Chapter 2
Utility and Rationality

2.1 The von Neumann–Morgenstern View on Economic Motivation

As we saw in the previous chapter, the link between income and wellbeing is complicated because they correlate only until the basic human needs are satisfied. Modern economic psychology has discovered that beyond that level, money changes its psychological function and becomes means for participation in social relationships, as well as for managing, and perhaps reformulating, one's personal identity within those relationships (Ahuvia 2008).

For the economist seeking to understand what motivates people in their choices there is too much psychology in that. More appreciated would be a simpler model that accounts for some essential aspects of human motivation and ignores everything else. The methodological ways to do so are endless and naturally, some are more promising than others are. Economics and decision science have a consensus that a very important achievement in that respect is the von Neumann–Morgenstern theory of expected utility (von Neumann and Morgenstern 1944, 1953). Over the decades after its publication, it has been upgraded and refined by thousands of scholars and this activity continues even in the 21st century. At the same time, a growing number of instances emerged that clearly showed how the theory was unable to explain real economic behaviour. Today the limits of its validity are well established. Although it gradually gives way to new theories integrating relevant psychological knowledge, it remains a cornerstone in the development of science. Historic respect as well as necessity leads us to discuss it briefly here.

Characterizing the economic agents, John von Neumann and Oscar Morgenstern took a position that was typical for their time, and in some sense remains widely adopted even nowadays. It was that in a deal, the consumer tries to obtain maximum utility, or pleasure, while the entrepreneur seeks maximum profit. Natural as this might look, however, even at the starting point the first serious difficulty appeared. The two centuries since Bernoulli had not been enough to establish what

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exactly *utility* was, how it should be defined scientifically, what its properties were, would it be possible to measure it, and how. The need for what today we call an operational definition was evident. In addition, it was necessary to merge theoretically the two kinds of economic agents' motivation—that of a consumer and entrepreneur—because there could not possibly be a fundamental difference between them, and therefore, no justification for more than a single construct existed. Consequently, the two scientists proposed the following definition-like formulation,

> […] the aim of all participants in the economic system, consumers as well as entrepreneurs, is money, or equivalently a single monetary commodity. This is supposed to be unrestrictedly divisible and substitutable, freely transferable and identical, even in the quantitative sense, with whatever "satisfaction" or "utility" is desired by each participant. (von Neumann and Morgenstern 1944, 2.1.1.)

Neither philosophers nor psychologists, von Neumann and Morgenstern thought it was enough to say that utility is simply what people aim for, with money serving as more or less an intermediary. Over the distance of many decades, we appreciate how they successfully maneuvered around this important conceptual question, yet we are left with the apprehension that utility must be more than just that. In particular, scholars such as Vorobyov (1970) noted that the "monetary commodity" as defined above, certainly does not possess all the properties of money in the real economy. On the other hand, true money does not possess all the characteristics in the above definition either. However, and that is the most important, the commodity in question was sufficiently like money, to be able to serve in that capacity in the new theory, and be identified with money terminologically.

In addition, we should be aware that von Neumann and Morgenstern developed a utility theory intended to serve as auxiliary to their main goal—a *theory of games*, which they planned to make the most adequate description of economic activity in general. Consequently, they introduced the different elements of their definition with the objective to meet the needs of different types of games in the construction they viewed as more important. This is an example how scientists can be careful and selective when preparing the grounds for an ambitious exploratory undertaking.

The new postulate about money and utility being identical was a useful simplification. In their book, the two scholars discussed various alternative ways for defining utility to make it either more plausible or more suitable for their purposes. (von Neumann and Morgenstern 1944, 66.1.1.–67.4.). Vorobyov (1970) suggested in retrospect that they could have introduced an *axiom* about the existence of utility with the above properties, but did not do so due to the controversial nature of the issue. Such an axiom might have become problematic to handle in practical terms, that is, in the mathematical developments of game theory. In addition, the simplification about the equivalence of utility and money, when understood only as a first approximation, avoids asking questions about the exact relationship between the two.

Considerable attention was paid by the two scholars to the question of utility's assessment. At the time, virtually the only certain thing was that if a person had chosen one alternative over another, then for her the first possessed more utility than the second did. An implicit assumption here was that the decision maker is always able to state which alternative they choose. Any measure of usefulness of the utility concept depended on the possibility to make comparisons, which was an accepted view in economics (Samuelson 1938), and one deeply rooted in the method of indifference curves. One unresolved issue at the time was if utility could indeed be measured.

The next assumption was that people are able to express their preferences not only among goods or services, but also among uncertain alternatives. The latter should be understood exactly as defined by us in the opening chapter—as compounds of outcomes with their probabilities. A justification was offered with the argument that uncertainty is inherent in any business activity: the agricultural output depends crucially on the weather conditions; the financial sector is influenced by political events on ongoing basis, etc. Consequently, a reasonable person should be able to make a preference between alternatives such as A and B defined as follows: A: gain of \$1000 with probability 0.50 and no gain (\$0) with the same probability; B: \$450 for sure. Possibly, she may state that she likes them in equal measure. A problem would occur only if, for some reason, the person is unable to make the choice. Another important assumption was that the person is always able to assess the utility of the risky alternative A on its own, in order to be able to compare it with B, the sure thing.

Therefore, this theory linked utilities not with mere goods or services, but with *probabilistic events* occurring in the future and associated with those goods or services. It has been noted (Vorobyov 1970) that here utility is measured with the help of something quite alien to it, as is the probability distribution. Apparently, this approach was influenced in part by the tradition of statistical physics, used by von Neumann as a methodological reference in the development of the new economic theory.

During the late 1940s and early 1950s, this marriage of utilities with probabilities was met with excitement among many economists and decision theorists, because it implied one important consequence—utility was proclaimed measurable. For a long time some of the leading scientists from different generations had been asking themselves if utility was, in their parlance, “ordinal” or “cardinal”, i.e. was it qualitative or quantitative. An example of the first would be the beauty of the songs in a rock music top twenty chart—we can neither say exactly how much better the first song is than the second, nor if the distance between the first two is bigger than that between the second and the third. As regards cardinal/quantitative variables, an example would be two opera tickets—we know which one is more expensive and exactly by how much. The question if utility could be measured, i.e. is it cardinal, was given a positive answer by the 19th century theorists of marginal utility. The next generation of economists, however, reversed that position, and then von Neumann and Morgenstern introduced yet another revision, this time armed with a new probabilistic definition and a system of axioms. In retrospect, we know that utility’s cardinality and measurability were eventually divorced, and later decades’ economics proclaimed utility not measurable once again.

2.2 Axioms and a Theorem in Expected Utility Theory

The axioms of the theory have provoked much interest and have been interpreted and reformulated many times over the decades since their publication. Here, we keep close to their original form. Our objective is to discuss them alongside the way they were subsequently tested by time, as the theory was extensively used in applications and challenged by some of them. It should be noted that the entire system of axioms described alternatives and utilities as perceived by the single person. No comparison of utilities among different individuals was ever implied.

I begin by introducing the following notation. First, p, q, and r are probabilities (real numbers between 0 and 1). Let U be a system of entities denoting abstract utilities $u_1, u_2, u_3, \ldots$ whose properties are to be clarified. For example, u_1 and u_2 can be the utilities of two mutually exclusive outcomes such as winning \$1000 or winning nothing, as was the content of A above. In U is defined a *relation of preference* ($\succ$) between two utilities, for instance $u_1 \succ u_2$ (that is, u_1 is preferred over u_2. Identical notation is: $u_2 \prec u_1$). In addition, it is assumed that a reasonable person is able to comprehend, and establish for herself a quantitative measure for the total utility of the entire risk-containing alternative. That is equivalent to defining the mathematical operation *combining* of utilities with their respective probabilities: $pu_1 + (1 - p)u_2$, where p and $1 - p$ are the probabilities of the two outcomes. The preference relation and the combining of utilities satisfy the following axioms:

A1. **Complete Ordering**. The relation $u_1 \succ u_2$ is a complete ordering[1] of U. Then

- A1a. For any two u_1 and u_2 one and only one of the following three relations holds: $u_1 = u_2$, $u_1 \succ u_2$, $u_2 \succ u_1$. (Here "=" stands for equal preference, also called indifference.)
- A1b. $u_1 \succ u_2$ and $u_2 \succ u_3$ imply $u_1 \succ u_3$. (Transitivity)

A2. **Ordering and Combining**.

- A2a. $u_1 \prec u_2$ implies that $u_1 \prec pu_1 + (1-p)u_2$ for every $p \in (0, 1)$.
- A2b. $u_1 \succ u_2$ implies that $u_1 \succ pu_1 + (1-p)u_2$ for every $p \in (0, 1)$.
- A2c. $u_1 \prec u_3 \prec u_2$ implies the existence of p, such that $pu_1 + (1-p)u_2 \prec u_3$.
- A2d. $u_1 \succ u_3 \succ u_2$ implies the existence of p, such that $pu_1 + (1-p)u_2 \succ u_3$.

A3. **Algebra of Combining**.

- A3a. $pu_1 + (1-p)u_2 = (1-p)u_2 + pu_1$. (Commutative property)
- A3b. $p(qu_1 + (1-q)u_2) + (1-p)u_2 = ru_1 + (1-r)u_2$ where $r = pq$. (Distributive property)

[1] I omit the mathematical treatment of the concept *complete ordering* and appeal to the educated reader's intuitive understanding, for example, about the set of the real numbers—their ordering is complete.

Let me now briefly comment on these axioms and consider some of their implications. It should be noted that in retrospect, it is easy to find fault with them, especially after decades of mounting empirical evidence. All the same, their formulation was a significant step forward in describing economic activity with scientific rigor. Moreover, at least in decision science, though not in economics, the idea that utility can be measured survived the test of time for much longer than the particular axiomatic system.

A1a states that the individual is always able to form an opinion about two alternatives and it can never happen, that she be unable to compare them. If she is unable to choose one, this means that both are equally attractive ($u_1 = u_2$), and by no means that she felt incompetent, or was objectively incompetent to compare them. This is an important limitation of the theory and obviously cannot always be satisfied in practice.

Transitivity, A1b, looks trivial, however it was violated so much and so often, that it became the Achilles' heel of expected utility theory. Indeed, if one prefers *A* to *B* and *B* to *C*, then one must always choose *A* over *C*. Of course, that is unrealistic in many decision situations. A straight example comes from consumer behaviour: goods and services in everyday life cannot be chosen as strictly, because there exists the need for diversity and the desire for changing pleasurable experiences.

Axiom A2a states that if u_2 is preferred over u_1, then each probabilistic combination of the two, which effectively amounts to at least having the chance to obtain u_2, will always be chosen over the less desired u_1. Axiom A2b is dual to A2a.

Axiom A2c essentially states that u_2 may be much desired, but if the chance for it to happen is small enough, its influence on the choice will be neglected. Axiom A2d is the dual of A2c. With these two axioms, von Neumann and Morgenstern in fact introduced a threshold for the perceptions of the agent, who is ignoring too small probabilities, in that case the chance of obtaining u_2. This is remarkable because it effectively states that the decision maker reacts to abstract concepts such as probability in exactly the same way as to sensory stimuli, for which Weber's Law has established the existence of a just noticeable difference (JND). Now, an unexpected implication of the new axioms is that people may be characterized by one more JND—that for too small probabilities. The two theorists never discussed the issue from that perspective in their book. They simply needed this property for further mathematical purposes but arrived, perhaps by chance, at the outskirts of an important psychology-related idea.

In addition, it must be stressed that there is a conceptual difference, but no contradiction, between the postulated human propensity to neglect too small probabilities, and hence, probabilistic outcomes as per A2c and A2d on the one hand, and the property of utility to be "unrestrictedly divisible" according to the theory's defining statement. Utility may be infinitely small, but we can perceive it in finite chunks only.

Axiom A3a states that it does not matter in which order the decision maker would approach u_1 and u_2. The same is true about the more complex combinations of utilities considered in A3b. Both these axioms have also proven problematic in later studies of actual human behaviour. The two authors had a presentiment about such a possibility, as is evident from their suggestion (von Neumann and Morgenstern 3.7.1) that A3b might contradict future empirical findings of a potentially "*much more refined system of psychology [...] than the one now available for the purposes of economics*". This is yet another example how a pioneering effort can be constrained by the scientific tools of the time.

Using the above axioms, von Neumann and Morgenstern have proven that there exists a mapping of the utilities on the set of real numbers up to a linear transformation. In other words, there exists a mapping between the abstract utility u and the real number v, with the function $v(u)$ having the following properties:

C1. $u_1 \succ u_2$ implies $v(u_1) \succ v(u_2)$.
C2. $v(pu_1 + (1-p)u_2) = pv(u_1) + (1-p)v(u_2)$.
C3. For each two $v_1(u_1)$ and $v_2(u_1)$ holds $v_2(u_1) = a.v_1(u_1) + b$, where a and b are fixed numbers and $a > 0$.

C1 states that the greater of two utilities is ascribed the greater of two numbers, that is, the mapping is monotone. According to C2, the nature of utility is such that the agent is able, loosely speaking, to evaluate the separate utilities of each probabilistic outcome in an alternative, and then combine them to assess the total utility of the alternative. Finally, C3 states that the mapping is determined up to a linear transformation. This means that no absolute zero of the utility is fixed, and no measurement unit is determined. von Neumann and Morgenstern formulated statements C1–C3 as a theorem and provided its detailed proof on about ten pages in an appendix to their book's third edition.

Because we will now accept as a postulate that utility can be numerically represented, for simplicity we abandon the notation $v(u_i)$, which was a shortened version of $v(u(x_i))$, and henceforth will write $u(x_i)$ instead, by which we will understand the utility of x_i. Furthermore, we can perform a natural generalization of C2 to account for not only two, but n possible mutually exclusive outcomes; then the utility of the entire alternative will be denoted as $U(x_1, p_1; \ldots; x_n, p_n)$. This brings us to the definition of utility via Eq. (1.3), which we introduced in Chap. 1 appealing to the reader's intuition instead of resorting to mathematical rigor.

Now we can write down the utility of any alternative, and for example, that of the previously discussed $A \equiv (\$1000, 50\,\%; \$0, 50\,\%)$ in the following way:

$$\begin{aligned} U(A) &= \sum_{i=1}^{n} p_i u(x_i) \\ &= 0.5u(1000) + 0.5u(0). \end{aligned}$$

After all this, we can recapitulate what we have done in the following set of statements. If a person chose A over B, then he/she: (1) was able to compare them; (2) for him or her, the first had more utility than the second; (3) the two utilities can be represented numerically; (4) each alternative's utility can be expressed with Eq. (1.3). Formally, all four statements are summarized in this way:

$$A \succ B \Leftrightarrow U(x_{A1}, p_{A1}; \ldots; x_{An}, p_{An}) > U(x_{B1}, p_{B1}; \ldots; x_{Bm}, p_{Bm}).$$

Here, n and m are the number of mutually exclusive outcomes in alternatives A and B respectively. In other words, we have traveled a long way and now are able almost to *compute* utilities of alternatives and then put them in algebraic inequalities to determine the better choice, much as a person is supposedly doing in her or his mind.

Fishburn (1989) has observed that there is a substantial difference between the approach of Daniel Bernoulli, and that of von Neumann and Morgenstern. The former considered utility as subjective for each individual and put it in a probabilistic context, but his analysis mostly emphasized the case of money received, or about to be received, with full certainty. In contrast, expected utility theory as developed by the latter two, deals with probabilistic alternatives, whereby the certain ones are a mere special case. The following example clarifies the issue. Take this equation:

$$u(\$350) = \frac{1}{2}u(\$1000) + \frac{1}{2}u(\$0).$$

In the von Neumann–Morgenstern interpretation the above means that a person considers equally attractive the sure \$350, on the one hand, and on the other, the risky choice of \$1000, to be gained with probability 0.50, or \$0 with the same probability. Now look at the equivalent algebraic equation:

$$u(\$350) - u(\$0) = u(\$1000) - u(\$350).$$

Its Bernoullian interpretation involves no probabilities but only sure gains. The agent derives utility from the first \$350 that is exactly equal to the utility of the subsequently received \$650, if she already had received the first \$350. Here we used the additional assumption that $u(\$0) = 0$.

2.3 Evolution of the Theory: Allais' Paradox

In the 1940s, many readers of Theory of Games and Economic Behaviour were impressed by the new horizons for exploration this volume outlined. Among other issues, one old question resurfaced: Now that utility was cardinal again, could it be possible to invent a method for measuring it, and then quantify the entire utility function of a decision maker? An important objective was to develop a procedure that would link the agent's revealed preferences—an established concept in economics due to Samuelson (1938)—with the probabilistic alternatives now at the centre of the attention. The matter had huge practical consequences. In particular, some economists were eager to use the new theory to understand why people are willing to buy insurance policies, i.e. to pay to eliminate certain risks, and at the same time engage in activities such as betting on sporting events, buying lottery tickets, and investing in stock shares—all of which amount to gambling, i.e., buying risks.

An article by Friedman and Savage (1948) from the University of Chicago made a pioneering contribution to the understanding of those issues. The two proposed a method eliciting the decision maker's preferences and leading to the construction of his or her utility function. They started with an example such as this: You can have an income of \$500 or \$1000 with probabilities p and $(1 - p)$ respectively. What value of p will make that alternative as attractive as a fixed income of \$600? Alternatively, one might fix the probability, e.g., $p = 0.30$, and then ask what would be the certainty equivalent of the risky alternative, in that case (\$500, 30 %; \$1000, 70 %)? Having obtained the answers to many such questions from a person, the analyst would be able to construct an empirical curve of that person's utility function. Friedman and Savage went further to suggest that having such knowledge about the preferences would make it possible to predict individual reactions in any such future situation involving risk. Remarkably, that idea remains at the core of all methods for eliciting personal utility curves even in the 21st century.

Attempting to explain why people are willing to spend money to avoid risk, and at the same time to acquire some forms of risk, the two scholars hypothesized a certain shape of the utility curve that would account for such behaviours. They suggested that utility as a mathematical function of income must consist of three segments as shown in Fig. 2.1. First, there is an initial region where the function is concave upward, which corresponds to risk avoiding.[2] That can be intuitively understood by recalling the St. Petersburg Paradox (Box 1.1): because of the diminishing marginal utility of money, people would prefer to sell their right to gamble and get a smaller but certain sum instead of a potentially huge but uncertain one. In this vein, a low-income agent would be willing to avoid risk and buy insurance, provided it is not too expensive. The second segment is convex, indicating risk seeking behaviour that in practical terms amounts to paying for the opportunity to gamble. Finally, in the third region, the costs of risky behaviour begin to look too great and gambling is again mostly avoided.

Friedman and Savage even speculated that the middle segment might relate to buying lotteries, offering a chance for a huge gain that would lift the lucky person out of their social class and put them in an upper class. Such behaviour is plausible, they mused, because, "*Men will and do take great risks to distinguish themselves […]*". In retrospect, we know that neither that particular suggestion, nor the

[2]Here I omit the mathematical proof of the equivalence of risk avoiding and concavity of the utility function. Interested readers are referred to Eeckhoudt and Gollier (1995) or Mas-Colell et al. (1995).

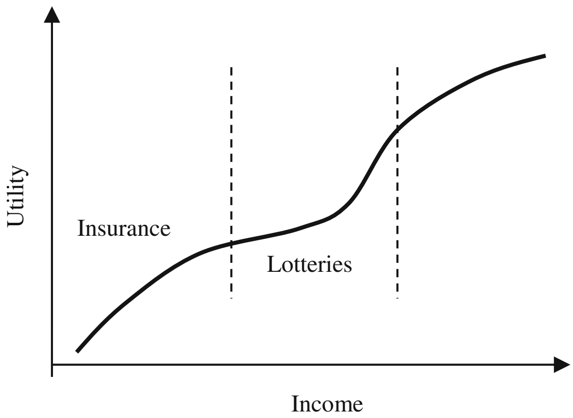


Fig. 2.1 The agent's utility curve according to Friedman and Savage (1948)

three-segment curve survived the test of reality, but this scientific field was just taking off, and it was the time of charming new ideas.

Further, in the middle of the 20th century it looked implausible that people had an apparatus in their brains to calculate risky utilities as a function of their income. In addition, no one could imagine a technology to look into one's brain and verify such a hypothesis. On the other hand, observing someone who always prefers *A* to *B*, *B* to *C* etc. regardless of how many times she has to choose, may lead us to believe that she has some brain mechanism doing utility maximization, which could be perfectly described by some mathematical function.

With such ideas in the air, it was certainly tempting to devise a real experiment aimed at measuring utility. Frederick Mosteller and Philip Nogee of Harvard University did exactly that (Mosteller and Nogee 1951). Their work went down in the history of decision science perhaps less recognized than it deserved, but it certainly contained a number of accomplishments. Probably the most important was the first empirical assessment of a personal utility function, as we understand it today. Two groups of subjects participated in their experiment—Harvard undergraduates and military professionals. Personal as well as between-group differences were hypothesized due to the participants' different social profiles: the students were younger, financially more privileged, and more optimistic about their future income. It turned out that these differences were indeed influencing the utility curves, but no clear profiles could be established.

Secondly, this was one of the earliest experiments ever, in which participants were paid in proportion to the results of their decisions, rather than for the time they spent in the laboratory, as was the norm for psychological experiments. In that sense, this was one of the earliest economic experiments conducted in history. It is interesting that even today economists remain skeptical about the quality of experimental work done by psychologists on the sole ground that money is not used in the proper way to motivate people. Of course, psychologists have their deep reasons to dismiss such an attitude.

Mosteller and Nogee's third achievement was the experimental confirmation of a result already established by other scientists regarding the way people comprehend probabilities. It had recently turned out that when choosing among risky alternatives, the human mind does not use mathematical probabilities straight as given, but "twists" them, that is, changes their values and eventually takes decisions using these altered "psychological", or "subjective" probabilities.

With regard to expected utility theory, very significant was the fourth result of the experiment. The participants violated much too often some of the axioms, in particular A2 (Ordering and Combining) and the requirement for stability of preferences, i.e. the transitivity axiom (A1b). Of course, no one expected totally consistent behaviour when choosing among alternatives. von Neumann and Morgenstern themselves had claimed that their system of axioms was just satisfactory. Friedman and Savage had called the transitivity requirement "an idealization". However, the Mosteller and Nogee data showed that expected utility theory would not be able, in its current form, to accommodate many of the newly established facts about real economic behaviour. In addition, that first experiment offered some support for, but at the same time cast some doubt on the Friedman–Savage three-segment utility curve.

The theory now shaken, it encouraged other scientists to question its validity on other accounts. The French economist Maurice Allais dealt it a substantial blow by showing that people violate the transitivity axiom A1b in a *systematic* way. His discovery remained in history as *Allais' Paradox*, which may be summarized as follows. Let a person have to choose between the following two alternatives:

$$A1 \equiv (\$1\,\text{m}; 100\,\%) \quad \text{and} \quad B1 \equiv \begin{pmatrix} \$1\,\text{m}; & 89\,\%; \\ \$5\,\text{m}; & 10\,\%; \\ \$0; & 1\,\% \end{pmatrix}.$$

Most often, he/she chooses the sure one million of $A1$. However, when the same person has to choose between these alternatives:

$$A2 \equiv \begin{pmatrix} \$1\,\text{m}; & 11\,\% \\ \$0; & 89\,\% \end{pmatrix} \quad \text{and} \quad B2 \equiv \begin{pmatrix} \$5\,\text{m}; & 10\,\% \\ \$0; & 90\,\% \end{pmatrix}$$

$B2$ is almost always the preferred option. However, the alternatives in the second problem are simply two dented versions of those in the first problem: indeed, to obtain $A2$ and $B2$ one must remove the opportunity to win \$1 m with probability 89 % from both $A1$ and $B1$. Taking away the same amount of utility from two competing alternatives should not change the decision maker's preferences between them. Let us use the main theorem (C1 and C2) as well as Eq. (1.3), with the natural assumption that zero gain has zero utility $u(0) = 0$, and also substitute $x_1 = \$1$ m, $x_2 = \$5$ m, $x_3 = \$0$, $p_1 = 89$ %, $p_2 = 10$ %, $p_3 = 1$ %. Then the preference of the majority of people, $A1 \succ B1$, can be expressed as an algebraic inequality:

$$p_1u(x_1) + (1 - p_1)u(x_1) > p_1u(x_1) + p_2u(x_2). \tag{2.1}$$

Similarly, from $A2 \prec B2$ we obtain:

$$(1 - p_1)u(x_1) < p_2u(x_2). \tag{2.2}$$

Since it is not possible to reverse the sign of Eq. (2.1) by subtracting from both sides the positive quantity $p_1u(x_1)$, it follows that Eqs. (2.1) and (2.2) cannot be simultaneously true. This implies that in the second problem people must prefer $A2$ to $B2$. In reality, an overwhelming majority chooses the opposite, and there lies the paradox.

Allais has demonstrated another kind of preference reversal, presented here with a slightly different set of alternatives than the original. When choosing between

$$A3 \equiv (\$3000; 100\%) \quad \text{and} \quad B3 \equiv \begin{pmatrix} \$4000; & 80\% \\ \$0; & 20\% \end{pmatrix},$$

people usually opt for $A3$. A fourfold reduction of the odds, in the first alternative from 100 % to 25 %, and in the second from 80 % to 20 %, would lead to the following new alternatives: $A4 \equiv$ (\$3000; 25 %) and $B4 \equiv$ (\$4000; 20 %). Instead of retaining their preferences, people again reverse them and en masse opt for $B4$.

Allais compiled a list of such problems and disseminated it among his students and colleagues. The majority of them fell in the traps and, by and large, violated the von Neumann–Morgenstern axioms. During a lunch at a 1952 scientific conference in Paris, Allais offered the problems to Leonard Savage himself, who also gave contradictory answers, much to his own amazement. After the initial shock, he decided to change some preferences to stay consistent with his own theory, rather than fall prey to Allais' paradox. Apparently problematic as a description of actual economic behaviour, expected utility theory, he believed, was still adequate as a normative theory (Shafer 1984).

2.4 New Issues

Since the discovery of Allais' paradox, the theory was destined to develop amid clashes of its proponents and its critics which continue to this day, although with diminished intensity. The debate was still in full force when in 1979 two important publications attracted attention. The first (Allais and Hagen 1979) was a collection of articles comprising a kind of recapitulation of the achievements and accumulated issues over the preceding three decades. In it, Allais gave an exposition of his own theory of decision-making under risk, while Morgenstern defended expected utility theory claiming that it was adequate descriptively, had a limited domain of applicability where it was accurate, and in that sense could even be compared to Newtonian mechanics. That volume contained also some empirical research by other contributors hinting at the psychological invasion that was to happen in the domain of decision science.

The second publication appeared in the leading journal *Econometrica* and presented *Prospect theory: an analysis of decision under risk* by psychologists Daniel Kahneman and Amos Tversky. By 2002, the year when Kahneman was awarded the Nobel Prize, this was the most cited article in that journal. It literally overhauled the scientific understanding of the way people take decisions, and remains highly influential to this day. I leave its detailed discussion to Chap. 3, and now continue with the developments around expected utility theory.

Its proponents adopted two different strategies to defend it, not counting the third one—of simply ignoring all criticisms. The first strategy was to formulate new axioms weakening the theory's strict requirements and reconnecting it with the observed economic behaviour. The second strategy was inspired by Savage's reaction to Allais' paradox (he revised his preferences to remain faithful to his principles). Born spontaneously, this tactic was later recognized as a potentially powerful method to achieve optimality in decision-making. This is how its followers reason: Indeed, one cannot always analyze the options available with utmost precision, and does not always make the best choice. However, such inefficiencies are caused by occasional inattentiveness, are accidental, and can be corrected. To this end, a more careful second thought would virtually always be sufficient.

Adding to these arguments from a recent perspective, we can say that in our time the agent can receive assistance by a computer-based decision support system, a consulting expert system, or something more modern in the same vein. Generally, the resulting procedure would involve reexamining the answers a person gave and finding out the contradictions among them. Then the latter—presumably a small number—will be revised to achieve a kind of "global consistency".

Of course, this will not be relevant to the thousands of minor decisions of little consequence we take on daily basis. Some of them can contradict our general pattern of activity but we may never become aware of that fact. Fortunately, almost all of our actions will be satisfactory, and even close enough to optimality due to inbuilt cognitive mechanisms.

The use of computer-aided consulting, however, requires a substantial effort in terms of time and financial means. Doing it makes sense only for important problems that are difficult to solve. Note that this is not the case when we seek professional advice in areas outside of our expertise—for example, when we need medical help, or assistance with new technology or software. Rather, the situation is much as we are the expert, but need to take into account a large variety of circumstances and have to use a computer model to help us. We can clarify this issue by adopting for a moment the point of view of the developers of such expert systems. For instance, an experienced physician might be asked to provide assessment on a multitude of medical cases because his knowledge will be put in a software to train students. However, the algorithm has discovered some inconsistencies among his answers and he has to reexamine them and introduce the needed corrections. After all, the value of the system will be in providing help with the really intricate cases.

This approach is most convincing when decision analysis tries to help solve important societal problems. When the optimal solution depends on a large number of criteria, the superiority of one alternative over the remaining is not obvious and may often be counterintuitive. In such cases it is said that decisions are not taken, but are computed. Pioneering research in that area was conducted in the 1960s at Harvard and MIT, with its high point being a book by Keeney and Raiffa (1976). Its topic was how to take decisions with multiple conflicting objectives especially when the latter are incommensurable. This may involve defining axiomatic systems to guide preferences among complex risk-containing alternatives. Then the researcher has to do theoretical work to find suitable special cases of otherwise intractable problems involving multidimensional utility functions. Finally, methods and procedures must be developed (Fishburn 1977) to extract those utility functions from the social values of the decision makers. The field was hot in the late 1970s as another book of the same school (Bell et al. 1977) summarized a number of subsequent theoretical contributions and practical cases. At the time, an already impressive list of applications existed. Such methods had been used in decisions involving: the construction of Mexico City Airport; selection of sites for nuclear reactors in the State of Washington; urban planning for the city of Darmstadt, Germany; resolving ecological issues around Japanese power plants; forestry management in New Brunswick, Canada; and many others. All these cases involved direct applications of expected utility theory and its extensions, and to this day serve as primary examples for rationally taken important decisions.

Now let us come back to the other strategy adopted by some theorists: weakening the original axioms to accommodate the most prominent violations of the theory. A huge step was made when the mathematical probabilities were replaced by "subjective" or "psychological" probabilities in the decision models. That idea was not new. As early as 1926, the Cambridge mathematician Frank Ramsey invented the concept of subjective probability, formulated axioms about it, and even suggested a way to measure it. Unfortunately, Ramsey died at the age of 26 and his ideas went into obscurity, even escaping the attention of von Neumann and Morgenstern (Fishburn 1989).

However, towards the end of the 1940s a lot of empirical evidence showed that people indeed distort numerical probabilities when thinking about them. Perhaps the first to discover this experimentally were Preston and Baratta (1948), who in 1948 observed and documented how gamblers in a lab game treated a mathematical probability of 0.75 as if it were equal to 0.61. Discrepancies existed also for other values, with people estimating objective probability reasonably adequately only around $p = 0.20$. In another study, Griffith investigated data from horseracing betting and established a corresponding point of equivalence at around 0.16. The Mosteller–Nogee experiment did not confirm any of these findings, but suggested an interval of 0.10–0.55 in which such an equivalence point might exist. All this was enough for Allais (1953) to use it as another argument against expected utility theory.

Naturally, the issue came across the attention of psychologists. One of them, Ward Edwards, the scientific mentor of Amos Tversky, discovered that in betting people prefer to deal with some probability values more often than with others. For example, they consistently liked bets with probability 0.50 but tended to avoid those with 0.75. Those preferences were reversed when losses were involved. Edwards suggested that not probability, but some extension of it, a kind of *weighted probability* should be incorporated in the analytic models.

Savage (1954) formulated a *subjective expected utility theory* for decision-making under uncertainty, in which he developed a system of axioms and explicitly used subjective probability. In the 1950s and 1960s, a new wave of mathematical models considered subjective probability $\pi(p)$ as a function of the mathematical probability p, whereby the empirical curve was built using a procedure eliciting a person's preferences. In a 1989 survey, Machina (1989) put together and compared at least five such models and their variations.

How objective and subjective probability are related, is clarified in sufficient measure in Kahneman and Tversky's prospect theory (1979, 1992). They needed that particular knowledge to define the general utility of an alternative (or prospect) with a new and more complex model, which was this:

$$U(x_1, p_1; \ldots; x_n, p_n) = \pi(p_1)u(x_1) + \ldots + \pi(p_n)u(x_n). \tag{2.3}$$

Comparing Eq. (2.3) with Eqs. (1.2) and (1.3) shows how the former introduces a further weakening of the mathematical expectation principle. What were once straight monetary gains and ordinary probabilities, were later diluted with Bernoulli's subjective utility of the gains, to be further complicated with subjective probabilities. This is a classical scientific approach—where a simple model cannot explain a phenomenon, a more sophisticated one is invented to do the job. As we will see, the innovation with introducing $\pi(p)$ helped resolve Allais' paradox, but opened the door for new paradoxes.

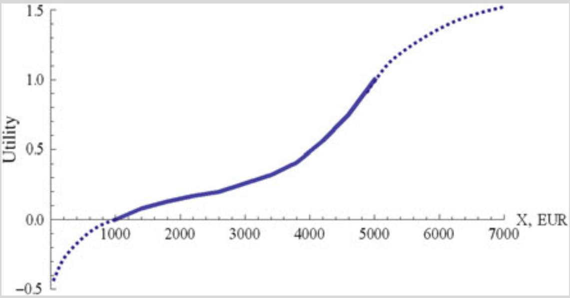
2.5 Empirical Assessment of Utility

For many decades, scientists have had a rough time trying to understand human choices not least because of the inherent problems with the idea for utility maximization. The fact remains, that we still do not have a sufficiently good understanding of what utility u is. It certainly helped that von Neumann and Morgenstern simplified matters by amalgamating money (the "monetary commodity") with utility, and in this way achieved a "satisfactory" and maybe even impressive account of many forms of economic behaviour. In fact, when the economic agent aims at receiving a substance both equivalent to money and bringing satisfaction, this is a very good combination of a theoretical and operational definition. Exactly that was what opened the way for experimental work and further advancement of not one, but two theories—of games, and of expected utility.

Today, however, a lot of new knowledge about humans as decision makers has accumulated and the vagueness of the utility construct is no longer acceptable. For pragmatic reasons, many scientists have adapted to its being measurable up to a linear transformation only. This circumvents the assessment problem and opens the door for the revealed preferences approach, avoiding the issue of utility's content. Indeed, the Friedman–Savage idea to interview a person about their preferences among probabilistic alternatives can be very constructive. A cleverly selected set of monetary gains, losses, and their probabilities can do a lot to uncover interesting facts about a person's utility function and even something about the motivation behind it.

In particular, changes in the curve's steepness may be quite illuminating. If it is natural to associate the curve's concavity with diminished marginal utility of money, and with risk-avoiding behaviour, then how a local curve convexity must be comprehended? In 18th century, Bernoulli himself gave an example with somebody jailed for unpaid debts. He suggested that a wealthy prisoner who needed two thousand ducats in addition to what he possessed to repurchase his freedom would derive more utility from that sum than somebody much poorer would. Under normal circumstances it will be, of course, the other way round—that money would bring more utility to the poor person. Graphically, this effect is a locally convex segment of the utility curve when approaching 2000 ducats, followed by the normal concave continuation.

Box 2.1 Individual utility curve with a local convex anomaly
Today we possess methods to obtain individual utility curves. In 2005, my colleague Yuri Pavlov from the Bulgarian Academy of Sciences and I conducted a computer-based experiment to characterize the utility curve of a German Professor of Economics. We used an iterative stochastic algorithm, developed by Dr. Pavlov (1989, 2005).



The experiment took place after the professor lectured undergraduates on the concepts of utility, marginal utility, and the First Law of Gossen. Then he engaged in our computer-based procedure and soon the result was available. Unexpectedly, the curve that emerged was quite different from what the theory would predict. Indeed, in the segment €1000–2500 it was concave (see the figure), but immediately afterwards it rose steeply and became convex up to €5000. Possibly, it could have continued that way, had we not stopped the procedure, considering inappropriate to subject the professor to further experimenting. Neither did we ask about the potential acquisition he had in mind. Economists, who saw that curve, were puzzled by its anomalous form and suggested that we draw two hypothetical concave segments with dotted lines around it, to make it look more in line with the established views.

The fictitious units on the y-axis of the figure in Box 2.1 remind us how problematic the measurement of utility is without a convincing operational definition. An influential study by Alchian (1953) produced the puzzling result that, proclaiming utility *measurable* did not help the economic analysis at all and was, moreover, methodologically unjustified. His main objection was that there existed no method for assessing the utility of a good in a market basket in conjunction with the utilities of the remaining goods in the basket. It was impossible to sum them up to obtain the total basket utility. (Will eating more chocolates, or meatballs, or both, bring to somebody more utility?) In addition, no prediction of somebody's future choice based on assessment of the alternatives' utilities was feasible.

The solution that economists found to this long-standing issue was to divorce cardinality from measurability. Alchian (1953) was quite clear about the latter, but it would have been too detrimental to discard the von Neumann–Morgenstern approach altogether, not least because of the benefit of the mathematical tools that came along with it. Gradually and quietly, economics decided that,

> Cardinality of a utility function is simply the mathematical property of uniqueness up to a linear transformation. […] cardinality and measurability […] were different concepts, and the former in no way implied the latter. (Lewin 1996)

In other words, measuring utility has no meaning, but putting it in mathematical models is useful. The kind that was actually suitable was given a specific name: *vNM utility* (von Neumann–Morgenstern utility). A neutral observer may find this slightly odd, but has to remember that progress must be made in one way or another, and in the social sciences such developments seem to be the norm rather than the exception. After all, as regards economics, Milton Friedman had concluded that a theory should be judged for its predictive capability rather than for the realism of its assumptions. From a 21st century perspective, it would appear that expected utility theory has reached its maximum potential, given the problematic operational definition of its fundamental concept *utility*.

2.6 Arrow–Pratt Formula for the Price of Risk

One of the most important achievements of utility theory was the quantification of the price a person is willing to pay to buy certain amount of risk, or alternatively, to get rid of a risk. In practical terms, such behaviour happens always when somebody indulges in the pleasure of gambling, or prefers to stay too much on the safe side.

Let a financier's job be to choose many times every day between $A \equiv$ (€1000, 50 %; €0, 50 %) and B: a sure gain of €450. If he consistently chooses A, at the end of a long period, say a year, he will have from each deal an average gain of around €500. Should he play it safe, he will have in each case €450. The question is more complicated if the decision maker faces the choice not on daily basis, but just once—then he might find it quite hard to decide. In both cases, however, we say that if he chooses A, he is probably *risk-neutral* while choosing B defines him as *risk-averse*, or a *riskophobe*. Anybody who practices some form of gambling, which technically means to pay more than the mathematical expectation of the risk-containing alternative, are termed *risk seeking*, or *riskophile*.

The price we are willing to pay to acquire a risk, or to get rid of a risk, is called *risk premium*. Independently of one another, in 1964, John Pratt (1964) and Kenneth Arrow (1965) came at almost identical results about the analytical definition of this quantity. Here we derive the Arrow–Pratt formula, following in part Eeckhoudt and Gollier (1995).

We use the following notation. Let W_0 be the initial wealth of a person who has to make a choice about a risk-containing alternative A, which could be the one above, but could be any alternative of the kind $A \equiv (x_1, p_1; \ldots; x_n, p_n)$. In fact, A can be not only a discrete variable, but a continuous one with a probability density function $f(x)$. Let it be defined in the interval $x \in [a, b]$ where a and b are the minimum and maximum possible gains (or losses). With no loss of generality, but for better intuitive understanding, we assume that the mathematical expectation of the alternative is positive.

Suppose that a person's wealth consists of two parts: one certain, W_0, as described already; and one risk containing, A, which may be shares, traded in a stock exchange. Then her total (final) wealth W_f is:

$$W_f = W_0 + A.$$

The mathematical expectation of A in the discrete case will be:

$$\mu \equiv E(A) = \sum_{i=1}^{n} p_i x_i, \tag{2.4}$$

while in the continuous case, given the domain $[a, b]$ in which it is defined, it is:

$$\mu \equiv E(A) = \int_a^b x f(x) dx. \tag{2.5}$$

Then the mathematical expectation of the total wealth W_f will be

$$E(W_f) = E(W_0 + A) = W_0 + E(A).$$

The person derives from that wealth W_f utility U, which can be described in the discrete case as

$$U(W_f) = \sum_{i=1}^{n} p_i u(W_0 + x_i)$$

and in the continuous case, as

$$U(W_f) = \int_a^b u(W_0 + x) f(x) dx. \tag{2.6}$$

For mathematical correctness, here we adopt the following notation. As usual, U denotes the utility of the entire risk-containing alternative; u is the utility of a single outcome, or the utility of the entire alternative when used in the integrand of equations with U in the *lhs*. The latter case refers only to Eqs. (2.6), (2.7), and (2.10).

Now suppose that the person is uncomfortable about the risk contained in the stock shares and would like to sell them. Naturally, after the transaction she must derive the same utility from her total wealth as before it. Let us denote this risk-free equivalent of her wealth as W^*. Because the total utility must not change, it is necessary that $u(W^*) = U(W_f)$. We substitute this in Eq. (2.6) and obtain:

$$U(W^*) = \int_a^b u(W_0 + x) f(x) dx. \tag{2.7}$$

The person will exchange the risk-containing wealth ($W_0 + A$) for the risk-free wealth W^*at a fair price. From her point of view, the price for A can be defined as:

$$P_a = W^* - W_0. \tag{2.8}$$

In fact, the "fair price" P_a fixes the minimum amount of money the person would be willing to accept for A to make the deal. A theorem (omitted here) states that if the person is risk neutral, her asking price P_a will be equal to the mathematical expectation of the risk-containing alternative A. Therefore, for her holds $P_a = E(A)$. If the person is a riskophobe (which she is in the example), or a riskophile, $P_a \neq E(A)$ will hold.

Now we are ready to define *risk premium* η with the following equation:

$$\eta = \mu - P_a, \tag{2.9}$$

where μ is defined by Eqs. (2.4) and (2.5). Apparently, the risk premium shows how much the person is willing to give out just to get rid of the risky stock. In our illustrative case $\eta > 0$, which means that $P_a < \mu$ or in other words, the person will accept a sum of money smaller than μ. To reiterate, because the person is a riskophobe, she is willing to get rid of the risky stock A and to this end will accept a fair price P_a, which is *less* than the objective mathematical expectation of A. Hypothetically, the person could be neutral to the risk and choose to ask for price exactly equal to the mathematical expectation, rendering the risk premium equal to zero. However, we are interested in quantifying η in the general case. To this end, we rewrite Eq. (2.8) as

$$W^* = W_0 + P_a$$

and with its *rhs* we substitute W^* in the utility function's argument in the *lhs* of Eq. (2.7). We obtain

$$U(W_0 + P_a) = \int_a^b u(W_0 + x)f(x)dx. \tag{2.10}$$

In general, regardless of her being risk neutral, riskophile, or riskophobe, if she received an offer below her P_a, she would reject it, lest her total utility diminished. Our objective is to determine that price P_a from Eq. (2.10). We use the fact that each mathematical function of certain properties (being n-times continuously differentiable etc.) can be approximated with a Taylor series. We recall that $\mu \equiv E(A)$ and expand the *lhs* of Eq. (2.10) around the point $(W_0 + \mu)$. Note that this particular point is in a sense the most objective assessment of the person's total wealth. A first-order approximation will give:

$$\begin{aligned} U(W_0 + P_a) &\approx U(W_0 + \mu) + (W_0 + P_a - (W_0 + \mu))U'(W_0 + \mu) \\ &\approx U(W_0 + \mu) + (P_a - \mu)U'(W_0 + \mu). \end{aligned} \tag{2.11}$$

Similarly, we expand $u(W_0 + x)$ from the integrand in Eq. (2.10), now reverting to its original notation $U(W_0 + x)$, around the point $(W_0 + \mu)$. Because $(W_0 + A)$ can vary around $(W_0 + \mu)$ a lot more than $(W_0 + P_a)$ can, now a higher-order approximation will be needed. That is why we will take the first three terms in the series:

$$U(W_0 + x) \approx U(W_0 + \mu) + (x - \mu)U'(W_0 + \mu) + \frac{(x - \mu)^2}{2!}U''(W_0 + \mu). \tag{2.12}$$

We have to substitute the integrand of Eq. (2.10) with the *rhs* expression of Eq. (2.12). In doing so we keep in mind that because of the definition of μ by Eq. (2.5), the following holds:

$$\int_a^b (x - \mu)f(x)dx = 0,$$

and also, we use the variance definition

$$\int_a^b (x-\mu)^2 f(x)dx = \sigma^2.$$

After the necessary transformations, for the *rhs* of Eq. (2.10) we obtain:

$$U(W_0+\mu) + \frac{\sigma^2}{2} U''(W_0+\mu). \tag{2.13}$$

Now, we substitute the *lhs* and *rhs* of Eq. (2.10) with the expressions from Eqs. (2.11) and (2.13) respectively, to obtain, after short transformations:

$$P_a - \mu \approx \frac{\sigma^2}{2} \frac{U''(W_0+\mu)}{U'(W_0+\mu)}. \tag{2.14}$$

Remembering the defining Eq. (2.9) for risk premium, and using Eq. (2.14), we finally obtain

$$\eta \approx \sigma^2 \left[-\frac{1}{2} \frac{U''(W_0+\mu)}{U'(W_0+\mu)} \right]. \tag{2.15}$$

Equation (2.15) is the Arrow–Pratt approximate formula for the risk premium. Let us discuss it briefly. First, the risk premium is proportional to σ^2, which may be considered as a measure of financial uncertainty, although a crude and rudimentary one. The proportionality means that the greater the risk, the higher the premium a risk-averse person will be willing to pay to get rid of it. In the same way, a riskophile will be willing to pay a *higher* premium for a *greater* risk, attracted either by the greater potential gain or simply by the prospect for more pleasure in gambling.

In Eqs. (2.14) and (2.15), the quantity

$$-U''(W_0+\mu)/U'(W_0+\mu)$$

is called *degree of absolute risk aversion*. It was introduced and initially studied in 1961 by Robert Schlaifer from Harvard University (Pratt 1995). It depends on $U(W_0)$ and is therefore unique for every person. Two people possessing the same wealth W_0 and finding themselves in the same risk-containing situation, will take different decisions (will pay different risk premium) because of their different utility functions. That quantity is also a *local* measure of risk aversion in the sense that it in no way characterizes the entire utility function.

Further, because Eq. (2.15) was derived using approximations it is applicable to cases where the outcomes in the risk-containing alternative are significantly smaller than the person's total wealth. The greater the risk, the greater the formula's error, with one exception, which is the risk neutral individual for whom the formula is always accurate and gives risk premium equal to zero.

Example Let us illustrate the use of the Arrow–Pratt formula with a numerical example. Imagine that the financier from the beginning of this section is considering alternative A ≡ (€1000, 50 %; €0, 50 %). Let his utility function in the interval [€0, €1,000,000] be given by the formula:

$$U(W_0) = W_0 + a\sin(W_0/a), \tag{2.16}$$

where a is a real positive constant. For $a = 10^5$, the financier's utility curve is shown in Fig. 2.2. Obviously, that kind of shape can be defined in infinitely many ways different from Eq. (2.16). Here we have a utility function with total wealth as argument, unlike the Friedman–Savage idea in Fig. 2.1, which plotted utility vs. income. I avoid the economics debate regarding which—income or wealth—should be the argument in the utility function, because it is irrelevant to the example with the particular alternative A. Psychologically, the difference for the decision maker cannot be very significant due to the very small amounts of money in A.

Now, if the financier has wealth W_0 amounting to €100,000 as shown in Fig. 2.2 by the first dotted line to the left, his utility function is concave there, and so he is expected to be risk averter ready to pay a premium to get rid of alternative A. Around €300,000 he may be even more inclined to avoid the risk (the dotted line in the middle), while for W_0 = €500,000 some appetite for risk may develop, as the convex curve intersected by the right dotted line suggests. In that case he may be willing to pay a premium when buying A.

Let us check these three suggestions applying the Arrow–Pratt formula. First, we have to calculate μ and σ^2 for the prospect A. We use Eq. (2.4) to obtain μ = 500, which then goes in $\sigma^2 = \sum_i (x_i - \mu)^2 p_i$ to compute σ^2 = 250, 000. Then we differentiate twice the utility function from Eq. (2.16):

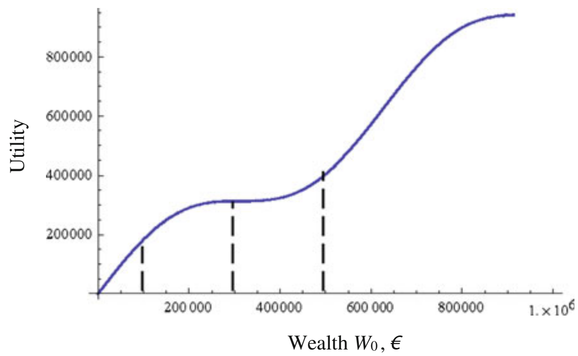


Fig. 2.2 A hypothetical utility curve of a financier

$$U'(W_0) = 1 + \cos(W_0/a),$$
$$U''(W_0) = -(1/a)\sin(W_0/a).$$

With these derivatives we substitute in Eq. (2.15) the respective quantities. We compute the risk premium values for the particular a, μ, σ^2 and the three discussed cases of W_0. The results are shown in Table 2.1.

The figures in Table 2.1 can be interpreted as follows. If the financier had a sure wealth of €100,000 and in addition, possessed A, he would have been ready to sell it for €499.31 (because $\mu = 500$). Were he three times richer, he would have been even more risk averse, willing to sell it for mere €481.73. However, with a personal wealth of half a million, but no possession of A, he would have offered €500.93 to buy it. He is a bit of a gambler now.

The Arrow–Pratt approach became widely used in a number of areas of economic theory. Only about a decade after it was published, it was already present in studies of demand in insurance and asset markets, taxation models, savings models (Ross 1981) and many other domains. Where it was found problematic, it was revised and extended, and also used as a reference point for similar approaches. There have been attempts even to estimate empirically the Arrow–Pratt coefficient of absolute risk aversion for various kinds of economic agents, including agricultural producers in some States in the US (Wilson and Eidman 1983). Hypotheses about the relative risk aversion (equal to the absolute, multiplied by the total wealth) of the representative investor in the US securities market have been empirically tested (Dieffenbach 1975).

Let us ask the question to what extent this type of analysis reflects people's actual behaviour. Even the most assertive economic applications have left the door open, admitting that empirical proofs have never been wholly satisfactory. As usual, psychologists had something to contribute to the understanding of the issue. It might be appropriate to discuss their point of view first by looking at the mere labels "riskophile", "riskophobe", "risk-neutral individual". Implicitly, they all suggest that people have a stable feature with regard to risk bearing, which is expressed somehow consistently in all situations. This personal characteristic ought to resemble the established in psychology Big Five personality traits: Openness, Conscientiousness, Extraversion, Agreeableness, and Neuroticism—they all comprise a statistically discovered robust model of human personality. While all of them change over a person's lifespan, they tend to be very stable during adulthood. Another, yet simpler comparison will be with the person's color of the eyes which is virtually unchanged throughout one's entire life.

Apparently, risk attitude has not proven to be such a stable construct. As we already saw with Allais' paradox, people's choices in risk-containing situations are far from consistent. Moreover, the same individual usually shows different degrees of

Table 2.1 Risk premium η for prospect A at three different levels of wealth

W_0	€100,000	€300,000	€500,000
η	€0.69	€18.27	–€0.93

risky behaviour in various aspects of their life—personal, professional, social etc. Therefore, trying to anticipate reactions to risk, one has to use measurement tools with tasks as similar as possible to the actual cases of interest. A number of studies have shown how specific and intricate that can be. For example, Weber et al. (2002) showed that real-world investment decisions were predicted well by psychometric instruments involving also investment decisions, but were predicted poorly when gambling tasks were used, although both experimental treatments involved monetary outcomes. In short, people's mindset regarding risk is highly context-dependent.

2.7 Experienced Utility and Future Utility

A further example of that context sensitivity is the distinction between two types of utility. A postulate in the von Neumann–Morgenstern theory was that the decision maker chooses among probabilistic alternatives with potential gains and losses, and expects the outcome in the near future (von Neumann and Morgenstern 1944). Quite a useful simplification, undoubtedly. It turned out, however, that people assess in entirely different ways the usefulness of past decisions as opposed to imminent prospects. Two very different attitudes play a role here. Past decisions have led to certain experiences, associated with pleasure, pain, and emotions. This is experienced utility. In contrast, present choices are governed by anticipation regarding future gains. Expected utility theory implicitly considers both types of utility being identical. However, this might be the case only for somebody with the gift of predicting precisely what pleasure, benefit or another type of advantage will the chosen prospect bring. But does such an ideal hedonist exist? The von Neumann–Morgenstern theory assumes that the answer is yes, although the two scholars never discussed it explicitly and there is no reason to believe that they ever gave it a thought. It looks more as we have here what Poincaré called a *hidden axiom*, one that is operating without the theoreticians being aware of its existence.

The problem with the ideal hedonist was noticed by James March (1978) and later elaborated by Tversky and Kahneman (1981) and Kahneman and Tversky (1984). Common sense tells us that such a character is impossible. A popular and vivid example (Kahneman and Tversky 1984) is the hungry man who has ordered five meals, only to discover how wrong he had been by the time the fifth one is served. It is indeed very hard to predict what exactly a choice will bring to us. The problem is even more complicated due to the importance of the context and wording used to describe the needs and the options available to satisfy them. Tversky and Kahneman (1981) suggested a pragmatic advice to deal with such situations: instead of asking "What do I want now?", it would be more useful to adopt a prognostic orientation and ask "What will I feel then?" Hopefully, with more experience the answers to the second question will tend to become more accurate.

In fact, as early as the 17th century, Blaise Pascal noticed the existence of the two types of utility in a rudimentary form as he observed how gamblers derive pleasure from the game independent of their actual gains. Sometimes other examples in this vein are given, like the mountain climbers who relish a dangerous experience bringing no apparent benefit. From a narrow-minded and formal point of view they behave irrationally because they prefer the 0.99 probability of surviving in the high mountain to the sure surviving by staying home. In a similar position is an employee who leaves a large corporation to become an entrepreneur. Obviously, the rationality argument is weak here because it fails to capture essential aspects of human motivation.

A formal treatment of the two types of utility was suggested in early 21st century (Frey et al. 2004; Menestrel 2001). Le Menestrel (2001) introduced axioms accounting for both the utility of the future benefit as well as that of the actual process leading to it. From his standpoint, rational is the person who is optimizing not only the result, but also the experience.

A variation of experienced utility was studied by Benz (2007) who introduced the term *procedural utility* to denote the quality of the encounters one has with the institutions in one's capacity as citizen, customer, employee, tax-payer, plaintiff etc. Court decisions were protested less often, he noticed, when the court procedures were perceived as fair, impartial, and respecting the arguments of the two sides.

2.8 Rationality Principles and Their Violation

The attempt at describing human choice by the mathematical expectation principle was found problematic in the case of the St. Petersburg game, which gave rise to a paradox. Daniel Bernoulli introduced expected utility and thus resolved the issue, at the same time making an important step in quantifying a feature of the subjective human nature. The theory's further development in the 20th century caused great interest that helped refine it, but at the same time drew the borders beyond which it was shown to be inaccurate. Allais' paradox served as the cornerstone there. The majority of mainstream economists disregarded this finding for about a quarter of a century, until in 1979 prospect theory provided a convincing explanation to it. Meanwhile, expected utility theory continued to develop in several directions, among them the introduction of subjective probability by Savage.

However, new paradoxes were discovered. In 1961, Ellsberg (1961) came across another violation of the tenets of the theory, this time of its subjective variant introduced by Savage. (Some historians have claimed that Ellsberg in fact revived an idea initially proposed by John Maynard Keynes about four decades earlier.) In essence, subjective expected utility theory considered the subjective probabilities as additive, and now that was found to be contradicting the actual behaviour of many people. To be more precise, for incompatible events A and B, i.e. $A \cap B = \varnothing$, the following equality about classical probabilities is true:

$$p(A \cup B) = p(A) + p(B).$$

It turned out from Ellsberg's work, however, that for subjective probabilities such an equality does not hold, and

$$\pi(A \cup B) \neq \pi(A) + \pi(B). \tag{2.17}$$

In particular, Ellsberg devised an experiment with coloured balls drawn from urns, and asked his subjects about their choices. For concrete events A, B, and C people en masse reacted in ways that can be mathematically described with these inequalities:

$$\pi(A) > \pi(B) \tag{2.18}$$

and

$$\pi(A \cup C) < \pi(B \cup C). \tag{2.19}$$

If Eq. (2.17) were equality, then Eq. (2.19) would imply $\pi(A) < \pi(B)$. That, however, cannot be simultaneously true with Eq. (2.18). It follows, therefore, that the probabilities people employ in their decisions are not additive. Subsequently, other theorists have introduced modifications overcoming the problem and resolving the paradox.

At present, it is hard to say how many theories explain which paradoxes regarding economic choice. Perhaps the best source for orientation is a set of publications by Birnbaum (1999, 2004, 2008). He outlined a number of most established and simple postulates, and studied how the influential decision theories relate to them. Let us have a look at his classification.

1. *Transitivity*. This principle coincides with Axiom A1b from the von Neumann–Morgenstern axiomatic system.
2. *Coalescing*. This means that prospects of the type (A, p_1; A, p_2; B, p_3) are unconsciously simplified to become (A, $p_1 + p_2$; B, p_3). For example, a businessman is facing the alternative ($50,000:50 %; $50,000:10 %; $0:40 %). This may be a competition for $50,000 offered by a small municipality to do some construction works. The contract can be won by a number of competitors, each with different probability for success. In that case, the businessperson can submit two separate offers via two different firms, both owned by him. Intuitively, he might reformulate for himself the situation and arrive at the following representation: ($50,000:60 %; $0:40 %). Now he will assess his chances using that particular form.
3. *Branch Independence*. If two alternatives contain a common element, it is disregarded when they are compared. For example, if a choice must be made between $(A, p_1; B, p_2; C, 1 - p_1 - p_2)$ and $(A, p_1; D, q_1; E, 1 - p_1 - q_1)$, they are simplified to become $(B, p_2; C, 1 - p_1 - p_2)$ and (D, q_1; E, $1 - p_1 - q_1$).

According to Birnbaum, these three principles underlie all others, including the stochastic dominance, on which I will comment shortly. Discussing Allais' paradox, Birnbaum talked about which theories explain it by retaining or violating the above postulates:

> One class of theories (including subjectively weighted utility theory and original prospect theory) retains branch independence but violates coalescing, and thereby violates stochastic dominance. Another class of theories (rank-dependent and rank- and sign-dependent utility theories including cumulative prospect theory) retains coalescing and stochastic dominance but violates branch independence. (Birnbaum 1999)

This quote from 1999 illustrates well the state of fragmentation in which decision analysis found itself at the time. Decades later, new theories, postulates, and conjectures have explained old paradoxes and have stumbled into new problems. However common it may be for mathematicians to introduce variations in their axiomatic systems to examine the implications, that approach may not be entirely useful for studying human behaviour. This can be understood for example, by analogy with the engineering sciences. What would happen if in machine building, architecture, or civil engineering the axioms of Euclid be reshuffled in the way Birnbaum describes above, just to suit the needs of any particular task?

2.8.1 Rational Behaviour in Context

In 2006, Jörg Rieskamp, Jerome Busemeyer, and Barbara Mellers reviewed the rationality principles and put them in the natural context of adaptive human behaviour. They demonstrated that the old interpretation of rational choice as revealing internal individual preferences is unsatisfactory. In contrast, viewing choice at least as much as the product of environmental demands and circumstances, makes seemingly contradictory or suboptimal decisions look perfectly rational. Summarizing decades of research, these authors came up with a hierarchy of postulates wherein each implies the subsequent ones, with some exceptions. The important question though, is to what extent real behaviour complies with them. These are the principles:

1. Consistency of choice (Invariance of preferences across situations)
2. Strong Stochastic Transitivity
3. Independence from Irrelevant Alternatives
4. Regularity
5. Weak Stochastic transitivity.

Let me review the principles in relation with the ways in which people violate them. *Consistency of* Choice demands that a person maintain their preferences, e.g. $A \succ B$ under all circumstances. A related postulate is transitivity: $A \succ B$ and $B \succ C$ imply $A \succ C$. Numerous laboratory experiments and field studies have discovered many violations of this demand for consistency. But is it irrational?

Simon (1955, 1978) was among the first to object to such a conclusion. In his view, when the situation demands a quick decision, it might be far more advantageous to react quickly, possibly at the expense of some inconsistency. Just as important is another objection: in a dynamically changing environment with new opportunities emerging every now and again, the decision maker might benefit more by experimenting with them, rather than sticking to her/his established preferences.

Strong Stochastic Transitivity is a postulate that in the mathematical sense weakens the requirement for consistency of choice. Since people cannot be consistent across all situations, perhaps they could be "roughly" consistent. That means keeping preferences stable if not always, then most of the time. Let $p(A|\{A, B\})$ be the probability of choosing A among A and B. Instead of judging people's preferences by observing their inconsistent choices, we might better be interested in their probabilistic decisions: do they prefer A to B in more than 50 % of cases, i.e.

$$p(A|\{A, B\}) \geq 0.5.$$

Formally, Strong Stochastic Transitivity for each three alternatives A, B, and C means that if

$$p(A|\{A, B\}) \geq 0.5,$$
$$p(B|\{B, C\}) \geq 0.5$$

hold simultaneously, then

$$p(A|\{A, C\}) \geq \max[p(A|\{A, B\}), p(B|\{B, C\})].$$

This inequality is easily understood when one considers the following example. If A is preferred to B in 60 % of the cases when the choice is between them, and B is preferred to C in 90 % of the cases, then A must be preferred to C in 90 % of the cases when the choice is between A and C.

Since the 1950s, numerous experiments have shown massive violations of Strong Stochastic Transitivity. Rieskamp et al. (2006) cite the established psychological explanation of this effect, which is that in direct comparisons people pay disproportionate attention to the aspects in which the alternatives are different, but underestimate the common features among them. This mechanism distorts the objective picture and leads to violations of the postulate.

The next principle, *Independence from Irrelevant Alternatives,* requires that the preference between A and B should not change when each of them is evaluated with respect to option C or D. Formally,

$$p(A|\{A, C\}) \geq p(B|\{B, C\})] \;\Rightarrow\; p(A|\{A, D\}) \geq p(B|\{B, D\})].$$

It has been proven (Tversky and Russo 1969) that the most popular version of this principle is equivalent to the previous one, Strong Stochastic Transitivity. In that context, or outside of it, the principle of Independence from Irrelevant Alternatives is also violated en masse.

Weak Stochastic Transitivity is a postulate demanding that if A is preferred to B in over 50 % of the cases, and B is preferred to C also in over 50 % of the cases, then in direct comparison, A will be chosen over C in more than 50 % of all cases. There is a lot of experimental evidence that people generally behave in line with this postulate, with violations only rarely occurring.

Tversky, however, studied how and when consistent and predictable preference intransitivities, i.e., violations of the principle, occur. He showed that when people face five alternatives such as those in Table 2.2, their choices usually are as follows: $A \succ B$, $B \succ C$, $C \succ D$, $D \succ E$. Instead of the expected overwhelming dominance of A over E, however, very often people choose the opposite: $E \succ A$.

This decision anomaly relates to the effect of the just noticeable difference discussed in Chap. 1 and in Sect. 2.2, regarding axioms A2c and A2d and small probabilities. People hardly perceive the differences in probability between any two adjacent alternatives, and their attention focuses on the monetary outcomes. However, that is no longer the case when A and E get compared, and now probability begins to matter so much that people react like this: $p(E|\{A, E\}) \geq 0.5$.

Regularity is the last principle from the classification of Rieskamp and colleagues. It states that adding an option to a set of options can never increase the probability of selecting one of the initial options. Let $n > m$ and let $\Omega_1 = \{A_1, \ldots, A_m\}$ be the initial set, to which new elements are added, to obtain $\Omega_2 = \{A_1, \ldots, A_m, A_{m+1}, \ldots, A_n\}$. Let A_i be an alternative from Ω_1 with probability to be chosen $p(A_i|\Omega_1)$. It is mathematically impossible to increase the chance for selecting A_i when new (competing) elements are added to the initial set. Therefore,

$$p(A_i|\Omega_1) \geq p(A_i|\Omega_2). \tag{2.20}$$

People may behave differently, however. If a newly added alternative is similar but worse than a particular alternative A_i from the original set, then the new one may actually increase the attractiveness of the original. For example, choosing a laptop between A and B may look like the case in Table 2.3. With all other features being identical, the only real differences are in weight and screen size.

If the customer's attention is drawn to a third laptop C with features as in Table 2.4, then option B suddenly rises in attractiveness. That is exactly the case when the inequality from Eq. (2.20) remains mathematically correct, but becomes descriptively wrong.

Table 2.2 Probabilistic alternatives

	A	*B*	*C*	*D*	*E*
Probability	7/24	8/24	9/24	10/24	11/24
Gain	\$5.00	\$4.75	\$4.50	\$4.25	\$4.00

Table 2.3 Features of two laptops

	A	*B*
Weight (kg)	3	1.5
Monitor size (diagonal) (in.)	15	13

Table 2.4 Features of three laptops

	A	B	C
Weight (kg)	3	1.5	1.8
Monitor size (diagonal) (in.)	15	13	13

Sen (1993, 1995, 1997) has added an important argument to the adequacy of violating the regularity principle. Under certain conditions—mostly of social nature—almost anybody will act "irrationally", as in the following example. In a thought experiment, someone would much appreciate taking apple X from the fruit basket, but would refrain to do so if the apple was the last remaining. She would gladly take it though, if there were two remaining: X and Y. Let us define the following events: $A = \{Take\ X\}$; $B = \{Take\ nothing\}$; $C = \{Take\ Y\}$. Then the two sets are:

$$\Omega_1 = \{Take\ X; Take\ nothing\}$$
$$\Omega_2 = \{Take\ X; Take\ Y; Take\ nothing\}.$$

Good manners will lead quite a lot of people to violate rationality (regularity) and be described by

$$p(A|\Omega_1) < p(A|\Omega_2),$$

although $\Omega_1 \subset \Omega_2$.

Beyond the social norms, inducing people to behave in such a way, there are a number of instances where no such clear-cut arguments exist. In other words, regularity may be violated under a wider range of circumstances. A psychological theory that explains this behaviour is Busemeyer and Townsend's (1993) Decision Field Theory. It posits that the mental mechanism of choice involves a dynamic process of retrieving, comparing, and integrating prospect utilities U_t, where t indicates time. It starts with a deliberation phase whereby each alternative A_i is associated with utility that is a random variable at any moment in time. All such quantities $U_t(A_i)$ are integrated and form dynamic states, in which the mind focuses on the various aspects of the options at hand and the decision to be taken. Alternatives compete with one another and the more similar they are, the more intense is the competition among them. In that case, the deliberation takes longer and violations of the regularity principle increase in number. In contrast, it has been shown in numerous studies (see for example Todorov et al. 2005) that rapid unreflective inferences can be very powerful in the decision-making process.

2.8.2 *The Dire Straits of Decision Analysis*

It seems that the rigorous mathematical approach to decision analysis has encountered unexpected difficulties. The idea that guided it all along—the maximization of subjective utility—has brought some rewards, but has remained just as elusive. In incremental steps, it was reformulated in line with the scientific concepts and methods of each time period. However, the available mathematical techniques, and in more recent time the computer modelling tools have always been helpful, but were never the single decisive factor for understanding people's decisions. Ever since Bernoulli, this science has advanced only when it has incorporated new knowledge about how the human mind works, with formal analysis playing second fiddle.

Influential economists have recognized the limits of the approach seeking only internal consistency of choice. As Amartya Sen had put it, this very idea was "confused" because,

> [...] there is no way of determining whether a choice function is consistent or not without referring to something external to choice behaviour (such as objectives, values, or norms). We have to re-examine the robustness of the standard results in this light. (Sen 1993)

In other words, preferences hardly abide by logical consistency, but surely depend on the personality and motivation of the decision maker, who is also influenced by circumstances and context, and may or may not remain faithful to the urge for maximizing utility. This position frustrates some researchers because they find most valuable the knowledge that quantifies and ultimately predicts behaviour.

Whether science will ever reach such a level of pragmatic and efficient understanding is unknown. Yet, it is certain that there is a long way to go until we even begin to have satisfactory methods to meet such goals. Every now and again, we discover a new area for research, in this case the subject of human decisions, only to realize how ignorant we are.

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