Stable Marriage and Roommates Problems with Restricted Edges: Complexity and Approximability

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Abstract. In the stable marriage and roommates problems, a set of agents is given, each of them having a strictly ordered preference list over some or all of the other agents. A matching is a set of disjoint pairs of mutually acceptable agents. If any two agents mutually prefer each other to their partner, then they block the matching, otherwise, the matching is said to be stable. We investigate the complexity of finding a solution satisfying additional constraints on restricted pairs of agents. Restricted pairs can be either *forced* or *forbidden*. A stable solution must contain all of the forced pairs, while it must contain none of the forbidden pairs.

Dias et al. [5] gave a polynomial-time algorithm to decide whether such a solution exists in the presence of restricted edges. If the answer is no, one might look for a solution close to optimal. Since optimality in this context means that the matching is stable and satisfies all constraints on restricted pairs, there are two ways of relaxing the constraints by permitting a solution to: (1) be blocked by as few as possible pairs, or (2) violate as few as possible constraints on restricted pairs.

Our main theorems prove that for the (bipartite) stable marriage problem, case (1) leads to NP-hardness and inapproximability results, whilst case (2) can be solved in polynomial time. For non-bipartite stable roommates instances, case (2) yields an NP-hard but (under some cardinality assumptions) 2-approximable problem. In the case of NP-hard problems, we also discuss polynomially solvable special cases, arising from restrictions on the lengths of the preference lists, or upper bounds on the numbers of restricted pairs.

1 Introduction

In the classical *stable marriage problem* (SM) [10], a bipartite graph is given, where one color class symbolises a set of men U and the other color class stands

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for a set of women W. Man u and woman w are connected by edge uw if they find one another mutually acceptable. Each participant provides a strictly ordered preference list of the acceptable agents of the opposite gender. An edge uw *blocks* matching M if it is not in M, but each of u and w is either unmatched or prefers the other to their partner. A *stable matching* is a matching not blocked by any edge. From the seminal paper of Gale and Shapley [10], we know that the existence of such a stable solution is guaranteed and one can be found in linear time. Moreover, the solutions form a distributive lattice [20]. The two extreme points of this lattice are called the *man-* and *woman-optimal stable matchings* [10]. These assign each man/woman their best partner reachable in any stable matching. Another interesting and useful property of stable solutions is the so-called Rural Hospitals Theorem. Part of this theorem states that if an agent is unmatched in one stable matching, then all stable solutions leave him unmatched [11].

One of the most widely studied extensions of SM is the *stable roommates problem* (SR) [10,14], defined on general graphs instead of bipartite graphs. The notion of a blocking edge is as defined above (except that it can now involve any two agents in general), but several results do not carry over to this setting. For instance, the existence of a stable solution is not guaranteed any more. On the other hand, there is a linear-time algorithm to find a stable matching or report that none exists [14]. Moreover, the corresponding variant of the Rural Hospitals Theorem holds in the roommates case as well: the set of matched agents is the same for all stable solutions [12].

Both SM and SR are widely used in various applications. In markets where the goal is to maximise social welfare instead of profit, the notion of stability is especially suitable as an optimality criterion [22]. For SM, the oldest and most common area of applications is employer allocation markets [24]. On one side, job applicants are represented, while the job openings form the other side. Each application corresponds to an edge in the bipartite graph. The employers rank all applicants to a specific job offer and similarly, each applicant sets up a preference list of jobs. Given a proposed matching M of applicants to jobs, if an employer-applicant pair exists such that the position is not filled or a worse applicant is assigned to it, and the applicant received no contract or a worse contract, then this pair blocks M. In this case the employer and applicant find it mutually beneficial to enter into a contract outside of M, undermining its integrity. If no such blocking pair exists, then M is stable. Stability as an underlying concept is also used to allocate graduating medical students to hospitals in many countries [23]. SR on the other hand has applications in the area of P2P networks [9].

Forced and forbidden edges in SM and SR open the way to formulate various special requirements on the sought solution. Such edges now form part of the extended problem instance: if an edge is *forced*, it must belong to a constructed stable matching, whilst if an edge is *forbidden*, it must not. In certain market situations, a contract is for some reason particularly important, or to the contrary, not wished by the majority of the community or by the central authority in control. In such cases, forcing or forbidding the edge and then seeking a stable solution ensures that the wishes on these specific contracts are fulfilled while stability is guaranteed. Henceforth, the term *restricted edge* will be used to refer either to a forbidden edge or a forced edge. The remaining edges of the graph are referred as *unrestricted edges*.

Note that simply deleting forbidden edges or fixing forced edges and searching for a stable matching on the remaining instance does not solve the problem of finding a stable matching with restricted edges. Deleted edges (corresponding to forbidden edges, or those adjacent to forced edges) can block that matching. Therefore, to meet both requirements on restricted edges and stability, more sophisticated methods are needed.

The attention of the community was drawn very early to the characterization of stable matchings that must contain a prescribed set of edges. In the seminal book of Knuth [20], forced edges first appeared under the term *arranged marriages*. Knuth presented an algorithm that finds a stable matching with a given set of forced edges or reports that none exists. This method runs in $O(n^2)$ time, where n denotes the number of vertices in the graph. Gusfield and Irving [12] provided an algorithm based on rotations that terminates in $O(|Q|^2)$ time, following $O(n^4)$ pre-processing time, where Q is the set of forced edges. This latter method is favoured over Knuth's if multiple forced sets of small cardinality are proposed.

Forbidden edges appeared only in 2003 in the literature, and were first studied by Dias et al. [5]. In their paper, complete bipartite graphs were considered, but the methods can easily be extended to incomplete preference lists. Their main result was the following (in the following theorem, and henceforth, m is the total number of edges in the graph).

Theorem 1 (Dias et al. [5]). *The problem of finding a stable matching in a* SM *instance with forced and forbidden edges or reporting that none exists is solvable in $O(m)$ time.*

While Knuth's method relies on basic combinatorial properties of stable matchings, the other two algorithms make use of *rotations*. We refer the reader to [12] for background on these. The problem of finding a stable matching with forced and forbidden edges can easily be formulated as a weighted stable matching problem (that is, we seek a stable matching with minimum weight, where the weight of a matching M is the sum of the weights of the edges in M). Let us assign all forced edges weight 1, all forbidden edges weight -1, and all remaining edges weight 0. A stable matching satisfying all constraints on restricted edges exists if and only if there is a stable matching of weight $|Q|$ in the weighted instance, where Q is the set of forced edges. With the help of rotations, maximum weight stable matchings can be found in polynomial time [6,7,15,19].

Since finding a weight-maximal stable matching in SR instances is an NP-hard task [6], it follows that solving the problem with forced and forbidden edges requires different methods from the aforementioned weighted transformation. Fleiner et al. [8] showed that any SR instance with forbidden edges can be converted into another stable matching problem involving ties that can be solved in $O(m)$ time [16] and the transformation has the same time complexity as well. Forced edges can easily be eliminated by forbidding all edges adjacent to them, therefore we can state the following result.

Theorem 2 (Fleiner et al. [8]). *The problem of finding a stable matching in an* SR *instance with forced and forbidden edges or reporting that none exists is solvable in $O(m)$ time.*

As we have seen so far, answering the question as to whether a stable solution containing all forced and avoiding all forbidden edges exists can be solved efficiently in the case of both SM and SR. We thus concentrate on cases where the answer to this question is no. What kind of approximate solutions exist then and how can we find them?

Our Contribution. Since optimality is defined by two criteria, it is straightforward to define approximation from those two points of view. In case BP, all constraints on restricted edges must be satisfied, and we seek a matching with the minimum number of blocking edges. In case CV, we seek a stable matching that violates the fewest constraints on restricted edges. The optimization problems that arise from each of these cases are defined formally in Sect. 2.

In Sect. 3, we consider case BP: that is, all constraints on restricted edges must be fulfilled, while the number of blocking edges is minimised. We show that in the SM case, this problem is computationally hard and not approximable within $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless $\mathsf{P} = \mathsf{NP}$. We also discuss special cases for which this problem becomes tractable. This occurs if the maximum degree of the graph is at most 2 or if the number of blocking edges in the optimal solution is a constant. We point out a striking difference in the complexity of the two cases with only forbidden and only forced edges: the problem is polynomially solvable if the number of forbidden edges is a constant, but by contrast it is NP-hard even if the instance contains a single forced edge. We also prove that when the restricted edges are either all forced or all forbidden, the optimization problem remains NP-hard even on very sparse instances, where the maximum degree of a vertex is 3.

Case CV, where the number of violated constraints on restricted edges is minimised while stability is preserved, is studied in Sect. 4. It is a rather straightforward observation that in SM, the setting can be modelled and efficiently solved with the help of edge weights. Here we show that on non-bipartite graphs, the problem becomes NP-hard, but 2-approximable if the number of forced edges is sufficiently large or zero. As in case BP, we also discuss the complexity of degree-constrained restrictions and establish that the NP-hardness results remain intact even for graphs with degree at most 3, while the case with degree at most 2 is polynomially solvable.

A structured overview of our results is contained in Table 1.

2 Preliminaries and Techniques

In this section, we introduce the notation used in the remainder of the paper and also define the key problems that we investigate later. A *Stable Marriage*

Table 1. Summary of results

	Stable marriage	Stable roommates
case BP: min # blocking edges	NP-hard to approximate within $n^{1-\varepsilon}$	NP-hard to approximate within $n^{1-\varepsilon}$
case CV: min # violated restricted edge constraints	solvable in polynomial time	NP-hard; 2-approximable if $\vert Q\vert$ is large or 0

instance (SM) $\mathcal{I} = (G, O)$ consists of a bipartite graph $G = (U \cup W, E)$ with n vertices and m edges, and a set O: the set of strictly ordered, but not necessarily complete preference lists. These lists are provided on the set of adjacent vertices at each vertex. The *Stable Roommates Problem* (SR) differs from SM in one sense: the underlying graph G need not be bipartite. In both SM and SR, a matching in G is sought, assigning each agent to at most one partner. An edge $uw \in E \setminus M$ *blocks* matching M if u is unmatched or it prefers w to its partner in M and w is unmatched or it prefers u to its partner in M. A matching that is not blocked by any edge is called *stable*.

As already mentioned in the introduction, an SR instance need not admit a stable solution. The number of blocking edges is a characteristic property of every matching. The set of edges blocking M is denoted by $bp(M)$. A natural goal is to find a matching minimising $|bp(M)|$. For convenience, the minimum number of edges blocking any matching of an instance $\mathcal{I}$ is denoted by $bp(\mathcal{I})$. Following the consensus in the literature, matchings blocked by $bp(\mathcal{I})$ edges are called *almost stable matchings*. This approach has a broad literature: almost stable matchings have been investigated in SM [3,13,18] and SR [1,2] instances.

All problems investigated in this paper deal with at least one set of restricted edges. The set of forbidden edges is denoted by P, while Q stands for the set of forced edges. We assume throughout the paper that $P \cap Q = \emptyset$. A matching M satisfies all constraints on restricted edges if $M \cap P = \emptyset$ and $M \cap Q = Q$.

In Fig. 1, a sample SM instance on four men and four women can be seen. The preference ordering is shown above or below the vertices. For instance,

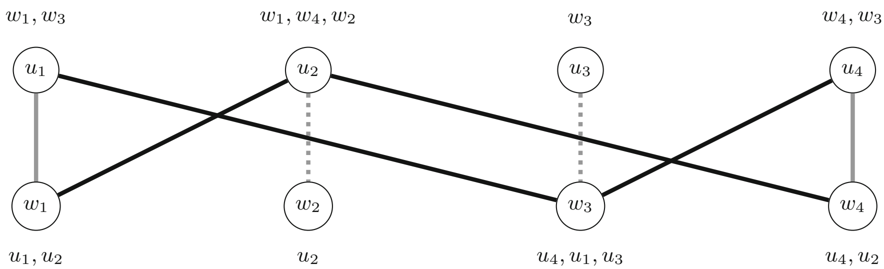

Fig. 1. A sample stable marriage instance with forbidden edges

vertex u_2 ranks w_1 best, then w_4, and w_2 last. The set of forbidden edges $P = \{u_2w_2, u_3w_3\}$ is marked by dotted gray edges. The unique stable matching $M = \{u_1w_1, u_2w_2, u_3w_3, u_4w_4\}$ contains both forbidden edges. Later on, we will return to this sample instance to demonstrate approximation concepts on it.

The first approximation concept (case BP described in Sect. 1) is to seek a matching M that satisfies all constraints on restricted edges, but among these matchings, it admits the minimum number of blocking edges. This leads to the following problem definition.

Problem 1. MIN BP SR RESTRICTED
Input: $\mathcal{I} = (G, O, P, Q)$; *an* SR *instance, a set of forbidden edges* P *and a set of forced edges* Q.
Output: A matching M *such that* $M \cap P = \emptyset$, $Q \subseteq M$ *and* $|bp(M)| \leq |bp(M')|$ *for every matching* M' *in* G *satisfying* $M' \cap P = \emptyset$, $Q \subseteq M'$.

Special attention is given to two special cases of MIN BP SR RESTRICTED: in MIN BP SR FORBIDDEN, $Q = \emptyset$, while in MIN BP SR FORCED, $P = \emptyset$. Note that an instance of MIN BP SR FORCED or MIN BP SR RESTRICTED can always be transformed into an instance of MIN BP SR FORBIDDEN by forbidding all edges that are adjacent to a forced edge. This transformation does not affect the number of blocking edges.

According to the other intuitive approximation concept (case CV described in Sect. 1), stability constraints need to be fulfilled, while some of the constraints on restricted edges are relaxed. The goal is to find a stable matching that violates as few constraints on restricted edges as possible.

Problem 2. SR MIN RESTRICTED VIOLATIONS
Input: $\mathcal{I} = (G, O, P, Q)$; *an* SR *instance, a set of forbidden edges* P *and a set of forced edges* Q.
Output: A stable matching M *such that* $|M \cap P| + |Q \setminus M| \leq |M' \cap P| + |Q \setminus M'|$ *for every stable matching* M' *in* G.

Just as in the previous approximation concept (referred to as case BP in Sect. 1), we separate the two subcases with only forbidden and only forced edges. If $Q = \emptyset$, SR MIN RESTRICTED VIOLATIONS is referred as SR MIN FORBIDDEN, while if $P = \emptyset$, the problem becomes SR MAX FORCED. In case BP, the subcase with only forced edges can be transformed into the other subcase, simply by forbidding edges adjacent to forced edges. This straightforward transformation is not valid for case CV. Suppose a forced edge was replaced by an unrestricted edge, but all of its adjacent edges were forbidden. A solution that does not contain the original forbidden edge might contain two of the forbidden edges, violating more constraints than the original solution. Yet most of our proofs are presented for the problem with only forbidden edges, and they require only slight modifications for the case with forced edges.

A powerful tool used in several proofs in our paper is to convert some of these problems into a weighted SM or SR problem, where the goal is to find a stable matching with the highest edge weight, taken over all stable matchings.

Irving et al. [15] were the first to show that the weighted SM can be solved in $O(n^4 \log n)$ time (where n is the number of vertices) if the weight function is monotone in the preference ordering, non-negative and integral. Feder [6,7] shows a method to drop the monotonicity requirement. He also presents the best known bound for the runtime of an algorithm for finding a minimum weight stable matching in SM: $O(n^2 \cdot \log(\frac{K}{n^2} + 2) \cdot \min\{n, \sqrt{K}\})$, where K is the weight of an optimal solution. Redesigning the weight function to avoid the monotonicity requirement using Feder's method can radically increase K. For weighted SR, finding an optimal matching is NP-hard, but 2-approximable, under the assumption of monotone, non-negative and integral weights [6]. These constraints restrict the practical use of Feder's results to a large extent. Fortunately, linear programming techniques allow the majority of the conditions to be dropped while retaining polynomial-time solvability. Weighted SM can be solved to optimality with arbitrary real-valued weight functions [19], and a 2-approximation for weighted SR can be found for every non-negative weight function [25].

In all discussed problems, n is the number of vertices and m is the number of edges in the graph underlying the particular problem instance. When considering the restriction of any of the above problems to the case of a bipartite graph SR is replaced by SM in the problem name. Finally, we note that all proofs can be found in the full version of the paper [4].

3 Almost Stable Matchings with Restricted Edges

In this section, constraints on restricted edges must be fulfilled strictly, while the number of blocking edges is minimised. Our results are presented in three subsections, and most of the results are given for MIN BP SM RESTRICTED. Firstly, in Sect. 3.1, basic complexity results are discussed. In particular, we prove that the studied problem MIN BP SM RESTRICTED is in general NP-hard and very difficult to approximate. Thus, restricted cases are analyzed in Sect. 3.2. First we assume that the number of forbidden, forced or blocking edges can be considered as a constant. Due to this assumption, two of the three problems that naturally follow from imposing these restrictions become tractable, but surprisingly, not all of them. Then, degree-constrained cases are discussed. We show that the NP-hardness result for MIN BP SM RESTRICTED holds even for instances where each preference list is of length at most 3, while on graphs with maximum degree 2, the problems become tractable. Finally, in Sect. 3.3 we mention the problem MIN BP SR RESTRICTED and briefly elaborate on how results established for the bipartite case carry over to the SR case.

3.1 General Complexity and Approximability Results

When minimising the number of blocking edges, one might think that removing the forbidden edges temporarily and then searching for a stable solution in the remaining instance leads to an optimal solution. Such a matching can only be blocked by forbidden edges, but as the upcoming example demonstrates, optimal solutions are sometimes blocked by unrestricted edges exclusively. In some instances, all almost stable solutions admit only non-forbidden blocking edges. Moreover, a man- or woman-optimal almost stable matching with forbidden edges does not always exist.

Let us recall the SM instance in Fig. 1. In the graph with edge set $E(G) \setminus P$, a unique stable matching exists: $M = \{u_1w_1, u_4w_4\}$. Matching M is blocked by both forbidden edges in the original instance. On the other hand, matching $M_1 = \{u_1w_1, u_2w_4, u_4w_3\}$ is blocked by exactly one edge: $bp(M_1) = u_4w_4$. Similarly, matching $M_2 = \{u_1w_3, u_2w_1, u_4w_4\}$ is blocked only by u_1w_1. Therefore, M_1 and M_2 are both almost stable matchings and $bp(\mathcal{I}) = 1$. One can easily check that M_1 and M_2 are the only matchings with the minimum number of blocking edges. They both are blocked only by unrestricted edges. Moreover, M_1 is better for u_1, w_1 and w_3, whereas M_2 is preferred by u_2, u_4 and w_4.

We now present two results demonstrating the NP-hardness and inapproximability of special cases of MIN BP SM RESTRICTED.

Theorem 3. MIN BP SM FORBIDDEN *and* MIN BP SM FORCED *are* NP*-hard.*

Theorem 4. *Each of* MIN BP SM FORBIDDEN *and* MIN BP SM FORCED *is not approximable within a factor of $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless* P = NP*.*

3.2 Bounded Parameters

Our results presented so far show that MIN BP SM RESTRICTED is computationally hard even if $P = \emptyset$ or $Q = \emptyset$. Yet if certain parameters of the instance or the solution can be considered as a constant, the problem can be solved in polynomial time. Theorem 5 firstly shows that this is true for MIN BP SM FORBIDDEN.

Theorem 5. MIN BP SM FORBIDDEN *is solvable in $O(n^2m^L)$ time, where $L = |P|$, which is polynomial if L is a constant.*

In sharp contrast to the previous result on polynomial solvability when the number of forbidden edges is small, we state the following theorem for the MIN BP SM FORCED problem.

Theorem 6. MIN BP SM FORCED *is* NP*-hard even if $|Q| = 1$.*

On the other hand, a counterpart to Theorem 5 holds in the case of MIN BP SM RESTRICTED if the number of blocking pairs in an optimal solution is a constant.

Theorem 7. MIN BP SM RESTRICTED *is solvable in $O(m^{L+1})$ time, where L is the minimum number of edges blocking an optimal solution, which is polynomial if L is a constant.*

Next we study the case of degree-constrained graphs, because for most hard SM and SR problems, it is the most common special case to investigate [2,13,21]. Here, we show that MIN BP SM RESTRICTED remains computationally hard even for instances with preference lists of length at most 3. On the other hand, the problem can be solved by identifying forbidden subgraphs when the length of preference lists is bounded by 2.

Theorem 8. MIN BP SM FORBIDDEN *and* MIN BP SM FORCED *are* NP*-hard even if each agent's preference list consists of at most 3 elements.*

Theorem 9. MIN BP SM RESTRICTED *is solvable in* $O(n)$ *time if each preference list consists of at most 2 elements.*

Even with the previous two theorems, we have not quite drawn the line between tractable and hard cases in terms of vertex degrees. The complexity of MIN BP SM RESTRICTED remains open for the case when preference lists are of length at most 2 on one side of the bipartite graph and they are of unbounded length on the other side. However we believe that this problem is solvable in polynomial time.

Conjecture 1. MIN BP SM RESTRICTED *is solvable in polynomial time if each woman's preference list consists of at most 2 elements.*

3.3 Stable Roommates Problem

Having discussed several cases of SM, we turn our attention to non-bipartite instances. Since SM is a restriction of SR, all established results on the NP-hardness and inapproximability of MIN BP SM RESTRICTED carry over to the non-bipartite SR case. As a matter of fact, more is true, since MIN BP SR RESTRICTED is NP-hard and difficult to approximate even if $P = \emptyset$ and $Q = \emptyset$ [1]. We summarise these observations as follows.

Remark 1. By Theorems 3 and 4, MIN BP SR FORBIDDEN and MIN BP SR FORCED are NP-hard and not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P = NP. Moreover Theorems 8 and 6 imply that MIN BP SR FORBIDDEN and MIN BP SR FORCED are NP-hard even if all preference lists are of length at most 3 or, in the latter case, $|Q| = 1$. Finally MIN BP SR RESTRICTED is NP-hard and not approximable within $n^{\frac{1}{2}-\varepsilon}$, for any $\varepsilon > 0$, unless P = NP, even if $P = \emptyset$ and $Q = \emptyset$ [1].

As for the polynomially solvable cases, the proofs of Theorems 5, 7 and 9 carry over without applying any modifications, giving the following.

Remark 2. MIN BP SR FORBIDDEN is solvable in polynomial time if $|P|$ is a constant. MIN BP SR RESTRICTED is solvable in polynomial time if the minimal number of edges blocking an optimal solution is a constant.

4 Stable Matchings with the Minimum Number of Violated Constraints on Restricted Edges

In this section, we study the second intuitive approximation concept. The sought matching is stable and violates as few constraints on restricted edges aspossible.

We return to our example that already appeared in Fig. 1. As already mentioned earlier, the instance admits a single stable matching, namely $M = \{u_1w_1, u_2w_2, u_3w_3, u_4w_4\}$. Since M contains both forbidden edges, the minimum number of violated constraints on restricted edges is 2.

As mentioned in Sect. 1, a weighted stable matching instance models SM MIN RESTRICTED VIOLATIONS.

Theorem 10. SM MIN RESTRICTED VIOLATIONS *is solvable in polynomial time.*

In the SR context, finding a minimum weight stable matching is NP-hard [6], so the above technique for SM does not carry over to SR. Indeed special cases of SR MIN RESTRICTED VIOLATIONS are NP-hard, as the following result shows.

Theorem 11. SR MIN FORBIDDEN *and* SR MAX FORCED *are* NP*-hard.*

In our proof, we reduce the Minimum Vertex Cover problem to these two problems. Minimum Vertex Cover is NP-hard and cannot be approximated within a factor of $2-\varepsilon$ for any positive ε, unless the Unique Games Conjecture is false [17]. The reduction also answers basic questions about the approximability of these problems. Since any vertex cover on K vertices can be interpreted as a stable matching containing K forbidden edges in SR MIN FORBIDDEN and vice versa, the $(2-\varepsilon)$-inapproximability result carries over. The same holds for the number of violated forced edge constraints in SR MAX FORCED. On the positive side, we can close the gap with the best possible approximation ratio if $Q = \emptyset$ or $|Q|$ is sufficiently large. To derive this result, we use the 2-approximability of weighted SR for non-negative weight functions [25]. Due to the non-negativity constraint, the case of $0 < |Q| < |M|$ remains open.

Theorem 12. *If* $|Q| \geq |M|$ *for a stable matching* M*, then* SR MIN RESTRICTED VIOLATIONS *is 2-approximable in polynomial time.*

When studying SR MAX FORCED, we measured optimality by keeping track of the number of violated constraints. One might find it more intuitive instead to maximise $|Q \cap M|$, the number of forced edges in the stable matching. Our NP-hardness proof for SR MAX FORCED remains intact, but the approximability results need to be revisited. In fact, this modification of the measure changes the approximability of the problem as well:

Theorem 13. *For* SR MAX FORCED*, the maximum of* $|Q \cap M|$ *cannot be approximated within* $n^{\frac{1}{2}-\varepsilon}$ *for any* $\varepsilon > 0$*, unless* $\mathsf{P} = \mathsf{NP}$*.*

We now turn to the complexity of SR MIN RESTRICTED VIOLATIONS and its variants when the degree of the underlying graph is bounded or some parameter of the instance can be considered as a constant.

Theorem 14. SR MIN FORBIDDEN *and* SR MAX FORCED *are* NP*-hard even if every preference list is of length at most 3.*

Theorem 15. SR MIN RESTRICTED VIOLATIONS *is solvable in $O(n)$ time if every preference list is of length at most 2.*

Theorem 16. SR MIN RESTRICTED VIOLATIONS *is solvable in polynomial time if the number of restricted edges or the minimal number of violated constraints is constant.*

5 Conclusion

In this paper, we investigated the stable marriage and the stable roommates problems on graphs with forced and forbidden edges. Since a solution satisfying all constraints need not exist, two relaxed problems were defined. In MIN BP SM RESTRICTED, constraints on restricted edges are strict, while a matching with the minimum number of blocking edges is searched for. On the other hand, in SR MIN RESTRICTED VIOLATIONS, we seek stable solutions that violate as few constraints on restricted edges as possible. For both problems, we determined the complexity and studied several special cases.

One of the most striking open questions is the approximability of SR MIN RESTRICTED VIOLATIONS if $0 < |Q| < |M|$. Our other open question is formulated as Conjecture 1: the complexity of MIN BP SM RESTRICTED is not known if each woman's preference list consists of at most 2 elements. A more general direction of further research involves the SM MIN RESTRICTED VIOLATIONS problem. We have shown that it can be solved in polynomial time, due to algorithms for maximum weight stable marriage. The following question arises naturally: is there a faster method for SM MIN RESTRICTED VIOLATIONS that avoids reliance on Feder's algorithm or linear programming methods?

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