is piecewise continuous in t and locally Lipschitz in x and u, is said to be input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for any $x(t_0)$ and for any input $u(\cdot)$ continuous and bounded on $[0, \infty)$, the solution exists for all $t \ge t_0 \ge 0$ and satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\Big(\|u_{[0,t)}\|_\infty\Big). \tag{2.46}$$

The following theorem establishes the equivalence between the existence of a Lyapunov-like function and the input-to-state stability.

Theorem 2.20 *Suppose that for the system (2.45) there exists a continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$\begin{aligned} &\gamma_1(\|x\|) \le V(t, x) \le \gamma_2(\|x\|), \\ &\|x\| \ge \rho(\|u_{[0,\infty)}\|_\infty) \Rightarrow \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \le -\gamma_3(\|x\|) \end{aligned} \tag{2.47}$$

where γ_1, γ_2, and ρ are class K_∞ functions and γ_3 is a class-$\mathcal{K}$ function. Then the system (2.45) is ISS with $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \rho$.

Proof If $x(t_0)$ is in the set

$$R_{t_0} = \{x \in \mathbb{R}^n \big| \|x\| \le \rho(\|u_{[0,\infty)}\|_\infty)\}, \tag{2.48}$$

then $x(t)$ remains within the set

$$S_{t_0} = \{x \in \mathbb{R}^n \big| \|x\| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho(\|u_{[0,\infty)}\|_\infty)\}, \tag{2.49}$$

for all $t \ge t_0$. Define $B = [t_0, T)$ as the time interval before $x(t)$ enters R_{t_0} for the first time. In view of the definition of R_{t_0} we have

$$\dot{V} \le -\gamma_3 \circ \gamma_2^{-1}(V), \forall t \in B. \tag{2.50}$$

Then, there exists a class-$\mathcal{KL}$ function β_v such that $V(t) \le \beta_v(V(t_0), t - t_0)$, $\forall t \in B$, which implies

$$\|x(t)\| \le \gamma_1^{-1}(\beta_v(\gamma_2(\|x(t_0)\|), t - t_0)) := \beta(\|x(t_0)\|, t - t_0), \forall t \in B. \tag{2.51}$$

On the other hand, by (2.49), we conclude that

$$\|x(t)\| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho(\|u_{[0,\infty)}\|_\infty) := \gamma(\|u_{[0,\infty)}\|_\infty) \tag{2.52}$$

for all $t \in [t_0, \infty]\backslash B$. Then by (2.51) and (2.52),

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma(\|u_{[0,\infty)}\|_\infty), \forall t \ge t_0 \ge 0. \tag{2.53}$$

By causality, we have

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma(\|u_{[0,t)}\|_\infty), \forall t \ge t_0 \ge 0. \tag{2.54}$$

A function V satisfying conditions (2.47) is called an ISS Lyapunov function.

2.4.4 Lyapunov's Direct Method

This section presents an extension of the Lyapunov function concept, which is a useful tool to design an adaptive controller for nonlinear systems. Assuming that the problem is to design a feedback control law $\alpha(x)$ for the time-invariant system:

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, \ u \in \mathbb{R}, \ f(0, 0) = 0, \tag{2.55}$$

such that the equilibrium $x = 0$ of the closed-loop system:

$$\dot{x} = f(x, \alpha(x)), \tag{2.56}$$

is globally asymptotically stable (GAS). Take the candidate Lyapunov function $V(x)$, in which derivative along the solutions of (2.56) satisfies $\dot{V}(x) \leq -W(x)$, where $W(x)$ is a positive definite function. We therefore need to find $\alpha(x)$ guarantee that for all $x \in \mathbb{R}^n$ such that

$$\frac{\partial V(x)}{\partial x} f(x, \alpha(x)) \leq -W(x). \tag{2.57}$$

This is a difficult problem. A stabilizing control law for (2.55) may exist but we may fail to satisfy (2.57) because of a poor choice of $V(x)$ and $W(x)$. A system for which a good choice of $V(x)$ and $W(x)$ exists is said to possess a control Lyapunov function (CLF). For systems affine in the control:

$$\dot{x} = f(x) + g(x)u, \; f(0) = 0, \tag{2.58}$$

the CLF inequality (2.57) becomes

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)\alpha(x) \leq -W(x), \tag{2.59}$$

If $V(x)$ is a CLF for (2.58), then a particular stabilizing control law $\alpha(x)$, smooth for all $x \neq 0$, is given by

$$u = \alpha(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f(x) + \sqrt{\left(\frac{\partial V}{\partial x} f(x)\right)^2 + \left(\frac{\partial V}{\partial x} g(x)\right)^4}}{\frac{\partial V}{\partial x} g(x)}, & \frac{\partial V}{\partial x} g(x) \neq 0, \\ 0, & \frac{\partial V}{\partial x} g(x) = 0. \end{cases}$$

It should be noted that (2.59) can be satisfied only if

$$\frac{\partial V}{\partial x} g(x) = 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0, \forall x \neq 0 \tag{2.60}$$

and that in this case (2.60) gives

$$W(x) = \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} > 0, \forall x \neq 0 \tag{2.61}$$

As a design tool, however, a main drawback of CLF is that for most nonlinear systems the CLF is usually unknown. Meanwhile, the task of finding an appropriate CLF may be as complex as that of designing a stabilizing feedback law in practical.

2.4.5 Barbalat-Like Lemmas

This section presents lemmas that are useful in investigating the convergence of time-varying systems.

If a function $f \in L_1$, it may not be bounded. On the contrary, if a function f is bounded, it is not necessary that $f \in L_1$. However, if $f \in L_1 \bigcap L_\infty$, then $f \in L_p$ for all $p \in [1, \infty)$. Moreover, $f \in L_p$ could not lead to $f \to 0$ as $t \to \infty$. If f is bounded, it can also lead to $f \to 0$ as $t \to \infty$. However, we have the following results.

Lemma 2.21 (Barbalat's lemma) *Consider the function $\phi : \mathbb{R}_+ \to \mathbb{R}$. If ϕ is uniformly continuous and* $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ *exists and is finite, then*

$$\lim_{t\to\infty} \phi(t) = 0. \tag{2.62}$$

Lemma 2.22 *Assume that a nonnegative scalar differentiable function $f(t)$ enjoys the following conditions:*

$$\left|\frac{d}{dt} f(t)\right| \le k_1 f(t) \tag{2.63}$$

$$\int_0^\infty f(t)dt \le k_2 \tag{2.64}$$

for all $t \ge 0$, where k_1 and k_2 are positive constants, then $\lim_{t\to\infty} f(t) = 0$.

Proof Integrating both sides of (2.63) gives

$$\begin{aligned} f(t) &\le f(0) + k_1 \int_0^t f(s)ds \le f(0) + k_1 k_2, \\ f(t) &\ge f(0) - k_1 \int_0^t f(s)ds \ge f(0) - k_1 k_2. \end{aligned} \tag{2.65}$$

These inequalities imply that $f(t)$ is a uniform bounded function. From (2.65) and the second condition in (2.63), we have that $f(t)$ is also bounded on the half axis $[0, \infty)$, i.e., $f(t) \le k_3$ with k_3 a positive constant. Hence $\left|\frac{d}{dt} f(t)\right| \le k_1 k_3$. Now assume that $\lim_{t\to\infty} f(t) \ne 0$. Then there exists a sequence of points t_i and a positive constant ϵ such that $f(t_i) \ge \epsilon, t_i \to \infty, i \to \infty, |t_i - t_{i-1}| > 2\epsilon/(k_1 k_3)$ and moreover $f(s) \ge \epsilon/2, s \in L_i = [t_i - \epsilon/(2k_1k_3), t_i + \epsilon/(2k_1k_3)]$. Since the segments L_i and L_j do not intersect for any i and j with $i \ne j$, we have

$$\int_0^\infty f(t)dt \ge \int_0^T f(t)dt \ge \sum_{t_i \le T} \int_{L_i} f(t)dt \ge \frac{\epsilon}{2}\frac{\epsilon}{k_1 k_3} M(T) \tag{2.66}$$

where $M(T)$ is the number of points t_i not exceeding T. Since $\lim_{T\to\infty} M(T) = \infty$, the integral $\int_0^\infty f(t)dt$ is divergent. This contradicts Condition 2 in (2.63). This contradiction proves the lemma.

Remark 2.23 Lemma 2.22 is different from Barbalat's Lemma 2.21. While Barbalat's lemma assumes that $f(t)$ is uniformly continuous, Lemma 2.22 assumes that $|\frac{d}{dt} f(t)|$is bounded by $k_1 f(t)$.

Corollary 2.24 *If $f(t)$ is uniformly continuous, such that $\int_0^\infty f(\tau)d\tau$ exists and is finite, then $f(t) \to 0$ as $t \to \infty$.*

Corollary 2.25 *If $f(t)$, $\dot{f}(t) \in L_\infty$, and $f(t) \in L_p$, for some $p \in [1, \infty)$, then $f(t) \to 0$ as $t \to \infty$.*

Corollary 2.26 *For the differentiable function $f(t)$, if $\lim_{t\to\infty} f(t) = k < \infty$ and $\ddot{f}(t)$ exists, then $\dot{f}(t) \to 0$ as $t \to \infty$.*

Corollary 2.27 *If $\lim_{t\to\infty} \int_0^\infty f^2(t)dt < \infty$ and $\dot{x}(t)$, $t \in [0, \infty)$, exists and bounded, then $f(t) \to 0$ as $t \to \infty$.*

Lemma 2.28 *If a scalar function $V(x, t)$ satisfies the following conditions:*

(i) $V(x, t)$ *is lower bounded*
(ii) $\dot{V}(x, t)$ *is negative semi-definite*
(iii) $\dot{V}(x, t)$ *is uniformly continuous in times*

then $\dot{V}(x, t) \to 0$ as $t \to \infty$.

Indeed, V then approaches a finite limiting value V_∞, such that $V_\infty \le V(x(0), 0)$ (this does not require uniform continuity). The above lemma then follows from Barbalat's lemma.

2.4.6 Lyapunov Theorem

The Lyapunov approach provides a rigorous method for addressing stability. The method is a generalization of the idea that if there is some "measure of energy" in a system, then we can study the rate of change of the energy of the system to ascertain stability. We review several concepts that are used in Lyapunov stability theory.

Definition 2.29 A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a locally positive definite function if for some $\varepsilon > 0$ and some continuous, strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$,

$$V(0, t) = 0, \quad \text{and } V(x, t) \ge \alpha\|x\|, \ \forall t \ge 0 \tag{2.67}$$

Definition 2.30 A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a positive definite function if it satisfies the conditions of Definition 2.29 and, additionally, $\alpha(p) \to \infty$ as $p \to \infty$.

Definition 2.31 A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is decrescent if for some $\varepsilon > 0$ and some continuous, strictly increasing function $\beta : \mathbb{R}_+ \to \mathbb{R}$,

$$V(x,t) \le \beta(\|x\|),\ \forall x \in \Omega,\ \forall t \ge 0 \tag{2.68}$$

Based on above definitions, by studying an appropriate energy function, the following theorem is provided to determine stability.

Theorem 2.32 (Lyapunov Theorem) *Any nonlinear dynamic system*

$$\dot{x} = f(x,t),\ x(0) = x_0 \tag{2.69}$$

with an the equilibrium point at the origin, let Ω be a ball of size around the origin, i.e., $\Omega = \{x : \|x\| \le \varepsilon, \varepsilon > 0\}$. If there exists a continuously differentiable function

$$V(0,t) = 0\ and\ V(x,t) > 0\ with\ x \neq 0$$

such that

$$\dot{V}(x,t) \le 0\ with\ x \in \Omega$$

the origin of system is stable.

Moreover, if

$$\dot{V}(x,t) < 0\ with\ x \in \Omega\ but\ x \neq 0$$

then the origin of system is asymptotically stable.

The function $V(x,t)$ is called the Lyapunov function.

The indirect method of Lyapunov uses the linearization of a system to determine the local stability of the original system.

Theorem 2.33 (Stability by linearization) *Consider the system $\dot{x} = f(x,t)$ and define*

$$A(t) = \frac{\partial f(x,t)}{\partial x} \tag{2.70}$$

with $x = 0$ to be the Jacobian matrix of $f(x,t)$ with respect to x, evaluated at the origin. It follows that for each fixed t,

$$f_1(x,t) = f(x,t) - A(t)x \tag{2.71}$$

approaches zero as x approaches zero. Assume

$$\lim_{\|x\| \to 0} \sup_{t \ge 0} \frac{\|f_1(x,t)\|}{\|x\|} = 0 \tag{2.72}$$

Further, If 0 is a uniformly asymptotically stable equilibrium point of

$$\dot{z} = A(t)z \tag{2.73}$$

then it is a locally uniformly asymptotically stable equilibrium point of $\dot{x} = f(x, t)$.

2.4.6.1 Invariant Set Theorems

Asymptotic stability of a control system is a very important property. However, the Lyapunov theorems are usually difficult to apply because frequently $\dot{V}$, the derivative of the Lyapunov function candidate, is only semi-definite. With the help of the invariant set theorems, asymptotic stability can still possibly be concluded for autonomous systems from LaSalle's invariance principle. The concept of an invariant set is a generalization of the concept of equilibrium point.

Definition 2.34 (α *limit set*) The set $\Omega \in \mathbb{R}^n$ is the α limit set of a trajectory $\omega(t, x_0, t_0)$ if for every $y \in \Omega$, there exists a strictly increasing sequence of times T such that $\omega(T, x_0, t_0) \to y$ as $T \to \infty$.

Definition 2.35 A set $\Omega \in \mathbb{R}^n$ is said to be an invariant set of the dynamic system $\dot{x} = f(x)$ if for all $y \in \Omega$ and $t_0 > 0$, we have $\omega(t, y, t_0) \in \Omega, \forall t > t_0$.

Theorem 2.36 (LaSalle's Theorem) *Let* Ω *be a compact invariant set* $\Omega = x \in \mathbb{R}^n : \dot{V}(x) = 0$ *and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a locally positive definite function such that on the compact set we have* $\dot{V}(x) \le 0$. *As* $t \to \infty$, *the trajectory tends to the largest invariant set inside* Ω; *i.e., its* α *limit set is contained inside the largest invariant set in* Ω. *In particular, if* Ω *contains no invariant sets other than* $x = 0$, *then* $V(x)$ *is asymptotically stable.*

Corollary 2.37 *Given the autonomous nonlinear system*

$$\dot{x} = f(x), x(0) = x_0 \tag{2.74}$$

and let the origin be an equilibrium point, $V(x) : \mathcal{N} \to \mathbb{R}$ *be a continuously differentiable positive definite function on a neighborhood* $\mathcal{N}$ *of the origin, such that* $\dot{V}(x) \le 0$ *in* $\mathcal{N}$, *then the origin is asymptotically stable if there is no solution that can stay forever in* $S = \{x \in \mathcal{N} \mid \dot{V}(x) = 0\}$, *other than the trial solution. The origin is globally asymptotically stable if* $\mathcal{N} = \mathbb{R}^n$ *and* $V(x)$ *is radially unbounded.*

2.5 Linear Matrix Inequalities

Some miscellaneous definitions and results involving matrices and matrix equations are presented in this section. These results will be used throughout the book, especially those related with the concept of linear matrix inequalities (or in short LMIs), which will play a very important role in the following chapters.

Definition 2.38 (*Generalized Inverse*) The generalized inverse (Moore–Penrose inverse) of a matrix A is the unique matrix A^+ such that

(i) $AA^+A = A$,
(ii) $A^+AA^+ = A^+$,
(iii) $(AA^+)^T = AA^+$,
(iv) $(A^+A)^T = A^+A$.

Lemma 2.39 (Schur Complements) *Consider a symmetric matrix A such that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}. \tag{2.75}$$

(i) $A < 0$ if and only if

$$\begin{cases} A_{22} < 0, \\ A_{11} - A_{12}A_{22}^{-1}A_{12}^T < 0, \end{cases} \tag{2.76}$$

or

$$\begin{cases} A_{11} < 0, \\ A_{22} - A_{12}^T A_{11}^{-1} A_{12} < 0. \end{cases} \tag{2.77}$$

(ii) $A \leq 0$ if and only if

$$\begin{cases} A_{22} \leq 0, \\ A_{12}(I - A_{22}^+ A_{22}) = 0, \\ A_{11} - A_{12}A_{22}^+ A_{12}^T \leq 0, \end{cases} \tag{2.78}$$

or

$$\begin{cases} A_{11} \leq 0, \\ A_{12}(I - A_{11}A_{11}^+) = 0, \\ A_{22} - A_{12}^T A_{11}^+ A_{12} \leq 0. \end{cases} \tag{2.79}$$

where I is the identity matrix with appropriate dimension.

Next, we present the definition of a linear matrix inequality.

Definition 2.40 A linear matrix inequality is any constraint that can be written or converted to

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_m F_m < 0, \tag{2.80}$$

where x_i are the variables, and the symmetric matrices F_i for $i = 1, \ldots, m$ are known.

The linear matrix inequality (2.80) is referred to as a strict linear matrix inequality. Also of interest is the nonstrict linear matrix inequality, where $F(x) \leq 0$. From the practical point of view, LMIs are usually presented as

$$f(X_1, \ldots, X_N) < g(X_1, \ldots, X_N), \tag{2.81}$$

where f and g are affine functions of the unknown matrices $X_1, \ldots, X_N$. Quadratic forms can usually be converted to affine ones using the Schur complements. Therefore, we will make no distinctions between quadratic and affine forms, or between a set of LMIs or a single one, and will refer to all of them as simply LMIs.

2.6 Stochastic Systems

2.6.1 Probability Preliminaries

A stochastic process $\underline{\mathbf{X}} = \{X(t), t \in T\}$ is a collection of random variables. That is, for each t in the index set T, $X(t)$ is a random variable. We often interpret t as time and call $X(t)$ the state of the process at time t. When the index set T is countable, $\underline{\mathbf{X}}$ is called a discrete-time stochastic process, while if T is a continuum, it is then called a continuous-time stochastic process.

A continuous-time stochastic process $\{X(t), t \in T\}$ is said to have independent increments if for all $t_0 < t_1 < t_2 < \cdots < t_n$, the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}) \tag{2.82}$$

are independent. It is said to possess stationary increments if $X(t+s) - X(t)$ has the same distribution for all t. That is, it possesses independent increments if the changes in the processes' value over nonoverlapping time intervals are independent; and it possesses stationary increments if the distribution of the change in value between any two points depends only on the distance between those points.

2.6.2 Continuous-Time Markov Chains

Consider a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ that takes on a finite or countable number of possible values. Unless otherwise mentioned, this set of possible values of the process will be denoted by the set of nonnegative integers $\{0, 1, 2, \ldots\}$. In this case, $X_n = i$ denotes that the process is in state i at time n. When the process is in state i, the value P_{ij} represents the probability that the process will be in state. Specifically,

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1, X_0 = i_0\} = P_{ij} \tag{2.83}$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$. Such a stochastic process is known as a Markov chain. In the above definition, for any Markov chain, the conditional distribution of any future state X_{n+1}, given the past states $X_0, X_1, \ldots, X_{n-1}$ and the present state x_n, is independent of the past states and depends only on the present state, which is called the Markovian property. A process with this property is said to be Markovian or a Markov process.

In analogy with the definition of a discrete-time Markov chain, a process $\{X_t, t \geq 0\}$ is called continuous-time Markov chain if for all $s, t \geq 0$, and nonnegative integers $i, j, X(\mu), 0 \leq \mu < s$,

$$\begin{aligned} P\{X(t+s) = j \mid X(s) = i, X(\mu) = x(\mu), 0 \leq \mu < s\} \\ = P\{X(t+s) = j \mid X(s) = i\}. \end{aligned}$$

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future state at time $t+s$, given the present state at s and all future states depend only on the current state while it is independent of the past ones.

Now let us introduce a Markovian jumping system, which is defined as

$$\dot{x}(t) = f(t, x(t), r(t)) \tag{2.84}$$

where $x \in \mathbb{R}^n$ is the state, $r : \mathbb{R}_+ \to \mathcal{S}$ is the continuous-time Markov chain with $\mathcal{S} \triangleq \{1, 2, \ldots, N\}$ being the discrete state, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^n$ is the nonlinear dynamic which satisfies the locally Lipschitz conditions. In general, to analyze the stability of system (2.84), the following infinitesimal operator is also needed.

Definition 2.41 For any given $V(x(t), t, r(t)) \in C(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$, associated with system (2.84), the infinitesimal operator $\mathcal{L}$, from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ to $\mathbb{R}$, can be described as follows:

$$\mathcal{L}V(x(t), t, i) = V_t(x(t), t, i) + V_x(x(t), t, i) f(t, x(t), i) + \sum_{j=1}^{N} \pi_{ij} V(x(t), t, j) \tag{2.85}$$

for any $i \in \mathcal{S}$, where

$$V_t(x(t), t, i) = \left(\frac{\partial V(x(t), t, i)}{\partial t} \right), \tag{2.86}$$

$$V_x(x(t), t, i) = \left(\frac{\partial V(x(t), t, i)}{\partial x_1}, \ldots, \frac{\partial V(x(t), t, i)}{\partial x_n} \right), \tag{2.87}$$

$$V_{xx}(x(t), t, i) = \left(\frac{\partial^2 V(x(t), t, i)}{\partial x_m \partial x_s} \right)_{n \times n}. \tag{2.88}$$

2.6.3 Stochastic Stability

2.6.3.1 Stability in Probability

In this section we shall investigate various types of stability for the d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \qquad \text{on } t \geq t_0. \tag{2.89}$$

Definition 2.42 The trivial solution of equation (2.89) is said to be:

1. stochastically stable or stable in probability if for every pair of $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\sigma = \sigma(\epsilon, r, t_0) > 0$ such that

$$P\{|x(t; t_0, x_0)| < r \ \forall\, t \geq t_0\} \geq 1 - \epsilon \tag{2.90}$$

   whenever $|x_0| < \sigma$. Otherwise, it is said to be stochastically unstable.
2. stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0, 1)$, there exists a $\sigma_0 = \sigma_0(\epsilon, t_0) > 0$ such that

$$P\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\} \geq 1 - \epsilon \tag{2.91}$$

   whenever $|x_0| < \delta_0$.
3. stochastically asymptotically stable in the large if it is stochastically stable and, moreover, for all $x_0 \in \mathbb{R}^d$

$$P\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\} = 1. \tag{2.92}$$

It should also be pointed out that when $g(x, t) \equiv 0$, these definitions reduce to the corresponding deterministic ones. We now extend the Lyapunov Theorem to the stochastic case.

Theorem 2.43 *If there exists a positive definite function $V(x, t) \in C(S_h \times [t_0, \infty); R_+)$ such that*

$$\mathcal{L}V(x, t) \leq 0 \tag{2.93}$$

for all $(x, t) \in S_h \times [t_0, \infty]$, then the trivial solution of equation (2.89) is stochastically stable.

Theorem 2.44 *If there exists a positive definite decrescent function $V(x, t) \in C(S_h \times [t_0, \infty); \mathbb{R}_+)$ such that $\mathcal{L}V(x, t)$ is negative definite, then the trivial solution of equation (2.89) is stochastically asymptotically stable.*

Theorem 2.45 *If there exists a positive definite decrescent radially unbounded function $V(x, t) \in C(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$ such that $\mathcal{L}V(x, t)$ is negative definite, then the trivial solution of equation (2.89) is stochastically asymptotically stable in the large.*

2.6.3.2 Almost Sure Exponential Stability

We first give the formal definition of the almost sure exponential stability.

Definition 2.46 The trivial solution of equation (2.89) is said to be almost surely exponentially stable if

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t; t_0, x_0)| < 0 \qquad \text{a. s.} \tag{2.94}$$

for all $x_0 \in \mathbb{R}^d$.

The left-hand side of (2.94) is called the sample Lyapunov exponents of the solution. We therefore see that the trivial solution is almost surely exponentially stable if and only if the sample Lyapunov exponents are negative. The almost sure exponential stability means that almost all sample paths of the solution will tend to the equilibrium position $x = 0$ exponentially fast. To establish the theorems on the almost sure exponential stability, we need to prepare a useful lemma. We assume that the existence-and-uniqueness is fulfilled and $f(0, t) \equiv 0$, $g(0, t) \equiv 0$.

Lemma 2.47 *For all $x_0 \neq 0$ in $\mathbb{R}^d$*

$$P\{x(t; t_0, x_0) \neq 0 \text{ on } t \geq t_0\} = 1. \tag{2.95}$$

That is, almost all the sample paths of any solution starting from a nonzero state will never reach the origin.

Theorem 2.48 *Assume that there exists a function $V \in C(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, and constants $p > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$, such that for all $x \neq 0$ and $t \geq t_0$,*

(i) $c_1|x|^p \leq V(x, t)$,
(ii) $\mathcal{L}V(x, t) \leq c_2 V(x, t)$,
(iii) $|V_x(x, t)g(x, t)|^2 \geq c_3 V^2(x, t)$.

Then

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \qquad \text{a. s.}$$

for all $x_0 \in \mathbb{R}^d$. In particular, if $c_3 > 2c_2$, the trivial solution of equation (2.89) is almost surely exponentially stable.

Corollary 2.49 *Assume that there exists a function $V \in C(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, and positive constants p, α, λ, such that for all $x \neq 0$, $t \geq t_0$,*

$$\alpha|x|^p \leq V(x, t) \qquad (\text{and}) \qquad \mathcal{L}V(x, t) \leq -\lambda V(x, t).$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t;t_0,x_0)| \le -\frac{\lambda}{p} \qquad a.\ s.$$

for all $x_0 \in \mathbb{R}^d$. In other words, the trivial solution of equation (2.89) is almost surely exponentially stable.

This corollary follows from Theorem 2.48 immediately by letting $c_1 = \alpha$, $c_2 = -\lambda$, and $c_3 = 0$. These results have given the upper bound for the sample Lyapunov exponents. Let us now turn to the study of the lower bound.

Theorem 2.50 *Assume that there exists a function $V \in C(\mathbb{R}^d \times [t_0,\infty);\mathbb{R}_+)$, and constants $p > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 > 0$, such that for all $x \ne 0$ and $t \ge t_0$,*

(i) $c_1|x|^p \ge V(x,t) > 0$,
(ii) $\mathcal{L}V(x,t) \ge c_2 V(x,t)$,
(iii) $|V_x(x,t)g(x,t)|^2 \le c_3 V^2(x,t)$.

Then

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t;t_0,x_0)| \ge -\frac{2c_2 - c_3}{2p} \qquad a.\ s.$$

for all $x_0 \in \mathbb{R}^d$. In particular, if $2c_2 > c_3$, then almost all the sample paths of $|x(t;t_0,x_0)|$ will tend to infinity, and we say in this case that the trivial solution of (2.89) is almost surely exponentially unstable.

2.6.3.3 Moment Exponential Stability

In this section we shall discuss the pth moment exponential stability for equation (2.89) and we shall always let $p > 0$. Let us first give the definition of the pth moment exponential stability.

Definition 2.51 The trivial solution of equation (2.89) is said to be pth moment exponentially stable if there is a pair of positive constants λ and C such that

$$\mathcal{E}|x(t;t_0,x_0)|^p \le C|x_0|^p e^{-\lambda(t-t_0)} \qquad \text{on } t \ge t_0 \tag{2.96}$$

for all $x_0 \in \mathbb{R}^d$. When $p = 2$, it is usually said to be exponentially stable in mean square.

Clearly, the pth moment exponential stability means that the pth moment of the solution will tend to 0 exponentially fast. The pth moment exponential stability and the almost sure exponential stability do not imply each other and additional conditions are required in order to deduce one from the other. The following theorem gives the conditions under which the pth moment exponential stability implies the almost sure exponential stability.

Theorem 2.52 *Assume that there is a positive constant K such that*

$$x^T f(x,t) \vee |g(x,t)|^2 \le K|x|^2 \qquad \forall\, (x,t) \in \mathbb{R}^d \times [t_0, \infty). \tag{2.97}$$

Then the pth moment exponential stability of the trivial solution of equation (2.89) implies the almost sure exponential stability.

Theorem 2.53 *Let $q > 0$. Assume that there is a function $V(x,t) \in C(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, and positive constants c_1, c_2, c_3, such that*

$$c_1|x|^q \le V(x,t) \le c_2|x|^q \qquad \text{and} \qquad \mathcal{L}V(x,t) \ge c_3 V(x,t) \tag{2.98}$$

for all $(x,t) \in \mathbb{R}^d \times [t_0, \infty]$. Then

$$\mathcal{E}|x(t; t_0, x_0)|^q \ge \frac{c_1}{c_2}|x_0|^q e^{c_3(t-t_0)} \qquad \text{on } t \ge t_0 \tag{2.99}$$

for all $x_0 \in \mathbb{R}^d$, and we say in this case that the trivial solution of equation (2.89) is qth moment exponentially unstable.

2.7 Time-Delay Systems

A prevalent model description for dynamical systems is the ordinary differential equations in the form of

$$\dot{x}(t) = f(t, x(t)). \tag{2.100}$$

In this description, the variables $x(t) \in \mathbb{R}^n$ are known as the state variables, and the differential equations characterize the evolution of the state variables with respect to time. A fundamental presumption on a system modeled as such is that the future evolution of the system is completely determined by the current value of the state variables. In other words, the value of the state variables $x(t)$, $t_0 \le t < \infty$, for any t_0, can be found once the initial condition

$$x(t_0) = x_0 \tag{2.101}$$

is known. Stability and control of dynamical systems modeled in ordinary differential equations have been an extensively developed subject of scientific learning. However, many dynamical systems cannot be properly modeled by an ordinary differential equation. In particular, for some particular classes of systems, the future evolution of the state variables $x(t)$ not only depends on their current value $x(t_0)$, but also on their past values, say $x(\xi)$, $t_0 - r \le \xi \le t_0$, $r > 0$. Such systems are called time-delay systems that may arise for a variety of reasons in many scientific disciplines including engineering, biology, ecology, and economics.

2.7.1 Functional Differential Equations

We can use functional differential equations to describe time-delay systems. When the past dependence is through the state variable and not the derivative of the state variable, we call the functional differential equations as retarded functional differential equations. When the delayed argument occurs in the derivative of the state variable as well as in the independent variable, we call them neutral functional differential equations. Because the retarded functional differential equations are more common, we discuss them below.

The simplest linear retarded functional differential equation has the form

$$\dot{x}(t) = Ax(t) + Bx(t-d) + f(t) \tag{2.102}$$

where A, B, and d are constants with $d > 0$, f is a given continuous function on $\mathbb{R}$, and x is a scalar. The following theorem specifies what is the initial value problem for Eq. (2.102).

Theorem 2.54 *If ϕ is a given continuous function on $[-d, 0]$, then there is a unique function $x(\phi, f)$ defined on $[-d, \infty]$ which coincides with ϕ on $[-d, 0]$ and satisfies Eq. (2.102) for $t \geq 0$. Of course, at $t = 0$, the derivative in Eq. (2.102) represents the right-hand derivative.*

Theorem 2.54 specifies the minimum amount of initial data—a function on the entire interval $[-d, 0]$—to find a solution $x(t)$ for (2.102).

If f is not continuous but only locally integrable on $\mathbb{R}$, then the same proof yields the existence of a unique solution $x(\phi, f)$. Of course, by a solution, we mean a function which satisfies Eq. (2.102) almost everywhere.

Theorem 2.55 *If $x(\phi, f)$ is the solution of Eq. (2.102) defined by Theorem 2.54, then the following assertions are valid.*

(i) $x(\phi, f)(t)$ has a continuous first derivative for all $t > 0$ and has a continuous derivative at $t = 0$ if and only if $\phi(\theta)$ has a derivative at $\theta = 0$ with

$$\dot{\phi}(0) = A\phi(0) + B\phi(-d) + f(0). \tag{2.103}$$

If f has derivatives of all orders, then $x(\phi, f)$ becomes smoother with increasing values of t.

(ii) If $B \neq 0$, then $x(\phi, f)$ can be extended as a solution of equation (2.102) on $[-d-\epsilon, \infty]$, $0 < \epsilon \leq d$, if and only if $\phi(\theta)$ has a continuous first derivative on $[-\epsilon, 0]$ and Eq. (2.103) is satisfied. Extension further to the left requires more smoothness of ϕ and f and additional boundary conditions.

The smoothing property (i) of Theorem 2.55 is precisely why retarded equations have a structure very similar to ordinary differential equations.

Lemma 2.56 *If μ and α are real-valued continuous functions on $[a, b]$, and $\beta \geq 0$ is integrable on $[a, b]$ with*

$$\mu(t) \leq \alpha(t) + \int_a^t \beta(s)\mu(s)ds, \qquad a \leq t \leq b,$$

then

$$\mu(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)\left[exp \int_s^t \beta(\tau)d\tau\right]ds, \qquad a \leq t \leq b.$$

If, in addition, α is nondecreasing, then

$$\mu(t) \leq \alpha(t)exp\left(\int_a^t \beta(s)ds\right), \qquad a \leq t \leq b.$$

Theorem 2.57 *Suppose $x(\phi, f)$ is the solution of equation (2.102) defined by Theorem 2.54. Then there are positive constants a and b such that*

$$|x(\phi, f)(t)| \leq ae^{bt}\left(|\phi| + \int_0^t |f(s)|ds\right), \qquad t \geq 0$$

where $|\phi| = sup_{-d\leq\theta\leq 0}|\phi(\theta)|$.

Since Eq. (2.102) is linear and solutions are uniquely defined by ϕ, the solution $x(\phi, 0)$ of the homogeneous equation

$$\dot{x}(t) = Ax(t) + Bx(t-d) \tag{2.104}$$

which coincides with ϕ on $[-d, 0]$ is linear in ϕ; that is, $x(\phi + \psi, 0) = x(\phi, 0) + x(\psi, 0)$ and $x(a\phi, 0) = ax(\phi, 0)$ for any continuous functions ϕ and ψ on $[-d, 0]$ and any scalar a. And, for $f = 0$, inequality (2.103) implies that $x(\phi, 0)(t)$ is continuous in ϕ for all t; that is, $x(\cdot, 0)(t)$ is a continuous linear functional on the space of continuous functions on $[-d, 0]$.

2.7.2 Stability of Time-Delay Systems

2.7.2.1 Stability Concepts

A functional differential equation details an evolution over a finite Euclidian space or a functional space. A general system with time delays is given by:

$$\dot{x}(t) = f(x_t, u(t)), \quad t \geq 0 \text{ a.e.}, \tag{2.105}$$

$$x(\tau) = \xi_0(\tau), \tau \in [-d, 0] \tag{2.106}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the input function, for $t \geq 0$ $x_t : [-d, 0] \rightarrow \mathbb{R}^n$ is the standard function given by $x_t(\tau) = x(t + \tau)$, d is the maximum involved delay, $f : C([-d, 0]; \mathbb{R}^n) \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the continuous function which is defined on $[-d, 0]$ and takes values in $\mathbb{R}^n$, $\xi_0 \in C([-d, 0]; \mathbb{R}^n)$. Without loss of generality, it is also assumed that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t, 0)$.

Definition 2.58 For the system (2.105), the trivial solution $x(t) = 0$ is said to be

- stable if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \epsilon$ for $t \geq t_0$.
- asymptotically stable if it is stable, and for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \xi) > 0$ such that $\|x_{t_0}\|_c < \delta_a$ implies $lim_{t\rightarrow\infty} x(t) = 0$ and $\|x(t)\| < \epsilon$ for $t \geq t_0$.
- uniformly stable if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of t_0.
- uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that $\|x_{t_0}\|_c < \delta$ implies $\|x(t)\| < \eta$ for $t \geq t_0 + T$ and $t_0 \in \mathbb{R}$.
- globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_a can be an arbitrarily large, finite number.

One should note that the stability notions herein are not at all different from their counterparts for systems without delay, modulo to the different assumptions on the initial conditions.

Definition 2.59
- If a time-delay system is asymptotically stable for any delay values belonging to $\mathbb{R}_+$, the system is said to be delay-independent asymptotically stable.
- If a time-delay system is asymptotically stable for all delay values belonging to a compact subset $\mathbf{D}$ of $\mathbb{R}_+$, the system is said to be delay-dependent asymptotically stable.
- For a delay-dependent asymptotically stable time-delay system, if the stability does not depend on the variation rate of delays or on the time derivative of delays, the system is said to be rate-independent asymptotically stable.
- For a delay-dependent asymptotically stable time-delay system, if the stability depends on the variation rate of delays or on the time derivative of delays, the system is said to be rate-dependent asymptotically stable.

2.7.2.2 Lyapunov–Krasovskii Theorem

For the time-delay systems, Lyapunov–Krasovskii type theorem plays a role in the analysis in both the input–output stability (corresponds to zero-state response) and the asymptotic stability (corresponds to zero-input response).

Theorem 2.60 (Lyapunov–Krasovskii Stability Theorem) *Suppose that* $x(0) = 0$ *is an equilibrium of the unforced state equation*

$$\dot{x}(t) = f(t, x_t), \tag{2.107}$$

$x(t) \in \mathbb{R}^n$, $f : \mathbb{R} \times C([-d, 0], \mathbb{R}^n) \to \mathbb{R}^n$ *and* $u, v, w : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, with* $u(0) = v(0) = 0$*. If there exists a continuous differentiable functional* $V : \mathbb{R} \times C \to \mathbb{R}$ *such that:*

$$u(||\phi(0)||) \leq V(t, \phi) \leq v(||\phi(0)||) \tag{2.108}$$

and:

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|) \tag{2.109}$$

then the zero solution of (2.107) is uniformly stable. If $w(s) > 0$ *for* $s > 0$ *, then it is uniformly asymptotically stable.*

2.7.2.3 Razumikhin Theorem

In the Lyapunov–Krasovskii theorem, the taken Lyapunov–Krasovskii functional requires the state variable $x(t)$ in the interval $[t - d, t]$. Note that, the necessitates in the manipulation of functionals make the application of the Lyapunov–Krasovskii theorem rather difficult. This difficulty may sometimes be circumvented using the Razumikhin-type theorem, an alternative result involving essentially only functions rather than functionals, made it available by Razumikhin.

Theorem 2.61 (Razumikhin Theorem) *Suppose that* $x(0) = 0$ *is an equilibrium of (2.107),* $f : \mathbb{R} \times C([-d, 0], \mathbb{R}^n) \to \mathbb{R}^n$ *and* $u, v, w : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, with* $u(0) = v(0) = 0$*,* v *strictly increasing. If there exists a continuous differentiable functional* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that:*

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|), \text{ for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \tag{2.110}$$

and the derivative of V *along the solution* $x(t)$ *of (2.105) satisfies*

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|), \text{ whenever } V(t + \theta, x(t + \theta)) \leq V(t, x(t)) \tag{2.111}$$

for $\theta \in [-d, 0]$*, then system (2.107) is uniformly stable.*

If, in addition, $w(s) > 0$ *for* $s > 0$*, and there exists a continuous nondecreasing function* $p(s) > s$ *for* $s > 0$ *such that condition (2.111) is strengthened to*

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \text{ if } V(t + \theta, x(t + \theta)) \leq p(V(t, x(t))) \tag{2.112}$$

for $\theta \in [-r, 0]$, *then system (2.107) is uniformly asymptotically stable.*

If, in addition, $lim_{s\to\infty} u(s) = \infty$, *then system (2.105) is globally uniformly asymptotically stable.*

2.7.2.4 Input-to-State Stability

Definition 2.62 The system (2.105) is said to be ISS if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input u, the solution exists for all $t \geq 0$ and furthermore it satisfies

$$|x(t)| \leq \beta(\|\xi_0\|_\infty, t) + \int_0^t \gamma(|u(s)|)ds. \tag{2.113}$$

A Lyapunov–Krasovskii methodology for studying the ISS of nonlinear time-delay systems is presented below.

Theorem 2.63 *If there exist a functional* $V : C([-d, 0]; \mathbb{R}^n) \to \mathbb{R}_+$, *functions* α_1, α_2 *of class* K_∞, *and functions* α_3, ρ *of class* $\mathcal{K}$ *such that:*

$$\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a), \forall \phi \in C([-d, 0]\mathbb{R}^n);$$
$$D^+V(\phi, u) \leq -\alpha_3(\|\phi\|_a), \forall \phi \in C([-d, 0]; \mathbb{R}^n), u \in \mathbb{R}^m : \|\phi\|_a \geq \rho(\|u_{[0,\infty)}\|_\infty);$$

then, the system (2.105) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha \circ \rho$.

2.7.3 Notes

Some of the materials presented in this chapter are not intended to be self-contained. Rather, they are prepared for readers to review the notations or switch to other references for detailed information. For linear algebra and system theory, readers are recommended to [1–6]. For linear matrix inequalities, readers are recommended to [7–9]. For stochastic systems, readers are recommended to [10–18]. For time-delay systems, readers are recommended to [19–27].

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