

Chapter 2

Frequency Domain

The time response of a complicated structure which is exposed to a number of forces is often extremely difficult to interpret. However, if the behavior of a linear mechanical system is studied in the frequency domain, it is possible to attribute the response at a certain frequency to a particular force or a resonance in the system. In this chapter some of the basic tools like Fourier transforms, frequency response and mobility functions used for analysis of signals are introduced. The response of simple systems excited by periodic as well as random forces are discussed. The simple one degree of freedom system is used to illustrate the basic procedures, when changing from the time to the frequency domain. The concepts discussed are later used for analysis of continuous systems. Some techniques for determining loss factors are also introduced.

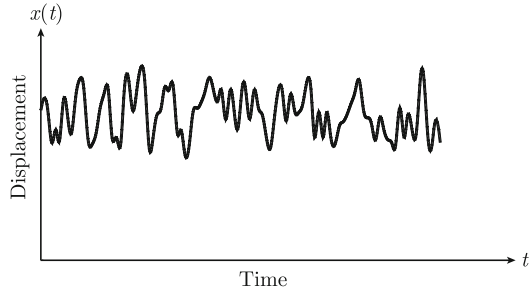
2.1 Introduction

The time functions hereto considered could be called deterministic. In other words, a mathematical expression can be formulated giving the instantaneous value of displacement, etc., of a mechanical system. The excitation of a linear 1-DOF system by means of a periodic force results in a response which in turn is periodic. This response can be expanded in a Fourier series. Each term in the series corresponds to a certain frequency. In the frequency domain this is equivalent to a line spectrum. Each line corresponds to a component in the Fourier series.

A force, response, or any other function could also be random. If a random function is stationary then the average

$$\bar{x} = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt \quad (2.1)$$

Fig. 2.1 The time response of a 1-DOF system excited by a random force



is independent of t_0 . The same should be true for all statistical properties of the signal. If a random process is stationary: its statistical properties do not vary, when computed over different samples, then the random process is ergodic. For a random process in general, it is not meaningful to consider the time history of the response. Figure 2.1 shows the time response of a 1-DOF system, excited by a random force. A direct interpretation of the time history is not easily deduced. However, the frequency response of the same signal can contain vital information within the mechanical system.

The actual excitation of typical mechanical systems is often due to a combination of harmonic and random forces. The discrete frequency spectrum of a periodic function becomes a continuous spectrum if the period T is extended to infinity. This means in the usual way that the Fourier series is transformed to a Fourier integral. Thus the function $x(t)$ can be expressed as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\omega t} d\omega \quad (2.2)$$

where $\hat{x}(\omega)$ is the FT (Fourier transform) of $x(t)$. Inversely $\hat{x}(\omega)$ can be defined as

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt \quad (2.3)$$

The result (2.3) is a consequence of the identity

$$\delta(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(\omega_0 - \omega)} dt \quad (2.4)$$

This is a very useful equality which is often utilized in the transformation from the time to the frequency domain and vice versa. The proofs of Eqs. (2.2) and (2.3) are left for Problem 2.13.

In principle, a signal $x(t)$ must be known in the entire time domain, i.e., from minus to plus infinity, for its Fourier transform to be defined. However, the signal

can be redefined as being equal to $x(t)$ in the time domain $-T/2 \leq t \leq T/2$ and zero elsewhere. Thereafter the necessary operations or transforms can be carried out. A certain error is introduced by means of this method. In practice, the so-called Fast Fourier Transform (FFT) technique is used for processing of signals. By means of this technique only a certain time history of a signal is used for calculation of the Fourier transform of the signal. The FFT technique leads to certain errors. However, it is well understood how to estimate and suppress any possible errors. Signal analysis in theory and practice is discussed in Refs. [6–9] and random vibrations in Refs. [10, 11].

2.2 Frequency Response

According to Eq. (1.39) the response $x(t)$ of a linear system, excited by a force $F(t)$ is given by the convolution integral

$$x(t) = \int_{-\infty}^{\infty} d\tau \cdot h(\tau) F(t - \tau) \quad (2.5)$$

where $h(t)$ is the response function of the system caused by a unit impulse at $t = 0$. Further, $F(t) = 0$ for $t < 0$.

The functions $h(t)$ and $x(t)$ have the Fourier transforms $H(\omega)$ and $\hat{F}(\omega)$, respectively. Thus,

$$H(\omega) = \int_{-\infty}^{\infty} dt \cdot e^{-i\omega t} \cdot h(t), \quad \hat{F}(\omega) = \int_{-\infty}^{\infty} dt \cdot e^{-i\omega t} \cdot F(t) \quad (2.6)$$

The function $F(t - \tau)$ can according to Eq. (2.2) be written as

$$F(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(u) \cdot e^{iu(t-\tau)} du \quad (2.7)$$

The FT of $x(t)$ as defined in Eq. (2.5) is

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} dt \cdot e^{-i\omega t} \cdot \int_{-\infty}^{\infty} d\tau \cdot h(\tau) F(t - \tau) \quad (2.8)$$

The definition (2.7) can now be inserted into Eq. (2.8). The order of integration is changed and the result (2.4), i.e., the FT of a Dirac function, is utilized as follows:

$$\begin{aligned}
\hat{x}(\omega) &= \int_{-\infty}^{\infty} dt \cdot e^{-i\omega t} \cdot \int_{-\infty}^{\infty} d\tau \cdot h(\tau) \frac{1}{2\pi} \int_{-\infty}^{\infty} du \cdot \hat{F}(u) \cdot e^{iu(t-\tau)} \\
&= \int_{-\infty}^{\infty} d\tau \cdot h(\tau) \int_{-\infty}^{\infty} du \cdot \hat{F}(u) \cdot e^{-iu\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \cdot e^{it(u-\omega)} \\
&= \int_{-\infty}^{\infty} d\tau \cdot h(\tau) \int_{-\infty}^{\infty} du \cdot \hat{F}(u) \cdot e^{-iu\tau} \delta(\omega - u) \\
&= \int_{-\infty}^{\infty} d\tau \cdot h(\tau) \hat{F}(\omega) \cdot e^{-i\omega\tau} \\
&= \hat{F}(\omega) \int_{-\infty}^{\infty} d\tau \cdot h(\tau) \cdot e^{-i\omega\tau} = \hat{F}(\omega) H(\omega)
\end{aligned} \tag{2.9}$$

The result means that for a linear mechanical system the FT of the response is equal to the product between the FT of the force and the FT of the response function. This last transform $H(\omega)$ is referred to as the frequency response function (FRF), or the transfer function of the mechanical system being excited.

The FRF of a 1-DOF system can be derived based on the equation of motion for the system. Returning to the time domain the equation of motion for a 1-DOF system excited by a unit pulse at $t = 0$ is

$$m\ddot{h} + c\dot{h} + k_0h = \delta(t) \tag{2.10}$$

where $h(t)$ is the response of a damped 1-DOF system excited by a unit pulse. The Fourier transform of $h(t)$ is defined in Eq. (2.6).

The equation of motion for the 1-DOF system, Eq. (2.10), is multiplied by $e^{-i\omega t}$ and integrated over time resulting in

$$\begin{aligned}
m \int_{-\infty}^{\infty} \ddot{h}(t) e^{-i\omega t} dt + c \int_{-\infty}^{\infty} \dot{h}(t) e^{-i\omega t} dt + k_0 \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \\
= \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt
\end{aligned} \tag{2.11}$$

The last integral on the left-hand side is according to definition equal to $k_0 H(\omega)$. The integral on the right-hand side is equal to unity. The first and the second integrals on the left-hand side are integrated by parts. Starting with the second integral the result is

$$\int_{-\infty}^{\infty} \dot{h}(t) e^{-i\omega t} dt = \left[h(t) e^{-i\omega t} \right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = i\omega H(\omega) \tag{2.12}$$

The function $h(t)$ is defined as being equal to zero for $t < 0$. The system has losses, consequently $h(t)$ approaches zero as $t \rightarrow \infty$. Therefore, the expression inside the bracket is equal to zero at both time limits. In a similar way the FT of $\ddot{h}(t)$ is found to be

$$\int_{-\infty}^{\infty} \ddot{h}(t) e^{-i\omega t} dt = (i\omega)^2 H(\omega) = -\omega^2 H(\omega)$$

Thus, based on the results defining the FTs of $h(t)$ and its time derivatives the frequency response function $H(\omega)$ is obtained as

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega c + k_0} = \frac{1}{m(\omega_0^2 - \omega^2 + 2i\beta\omega)} \quad (2.13)$$

whereas before $\omega_0^2 = k_0/m$ and $\beta = c/(2m)$. For a forced linear 1-DOF system the equation of motion is

$$m\ddot{x} + c\dot{x} + k_0x = F(t) \quad (2.14)$$

The Fourier transform of $x(t)$ is, according to Eq. (2.9), given by $\hat{x} = \hat{F}H(\omega)$ or

$$\hat{x}(\omega) = \hat{F}(\omega)H(\omega) = \frac{\hat{F}(\omega)}{m(\omega_0^2 - \omega^2 + 2i\beta\omega)} \quad (2.15)$$

It is interesting now to note that exactly the same result is obtained by making the following substitutions in Eq. (2.14)

$$x(t) \rightarrow \hat{x}(\omega) \cdot e^{i\omega t}, \quad F(t) \rightarrow \hat{F}(\omega) \cdot e^{i\omega t} \quad (2.16)$$

Consequently, if for a linear system, it is assumed that the time dependence for force and displacement is $\exp(i\omega t)$, then the resulting amplitude of the displacement is identical to the FT of the displacement if the amplitude of the force is equal to its FT.

The basic Eq. (2.10) can according to the discussion in Sect. 1.6, Eq. (1.82), also be written as

$$m\ddot{h} + kh = \delta(t), \quad k = k_0(1 + i\delta_0)$$

The frequency response function $H(\omega)$ is for this case obtained as

$$H(\omega) = \frac{1}{-m\omega^2 + k} = \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \quad (2.17)$$

where $\omega_0^2 = k_0/m$. The lossfactor is in Eq. (1.81) given as $2\omega\beta/\omega_0^2$. Since the quantities β and ω_0^2 are real and positive, the loss factor δ_0 must be negative for $\omega < 0$. This change of sign is of importance whenever $H(\omega)$ or any other function of δ_0 is part of a Fourier integral. This is demonstrated in Problem 2.3.

In conclusion, the FT of the response of a 1-DOF system is readily defined if the FT of the force exciting the system is known. Thus, if for a 1-DOF system the equation of motion is $m\ddot{x} + kx = F(t)$, then by making the substitutions $x(t) \rightarrow \hat{x}(\omega) \cdot e^{i\omega t}$ and $F(t) \rightarrow \hat{F}(\omega) \cdot e^{i\omega t}$ the FT of the response is

$$\hat{x}(\omega) = \hat{F}(\omega)H(\omega) = \frac{\hat{F}(\omega)}{-m\omega^2 + k} = \frac{\hat{F}(\omega)}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \quad (2.18)$$

The FT of the response is the product of two functions, one, the frequency response function $H(\omega)$, depends on the frequency and the material parameters of the system. The other function, $\hat{F}$, is determined by the characteristics and time history of the force.

2.3 Correlation Functions

The correlation functions are of great importance when dealing with random and other stationary processes. If for example a structure is excited by several forces, then the response clearly depends on how these forces are correlated. Furthermore, the characteristics of any stationary signal can be determined by correlating the signal at one instant with the same signal at another time.

The autocorrelation function $R_{xx}(\tau)$ for the signal $x(t)$ is defined as the time average of the product of the signal at time t and the same signal at time $t + \tau$ or

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t + \tau)dt \quad (2.19)$$

This integral can be symbolized by

$$R_{xx}(\tau) = E [x(t)x(t + \tau)]$$

The symbol E stands for the expected value of the expression inside the bracket. In a similar way the cross-correlation function $R_{xy}(\tau)$ between the signals $x(t)$ and $y(t)$ is defined as

$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t + \tau)dt = E [x(t)y(t + \tau)] \quad (2.20)$$

As a first example, consider the autocorrelation function for a random signal. If the mean value of this random signal $x(t)$ is zero, then for large values of τ the autocorrelation must be zero; because of the random nature of the signal, $x(t)$ and $x(t + \tau)$ are quite unrelated. The product of the two signals is sometimes positive and at other times negative. The total average over time is therefore equal to zero. For the other extreme case, $\tau = 0$, the definition (2.19) gives

$$R_{xx}(0) = E[x^2(t)] \quad (2.21)$$

This is quite simply the mean square value of x . This function is not equal to zero. In order to satisfy these requirements the autocorrelation function for an ideally random signal is defined as

$$R_{xx}(\tau) = a \cdot \delta(\tau) \quad (2.22)$$

where a is a positive nonzero constant. Thus, $R_{xx}(\tau) = 0$ for $\tau \neq 0$ for an ideal random signal. The significance of the parameter a is discussed in connection with the derivation of Eq. (2.39).

For the harmonic function $x(t) = A \sin \omega t$ the autocorrelation function is obtained from Eq. (2.19) as

$$R_{xx}(\tau) = \frac{A^2}{T} \int_0^T \sin \omega t \cdot \sin[\omega(t + \tau)] dt = \frac{A^2}{2} \cos \omega \tau \quad (2.23)$$

For a harmonic function it is sufficient to take a time average over one period T where $\omega T = 2\pi$. For the signal $x(t) = A \cos \omega t$ the autocorrelation function is the same as for the sine function. Thus, the autocorrelation function of a harmonic signal is a harmonic function with the same period as the signal.

If a signal is composed of the random function $\xi(t)$ with a mean value equal to zero and, say, a harmonic function $h(t)$ then for $x(t) = h(t) + \xi(t)$ the autocorrelation function is

$$\begin{aligned} R_{xx}(\tau) &= E[x(t)x(t + \tau)] = E[h(t)h(t + \tau)] \\ &\quad + E[\xi(t)h(t + \tau)] + E[h(t)\xi(t + \tau)] + E[\xi(t)\xi(t + \tau)] \\ &= R_{hh}(\tau) + R_{\xi\xi}(\tau) = R_{hh}(\tau) + a \cdot \delta(\tau) \end{aligned} \quad (2.24)$$

The time average of the product between a random and a harmonic function is equal to zero. The cross terms in Eq. (2.24) are consequently eliminated.

For two stationary processes $x(t)$ and $y(t)$, random or harmonic, the cross-correlation function can be written in two ways

$$R_{xy}(\tau) = E[x(t)y(t + \tau)] \quad \text{or} \quad R_{xy}(\tau) = E[x(t - \tau)y(t)] \quad (2.25)$$

The derivatives with respect to τ should be the same for the two expressions. This means that

$$R'_{xy} = \frac{dR_{xy}(\tau)}{d\tau} = E[x(t)y'(t + \tau)] = E[-x'(t - \tau)y(t)]$$

For $\tau = 0$ the last two expressions give

$$E[x(t)y'(t)] = E[-x'(t)y(t)] \quad \text{or} \quad E[x\dot{y}] + E[\dot{x}y] = 0$$

If $x(t) = y(t)$ this result can only be satisfied if

$$E [x(t)\dot{x}(t)] = 0 \quad (2.26)$$

In a similar way it can be shown that

$$E [\dot{x}(t)\ddot{x}(t)] = 0 \quad (2.27)$$

If $x(t)$ is stationary and describes the displacement of a structure and $\dot{x}(t)$ is the velocity at that point, then the time average of the product between velocity and displacement is equal to zero. The time average of the velocity and acceleration at that same point is also equal to zero. These relationships are important when dealing with statistical energy methods. It should be noted that the time average of the product between displacement and acceleration in general is different from zero. The average of this product can be written as

$$E [x(t)\ddot{x}(t)] = \left[\frac{d^2 R_{xx}(\tau)}{d\tau^2} \right]_{\tau=0} \quad (2.28)$$

The time average of the velocity squared is

$$E [\dot{x}^2(t)] = - \left[\frac{d^2 R_{xx}(\tau)}{d\tau^2} \right]_{\tau=0} \quad (2.29)$$

The derivation of the last expression is left for Problem 2.14.

2.4 Spectral Density

A transfer from the time domain to the frequency domain naturally leads to the FT of the correlation functions. Starting with the cross-correlation function $R_{xy}(\tau)$ its FT is according to Eqs. (2.3) and (2.20)

$$\begin{aligned} FT \{R_{xy}(\tau)\} &= \int_{-\infty}^{\infty} d\tau \cdot e^{-i\omega\tau} E[x(t)y(t+\tau)] \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau \cdot e^{-i\omega\tau} \int_{-T/2}^{T/2} dt x(t)y(t+\tau) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau \int_{-T/2}^{T/2} dt \cdot e^{i\omega t} x(t) e^{-i\omega\tau - i\omega t} y(t+\tau) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} dt \cdot e^{i\omega t} x(t) \int_{-T/2}^{T/2} d\xi \cdot e^{-i\xi t} y(\xi) \\ &= \lim_{T \rightarrow \infty} \frac{\hat{x}^*(\omega)\hat{y}(\omega)}{T} \end{aligned} \quad (2.30)$$

In deriving the result ξ has been set to $\xi = t + \tau$. Further, the FT of $x(t)$ is given by $\hat{x}(\omega)$ as defined in Eq. (2.3). According to the same expression it follows that

$$\hat{x}^*(\omega) = \int_{-\infty}^{\infty} dt \cdot x(t) \cdot e^{i\omega t} \quad (2.31)$$

This expression is utilized to derive Eq. (2.30). The result given in Eq. (2.30) is again obtained using the expression $R_{xy}(\tau) = E[x(t)y(t + \tau)] = E[x(t - \tau)y(t)]$.

The FT of the cross-correlation function is referred to as the cross-spectral density $S_{xy}(\omega)$ and is defined as

$$S_{xy}(\omega) = S_{xy}(2\pi f) = \lim_{T \rightarrow \infty} \frac{\hat{x}(\omega)\hat{y}^*(\omega)}{T} \quad (2.32)$$

where f is the frequency. The FT of the autocorrelation function is immediately obtained by substituting $y(t + \tau)$ by $x(t + \tau)$ into Eq. (2.30). Thus,

$$S_{xx}(\omega) = FT \{R_{xx}(\tau)\} = \lim_{T \rightarrow \infty} \frac{|\hat{x}(\omega)|^2}{T} \quad (2.33)$$

The quantity $S_{xx}(\omega)$ is the power spectral density or autospectral density of the signal $x(t)$.

The results can be summarized as

$$\begin{aligned} R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega \\ S_{xx}(\omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau \\ R_{xy}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega\tau} d\omega \\ S_{xy}(\omega) &= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (2.34)$$

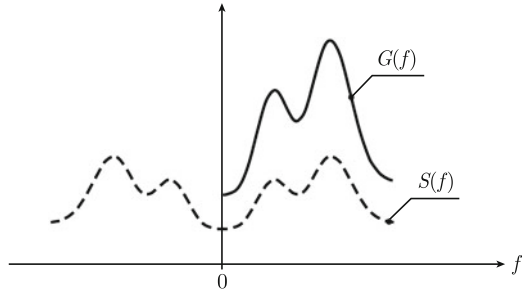
The spectral densities have some properties which are useful in various mathematical manipulations. From the hypothesis of a stationary process it follows that the correlation functions satisfy the equalities

$$R_{xx}(\tau) = R_{xx}(-\tau), \quad R_{xy}(\tau) = R_{yx}(-\tau) \quad (2.35)$$

From these symmetry properties for the stationary correlation functions and the results (2.33) and (2.34) it follows that

$$S_{xx}(\omega) = S_{xx}(-\omega), \quad S_{xy}(\omega) = S_{yx}(-\omega) \quad (2.36)$$

Fig. 2.2 One-sided and two-sided spectral density functions



Consequently $S_{xx}(\omega)$ is an even function. It is therefore possible to define the physically realizable one-sided spectral density $G_{xx}(\omega)$ as

$$G_{xx}(\omega) = 2S_{xx}(\omega) \text{ for } \omega \geq 0 \text{ otherwise zero} \quad (2.37)$$

The one-sided and two sided spectral densities are illustrated in Fig. 2.2. In a similar way the physically realizable one-sided cross-spectral density $G_{xy}(\omega)$ is defined as

$$G_{xy}(\omega) = 2S_{xy}(\omega) \text{ for } \omega \geq 0 \text{ otherwise zero}$$

However, $G_{xy}(\omega)$ is not a real function which is the case for $G_{xx}(\omega)$. The real part of $G_{xy}(\omega)$ is symmetric. The importance of the real part of $G_{xy}(\omega)$ will later be demonstrated. For future reference the definitions above yield for $\omega \geq 0$

$$\text{Re}[G_{xy}(\omega)] = 2\text{Re}[S_{xy}(\omega)] = 2\text{Re}[S_{yx}(\omega)] \quad (2.38)$$

2.5 Examples of Spectral Density

It has previously been pointed out that the autocorrelation function for a completely stationary random signal $x(t)$ or white noise can be written as $R_{xx}(\tau) = a \cdot \delta(\tau)$. Thus, the corresponding power spectral density for the signal is obtained from Eq. (2.34) as

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} a \cdot \delta(\tau) e^{-i\omega\tau} d\tau = a \text{ for all } \omega, \quad G_{xx}(\omega) = 2a \text{ for } \omega \geq 0 \quad (2.39)$$

The power spectral density for a completely random signal is constant and thus independent of frequency. This is the definition of white noise—every frequency component is equally represented. However, this type of signal can not be physically reproduced. For a completely random signal $x(t)$, the time average of $x^2(t)$,

or $R_{xx}(\tau) = a \cdot \delta(\tau)$ is infinite for $\tau = 0$. The energy required to reproduce a completely random signal would be infinite. All physically generated “random” signals have in fact only a random character within a certain frequency band. Still, in most mathematical formulations, the definitions (2.22) and (2.39) are preferably used.

For a harmonic signal $x(t) = A \cdot \sin \omega_0 t$ or $x(t) = A \cdot \cos \omega_0 t$ the autocorrelation function is given by Eq. (2.23) as

$$R_{xx}(\tau) = \frac{A^2}{2} \cos \omega_0 \tau \quad (2.40)$$

According to definition the power spectral density of this signal is

$$\begin{aligned} S_{xx}(\omega) &= \frac{A^2}{2} \int_{-\infty}^{\infty} \cos \omega_0 \tau \cdot e^{-i\omega \tau} d\tau \\ &= \frac{A^2}{4} \int_{-\infty}^{\infty} d\tau (e^{i\omega_0 \tau - i\omega \tau} + e^{-i\omega_0 \tau - i\omega \tau}) \\ &= \frac{\pi A^2}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \end{aligned} \quad (2.41)$$

The two-sided power spectral density of a harmonic signal, angular frequency ω_0 , has two frequency components corresponding to $\omega = \pm \omega_0$. The one-sided spectral density is

$$G_{xx}(\omega) = \pi A^2 \delta(\omega - \omega_0) \text{ for } \omega \geq 0 \quad (2.42)$$

The expression (2.42) can also be written as

$$G_{xx}(2\pi f) = A^2/2 \cdot \delta(f - f_0) \text{ for } f \geq 0$$

This is a consequence of $\delta(kt - kt_0) = \delta(t - t_0)/k$.

For a harmonic signal of the form

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t), \quad \omega_n = \frac{2\pi n}{T}$$

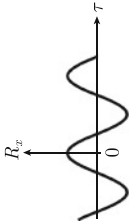
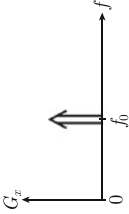
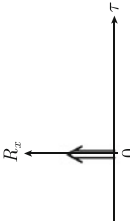
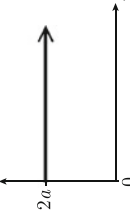
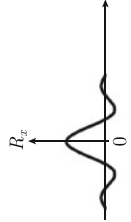
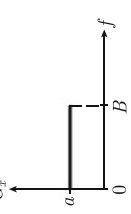
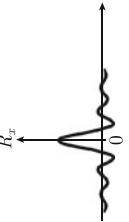
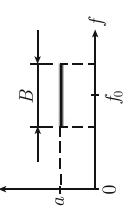
the autocorrelation function is

$$R_{xx}(\tau) = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \left(\frac{a_n^2 + b_n^2}{2}\right) \cos(\omega_n \tau) \quad (2.43)$$

Based on the results Eqs. (2.39) and (2.40) the one-sided power spectral density of the signal is

$$G_{xx}(\omega) = \frac{\pi a_0^2}{2} \cdot \delta(\omega) + \sum_{n=1}^{\infty} \pi (a_n^2 + b_n^2) \delta(\omega - \omega_n) \quad (2.44)$$

Table 2.1 Some special stationary autocorrelation function and related one side spectral density function

Types	Autocorrelation Function	(One-Sided) Power Spectral Density Function
Sine wave	 $R_x(\tau) = \frac{X^2}{2} \cos(2\pi f_0 \tau)$	 $G_x(f) = \frac{X^2}{2} \delta(f - f_0)$
White noise	 $R_x(\tau) = a\delta(\tau)$	 $G_x(f) = 2a, f \geq 0; \text{ otherwise zero}$
Low-pass, white noise	 $R_x(\tau) = aB \frac{\sin(2\pi B\tau)}{2\pi B\tau}$	 $G_x(f) = a, 0 \leq f \leq B; \text{ otherwise zero}$
Band-pass, white noise	 $R_x(\tau) = aB \frac{\sin(\pi B\tau)}{\pi B\tau} \cos(2\pi f_0 \tau)$	 $G_x(f) = a, 0 < f_0 - (B/2) \leq f \leq f_0 + (B/2); \text{ otherwise zero}$

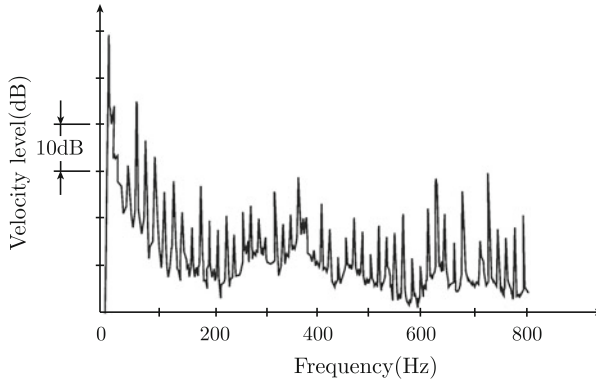


Fig. 2.3 Measured velocity level on foundation of a Diesel engine

The result is a line spectrum as opposed to the continuous spectrum shown in Fig. 2.2. In the spectrum there is a line for $f = \omega_n/(2\pi) = n/T$, where $n = 0, 1, 2, \dots$. Table 2.1 presents some special stationary autocorrelation functions and related one sided spectral density functions. The autocorrelation functions for low-pass and band-pass white noise are discussed in Problem 2.5.

A spectrum from a real source and in particular, a source like a machine rotating at a constant rpm. is generally built up of continuous and line spectra. One example is shown in Fig. 2.3. The power spectral density of the velocity is G_{vv} . A reference function is given by G_{ref} . The velocity level L_v in dB or rather $L_v = 10 \cdot \log(G_{vv}/G_{\text{ref}})$ is shown as function of frequency. The velocity level is measured on the foundation on which a ship Diesel engine is resiliently mounted. The engine is running at 960 rpm. Corresponding to a rotation frequency of 16 Hz. Distinct peaks or lines are shown in the spectrum for the frequencies $f_n = n \cdot 16$ Hz for n an integer.

Even a simple measured noise spectrum can often be used to identify the contributions from various noise sources. If at a certain position away from a large vibrating structure the sound pressure induced by a number of sources is analyzed with a good frequency resolution, the main noise sources could be identified based on a knowledge of the operation of the source. One example is illustrated in Fig. 2.4. The noise was radiated from the hull of a catamaran into the water. The sound pressure level, or rather the quantity $L_p = 10 \cdot \log(G_{pp}/G_{\text{ref}})$, has been recorded at a position in the water. The one-sided power spectral density of the pressure is given by G_{pp} . As in the previous case the rpm of the main engine is 960, corresponding to 16 cycles/s. The first five harmonics, denoted ME₁ through ME₅, are quite distinct in the spectrum as shown in Fig. 2.4.

The auxiliary engine is running at 1460 rpm. The first few harmonics, AE₁, etc., are approximately at $n \times 24$ Hz. The rpm of the six-bladed pump wheel of the water jet is 455, with the first, WJ₁, and second, WJ₂, harmonics at 46 and 92 Hz respectively. The frequency of the first harmonic is given by the number of rounds per second of the shaft times the number of propeller blades. The second harmonics AE₂ induced

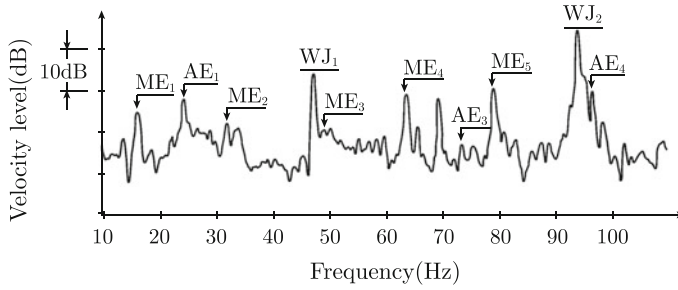


Fig. 2.4 Sound pressure level measured in the water close to a catamaran

by the auxiliary engine cannot clearly be distinguished from the dominating first harmonic WJ_1 from the water jet.

In the high frequency region, the higher harmonics of the various sources tend to merge and the sources cannot be readily identified by this simple technique, more sophisticated methods must be used. For multiple-input linear systems, the so-called partial coherence function gives a quantitative indication of the degree of linear dependence between, for example, the vibration velocity of a source and the measured sound pressure level at a certain point. The method is described in for example Ref. [8].

2.6 Coherence

The so-called coherence function $\gamma_{xy}^2(\omega)$ can be used as a measure of the quality of two observations $x(t)$ and $y(t)$. The coherence function is defined as

$$\gamma_{xy}^2(\omega) = \frac{|G_{xy}(\omega)|^2}{G_{xx}(\omega)G_{yy}(\omega)} \quad (2.45)$$

The coherence function is always real and positive and can vary between zero and unity. If the coherence function is equal to zero for $\omega = \omega_0$ the signals $x(t)$ and $y(t)$ are said to be incoherent or uncorrelated at the frequency $f = \omega_0/(2\pi)$. For the coherence function to be equal to unity $x(t)$ and $y(t)$ must be fully coherent or correlated. This, for example, is the case if $y(t) = q \cdot x(t)$, where q is a constant.

The applicability of the coherence function can be demonstrated by considering a linear 1-DOF system. A force $F(t)$ is exciting the mass of the system. However, the response of the mass is not only determined by the force $F(t)$, but is also influenced by a secondary force $K(t)$. The FT of the response $x(t)$ of the mass is determined by the frequency response function $H(\omega)$. Thus,

$$\hat{x} = (\hat{F} + \hat{K})H \quad (2.46)$$

The power spectral density of the response is according to the definition (2.33) equal to

$$\begin{aligned} S_{xx}(\omega) &= \lim_{T \rightarrow \infty} \frac{|\hat{x}(\omega)|^2}{T} = \lim_{T \rightarrow \infty} \frac{(\hat{F} + \hat{K})(\hat{F}^* + \hat{K}^*) |H(\omega)|^2}{T} \\ &= (S_{FF} + S_{FK} + S_{KF} + S_{KK}) |H|^2 \end{aligned}$$

If the forces K and F are completely uncorrelated then the cross-spectral densities S_{FK} and S_{KF} are equal to zero. In this case

$$S_{xx}(\omega) = (S_{FF} + S_{KK}) \cdot |H|^2 \quad \text{and} \quad G_{xx}(\omega) = (G_{FF} + G_{KK}) \cdot |H|^2 \quad (2.47)$$

The cross-power spectral density G_{Fx} is equal to $G_{FF}H$. The coherence function with respect to the force $F(t)$ and the displacement $x(t)$ is thus,

$$\begin{aligned} \gamma_{Fx}^2(\omega) &= \frac{|G_{Fx}(\omega)|^2}{G_{xx}(\omega)G_{FF}(\omega)} = \frac{|G_{FF}|^2 |H|^2}{(G_{FF} + G_{KK})G_{FF} |H|^2} \\ &= \frac{|G_{FF}|^2}{(G_{FF} + G_{KK})G_{FF}} < 1 \end{aligned} \quad (2.48)$$

The spectral densities G_{FF} and G_{KK} are positive and real. The coherence function is consequently equal to unity only if G_{KK} is zero. Otherwise the coherence function is less than unity. This is the case whenever:

- (i) The measurements are influenced by extraneous noise;
- (ii) The system is not linear;
- (iii) The response is due to an input $F(t)$ as well as other inputs.

The coherence function is thus a measure of the quality of two observations.

2.7 Time Averages of Power and Energy

It is often of great interest to determine the time averages of kinetic and potential energies of a mechanical system. These energies can be determined as functions of the FT of displacement and velocity. The time average of the square of a signal $x(t)$ is according to Eq. (2.34) equal to

$$E[x^2] = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = \frac{1}{2\pi} \int_0^{\infty} G_{xx}(\omega) d\omega \quad (2.49)$$

The time average of the power input to a system is equal to the average of the product between force and velocity, or in a general way, between the signals $x(t)$ and $y(t)$. Thus,

$$E[xy] = R_{xy}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) d\omega$$

From Eqs. (2.36) and (2.32) it follows that

$$S_{xy}(-\omega) = S_{yx}(\omega) \quad \text{and} \quad S_{yx}(\omega) = S_{xy}^*(\omega)$$

Consequently,

$$\begin{aligned} E[xy] &= R_{xy}(0) = \frac{1}{2\pi} \int_0^{\infty} [S_{xy}(\omega) + S_{yx}(\omega)] d\omega \\ &= \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} S_{xy}(\omega) d\omega = \frac{1}{2\pi} \operatorname{Re} \int_0^{\infty} G_{xy}(\omega) d\omega \end{aligned} \quad (2.50)$$

It can be stated that the functions $G_{xx}(\omega)$ and $G_{xy}(\omega)$ represent the time averages $E[x^2]$ and $E[xy]$ in the frequency domain.

Again returning to the simple mass-spring system, the transfer function $H(\omega)$ is from Eq. (2.17)

$$H(\omega) = \frac{1}{-m\omega^2 + k} = \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \quad (2.51)$$

The FT of the force is $\hat{F}(\omega)$, consequently and according to Eq. (2.15) the FT of the response is $\hat{x} = \hat{F}H$. The power spectral density of the displacement is

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{|\hat{x}|^2}{T} = \lim_{T \rightarrow \infty} \frac{|H|^2 |\hat{F}|^2}{T} = |H|^2 S_{FF}(\omega) \quad (2.52)$$

As defined the FT of the resulting velocity is $\hat{x} = i\omega\hat{x} = \hat{v}$. Thus,

$$S_{vv}(\omega) = \omega^2 |H|^2 S_{FF}(\omega) \quad \text{or} \quad G_{vv}(\omega) = \omega^2 |H|^2 G_{FF}(\omega) \quad (2.53)$$

For a force defined by $F(t) = F_0 \sin \omega_1 t$ the corresponding power spectral density of the force is $G_{FF}(\omega) = \pi F_0^2 \cdot \delta(\omega - \omega_1)$. Consequently the spectral density has only one frequency component. The force is exciting a 1-DOF system, represented by the frequency response function $H(\omega)$. The resulting time average of the velocity squared is

$$\begin{aligned} \bar{v}^2 &= R_{vv}(0) \\ &= \frac{1}{2\pi} \int_0^{\infty} G_{vv}(\omega) d\omega \\ &= \frac{1}{2} \int_0^{\infty} |\omega H(\omega)|^2 \cdot F_0^2 \cdot \delta(\omega - \omega_1) d\omega \end{aligned}$$

$$\begin{aligned}
&= |\omega_1 H(\omega_1)|^2 \frac{F_0^2}{2} \\
&= \frac{\omega_1^2 F_0^2}{2m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]}
\end{aligned} \tag{2.54}$$

This is the same result as obtained in Sect. 1.4.

The time average of the power input to the system is—compare Eq. (2.50)

$$\bar{\Pi} = E[F(t)v(t)] = R_{Fv}(0) = \frac{1}{2\pi} \text{Re} \int_0^\infty G_{Fv}(\omega) d\omega \tag{2.55}$$

The cross-spectral density G_{Fv} is

$$G_{Fv}(\omega) = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^* \hat{v}}{T} = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^*(i\omega) \hat{x}}{T} = 2i\omega S_{Fx} = i\omega G_{Fx} \tag{2.56}$$

The FT of the response is $\hat{x} = \hat{F}H$. For a force defined by $F(t) = F_0 \sin \omega_1 t$ and $G_{FF}(\omega) = \pi F_0^2 \cdot \delta(\omega - \omega_1)$, the cross-power spectral density between force and velocity is

$$G_{Fv}(\omega) = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^* \hat{v}}{T} = \lim_{T \rightarrow \infty} 2 \frac{|\hat{F}|^2 (i\omega) H(\omega)}{T} = i\omega G_{FF} H(\omega) \tag{2.57}$$

The frequency response function for the simple mass-spring system is defined in Eq. (2.17). Thus,

$$\begin{aligned}
\text{Re} G_{Fv}(\omega) &= \text{Re} \left\{ \frac{i\omega G_{FF}(\omega)}{m[(\omega_0^2 - \omega^2) + i\omega_0^2 \delta_0]} \right\} \\
&= \frac{\pi \omega_0^2 \omega \delta_0 F_0^2}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} \delta(\omega - \omega_1)
\end{aligned} \tag{2.58}$$

From Eq. (2.55) it follows that

$$\bar{\Pi} = \frac{\omega_0^2 \omega_1 \delta_0 F_0^2}{2m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \tag{2.59}$$

The same procedure can be repeated for white noise excitation of the mass of the simple mass-spring system. For white noise excitation the autocorrelation function and the power spectral densities are

$$R_{FF}(\tau) = a \cdot \delta(\tau)$$

where

$$a = \lim_{T \rightarrow \infty} \frac{|\hat{F}|^2}{T} = S_{FF}(\omega)$$

$$G_{FF}(\omega) = 2a$$

Following the discussion above this means that

$$\begin{aligned} \bar{v}^2 &= \frac{1}{2\pi} \int_0^\infty G_{vv}(\omega) d\omega = \frac{1}{2\pi} \int_0^\infty \omega^2 G_{FF}(\omega) |H(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega^2 G_{FF}(\omega)}{m^2[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} = \frac{a}{\pi m^2} \cdot I \end{aligned} \quad (2.60)$$

where

$$I = \int_0^\infty d\omega \frac{\omega^2}{[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]}$$

This type of integral appears quite often in analysis of dynamic problems. For future reference it is convenient to formulate a general solution to this type of integral. Thus, consider the contour integral

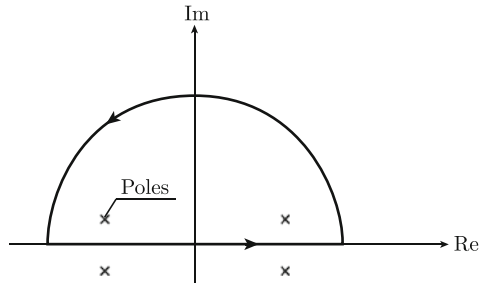
$$J = \oint \frac{g(z) dz}{(z^2 - \omega_0^2)^2 + (\omega_0^2 \delta_0)^2} = \oint \frac{g(z) dz}{h(z)}$$

The integration in the complex plane is along the path shown in Fig. 2.5.

The function $g(z)$ is analytic and regular in the domain. The integrand has two poles, $z_1 = \omega_0(1 + i\delta/2)$ and $z_2 = -\omega_0(1 - i\delta/2)$, in the upper half of the complex plane. It is assumed that $0 < \delta_0 \ll 1$. According to the theorem of residues and Cauchy's theorem—see Ref. [5]—the integral J is equal to

$$J = \oint \frac{g(z) dz}{h(z)} = 2\pi i \sum_n \frac{g(z_n)}{h'(z_n)} = 2\pi i \sum_n \frac{g(z_n)}{4z_n(z_n^2 - \omega_0^2)} \quad (2.61)$$

Fig. 2.5 Path of integration in complex plane



The summation is made over all poles inside the closed path shown in Fig. 2.5. For $\delta_0 \ll 1$ the result is for $g(\omega) = g(-\omega)$ quite simply equal to

$$J = \frac{\pi g(\omega_0)}{\omega_0^3 \delta_0} \quad (2.62)$$

The result is the sum of the integral along the real axis and along the semicircle C , or

$$J = \oint \frac{g(z)dz}{h(z)} = \int_{-\infty}^{\infty} \frac{g(z)dz}{h(z)} + \int_C \frac{g(z)dz}{h(z)}$$

If the functions $g(z)$ and $h(z)$ are such that the integral along the path C goes to zero as the radius of the circle goes to infinity, i.e., for $z = R \cdot \exp(i\varphi)$ and $R \rightarrow \infty$, then for $\delta_0 \ll 1$

$$J = \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} = \frac{\pi \cdot g(\omega_0)}{\omega_0^3 \delta_0} \quad (2.63)$$

Returning to Eq. (2.60) the integral I is now readily solved. For $g(\omega) = \omega^2$ the result given by Eqs. (2.60) and (2.63) is $I = \pi / (2\omega_0 \delta_0)$. Thus, for white noise excitation the time average of the velocity squared, Eq. (2.60), is

$$\begin{aligned} \bar{v}^2 &= \frac{a}{2m^2 \omega_0 \delta_0} \\ &= \frac{G_{FF}}{4m^2 \omega_0 \delta_0} \end{aligned} \quad (2.64)$$

The time average of the kinetic energy is consequently

$$\bar{T} = \frac{m \bar{v}^2}{2} = \frac{G_{FF}}{8m \omega_0 \delta_0} \quad (2.65)$$

The time average of the potential energy for the mass-spring system is, for white noise excitation and based on Eq. (2.63), equal to

$$\begin{aligned} \bar{U} &= \frac{\bar{x}^2 k_0}{2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{k_0 S_{FF}}{2m^2 [(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} \\ &= \frac{S_{FF}}{4m \omega_0 \delta_0} \\ &= \frac{G_{FF}}{8m \omega_0 \delta_0} \end{aligned} \quad (2.66)$$

For white noise excitation of a 1-DOF system $\bar{T}$ and $\bar{U}$ are equal. This is not the case for harmonic excitation. In this case equality holds only at resonance.

The time average of the power input to the linear 1-DOF system is according to Eqs. (2.55) and (2.58) equal to

$$\bar{\Pi} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \cdot \omega S_{FF}(\omega) \cdot \text{Im}H(\omega)$$

The frequency response function $H(\omega)$ can be defined as in Eq. (2.13) or in (2.17). In the last case $\text{Im}H$ is

$$\text{Im}H(\omega) = -\frac{\omega_0^2 \delta_0}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]}$$

The loss factor δ_0 is equal to $2\omega\beta/\omega_0^2$ as defined in Eq. (1.81). According to this definition, δ_0 is negative for $\omega < 0$ and positive for $\omega > 0$. Considering this, the time average of the power input to the system can be solved by means of a contour integration of the expression above. Compare the discussion at the end of Sect. 2.2. The alternative approach for solving the integral is to use Eq. (2.17) defining $H(\omega)$ as demonstrated in Problem 2.12. In the first case the time average of the input power for white noise excitation of the system is

$$\begin{aligned} \bar{\Pi} &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \cdot \omega S_{FF}(\omega) \text{Im}H(\omega) \\ &= S_{FF} \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot \frac{\omega \omega_0^2 \delta_0}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} d\omega \\ &= \frac{S_{FF}}{2m} \\ &= \frac{G_{FF}}{4m} \end{aligned} \tag{2.67}$$

Considering the results shown in Eqs. (2.65)–(2.67) the following important result is obtained:

$$\bar{\Pi} = \omega_0 \delta_0 (\bar{T} + \bar{U}) = m \bar{v}^2 \omega_0 \delta_0 \tag{2.68}$$

Thus, for white noise excitation of a linear mechanical system, the time average of the total energy of the system is proportional to the time average of the input power to the system and inversely proportional to its losses. The same result is not valid for harmonic excitation since for that case equality does not hold for the kinetic and potential energies. However, as in the case with white noise excitation the average velocity squared of the system is proportional to the input power and inversely proportional to the losses as demonstrated in Eq. (1.65). The last part of the expression (2.68) can be written in an alternative way since $\omega_0^2 = k_0/m$. Thus,

$$\bar{v}^2 = \frac{\bar{\Pi}}{\delta_0 \sqrt{mk_0}} \quad (2.69)$$

For a fixed power input to a system the velocity of the mass can be decreased if the stiffness, mass, or lossfactor is increased. In most practical cases it can be more or less impossible to increase mass or stiffness due to some other constraints. However, the losses can in general be altered as discussed in Chap. 5. These conclusions are somewhat deceptive since the input power to a system seldom is constant. If a 1-DOF system is excited by a harmonic force the average velocity squared is almost independent of the losses if the frequency of the harmonic force is somewhat different from the natural frequency of the system as shown in Eq. (2.54). Consequently, for this particular case added losses to the system do not significantly decrease the velocity of the mass. However, for white noise excitation the time average of the velocity squared is inversely proportional to the loss factor, Eq. (2.64). For this case added losses will always decrease the velocity of the mass.

2.8 Frequency Response and Point Mobility Functions

From Eq. (2.17) the frequency response function $H(\omega)$ for a linear 1-DOF system or quite simply a mass-spring system is

$$\begin{aligned} H(\omega) &= \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \\ &= \frac{(\omega_0^2 - \omega^2) - i\omega_0^2\delta_0}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta_0)^2]} = |H(\omega)| e^{i\varphi_1} \end{aligned} \quad (2.70)$$

where

$$|H(\omega)| = \frac{1}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta_0)^2]^{1/2}}, \quad \varphi_1 = -\arctan \left[\frac{\omega_0^2\delta_0}{(\omega_0^2 - \omega^2)} \right] \quad (2.71)$$

The loss factor δ_0 can be a function of frequency as previously discussed. For example, if the losses are viscous then $\delta_0 = c\omega/k_0$, where c and k_0 are constants as defined in Chap. 1. For viscous losses $|H|$ has a maximum for $\omega = \sqrt{\omega_0^2 - (c/m)^2/2}$. In most practical cases the losses are small, which means that with a sufficient degree of accuracy the maximum of the absolute value of the frequency response function is obtained for $\omega = \omega_0$. If the loss factor is independent of frequency, then the maximum value is obtained exactly for $\omega = \omega_0$ and the value is

$$|H(\omega)|_{\max} = \frac{1}{m\omega_0^2\delta_0}$$

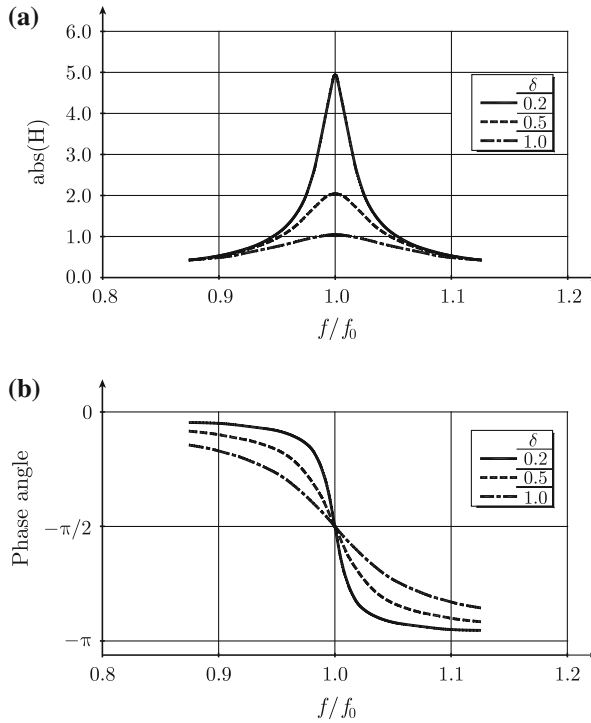


Fig. 2.6 Absolute value of frequency response function and phase angle for different loss factors

In Fig. 2.6 the function $|H|$ and φ_1 are shown for some different values of δ_0 . The limiting values are

$$\begin{aligned} |H(\omega)| &\rightarrow 1/k_0, & \varphi_1 &\rightarrow 0 & \text{as } \omega &\rightarrow 0 \\ |H(\omega)| &\rightarrow |H(\omega)|_{\max}, & \varphi_1 &\rightarrow -\pi/2 & \text{as } \omega &\rightarrow \omega_0 \end{aligned}$$

$$|H(\omega)| \rightarrow 1/(m\omega^2), \quad \varphi_1 \rightarrow -\pi \quad \text{as } \omega \rightarrow \infty$$

For a loss factor δ_0 of the order 0.1 the maximum at resonance is not very distinct. However, a loss factor of 0.1 is fairly high. Typical loss factors for various materials are presented in Chap. 3. Based on measurements of frequency response functions the resonance frequency for a heavily damped system is most readily determined from a phase diagram as shown in Fig. 2.6. The resonance is at the frequency for which the phase shift is $-\pi/2$.

The frequency response function has an interesting characteristic which is apparent if the function is plotted in a Nyquist diagram i.e. the imaginary part of H is plotted as a function of the real part of H . According to Eq. (2.70)

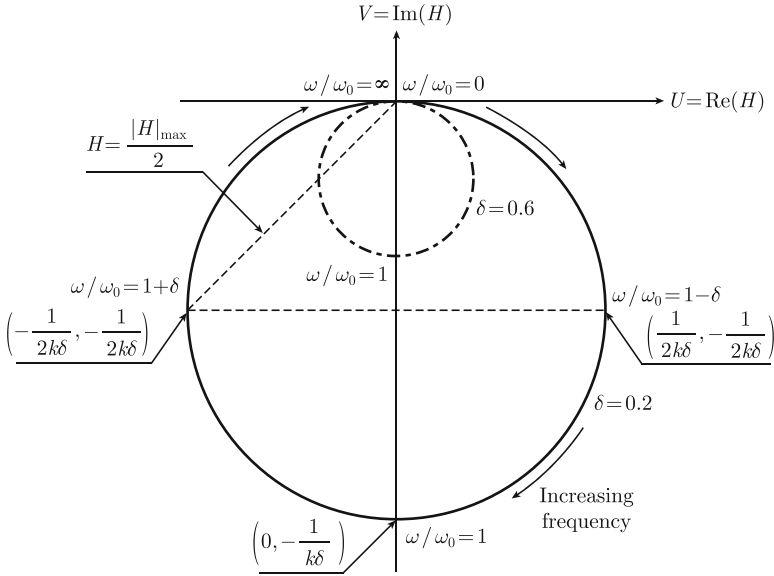


Fig. 2.7 Two Nyquist plots of transfer functions

$$\begin{aligned} \operatorname{Re} H(\omega) = U &= \frac{(\omega_0^2 - \omega^2)}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} \\ \operatorname{Im} H(\omega) = V &= \frac{-\omega_0^2 \delta_0}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} \end{aligned} \quad (2.72)$$

The parameters U and V fulfil the equation for a circle, i.e.,

$$U^2 + (V + V_0)^2 = V_0^2, \quad V_0 = \frac{1}{2m\omega_0^2 \delta_0} = \frac{1}{2k_0 \delta_0} \quad (2.73)$$

The radius of the circle is inversely proportional to the loss factor. Two examples of Nyquist diagrams of H are given in Fig. 2.7. For systems with several degrees of freedom similar curves but not exact circles are obtained for frequencies at and very close to each resonance frequency. The distortion of the circular shape for a multi-degree of freedom system depends on how close the resonance frequencies are. For a large spacing the shape of the curve is almost a perfect circle. Note that when $|H| = |H|_{\max}/2$ then $U = \pm V$ as indicated in Fig. 2.7.

The point mobility function $Y(\omega)$ is related to the frequency response function. If a force F , FT $\hat{F}$, is applied to a certain point of a dynamical system and if the resulting FT of the velocity in that particular point is $\hat{v}$, then the point mobility $Y(\omega)$ is defined as:

$$Y(\omega) = \frac{\hat{v}(\omega)}{\hat{F}(\omega)} \quad (2.74)$$

It has already been shown that the FT of the deflection of the point at which a force F , FT $\hat{F}$, is applied is given by $\hat{x} = \hat{F}H$. It has also been demonstrated that the FT of the velocity in that point is $\hat{v} = i\omega\hat{x}$. Based on these two expressions and Eq. (2.74) the point mobility is written as

$$Y(\omega) = \frac{\hat{v}(\omega)}{\hat{F}(\omega)} = \frac{i\omega\hat{x}(\omega)}{\hat{F}(\omega)} = i\omega H(\omega) \quad (2.75)$$

Accordingly, for a simple mass-spring system mounted on an infinitely stiff foundation the point mobility of the mass is obtained from Eqs. (2.75) and (2.70)

$$\begin{aligned} Y(\omega) &= \frac{i\omega}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \\ &= \frac{\omega\omega_0^2\delta_0 + i\omega(\omega_0^2 - \omega^2)}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta_0)^2]} = |Y(\omega)| e^{i\varphi_2} \end{aligned} \quad (2.76)$$

The absolute value of the mobility and the phase angle are

$$\begin{aligned} |Y(\omega)| &= \frac{|\omega|}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta_0)^2]^{1/2}}, \\ \varphi_2 &= -\arctan \left[\frac{(\omega^2 - \omega_0^2)}{\omega_0^2\delta_0} \right] = \varphi_1 + \frac{\pi}{2} \end{aligned} \quad (2.77)$$

In Fig. 2.8 the point mobility and phase angle are given for a few cases for which the loss factor has been varied. Compare the corresponding results for the frequency response function, Fig. 2.6.

Again, some limiting values

$$\begin{aligned} |Y(\omega)| &\rightarrow \omega/k_0, & \varphi_2 &\rightarrow \pi/2 & \text{as } \omega &\rightarrow 0 \\ |Y(\omega)| &\rightarrow |Y(\omega)|_{\max}, & \varphi_2 &\rightarrow 0 & \text{as } \omega &\rightarrow \omega_0 \\ |Y(\omega)| &\rightarrow 1/(m\omega), & \varphi_2 &\rightarrow -\pi/2 & \text{as } \omega &\rightarrow \infty \end{aligned}$$

In the first case, $\omega \rightarrow 0$, the absolute value of the point mobility can be said to be stiffness controlled. In the second case, i.e., in the high frequency region $|Y(\omega)|$ is mass controlled. When the mass and the stiffness of the spring approach infinity, the point mobility tends to zero.

The point mobility of a mass-spring system or, in fact, any system is a very important parameter, which determines the power input to the system. Returning to Eqs. (2.55) and (2.56) it is demonstrated that the cross-power spectral density G_{Fv} is

$$G_{Fv}(\omega) = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^* \hat{v}}{T}$$

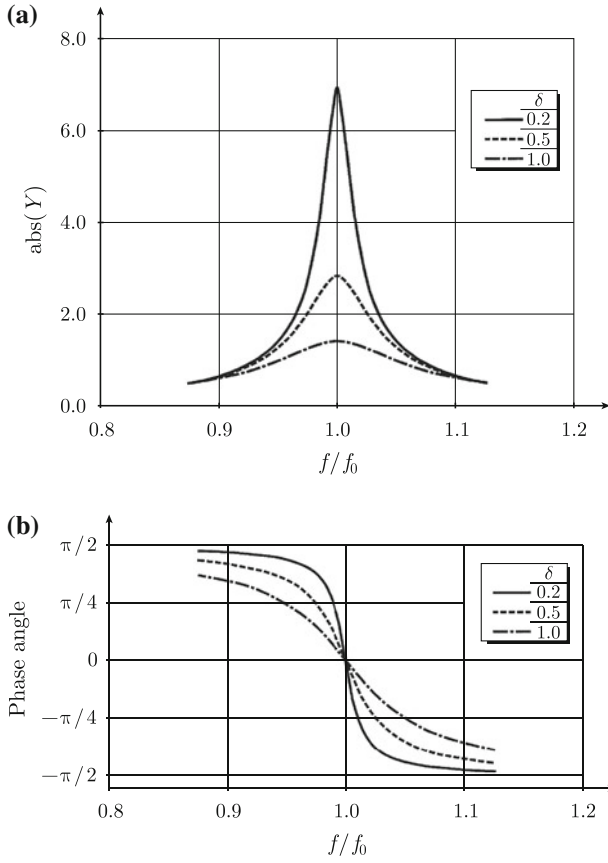


Fig. 2.8 Absolute value of point mobility and corresponding phase angle as function of frequency for different loss factors

However, $\hat{v} = \hat{F}Y$. Thus,

$$G_{Fv}(\omega) = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^* \hat{v}}{T} = \lim_{T \rightarrow \infty} 2 \frac{\hat{F}^* \hat{F} Y}{T} = G_{FF} Y \quad (2.78)$$

This expression is the same as Eq.(2.57) considering that $Y = i\omega H$. The time average of the power input to the system is according to Eq.(2.55) given by

$$\bar{\Pi} = \frac{1}{2\pi} \text{Re} \int_0^\infty G_{Fv}(\omega) d\omega = \frac{1}{2\pi} \int_0^\infty \text{Re} Y \cdot G_{FF}(\omega) d\omega \quad (2.79)$$

Thus, the time average of the power input to a system is a function of the real part of the point mobility of the structure at the excitation point.

In all the previous examples it has been assumed that the point mobility of the foundation on which a mass-spring system is mounted is equal to zero, i.e., the foundation is infinitely stiff. If however this point mobility is known from measurements or calculations, the resulting motion of the mass can be derived as functions of the mobility at the point on the foundation where the spring is mounted.

Thus, consider the system shown in Fig. 1.11. The equation of motion for the mass m is

$$m\ddot{x} + k(x - y) = F \quad (2.80)$$

The force on the foundation is $F_f = k(x - y)$. The spring constant k is complex or $k = k_0(1 + i\delta_0)$. In the equation above the deflections of mass and foundation are x and y respectively. The problem is solved by making the following transformations:

$$x \rightarrow \hat{x} \cdot e^{i\omega t}, \quad y \rightarrow \hat{y} \cdot e^{i\omega t}, \quad F \rightarrow \hat{F} \cdot e^{i\omega t}$$

This type of transformation was discussed in Sect. 2.2. Consequently $\hat{x}$ is the FT of x and so on. The velocity of the foundation depends on the force F exciting it and the point mobility at the excitation point. The FT of the velocity of the foundation is

$$\hat{y} = i\omega \hat{y} = \hat{F}_f \cdot Y = kY(\hat{x} - \hat{y}) \quad (2.81)$$

where Y is its point mobility of the foundation. The displacements of mass and foundation are given by two equations

$$\hat{x}(-m\omega^2 + k) - k\hat{y} = \hat{F} \quad \text{and} \quad i\omega \hat{y} = kY(\hat{x} - \hat{y}) \quad (2.82)$$

These equations are obtained after inserting the transformations above in Eq. (2.80). From Eq. (2.82) the FTs of the displacements $\hat{x}$ and $\hat{y}$ are

$$\hat{x} = \frac{\hat{F}(kY + i\omega)}{-m\omega^2 kY + i\omega k - i\omega^3 m}, \quad \hat{y} = \frac{\hat{F}kY}{-m\omega^2 kY + i\omega k - i\omega^3 m} \quad (2.83)$$

In the limiting case for an infinitely stiff foundation $Y = 0$, the standard solution for $\hat{x}$ is obtained. With no foundation at all, Y is infinite and $\hat{x} = -\hat{F}/(m\omega^2)$. This is also the response of a completely free mass.

From Eq. (2.83) it is obvious that the resonance frequencies for the system are functions of Y . Depending on the point mobility the system can now have multi degrees of freedom. For $\text{Re}Y > 0$ there is an energy flow to the foundation that increases the losses of the mass-spring system. Point and transfer mobilities for beams and plates are discussed in subsequent chapters.

2.9 Loss Factor

The loss factor for a mechanical system is clearly a very important parameter. Detailed knowledge of this factor is essential in order to describe the motion and energy balance of any system. The loss factor is often measured by means of either of four methods:

- (i) half band width;
- (ii) reverberation time;
- (iii) power injection;
- (iv) Nyquist plot.

The last technique was reviewed in Sect. 2.8. In addition, the modal analysis method can be used as discussed in Ref. [12].

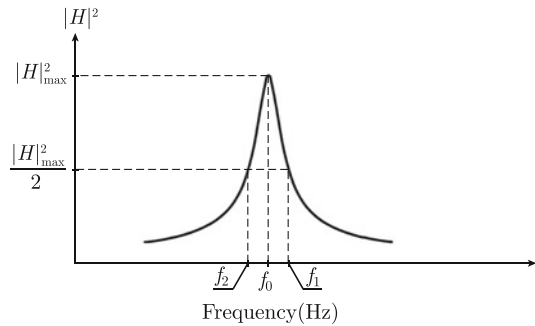
For a hypothetical 1-DOF system the four methods are straightforward. So let us again return to the simple mass-spring system with mass m and complex spring constant $k = k_0(1 + i\delta)$. The frequency response function H for a 1-DOF system relating force and displacement can be determined from measurements of the cross-spectral density G_{Fx} between force and displacement and the power spectral density G_{FF} of the force function. According to definition the frequency response function is obtained as

$$H(\omega) = \frac{G_{Fx}(\omega)}{G_{FF}(\omega)} \quad (2.84)$$

One example of a frequency response function or $|H| = |\hat{x}/\hat{F}|$ is shown in Fig. 2.9. Compare also Fig. 2.6. The resonance frequency f_0 , angular frequency $\omega_0 = 2\pi f_0$, is the frequency for which $|H|$ has a maximum and when the phase angle is equal to $-\pi/2$ as shown in Fig. 2.6. The energy of a 1-DOF system is proportional to $|H|^2$. A frequently used concept for the characterization of a system is to define the angular frequencies ω_1 and ω_2 for which

$$|H(\omega_1)|^2 = |H(\omega_2)|^2 = |H(\omega_0)|^2 / 2 = |H|_{\max}^2 \quad (2.85)$$

Fig. 2.9 Determination of loss factor based on the half band method



These angular frequencies satisfying Eq. (2.85) can be derived from Eq. (2.71) by considering the denominator in the expression for $|H(\omega)|$. Thus,

$$(\omega_0^2 - \omega_{1,2}^2)^2 + (\omega_0^2 \delta_0)^2 = 2(\omega_0^2 \delta_0)^2$$

The roots are $\omega_{1,2} = \omega_0 \sqrt{1 \pm \delta_0}$. For $\delta \ll 1$, which is generally the case, the roots can be simplified as

$$\omega_1 = \omega_0(1 + \delta_0/2), \quad \omega_2 = \omega_0(1 - \delta_0/2) \quad (2.86)$$

Consequently

$$\delta_0 = \frac{\omega_1 - \omega_2}{\omega_0} = \frac{f_1 - f_2}{f_0} \quad (2.87)$$

The frequency difference $\Delta f = f_1 - f_2$ is often referred to as the half bandwidth of the resonance peak.

Thus, if the frequency response function for a system can be determined through measurements as function of frequency, then the resonance frequency and the loss factor can be estimated. The half bandwidth of the resonance peak is equal to the width of the peak when the maximum absolute value of the frequency response function is reduced by a factor $\sqrt{2}$. The loss factor obtained from Eq. (2.87) is the loss factor at the frequency f_0 . The quality of the estimate depends on the resolution of the frequency analyser used. For continuous systems or in fact real structures excited by a force there are an infinite number of resonances. The various resonance peaks can be extremely difficult to separate even in the low frequency region. The half bandwidth technique is best suited for simple systems like beams. Even so, only approximately the first ten resonances can usually be identified. For a plate, resonances are very close or even coinciding making the technique less useful.

It is interesting to return to the Nyquist diagram of the frequency response function shown in Fig. 2.6. The angular frequencies for which $\text{Re}H = \pm \text{Im}H$ are equal to ω_1 and ω_2 given in Eq. (2.86). This follows directly from the discussion in Sect. 2.3. The Nyquist plot can be used also for continuous systems. However, the proximity of resonance frequencies makes the determination of loss factors difficult.

The reverberation time method is based on the fact that free vibrations of a system decay with time. If a system is excited by a constant force which at a certain time is turned off or if the system is excited by an impulse, for example a hammer blow, then the motion of the system is determined by free vibrations. For lightweight structures a gentle knock by the fist is often sufficient to excite the structures. For heavy solid structures a hammer blow might be required. At other times a shaker is needed. The response of a 1-DOF system excited by an impulse is given by Eq. (1.79). The kinetic energy of the system is decaying as $\exp(-\omega_0 \delta_0 t)$ as shown in Eq. (1.80). Thus,

$$\bar{v}^2 = C \cdot e^{-\omega_0 \delta_0 t} \quad (2.88)$$

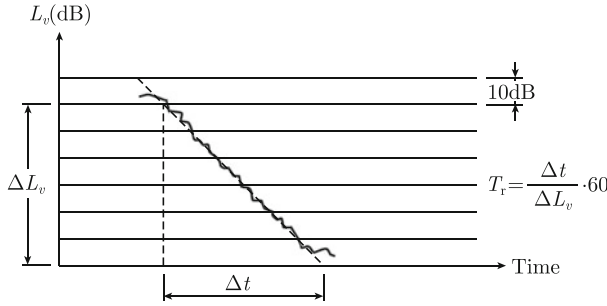


Fig. 2.10 Decay curve for measurement of reverberation time

where C is some constant. The velocity level of the mass as a function of time is

$$L_v(t) = 10 \log \left(\frac{\bar{v}^2}{v_{\text{ref}}^2} \right) = L_v(0) - 10\omega_0\delta_0 t \log e \quad (2.89)$$

The velocity level at $t = 0$ is $L_v(0)$. An example of a measured decay curve is given in Fig. 2.10. The decay curve is fairly smooth if ω_0 is sufficiently large. The reverberation time T_r in seconds is defined as the time during which the velocity level is decreased 60 dB. Thus, from Eq. (2.89) $10\omega_0\delta_0 T_r \log e = 60$.

The loss factor is consequently given by

$$\delta_0 = \frac{6}{2\pi f_0 T_r \log e} \approx \frac{2.2}{f_0 T_r} \quad (2.90)$$

Very often it is not possible to record the decay for 60 dB. This could for example be due to noise in the system or a broken decay curve as shown in Fig. 2.11. The general procedure is therefore to measure the time for the velocity level to decrease from say the 5 dB level to the 20 dB level below the maximum. The reverberation time is

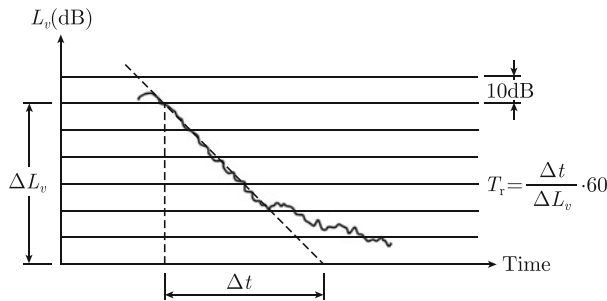


Fig. 2.11 Broken decay curve

thereafter obtained through extrapolation. The reverberation time and thus the loss factor can only be determined at a natural frequency of the system. For continuous systems the reverberation time and loss factors are determined for frequency bands, typically 1/3 octave bands. The resulting loss factor then represents some frequency average over the band width since each band could include a number of natural frequencies.

The power injection method has already been discussed in Sect. 2.7. If a constant force is acting on a mass of a mass-spring system the resulting velocity of the mass is a function of the loss factor. The result given in Eq. (2.68) means that if the time average of the input power $\bar{\Pi}$ and the time average of the velocity squared $\bar{v}^2$ are known then for a 1-DOF system

$$\delta_0 = \frac{\bar{\Pi}}{m\omega_0\bar{v}^2} \quad (2.91)$$

From Eq. (2.54) the power spectral density G_{vv} for the velocity is given as

$$G_{vv} = |H|^2 \omega^2 G_{FF}$$

The real part of the cross-spectral density G_{Fv} is according to Eq. (2.57) equal to

$$\text{Re}G_{Fv} = -\omega G_{FF} \text{Im}H$$

According to Eq. (2.17) the imaginary part of the frequency response function H for a 1-DOF system is

$$\begin{aligned} \text{Im}H(\omega) &= \text{Im} \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2\delta_0]} \\ &= \frac{-\omega_0^2\delta_0}{m[(\omega_0^2 - \omega^2)^2 + (\omega_0^2\delta_0)^2]} = -m\omega_0^2\delta_0 |H|^2 \end{aligned}$$

From these expressions it follows that

$$\delta_0 = \frac{\omega}{m\omega_0^2} \cdot \frac{\text{Re}G_{Fv}}{G_{vv}} \quad (2.92)$$

The methods discussed above can also be used to determine the loss factors for systems with multidegrees of freedom. For plate structures the half bandwidth method is difficult to use in the high frequency range since the resonance peaks can be closely linked or spaced. It can therefore be impossible to determine the necessary frequencies f_0 , f_1 and f_2 for all but a few resonances.

The reverberation time technique is more versatile, when as is generally the case, an average loss factor is required for a certain frequency band. The structure is excited by white noise within a frequency band while the time decay of the velocity

level within this frequency band is recorded. This band could contain one or several resonances. This could result in a broken decay curve of the type shown in Fig. 2.10. In a case like this the upper part of the decay curve should always be used, in order to get a proper average of the losses within the band. It is consequently of great importance to have a visual display of the reverberation curve.

The power injection method can always be used for any type of structure as long as this structure is not connected to any other construction. If this is the case the vibrating mass can not be determined. The resulting loss factor includes losses to adjoining structures. However, this is always the case whenever loss factors are measured. Detailed studies indicate that the use of the reverberation time technique ensures the most reliable results as compared to the other three procedures.

2.10 Response of a 1-DOF System, A Summary

A number of different representations have been used to describe the response of a simple 1-DOF system. For the sake of completeness it is therefore appropriate to make a comparison between the different techniques.

1. Harmonic force excitation

A 1-DOF system is excited by a harmonic force $F(t) = F_0 \sin \omega_1 t$. The equation of motion of the system is

$$m\ddot{x} + c\dot{x} + k_0x = F_0 \sin \omega_1 t$$

According to Eq. (1.57) the response of the mass is given by

$$\begin{aligned} x(t) &= A_1 \sin \omega_1 t + A_2 \cos \omega_1 t = A_0 \sin(\omega_1 t + \varphi) \\ &= \frac{F_0[(\omega_0^2 - \omega_1^2) \sin \omega_1 t - (\omega_0^2 \delta_0) \cos \omega_1 t]}{m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \end{aligned} \quad (2.93)$$

where A_0 , A_1 and A_2 and φ are defined in Eq. (1.60). The time average, over the period T , of the velocity squared is

$$\bar{v}^2 = \frac{1}{T} \int_0^T dt \dot{x}^2 = \frac{\omega_1^2 F_0^2}{2m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.94)$$

The time average of the force squared is $\bar{F}^2 = |F_0|^2 / 2$. Thus, Eq. (2.94) can also be written as

$$\bar{v}^2 = \bar{F}^2 \frac{\omega_1^2}{m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.95)$$

The time average of the input power to the system is

$$\begin{aligned}\bar{\Pi} &= \frac{1}{T} \int_0^T dt F \dot{x} = \frac{\omega_1 \omega_0^2 \delta_0 F_0^2}{2m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \\ &= \bar{F}^2 \frac{\omega_1 \omega_0^2 \delta_0}{m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]}\end{aligned}\quad (2.96)$$

2. Harmonic force excitation, complex notation

The equation of motion for the system is given by Eq. (1.82) as

$$m\ddot{x} + kx = F(t)$$

where

$$F(t) = F_0 \exp(i\omega_1 t), \quad k = k_0(1 + i\delta)$$

The response, having the same time dependence as the force, is

$$x = x_0 e^{i\omega_1 t} = \frac{F_0 e^{i\omega_1 t}}{k - m\omega_1^2} = \frac{F_0 e^{i\omega_1 t}}{m[\omega_0^2(1 + i\delta_0) - \omega_1^2]} \quad (2.97)$$

The time average of the absolute value of the force squared is $|\bar{F}|^2 = |\bar{F}_0|^2/2$ and the time average of the absolute value of the velocity squared is

$$\begin{aligned}|\bar{v}^2| &= \frac{|\dot{x}|^2}{2} = \frac{|F_0|^2 \omega_1^2}{2m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \\ &= \bar{F}^2 \frac{\omega_1^2}{m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]}\end{aligned}\quad (2.98)$$

The time average of the input power to the system is

$$\begin{aligned}\bar{\Pi} &= \frac{1}{2} \text{Re}(F \cdot v^*) = \frac{|F_0|^2 \omega_1 \omega_0^2 \delta_0}{2m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \\ &= \frac{|\bar{F}|^2 \omega_1 \omega_0^2 \delta_0}{m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]}\end{aligned}\quad (2.99)$$

Thus, as in the previous case, Eq. (2.98) is identical to Eq. (2.95) and Eq. (2.99) to (2.96).

Equation (2.99) can also be written as

$$\bar{\Pi} = \frac{1}{2} \text{Re}(F \cdot v^*) = \frac{|F_0|^2}{2} \text{Re} Y \quad (2.100)$$

where Y is the mobility at the excitation point.

3. Fourier transform

In the time domain the equation of motion of the system is as before written as

$$m\ddot{x} + kx = F(t)$$

where

$$F(t) = F_0 \exp(i\omega t), \quad k = k_0(1 + i\delta)$$

By making the substitutions $F \rightarrow \hat{F}e^{i\omega t}$ and $x \rightarrow \hat{x}e^{i\omega t}$ in the equation of motion, then if $\hat{F}$ is the Fourier transform of $F(t)$, it follows that $\hat{x}$ is the FT of x as discussed in Sect. 2.2. Thus

$$\hat{x} = \frac{\hat{F}}{k - m\omega^2} = \frac{\hat{F}}{m[\omega_0^2(1 + i\delta_0) - \omega^2]} = \hat{F} \cdot H(\omega) \quad (2.101)$$

The Fourier transform of the velocity is $\hat{v} = i\omega\hat{x} = i\omega\hat{F}H$. Thus, the power spectral density of the velocity is obtained as

$$\begin{aligned} S_{vv}(\omega) &= \lim_{T \rightarrow \infty} \frac{|\hat{v}|^2}{T} = \lim_{T \rightarrow \infty} \frac{\omega^2 |\hat{F}|^2 |H|^2}{T} = S_{FF}(\omega) \omega^2 |H|^2 \\ &= S_{FF}(\omega) \frac{\omega^2}{m^2[(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \delta_0)^2]} \end{aligned} \quad (2.102)$$

For harmonic excitation $F(t) = F_0 \sin \omega_1 t$ the power spectral density of the force is

$$S_{FF}(\omega) = \frac{\pi F_0^2 \cdot [\delta(\omega - \omega_1) + \delta(\omega + \omega_1)]}{2} \quad (2.103)$$

The time average of the velocity squared is obtained from Eqs. (2.101) and (2.103) as

$$|\bar{v}|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{vv}(\omega) d\omega = \frac{\omega_1^2 F_0^2}{2m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.104)$$

The time average of the force squared is

$$|\bar{F}|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{FF}(\omega) d\omega = \frac{|F_0|^2}{2}$$

Thus,

$$|\bar{v}|^2 = \frac{\omega_1^2 |\bar{F}|^2}{m^2[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.105)$$

The power spectral density of the input power is given as

$$G_{\Pi} = G_{FF} \text{Re}(i\omega H) = G_{FF} \text{Re}(Y) = G_{FF} \frac{\omega_0^2 \omega_1 \delta_0}{m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.106)$$

The time average of the power in put is

$$\bar{\Pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{\Pi}(\omega) d\omega = \frac{|\bar{F}|^2 \omega_0^2 \omega_1 \delta_0}{m[(\omega_0^2 - \omega_1^2)^2 + (\omega_0^2 \delta_0)^2]} \quad (2.107)$$

Thus, using any of the three methods the results are the same with respect to the time averages of velocity squared and input power. The results support the convention introduced in Sect. 1.6 with respect to complex notations.

Problems

2.1 Determine the FT of the function

$$\begin{aligned} h(t) &= \exp(-\beta t) \cdot \sin(\omega_n t) / (m\omega_n) \text{ for } t \geq 0 \\ h(t) &= 0 \text{ for } t < 0 \end{aligned}$$

where $\omega_n^2 = \omega_0^2 - \beta^2$ and $\beta = \omega_0^2 \delta / (2\omega) > 0$.

2.2 A periodic signal $x(t) = x(t + T)$ is a function of time as

$$x(t) = A \text{ for } 0 \leq t \leq T/2 \text{ and } x(t) = 0 \text{ for } T/2 < t < T$$

Determine the autocorrelation function and power spectral density of the signal.

2.3 The frequency response function $H(\omega)$ of a 1-DOF system is

$$H(\omega) = \frac{1}{-m\omega^2 + k} = \frac{1}{m[(\omega_0^2 - \omega^2) + i\omega_0^2 \delta]}$$

Show that the inverse FT of H is equal to

$$h(t) = \exp(-\omega_0 t \delta / 2) \cdot \sin(\omega_0 t) / (m\omega_0)$$

2.4 The mass of a mass-spring system is excited by the force defined in example 2.2. Determine the time average of the velocity squared for the mass m . The spring constant is

$$k = k_0(1 + i\delta).$$

2.5 Determine the autocorrelation functions for band-pass white noise and low-pass white noise. In the first case $G_{xx}(f) = a$ for $0 \leq f_0 - B/2 \leq f \leq f_0 + B/2$ and in the second case $G_{xx}(f) = a$ for $0 \leq f \leq B$.

2.6 A force $F(t)$ is applied to the mass m of a mass-spring system. The complex spring constant is given by $k = k_0(1 + i\delta)$. The force is $F(t) = A \sin(\omega_1 t) + \xi(t)$, where $\xi(t)$ is a random signal with the one sided power spectral density $G_{\xi\xi} = A^2/(2\omega_0)$ where $\omega_0^2 = k_0/m$. Determine the time average of the velocity squared of the mass.

2.7 Determine the time averages of the potential and kinetic energies of a mass-spring system for which the mass is excited by a force $F(t) = F_0 \cdot \sin(\omega_1 t)$.

2.8 Determine the time average of the velocity squared of the mass of a lightly damped mass-spring system excited by a force characterized by an exponential autocorrelation function, i.e., having a one-sided power spectral density

$$G_{FF}(\omega) = \frac{4a}{a^2 + \omega^2}.$$

2.9 A mass-spring system is mounted on a foundation as shown in Fig. 1.1. The point mobility of the foundation is Y_f . Determine the point mobility Y in the excitation point.

2.10 For the system described in Problem 2.9 determine the one sided power spectral density of the power transmitted to the foundation. The power spectral density of the force exciting the mass is constant and equal to G_{FF} . The point mobility of the foundation is Y_f . Determine also the time average of the power input to the foundation if Y_f is real and much smaller than unity and in addition independent of frequency.

2.11 The mass of a mass-spring system is excited by a force $F(t)$, with the one-sided power spectral density G_{FF} . The response of the mass is $z(t) = x(t) + y(t)$ where $y(t)$ is due to extraneous and random noise. The one-sided power spectral density of the random signal y is G_{yy} . The response x due to the force can be written as $x = HF$ where H is the frequency response function for the system. Determine the coherence function between the force F and the displacement z .

2.12 Determine the time average of the power input

$$\bar{\Pi} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \cdot S_{FF}(\omega) \cdot \text{Im}(H)$$

when the frequency response function is defined according to Eq. (2.17) as

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega c + k_0} = \frac{1}{m(\omega_0^2 - \omega^2 + 2i\beta\omega)}$$

2.13 A function $x(t)$ is written $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$ in the time interval $-T/2 \leq t \leq T/2$. Show that as $T \rightarrow \infty$ the function can be written in integral form as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \cdot \hat{x}(\omega) e^{i\omega t} \quad \text{where} \quad \hat{x}(\omega) = \int_{-\infty}^{\infty} dt \cdot x(t) e^{-i\omega t}$$

2.14 Show that $E[\dot{x}^2(t)] = - \left[\frac{d^2 R_{xx}}{d\tau^2} \right]_{\tau=0}$.

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