Nonlinear Scalarizations of Set Optimization Problems with Set Orderings

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Abstract This paper concerns with scalarization processes of set-valued optimization problems, whose objective space is a Hausdorff locally convex topological linear space and the preferences between the objective values are stated through set orderings. To be precise, general necessary and sufficient optimality conditions for minimal and weak minimal solutions of these optimization problems are obtained by dealing with abstract scalarization mappings that satisfy certain order preserving and order representing properties. Then these conditions are applied to well-known scalarization mappings in set optimization. This approach extends and unifies the main nonlinear scalarization results of the literature on set optimization problems with set orderings.

Keywords Set-valued optimization · Set relations · Nonlinear scalarization

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1 Introduction

Scalarization methods are one of the most powerful mathematical tools to study set-valued optimization problems with set orderings (see [1, 7, 13–15, 17–19, 31–34, 36, 38–41, 43]). As an example, let us observe that they have been successfully used to obtain minimal element theorems (see [13, 17–19, 41]), Ekeland variational principles (see [7, 13, 15, 17, 18]), well-posedness properties (see [14, 43]), stability results (see [15]), scalar representations without convexity assumptions (see [19]), nonconvex separation type theorems and alternative theorems (see [1, 38]), Takahashi type minimization theorems (see [1]) and optimality conditions through solutions of associated scalar optimization problems (see [1, 11, 19, 34, 36, 40]).

To the best of our knowledge, the first scalarization mappings for set-valued optimization problems with set orderings were introduced by Hamel and Löhne [17] (see also [18]), Nishizawa et al. [39] and Ha [15]. The mappings defined by Hamel and Löhne, and Nishizawa et al. extend the so-called Gerstewitz's nonconvex separation functional (see [8–10]) in order to deal with set orderings through the usual order in $\mathbb{R} \cup \{\pm\infty\}$. The approach due to Hamel and Löhne is more general, since they consider a fixed set that plays the role of "parameter". This fact is crucial in order to characterize minimal solutions of set-valued optimization problems with set orderings through solutions of associated scalar optimization problems (see [1, 34]). The scalarization mappings due to Nishizawa et al. do not consider this parameter and they characterize nondominated solutions (see [40]), a particular type of minimal solutions. On the other hand, Ha [15] generalized the well-known weighting scalarization method, extensively used in convex vector optimization problems.

The ideas, concepts and mathematical tools introduced by Hamel and Löhne in [17, 18] have motivated a lot of new contributions for scalarizing set-valued optimization problems with set orderings. In [19], Hernández and Rodríguez-Marín introduced a nonlinear scalarization mapping, studied its properties in deep and, for the first time in the literature, they characterized minimal and weak minimal solutions of set-valued optimization problems with set orderings via solutions of associated scalar optimization problems. Some new properties of this scalarization mapping have been stated in [43], where it has been used to derive well-posedness properties of set-valued optimization problems with set orderings.

Recently and inspired by this approach, Gutiérrez et al. [14] derived new properties of the scalarization mappings due to Hamel and Löhne, generalized the well-posedness properties obtained in [43], and characterized minimal and strict minimal solutions of set-valued optimization problems with set orderings via scalarization. Also, Gutiérrez et al. [13] defined a sup-inf type scalarization mapping and via this mapping they derived approximate strict minimal element theorems and approximate versions of the Ekeland variational principle for set orderings.

On the other hand, Kuwano et al. [31] and Araya [1] introduced scalarization schemes that unify several nonlinear scalarization mappings introduced in the literature and allow to characterize via scalarization minimal solutions of set-valued optimization problems by considering different set orderings (see [34]). Moreover, Maeda [36] characterized via the scalarization mappings due to Hamel and Löhne new concepts of solution based on set orderings motivated by fuzzy mathematical programming problems (see [35]).

This work is structured as follows. In Sect. 2 we introduce the set-valued optimization problem and the basic notations. Moreover, some technical results on topological properties of the conic extension of a set and about the set orderings induced by an open ordering cone are stated. In Sect. 3 we introduce the order representing and monotonicity properties for mappings defined in the power set of a vector space. This kind of properties have been widely used in vector optimization to characterize minimal solutions through scalarization. Moreover, we study in deep some properties of the nonlinear scalarization mappings introduced by Hamel and Löhne [17, 18] and Gutiérrez et al. [13]. In particular, we analyze when these scalarization mappings satisfy the mentioned order representing and monotonicity properties, and we prove that the scalarization mapping due to Hernández and Rodríguez-Marín [19] coincides with the scalarization mapping due to Hamel and Löhne. Finally, in Sect. 4, we characterize the minimal and weak minimal solutions of set-valued optimization problems with set orderings by solutions of scalar optimization problems defined via generic scalarization mappings that satisfy order representing and monotonicity properties. We show that these "implicit" characterizations can be done "explicit" by using Hamel and Löhne scalarization mapping and, in general, by considering any scalarization mapping that satisfies the required order representing and monotonicity properties. The results obtained in Sects. 3 and 4 extend and clarify some results of the literature.

2 Preliminaries

Let Y be a Hausdorff locally convex topological linear space. The topological dual space of Y is denoted by Y^*, and the duality pairing by $\langle y^*, y\rangle$, $y^* \in Y^*$, $y \in Y$. We denote by $\operatorname{int} M$, $\operatorname{cl} M$ and $\operatorname{cone} M$ the interior, the closure and the cone generated by a set $M \subset Y$, and we say that M is solid if $\operatorname{int} M \neq \emptyset$. The ordering cone of Y is denoted by D, which is assumed to be proper (i.e., $D \neq Y$), closed, solid and convex. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

The positive polar cone of D is denoted by D^+, i.e.,

$$D^+ := \{\lambda \in Y^* : \langle \lambda, d\rangle \geq 0, \forall\, d \in D\}.$$

For each $q \in \operatorname{int} D$ we denote

$$D^+(q) := \{\lambda \in D^+ : \langle \lambda, q\rangle = 1\}.$$

It is well-known (see, for instance, [10]) that $D^+(q)$ is compact in the weak star topology, convex and $\operatorname{cone} D^+(q) = D^+$.

We denote the Minkowski sum of two nonempty sets $M_1, M_2 \subset Y$ by $M_1 + M_2$, i.e.,

$$M_1 + M_2 := \{y_1 + y_2 : y_1 \in M_1, y_2 \in M_2\}.$$

Moreover, we assume that $M + \emptyset = \emptyset + M = \emptyset$, for all $M \subset Y$, and for each $y \in Y$, $y + M$ (resp. $M + y$) denotes $\{y\} + M$ (resp. $M + \{y\}$). The following topological properties on the conic extension of a set will be used in the paper. Part (a) was stated in [4, Lemma 2.5] and so its proof is omitted.

Proposition 2.1 *Consider a nonempty set $M \subset Y$. We have that*

(a) $\operatorname{int}\operatorname{cl}(M + D) = M + \operatorname{int} D$,
(b) $\operatorname{cl}(M + D) + \operatorname{int} D \subset M + \operatorname{int} D$.

Proof Let us proof part (b). It is clear that

$$\operatorname{cl}(M + D) + \operatorname{int} D \subset \operatorname{cl}(M + D) + D = \operatorname{cl}(M + D).$$

Then, by part (a) it follows that

$$\operatorname{int}(\operatorname{cl}(M + D) + \operatorname{int} D) \subset M + \operatorname{int} D$$

and the proof is completed. □

Recall that a set $M \subset Y$ is D-bounded if for each neighborhood U of zero in Y there exists $\alpha > 0$ such that $M \subset \alpha U + D$. M is D-compact if any cover of M of the form $\{U_i + D : U_i \text{ is open}\}$ admits a finite subcover. Observe that the family $\{V_i + \operatorname{int} D : V_i \subset Y\}$ fits with this form, since $V_i + \operatorname{int} D = (V_i + \operatorname{int} D) + D$, and the sets $U_i := V_i + \operatorname{int} D$ are open. Analogously, we say that M is D-closed if $M + D$ is closed.

On the other hand, M is D-proper if $M + D \neq Y$. Cone properness is a kind of boundedness weaker than the cone boundedness (see [14]). Next we recall an important characterization of this property.

Lemma 2.2 *[14, Theorem 3.6] A nonempty set $M \subset Y$ is D-proper if and only if there is not an element $e \in \operatorname{int} D$ such that $-e + M \subset M + D$.*

In this paper we study the following set optimization problem:

$$\operatorname{Min}\{F(x) : x \in S\}, \tag{P}$$

where the objective mapping $F : X \to 2^Y$ is set-valued, the decision space X is an arbitrary set, and the feasible set $S \subset X$ is nonempty. In [2, 3, 16, 20, 24] the reader can find some practical problems which are modeled by set optimization problems.

We denote

$$\operatorname{Dom} F := \{x \in X : F(x) \neq \emptyset\}$$

and we suppose that F is proper in S, i.e., $\mathrm{Dom}F \cap S \neq \emptyset$. We say that F is D-proper (resp. D-compact, D-closed) valued in S if $F(x)$ is D-proper (resp. D-compact, D-closed), for all $x \in S$.

To solve this problem one needs to discriminate between the objective values $F(x)$, $x \in S$. We model this task via the following well-known set orderings (see [25–30]), where $K \in \{D, \mathrm{int}\, D\}$:

$$\begin{aligned} M_1, M_2 \subset Y,\ \ M_1 \precsim_K^l M_2 &\iff M_2 \subset M_1 + K, \\ M_1 \precsim_K^u M_2 &\iff M_1 \subset M_2 - K, \\ M_1 \backsim_K^j M_2 &\iff M_1 \precsim_K^j M_2 \text{ and } M_2 \precsim_K^j M_1, \\ M_1 \prec_K^j M_2 &\iff M_1 \precsim_K^j M_2 \text{ and } M_1 \not\backsim_K^j M_2 \quad (j \in \{l, u\}). \end{aligned}$$

Remark 2.3 (a) Let $K \in \{D, \mathrm{int}\, D\}$. The following equivalences are clear:

$$\begin{aligned} M_1 \backsim_K^l M_2 &\iff M_2 \backsim_K^l M_1, \\ M_1 \precsim_K^u M_2 &\iff M_2 \precsim_{-K}^l M_1, \\ M_1 \backsim_K^u M_2 &\iff M_1 \backsim_{-K}^l M_2, \\ M_1 \prec_K^u M_2 &\iff M_2 \prec_{-K}^l M_1, \\ M_1 \prec_K^l M_2 &\iff M_2 \subset M_1 + K \text{ and } M_1 \not\subset M_2 + K, \\ M_1 \prec_K^u M_2 &\iff M_1 \subset M_2 - K \text{ and } M_2 \not\subset M_1 - K. \end{aligned}$$

Moreover,

$$\begin{aligned} M_1 \backsim_D^l M_2 &\iff M_1 + D = M_2 + D, \\ M_1 \backsim_D^u M_2 &\iff M_1 - D = M_2 - D. \end{aligned}$$

These two statements could be false for the relations $\backsim_{\mathrm{int}\, D}^l$ and $\backsim_{\mathrm{int}\, D}^u$. For example, consider the following data: $Y = \mathbb{R}^2$, $D = \mathbb{R}_+^2$, $M_1 = \mathrm{int}\, \mathbb{R}_+^2$, $M_2 = \mathbb{R}_+^2$. It is easy to check that $M_1 + \mathrm{int}\, D = M_2 + \mathrm{int}\, D$ and $M_2 \prec_{\mathrm{int}\, D}^l M_1$.

(b) Let us observe that there exist in the literature a lot of set relations from which one can model the preferences between the objective values of the problem (P) (see [6, 23]).

In the next lemma we state two technical properties of the relation $\precsim_{\mathrm{int}\, D}^l$, which will be used in the paper.

Lemma 2.4 *Consider $q \in \mathrm{int}\, D$ and two nonempty sets $A, M \subset Y$. The following statements hold:*

(a) If A is D-bounded, then there exists $t \in \mathbb{R}$ such that $M + tq \precsim_{\mathrm{int}\, D}^l A$.
(b) If A is D-compact and $M \precsim_{\mathrm{int}\, D}^l A$, then there exists $t > 0$ such that $M + tq \precsim_{\mathrm{int}\, D}^l A$.

Proof (a) Let us consider an arbitrary point $y \in M$. It is obvious that $A - y$ is D-bounded and $-q + \operatorname{int} D$ is a neighborhood of zero in Y. Then there exists $\alpha > 0$ such that $A - y \subset \alpha(-q + \operatorname{int} D) + D = -\alpha q + \operatorname{int} D$. Therefore, $A \subset M - \alpha q + \operatorname{int} D$ and the proof of part (a) is finished.

(b) As $A \subset M + \operatorname{int} D$ and $M + \operatorname{int} D$ is an open set, for each $y \in A$ there exists $t_y > 0$ such that $y - t_y q \in M + \operatorname{int} D$, and it follows that

$$A \subset \bigcup_{y \in A} (t_y q + M + \operatorname{int} D).$$

Since A is D-compact we deduce that there exist $\{y_1, y_2, \ldots, y_n\} \subset A$ such that

$$A \subset \bigcup_{i=1}^{n} (t_{y_i} q + M + \operatorname{int} D).$$

By considering $t := \min\{t_{y_i} : i = 1, 2, \ldots, n\} > 0$ we obtain

$$A \subset tq + M + \operatorname{int} D,$$

and the proof is completed. □

The set orderings $\precsim^l$ and $\precsim^u$ define the preferences on the feasible points via their objective values. The concepts of solution of problem (P) were introduced according with these preferences (see [19, 25–29]) and the classical minimality notion in the theory of ordered sets.

Definition 2.5 Consider $j \in \{l, u\}$. A point $x_0 \in S$ is a j-minimal (resp. weak j-minimal) solution of problem (P), denoted by $x_0 \in \mathrm{M}^j(F, S)$ (resp. $x_0 \in \mathrm{WM}^j(F, S)$), if

$$x \in S,\ F(x) \precsim_D^j F(x_0) \Rightarrow F(x_0) \precsim_D^j F(x)$$
$$(\text{resp. } x \in S,\ F(x) \precsim_{\operatorname{int} D}^j F(x_0) \Rightarrow F(x_0) \precsim_{\operatorname{int} D}^j F(x)).$$

Remark 2.6 (a) Let $G : X \to 2^Y$ be such that $F(x) = G(x) + D$ (resp. $F(x) = G(x) - D$), for all $x \in S$. It is easy to check that $\mathrm{M}^l(F, S) = \mathrm{M}^l(G, S)$ and $\mathrm{WM}^l(F, S) = \mathrm{WM}^l(G, S)$ (resp. $\mathrm{M}^u(F, S) = \mathrm{M}^u(G, S)$ and $\mathrm{WM}^u(F, S) = \mathrm{WM}^u(G, S)$).

(b) Let us recall that the first concepts of solution of problem (P) introduced in the literature did not use set relations. They consider feasible points whose images contain minimal points with respect to the whole image of the objective mapping and the ordering induced by the cone D (see [5, 21]).

As F is proper in S, it follows that $\mathrm{M}^l(F, S) \subset \mathrm{Dom} F \cap S$ and $\mathrm{WM}^l(F, S) \subset \mathrm{Dom} F \cap S$. Analogously, if $S \backslash \mathrm{Dom} F \neq \emptyset$ then $\mathrm{M}^u(F, S) = \mathrm{WM}^u(F, S) = S \backslash \mathrm{Dom} F$. Therefore, without losing generality, in the sequel we assume that $S \subset \mathrm{Dom} F$. Moreover, we denote $\mathcal{Y} := 2^Y \backslash \{\emptyset\}$ and $\mathcal{F} := \{F(x) : x \in S\}$.

Let us observe that if there exists $x \in S$ such that $F(x)$ is not D-proper, then

$$\mathrm{M}^l(F, S) = \mathrm{WM}^l(F, S) = \{z \in S : F(z) + D = Y\}.$$

On the other hand, if there exists $x \in S$ such that $F(x)$ is $-D$-proper, then $F(z)$ is $-D$-proper, for all $z \in \mathrm{M}^u(F, S)$ and for all $z \in \mathrm{WM}^u(F, S)$. Therefore, for solving problem (P) in the sense of the l-minimality or weak l-minimality (resp. u-minimality or weak u-minimality), one could assume without loss of generality that the objective mapping F is D-proper valued (resp. $-D$-proper valued) in S.

3 Scalarization Processes

The scalarization processes are among the most important techniques to study problem (P). They relate the solutions of problem (P) with solutions of associated scalar optimization problems. Usually, these associated scalar optimization problems are defined by the composition of the objective mapping F with the elements of a parametric family $\{\varphi_p\}_{p\in\mathcal{P}}$ of extended real-valued mappings $\varphi_p : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$, where $\mathcal{P}$ is an index set (see [11, 19]). Then, the scalarization processes relate the minimal and weak minimal solutions of problem (P) with the solutions of the following scalar optimization problems:

$$\mathrm{Min}\{(\varphi_p \circ F)(x) : x \in S\}. \tag{P_{φ_p}}$$

We denote the set of solutions of problem (P_{φ_p}) by $\mathrm{S}(\varphi_p \circ F, S)$. Let us recall that a point $x_0 \in S$ is a strict solution of problem (P_{φ_p}) if $\varphi_p(F(x_0)) < \varphi_p(F(x))$, $\forall\, x \in S\backslash\{x_0\}$, i.e., if $\mathrm{S}(\varphi_p \circ F, S) = \{x_0\}$.

Let $\mathcal{M} \subset \mathcal{Y}$, we say that an extended real-valued mapping $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ is proper if $\varphi(M) > -\infty$, for all $M \in \mathcal{M}$, and

$$\mathrm{Dom}\varphi := \{M \in \mathcal{M} : \varphi(M) < +\infty\} \neq \emptyset.$$

It is well-known between practitioners and researchers in vector optimization that a scalarization mapping is useful to characterize the solutions of a vector optimization problem through the solutions of the associated scalar optimization problem whenever it is monotone and satisfies the so-called order representing property (see [12, 37, 42]). Next we extend these properties to problem (P).

Definition 3.1 Let $\varphi : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$, $A \in \mathcal{M}$ and $j \in \{l, u\}$.

(a) φ is order $\precsim^l_D$-representing (resp. $\precsim^u_D$-representing) at A if

$$\{M \in \mathcal{M} : \varphi(M) \le \varphi(A)\} \subset \{M \in \mathcal{M} : M \precsim^l_D A\}$$
$$(\text{resp. } \{M \in \mathcal{M} : \varphi(A) \le \varphi(M)\} \subset \{M \in \mathcal{M} : A \precsim^u_D M\}).$$

(b) φ is strictly order $\precsim_D^l$-representing (resp. strictly $\precsim_D^u$-representing) at A if

$$\{M \in \mathcal{M} : \varphi(M) < \varphi(A)\} \subset \{M \in \mathcal{M} : M \prec_D^l A\}$$
$$(\text{resp. } \{M \in \mathcal{M} : \varphi(A) < \varphi(M)\} \subset \{M \in \mathcal{M} : A \prec_D^u M\}).$$

(c) φ is $\precsim_D^l$-monotone (resp. $\precsim_D^u$-monotone) at A if

$$M \in \mathcal{M},\ M \precsim_D^l A \quad \Rightarrow \quad \varphi(M) \leq \varphi(A)$$
$$(\text{resp. } M \in \mathcal{M},\ A \precsim_D^u M \quad \Rightarrow \quad \varphi(A) \leq \varphi(M)).$$

(d) φ is strictly $\precsim_D^l$-monotone (resp. strictly $\precsim_D^u$-monotone) at A if

$$M \in \mathcal{M},\ M \prec_D^l A \quad \Rightarrow \quad \varphi(M) < \varphi(A)$$
$$(\text{resp. } M \in \mathcal{M},\ A \prec_D^u M \quad \Rightarrow \quad \varphi(A) < \varphi(M)).$$

(e) φ is $\precsim_D^j$-monotone (resp. strictly $\precsim_D^j$-monotone) on $\mathcal{M}$ if φ is $\precsim_D^j$-monotone (resp. strictly $\precsim_D^j$-monotone) at A, for all $A \in \mathcal{M}$.

In the literature one can find several scalarization processes to deal with problem (P) without assuming any convexity assumption (see [1, 13, 14, 17–19, 31, 33, 34, 36, 39, 40, 43]). All of them generalize the so-called Gerstewitz scalarization mapping $s_q : Y \to \mathbb{R}$ (see [8–10]):

$$s_q(y) := \inf\{t \in \mathbb{R} : y \in tq - D\}, \quad \forall\, y \in Y,$$

where q is an arbitrary point in int D. This mapping has been extensively used for scalarizing vector optimization problems (see [5, 10] and the references therein). In particular, it follows that (see [5, Proposition 1.53]):

$$s_q(y) = \max\{\langle \lambda, y \rangle : \lambda \in D^+(q)\}, \quad \forall\, y \in Y. \tag{1}$$

Motivated by the scalarization processes due to Hamel and Löhne, we consider the following families $\{\Phi_{q,F(x)}^{j,D}\}_{x \in S}$, $j \in \{l, u\}$, $q \in \operatorname{int} D$, of scalarization mappings of problem (P) (see [1, 14, 17, 18, 34]). For each $A \in \mathcal{Y}$, $\Phi_{q,A}^{j,D} : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$,

$$\Phi_{q,A}^{l,D}(M) := \inf \Lambda_{q,A}^{l,D}(M)$$

$$\Phi_{q,A}^{u,D}(M) := \sup \Lambda_{q,A}^{u,D}(M)$$

where

$$\Lambda_{q,A}^{l,D}(M) := \{t \in \mathbb{R} : M \precsim_D^l tq + A\},$$
$$\Lambda_{q,A}^{u,D}(M) := \{t \in \mathbb{R} : tq + A \precsim_D^u M\},$$

and we assume the usual conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. It is clear that

$$t \in \Lambda^{l,D}_{q,A}(M) \Rightarrow [t, \infty) \subset \Lambda^{l,D}_{q,A}(M)$$

and then, one of the following cases happens: $\Lambda^{l,D}_{q,A}(M) = \emptyset$, $\Lambda^{l,D}_{q,A}(M) = \mathbb{R}$ (i.e., $\Phi^{l,D}_{q,A}(M) = -\infty$) or $\Lambda^{l,D}_{q,A}(M) \neq \emptyset$, $\Lambda^{l,D}_{q,A}(M) \neq \mathbb{R}$ (i.e., $\Phi^{l,D}_{q,A}(M) \in \mathbb{R}$ and $\Lambda^{l,D}_{q,A}(M) = [\Phi^{l,D}_{q,A}(M), +\infty)$ or $\Lambda^{l,D}_{q,A}(M) = (\Phi^{l,D}_{q,A}(M), +\infty)$).

On the other hand, it is easy to check that

$$-\Lambda^{u,D}_{q,A}(M) = \Lambda^{l,-D}_{-q,A}(M) = \{t \in \mathbb{R} : M \precsim^{l}_{-D} t(-q) + A\}, \quad \forall\, M \in \mathcal{Y},$$

(observe that $\Lambda^{l,-D}_{-q,A}(M)$ is defined by $-D$ instead of D) and so we have that

$$\Phi^{u,D}_{q,A}(M) = -\Phi^{l,-D}_{-q,A}(M), \quad \forall\, M \in \mathcal{Y}. \tag{2}$$

Thus, in the sequel, the statements on the scalarization process $\Phi^{u,D}_{q,A}$ are not proved, since they follow easily from the corresponding statements on the scalarization process $\Phi^{l,D}_{q,A}$ by the equivalences of Remark 2.3(a) and relation (2). Let us observe that $\Phi^{l,D}_{q,A}$ and $\Phi^{u,D}_{q,A}$ reduce to the scalarization mappings introduced by Nishizawa et al. in [39] by taking $A = \{0\}$, i.e., by removing the "parameter" A.

In the next four results we collect some of the main properties of these scalarization processes. Proposition 3.2 and Theorem 3.5 extend Proposition 4.1 and Theorem 4.2 of [14]. In order to simplify the notation we write $\Lambda^{l}_{q,A}$, $\Lambda^{u}_{q,A}$, $\Phi^{l}_{q,x}$ and $\Phi^{u}_{q,x}$ instead of $\Lambda^{l,D}_{q,A}$, $\Lambda^{u,D}_{q,A}$, $\Phi^{l,D}_{q,F(x)}$ and $\Phi^{u,D}_{q,F(x)}$, respectively. Let us recall that mappings $\Phi^{l}_{q,x}$ and $\Phi^{u}_{q,x}$ are defined on $\mathcal{Y} = 2^{Y}\backslash\{\emptyset\}$.

Proposition 3.2 *Let $q \in \operatorname{int} D$, $x \in S$ and $M \in \mathcal{Y}$.*

(a) $\Phi^{l}_{q,x}(M) = -\infty$ *if and only if M is not D-proper.*
(b) If $F(x)$ is not D-proper, then

$$\Phi^{l}_{q,x}(M) = \begin{cases} +\infty & \textit{if } M \textit{ is } D\textit{-proper}, \\ -\infty & \textit{otherwise}. \end{cases}$$

(c) Suppose that $F(x)$ is D-bounded. Then $\Phi^{l}_{q,x}(M) < +\infty$.

Proof (a) If $\Phi^{l}_{q,x}(M) = -\infty$, then $\Lambda^{l}_{q,F(x)}(M) = \mathbb{R}$ and $M \precsim^{l}_{D} tq + F(x)$, for all $t \in \mathbb{R}$. In other words,

$$\bigcup_{t\in\mathbb{R}} (tq + F(x)) \subset M + D. \tag{3}$$

As $q \in \operatorname{int} D$, it is easy to check that $\bigcup_{t\in\mathbb{R}}(tq + D) = Y$. Then, by (3) we see that

$$Y = Y + F(x) = \bigcup_{t \in \mathbb{R}} (tq + D) + F(x) \subset M + D + D = M + D$$

and M is not D-proper. Reciprocally, if M is not D-proper, then it is obvious that $M \precsim^l_D tq + F(x)$, for all $t \in \mathbb{R}$, and so $\Phi^l_{q,x}(M) = -\infty$.

(b) Suppose that $F(x)$ is not D-proper and let $M \in \mathcal{Y}$. If M is not D-proper, then by part (a) we deduce that $\Phi^l_{q,x}(M) = -\infty$. If M is D-proper and $\Lambda^l_{q,F(x)}(M) \neq \emptyset$, then there exists $t \in \mathbb{R}$ such that $tq + F(x) \subset M + D$, and so

$$Y = tq + F(x) + D \subset M + D + D = M + D,$$

that is a contradiction. Thus $\Lambda^l_{q,F(x)}(M) = \emptyset$ and so $\Phi^l_{q,x}(M) = +\infty$, which finishes the proof of part (b).

(c) As $F(x)$ is D-bounded and $q \in \operatorname{int} D$, by Lemma 2.4($a$), there exists $t \in \mathbb{R}$ such that $F(x) + tq \subset M + D$. Then $t \in \Lambda^l_{q,F(x)}(M)$, and as $\Lambda^l_{q,F(x)}(M) \neq \emptyset$ it follows that $\Phi^l_{q,x}(M) < +\infty$, which finishes the proof. □

The next similar properties on $\Phi^u_{q,x}$ are direct consequences of Proposition 3.2 and relation (2).

Proposition 3.3 *Let $q \in \operatorname{int} D$, $x \in S$ and $M \in \mathcal{Y}$. Then:*

(a) $\Phi^u_{q,x}(M) = +\infty$ *if and only if M is not $-D$-proper.*
(b) If $F(x)$ is not $-D$-proper, then

$$\Phi^u_{q,x}(M) = \begin{cases} -\infty & \text{if } M \text{ is } -D\text{-proper}, \\ +\infty & \text{otherwise}. \end{cases}$$

(c) Suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}(M) > -\infty$.

Remark 3.4 The sufficient condition of Proposition 3.2(a) and Proposition 3.2(c) reduce to [36, Theorem 3.1(i),(ii)] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that M is D-bounded. Analogously, the sufficient condition of Proposition 3.3(a) and Proposition 3.3(c) reduce to [36, Theorem 3.1(iv), (iii)] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that M is $-D$-bounded.

Theorem 3.5 *Consider $q \in \operatorname{int} D$, $x \in S$, and suppose that $F(x)$ is D-proper.*

(a) $\Phi^l_{q,x}(F(x)) = 0$.
(b) If F is D-proper valued in S, then $\Phi^l_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is proper.
(c) $\Phi^l_{q,x}(M + tq) = \Phi^l_{q,x}(M) + t$, *for all $M \in \mathcal{Y}$ and $t \in \mathbb{R}$.*
(d) $\Phi^l_{q,x}$ *is $\precsim^l_D$-monotone on $\mathcal{Y}$.*
(e) Let $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It follows that

$$\Phi^l_{q,x}(M) \leq t \iff F(x) \subset -tq + \operatorname{cl}(M + D). \tag{4}$$

(f) $\Phi^l_{q,x}$ *is strictly order $\precsim^l_{\operatorname{int} D}$-representing at $F(x)$.*

(g) *Consider $Q, M \in \mathcal{Y}$ such that Q is D-compact and $M \precsim^l_{\mathrm{int}\, D} Q$, and suppose that $F(x)$ is D-bounded. Then $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(Q)$.*

(h) *Let $\mathcal{M} \subset \mathcal{Y}$ be a family of D-compact sets and suppose that $F(x)$ is D-bounded. Then $\Phi^l_{q,x}$ is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone on $\mathcal{M}$.*

Proof (a) It is clear that $\Phi^l_{q,x}(F(x)) \le 0$, since $0 \in \Lambda^l_{q,F(x)}(F(x))$. If $\Phi^l_{q,x}(F(x)) < 0$, then there exists $t > 0$ such that $F(x) \precsim^l_D -tq + F(x)$, i.e., $-tq + F(x) \subset F(x) + D$. By Lemma 2.2 it follows that $F(x)$ is not D-proper, which is a contradiction. Thus $\Phi^l_{q,x}(F(x)) = 0$ and the proof of part (a) is completed.

(b) As $F(z)$ is D-proper for all $z \in S$, by Proposition 3.2(a) we deduce that $\Phi^l_{q,x}(F(z)) > -\infty$, for all $z \in S$. Moreover, by part (a) we have that $F(x) \in \mathrm{Dom}\Phi^l_{q,x}$. Then $\Phi^l_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is proper.

(c) Consider $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It is easy to check that $\Lambda^l_{q,F(x)}(M + tq) = \Lambda^l_{q,F(x)}(M) + t$. Then, $\Phi^l_{q,x}(M + tq) = +\infty$ if and only if $\Phi^l_{q,x}(M) = +\infty$, since $\Lambda^l_{q,F(x)}(M + tq) = \emptyset$ if and only if $\Lambda^l_{q,F(x)}(M) = \emptyset$. On the other hand, if $\Phi^l_{q,x}(M) < +\infty$, then

$$\Phi^l_{q,x}(M + tq) = \inf \Lambda^l_{q,F(x)}(M + tq) = \inf \Lambda^l_{q,F(x)}(M) + t = \Phi^l_{q,x}(M) + t$$

and part (c) is proved.

(d) Consider $A, M \in \mathcal{Y}$ such that $M \precsim^l_D A$. As the relation $\precsim^l_D$ is transitive, it is easy to check that $\Lambda^l_{q,F(x)}(A) \subset \Lambda^l_{q,F(x)}(M)$ and so $\Phi^l_{q,x}(M) \le \Phi^l_{q,x}(A)$. Thus, $\Phi^l_{q,x}$ is $\precsim^l_D$-monotone at A, for all $A \in \mathcal{Y}$.

(e) It follows that

$$\begin{aligned}\Phi^l_{q,x}(M) \le t &\Rightarrow t + \varepsilon \in \Lambda^l_{q,F(x)}(M), \quad \forall\, \varepsilon > 0\\ &\Rightarrow F(x) + (t + \varepsilon)q \subset M + D, \quad \forall\, \varepsilon > 0\\ &\Rightarrow y + \varepsilon q \in -tq + M + D, \quad \forall\, \varepsilon > 0, \forall\, y \in F(x).\end{aligned}$$

Then, by taking the limit when $\varepsilon \downarrow 0$ we have that

$$\Phi^l_{q,x}(M) \le t \Rightarrow y \in \mathrm{cl}\,(-tq + M + D) = -tq + \mathrm{cl}\,(M + D), \quad \forall\, y \in F(x),$$

and the necessary condition of part (e) is proved. Reciprocally, if $y \in -tq + \mathrm{cl}\,(M + D)$ for all $y \in F(x)$, then by Proposition 2.1(b) we have

$$\begin{aligned}y + \varepsilon q \in -tq + \mathrm{cl}\,(M + D) + \varepsilon q &\subset -tq + \mathrm{cl}\,(M + D) + \mathrm{int}\, D\\ &\subset -tq + M + \mathrm{int}\, D, \quad \forall\, \varepsilon > 0, \forall\, y \in F(x).\end{aligned}$$

Therefore, $t + \varepsilon \in \Lambda^l_{q,F(x)}(M)$ for all $\varepsilon > 0$, and so $\Phi^l_{q,x}(M) \le t$.

(f) Let us consider $M \in \mathcal{Y}$ such that $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(F(x))$. By part (a) we see that $\Phi^l_{q,x}(M) < 0$ and there exists $t > 0$ such that $-tq + F(x) \subset M + D$. Thus,

$$F(x) \subset M + (tq + D) \subset M + \operatorname{int} D$$

and we see that $M \precsim^l_{\operatorname{int} D} F(x)$. If $F(x) \precsim^l_{\operatorname{int} D} M$, then it is obvious that $F(x) \precsim^l_D M$ and as $\Phi^l_{q,x}$ is $\precsim^l_D$-monotone on $\mathcal{Y}$ we deduce that $\Phi^l_{q,x}(F(x)) \leq \Phi^l_{q,x}(M)$, which is a contradiction. Therefore $M \prec^l_{\operatorname{int} D} F(x)$, which finishes the proof of part (f).

(g) As Q is D-compact it follows that Q is D-proper, and by Proposition 3.2(a) we have that $\Phi^l_{q,x}(Q) > -\infty$. If M is not D-proper, then we see that $\Phi^l_{q,x}(M) = -\infty$, and so $\Phi^l_{q,x}(M) < \Phi^l_{q,x}(Q)$.

Suppose that M is D-proper. By Lemma 2.4(b) it follows that there exists $t > 0$ such that $M + tq \precsim^l_D Q$. As M is D-proper and $F(x)$ is D-bounded, by Proposition 3.2(a), (c) we see that $\Phi^l_{q,x}(M) \in \mathbb{R}$. Then, by parts ($c$) and ($d$) we deduce that

$$\Phi^l_{q,x}(M) < t + \Phi^l_{q,x}(M) = \Phi^l_{q,x}(M + tq) \leq \Phi^l_{q,x}(Q).$$

Part (h) is a direct consequence of part (g). □

Similar results for $\Phi^u_{q,x}$ are collected without proof in the next theorem.

Theorem 3.6 *Consider $q \in \operatorname{int} D$, $x \in S$, and suppose that $F(x)$ is $-D$-proper.*

(a) $\Phi^u_{q,x}(F(x)) = 0$.
(b) If F is $-D$-proper valued in S and $F(x)$ is $-D$-bounded, then $\Phi^u_{q,x} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is finite valued.
(c) $\Phi^u_{q,x}(M + tq) = \Phi^u_{q,x}(M) + t$, *for all $M \in \mathcal{Y}$ and $t \in \mathbb{R}$.*
(d) $\Phi^u_{q,x}$ *is $\precsim^u_D$-monotone on $\mathcal{Y}$.*
(e) Let $M \in \mathcal{Y}$ and $t \in \mathbb{R}$. It follows that

$$\Phi^u_{q,x}(M) \geq t \iff F(x) \subset -tq + \operatorname{cl}(M - D).$$

(f) $\Phi^u_{q,x}$ *is strictly order $\precsim^u_{\operatorname{int} D}$-representing at $F(x)$.*
(g) Consider $Q, M \in \mathcal{Y}$ such that Q is $-D$-compact and $Q \precsim^u_{\operatorname{int} D} M$, and suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}(Q) < \Phi^u_{q,x}(M)$.
(h) Let $\mathcal{M} \subset \mathcal{Y}$ be a family of $-D$-compact sets and suppose that $F(x)$ is $-D$-bounded. Then $\Phi^u_{q,x}$ is strictly $\precsim^u_{\operatorname{int} D}$-monotone on $\mathcal{M}$.

Remark 3.7 (a) Parts (c) and (d) in Theorem 3.5 have been stated in different papers (see [18, Theorem 3.1(ii), (iii)], [1, Theorem 3.2(iii), (v)], [34, Proposition 3.2]). Theorem 3.5(d) reduces to [36, Theorem 3.2, statement (14)] by considering $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$. Analogously, parts (c) and (d) in Theorem 3.6 have been obtained in [18, Corollary 3.2], [1, Theorem 3.7(iii), (v)], [34, Proposition 3.2], and Theorem 3.6(d) reduces to [36, Theorem 3.2, statement (13)] by considering $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$.

On the other hand, Theorems 3.5(g) and 3.6(g) reduce to statements (18) and (17) of [36, Theorem 3.3], respectively, by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and by assuming that Q and M are compact. Analogously, Theorems 3.5(a), (f), (g) and 3.6(a), (f), (g)

reduce to statements (i)–(v) of [36, Theorem 3.4] by considering $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$ and the class of all nonempty and compact sets of $\mathbb{R}^n$.

(b) Proposition 3.2 and Theorem 3.5 extend [18, Theorem 3.1]. Analogously, Proposition 3.3 and Theorem 3.6 extend [18, Corollary 3.2].

In [18, Theorem 3.1], the authors prove that $\Phi^l_{q,x}(M) < +\infty$, for all $M \in \mathcal{Y}$, $M \precsim^l_D F(x)$. In Theorem 3.5 we obtain $\Phi^l_{q,x}(M) < +\infty$, for all $M \in \mathcal{Y}$ whenever $F(x)$ is D-bounded. On the other hand, in [18, Theorem 3.1], the authors prove that $\Phi^l_{q,x}$ is lower bounded on $\mathcal{M} \subset \mathcal{Y}$ whenever the elements of $\mathcal{M}$ are lower $\precsim^l_D$-bounded by a bounded set A, i.e., whenever there exists a (topological) bounded set A such that $A \precsim^l_D M$, for all $M \in \mathcal{M}$. Moreover, let us observe that [18, Theorem 3.1] can be applied if the ordering cone D is not solid whenever $D\backslash(-D) \neq \emptyset$.

(c) Proposition 3.2 and Theorem 3.5 (resp. Proposition 3.3 and Theorem 3.6) extend and clarify [1, Theorem 3.2] (resp. [1, Theorem 3.7]). To be precise, in [1, Theorem 3.2] the mapping $\Phi^l_{q,A}$ is denoted by $h^l_{\inf}(\cdot\,; A)$ and it is defined by a point $q \in D\backslash(-D)$ instead of $q \in \operatorname{int} D$. Under these assumptions, Theorem 3.2(i) of [1] states that $h^l_{\inf}(M; A) > -\infty$, for all D-proper sets $M \in \mathcal{Y}$. However, this statement could be wrong if $q \notin \operatorname{int} D$ (compare with Proposition 3.2(a)), as it is showed in the following example. Consider $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 0)$ $A = \{(0, 0)\}$ and $M = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$. It is clear that M is D-proper and $h^l_{\inf}(M; A) = -\infty$.

On the other hand, for each $t \in \mathbb{R}$, Theorem 3.2(ii) of [1] states that

$$h^l_{\inf}(M; A) \leq t \iff A + tq \subset M + D,$$

but this equivalence is true whenever M is D-closed (compare with (4)), as the following example shows. Let $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 1)$, $A = \{(0, 0)\}$ and $M = \operatorname{int} \mathbb{R}^2_+$. It is clear that $h^l_{\inf}(M; A) = 0$ and $A \not\subset M + D$. Analogously, the sufficient condition of [1, Theorem 3.2(xi)] and [1, Theorem 3.2(xii)] could not be true for non-D-compact sets (compare with parts (g) and (h) of Theorem 3.5). Indeed, consider $Y = \mathbb{R}^2$, $D = M = \mathbb{R}^2_+$, $q = (1, 1)$ and $A = \operatorname{int} \mathbb{R}^2_+$. It is obvious that $M \precsim^l_{\operatorname{int} D} A$, but $h^l_{\inf}(M; A) = h^l_{\inf}(A; A) = 0$. Moreover, the necessary condition of [1, Theorem 3.2(xi)] can be generalized as follows:

$$A, M \in \mathcal{Y}, t \in \mathbb{R}, \quad h^l_{\inf}(M; A) < t \Rightarrow tq + A \subset M + \operatorname{int} D, \tag{5}$$

i.e., the assumptions on the D-properness and D-closedness of M can be removed. Let us check (5). If $h^l_{\inf}(M; A) < t$ there exists $\varepsilon > 0$ such that $h^l_{\inf}(M; A) < t - \varepsilon$. Thus

$$tq + A \subset (t - \varepsilon)q + A + \varepsilon q \subset M + (\varepsilon q + D) \subset M + \operatorname{int} D$$

and statement (5) is proved.

(d) Theorem 3.5(a) (resp. Theorem 3.6(a)) has been stated in [34, Proposition 3.3] for any nonempty set $F(x) \subset Y$, i.e., without assuming that $F(x)$ is D-proper (resp. $-D$-proper). However, Proposition 3.2 (resp. Proposition 3.3) shows that this assumption cannot be removed.

Analogously, Theorem 3.5(g) (resp. Theorem 3.6(g)) has been derived in [34, Proposition 3.6] by assuming that Q is D-closed (resp. $-D$-closed) instead of D-compact (resp. $-D$-compact). The following data show that the D-closedness is not sufficient to satisfy Theorem 3.5(g) (a similar example can be proposed in order to show that the $-D$-closedness is not sufficient to satisfy Theorem 3.6(g)). Consider $Y = \mathbb{R}^2$, $D = M = \mathbb{R}^2_+$, $q = (1, 1)$ and $F(x) = Q = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_2 = 1/y_1\}$. It is easy to check that Q is D-closed, $M \precsim^l_{\text{int}\, D} Q$ and $\Phi^l_{q,x}(M) = \Phi^l_{q,x}(Q) = 0$.

Next we show an equivalent representation of the mapping $\Phi^l_{q,x}$.

Theorem 3.8 *Consider $x \in S$. We have that*

$$\Phi^l_{q,x}(M) = \sup_{y\in F(x)} \Phi^l_{q,\{y\}}(M), \quad \forall\, M \in \mathcal{Y}. \tag{6}$$

Proof Let us define the mapping $H_q : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$,

$$H_q(M_1, M_2) := \sup_{y\in M_2} \Phi^l_{q,\{y\}}(M_1), \quad \forall\, M_1, M_2 \in \mathcal{Y},$$

and consider $r \in \mathbb{R}$. Let us prove

$$H_q(M, F(x)) \leq r \iff \Phi^l_{q,x}(M) \leq r. \tag{7}$$

Indeed, by (4) we have that

$$\begin{aligned} H_q(M, F(x)) \leq r &\iff \Phi^l_{q,\{y\}}(M) \leq r, \quad \forall\, y \in F(x), \qquad (8)\\ &\iff y \in -rq + \text{cl}\,(M + D), \quad \forall\, y \in F(x),\\ &\iff F(x) + rq \subset \text{cl}\,(M + D). \qquad (9) \end{aligned}$$

Suppose that $H_q(M, F(x)) \leq r$ and consider an arbitrary $\varepsilon > 0$. Then by (9) and Proposition 2.1(b) we see that

$$F(x) + (r + \varepsilon)q \subset \text{cl}\,(M + D) + \text{int}\, D \subset M + D, \quad \forall\, \varepsilon > 0,$$

and so $r + \varepsilon \in \Lambda^l_{q,F(x)}(M)$, for all $\varepsilon > 0$. Thus $\Phi^l_{q,x}(M) \leq r$.

Reciprocally, if $\Phi^l_{q,x}(M) \leq r$ then $r + \varepsilon \in \Lambda^l_{q,F(x)}(M)$, for all $\varepsilon > 0$ and we have $F(x) + (r + \varepsilon)q \subset M + D$, for all $\varepsilon > 0$. By (8) and (9) we deduce that $H_q(M, F(x)) \leq r + \varepsilon$, for all $\varepsilon > 0$ and it follows that $H_q(M, F(x)) \leq r$.

By (7) we have that

$$\begin{aligned} \Phi^l_{q,x}(M) = -\infty &\iff H_q(M, F(x)) = -\infty,\\ \Phi^l_{q,x}(M) = +\infty &\iff H_q(M, F(x)) = +\infty, \end{aligned}$$

and also $\Phi^l_{q,x}(M) = H_q(M, F(x))$ whenever $\Phi^l_{q,x}(M) \in \mathbb{R}$ and $H_q(M, F(x)) \in \mathbb{R}$, which finishes the proof. $\square$

Remark 3.9 The scalarization mapping H_q was introduced as G_{-q} in [19] by assuming that the ordering cone D is pointed. Theorem 3.8 shows that it coincides with the scalarization mapping $\Phi^l_{q,A}$ due to Hamel and Löhne (see [17, 18]).

Let us observe that some properties stated in Theorem 3.5 were obtained implicitly in different results of [19] via the formulation H_q and by assuming additional hypotheses. To be precise, part (*a*) of Theorem 3.5 reduces to [19, Theorem 3.10(i)] by assuming that $F(x)$ is D-closed; Part (*d*) extends [19, Theorem 3.8(v)] to nonempty sets which are not D-proper; Parts (*f*) and (*g*) extend [19, Corollary 3.11(i)] to sets $A, B \in \mathcal{Y}$, where A could not be D-compact.

The following scalarization process $\{\Psi^l_{q,x}\}_{x\in S}$, $\Psi^l_{q,x} : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$ for all $x \in S$, was introduced in [13] to study approximate versions of the Ekeland variational principle in set-valued optimization problems. Consider $q \in \operatorname{int} D$ and the mapping $\xi_q : \mathcal{Y} \times Y \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\xi_q(M, y) := \inf_{z\in M}\{\max\{\langle\lambda, z - y\rangle : \lambda \in D^+(q)\}\}, \quad \forall\, M \in \mathcal{Y}, y \in Y.$$

Then, for each $x \in S$,

$$\Psi^l_{q,x}(M) := \sup\{\xi_q(M, y) : y \in F(x)\}, \quad \forall\, M \in \mathcal{Y}.$$

If $F = f$, where $f : X \to Y$ (i.e., F is single-valued), then $\Psi^l_{q,x} : Y \to \mathbb{R}$ reduces to the Gerstewitz scalarization mapping. Indeed, it is clear that

$$\xi_q(\{z\}, y) := \max\{\langle\lambda, z - y\rangle : \lambda \in D^+(q)\}, \quad \forall\, z, y \in Y,$$

and by (1) we see that

$$\begin{aligned}\Psi^l_{q,x}(\{z\}) = \xi_q(\{z\}, f(x)) &= \max\{\langle\lambda, z - f(x)\rangle : \lambda \in D^+(q)\}\\ &= s_q(z - f(x)), \quad \forall\, z \in Y.\end{aligned}$$

Moreover, in view of the definition it is clear that

$$\Psi^l_{q,x}(M) = \sup_{y\in F(x)} \inf_{z\in M} s_q(z - y), \quad \forall\, M \in \mathcal{Y}. \tag{10}$$

The following theorem shows that $\Psi^l_{q,x}$ is a reformulation of $\Phi^l_{q,x}$.

Theorem 3.10 *Consider $q \in \operatorname{int} D$ and $x \in S$. Then the mappings $\Psi^l_{q,x}$ and $\Phi^l_{q,x}$ coincide.*

Proof For each $y \in Y$ and $M \in \mathcal{Y}$ we have that:

$$\begin{aligned}\Phi^l_{q,\{y\}}(M) &= \inf\{t \in \mathbb{R} : M \precsim^l_D tq + y\}\\ &= \inf\{t \in \mathbb{R} : tq + y \in M + D\}\\ &= \inf_{z\in M} \inf\{t \in \mathbb{R} : z - y \in tq - D\}\\ &= \inf_{z\in M} s_q(z - y)\end{aligned}$$

and the proof follows by (6) and (10). □

By Remark 3.9 and Theorem 3.10 we see that the mapping G_{-q} by Hernández and Rodríguez-Marín, the mapping $\Phi^l_{q,x}$ due to Hamel and Löhne and the mapping $\Psi^l_{q,x}$ introduced by ourselves are the same function.

4 Minimality Conditions Through Scalarization

Next we obtain necessary and sufficient conditions for weak l-minimal and l-minimal solutions of problem (P) by scalarization processes $\{\varphi_x\}_{x\in S}$ such that for each $x \in S$, the mapping $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ satisfies certain order representing and monotonicity properties at $F(x)$. For each $x_0 \in S$ we denote

$$\begin{aligned}E(x_0) &= \{x \in S : F(x) \sim^l_D F(x_0)\},\\ S(x_0) &= (S \backslash E(x_0)) \cup \{x_0\}.\end{aligned}$$

First we derive necessary l-minimality conditions by using order representing mappings.

Theorem 4.1 *Let $\{\varphi_x\}_{x\in S}$ be a scalarization process, $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$, for all $x \in S$.*

(a) Let $x_0 \in S$ and suppose that φ_{x_0} is strictly order $\precsim^l_{\mathrm{int}\,D}$-representing at $F(x_0)$. If $x_0 \in WM^l(F, S)$, then $x_0 \in S(\varphi_{x_0} \circ F, S)$.
(b) Let $x_0 \in S$ and suppose that φ_{x_0} is order $\precsim^l_D$-representing at $F(x_0)$. If $x_0 \in M^l(F, S)$, then $S(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$. If additionally φ_{x_0} is $\precsim^l_D$-monotone on $\mathcal{F}$, then $S(\varphi_{x_0} \circ F, S) = E(x_0)$.

Proof (a) Suppose that $x_0 \notin \mathrm{S}(\varphi_{x_0} \circ F, S)$. Then there exists $x \in S$ such that $\varphi_{x_0}(F(x)) < \varphi_{x_0}(F(x_0))$. As φ_{x_0} is strictly order $\precsim^l_{\mathrm{int}\,D}$-representing at $F(x_0)$, it follows that $F(x) \prec^l_{\mathrm{int}\,D} F(x_0)$, which is a contradiction since $x_0 \in \mathrm{WM}^l(F, S)$.

(b) Consider $x \in S(x_0)$, $x \neq x_0$. It follows that $x \in S$, $F(x) \nsim^l_D F(x_0)$ and since $x_0 \in \mathrm{M}^l(F, S)$ we have $F(x) \not\precsim^l_D F(x_0)$. Thus, as φ_{x_0} is order $\precsim^l_D$-representing at $F(x_0)$ we deduce that $\varphi_{x_0}(F(x)) > \varphi_{x_0}(F(x_0))$ and so $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$.

If additionally φ_{x_0} is $\precsim^l_D$-monotone, then

$$\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\} \iff \mathrm{S}(\varphi_{x_0} \circ F, S) = E(x_0) \tag{11}$$

and so the last statement of part (b) is a direct consequence of the first one. Let us prove equivalence (11). The sufficient condition is trivial. On the other hand, suppose that $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ and consider $x \in E(x_0)$. Since $F(x) \precsim^l_D F(x_0)$, $F(x_0) \precsim^l_D F(x)$ and φ_{x_0} is $\precsim^l_D$-monotone on $\mathcal{F}$, we have that $\varphi_{x_0}(F(x)) = \varphi_{x_0}(F(x_0))$ and so $\mathrm{S}(\varphi_{x_0} \circ F, S) = E(x_0)$, which finishes the proof. $\square$

In the following theorem we obtain sufficient conditions for weak l-minimal and l-minimal solutions of problem (P) by solutions and strict solutions of scalarization processes $\{\varphi_x\}_{x \in S}$ such that for each $x \in S$, the mapping $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone and $\precsim^l_D$-monotone at $F(x)$, respectively.

Theorem 4.2 *Let $\{\varphi_x\}_{x \in S}$ be a scalarization process, where $\varphi_x : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$, for all $x \in S$.*

(a) Let $x_0 \in S$ be such that φ_{x_0} is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone at $F(x_0)$. If $x_0 \in S(\varphi_{x_0} \circ F, S)$ then $x_0 \in WM^l(F, S)$.
(b) Suppose that φ_x is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone on $\mathcal{F}$, for all $x \in S$. Then

$$\bigcup_{x \in S} S(\varphi_x \circ F, S) \subset WM^l(F, S).$$

(c) Let $x_0 \in S$ be such that φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$. If $S(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ then $x_0 \in M^l(F, S)$.

Proof Let us prove parts (a) and (c), since the proof of part (b) is similar to the proof of part (a).

(a) Suppose that $x_0 \notin \mathrm{WM}^l(F, S)$. Then there exists $x \in S$ such that $F(x) \prec^l_{\mathrm{int}\, D} F(x_0)$. As φ_{x_0} is strictly $\precsim^l_{\mathrm{int}\, D}$-monotone at $F(x_0)$ we deduce that $\varphi_{x_0}(F(x)) < \varphi_{x_0}(F(x_0))$, which is a contradiction.

(c) Suppose that $x_0 \notin \mathrm{M}^l(F, S)$. Then there exists $x \in S$ such that $F(x) \prec^l_D F(x_0)$, i.e., $F(x) \precsim^l_D F(x_0)$ and $F(x) \not\sim^l_D F(x_0)$, and so $x \in S(x_0)\backslash\{x_0\}$. We have that $\varphi_{x_0}(F(x)) \leq \varphi_{x_0}(F(x_0))$, since φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$. As $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$, it follows that $x = x_0$, that is a contradiction and the proof is completed. $\square$

Remark 4.3 Let us observe that if φ_{x_0} is $\precsim^l_D$-monotone at $F(x_0)$ and $\mathrm{S}(\varphi_{x_0} \circ F, S) = \{x_0\}$ (i.e., x_0 is a strict solution of problem (P_{φ_p}) with $\varphi_p = \varphi_{x_0}$), then $\mathrm{S}(\varphi_{x_0} \circ F, S(x_0)) = \{x_0\}$ and by Theorem 4.2 it follows that $x_0 \in \mathrm{M}^l(F, S)$.

By applying Theorems 4.1 and 4.2 to the mappings $\{\Phi^l_{q,x}\}_{x \in S}$ one obtains the following characterizations for weak l-minimal and l-minimal solutions of problem (P). Moreover, the same approach can be done to characterize weak u-minimal and u-minimal solutions of problem (P) by considering the scalarization mapping $\Phi^u_{q,x}$ and the set orderings $\precsim^u_D$ and $\precsim^u_{\mathrm{int}\, D}$ instead of $\precsim^l_D$ and $\precsim^l_{\mathrm{int}\, D}$, respectively. In fact, this approach can be extended to other set orderings (see [11]).

Corollary 4.4 *Let $q \in \mathrm{int}\, D$. The following statements hold:*

(*a*) *Assume that F is D-compact valued in S. Then, for each $x_0 \in X$ it follows that*

$$x_0 \in WM^l(F, S) \iff x_0 \in S(\Phi^l_{q,x_0} \circ F, S).$$

Moreover,

$$WM^l(F, S) = \bigcup_{x \in S} S(\Phi^l_{q,x} \circ F, S). \tag{12}$$

(*b*) *Let $x_0 \in S$ such that $F(x_0)$ is D-proper, and assume that F is D-closed valued in S. Then*

$$x_0 \in M^l(F, S) \iff S(\Phi^l_{q,x_0} \circ F, S) = E(x_0).$$

Proof (*a*) By Theorem 3.5 it follows that $\Phi^l_{q,x}$ is strictly order $\precsim^l_{\text{int } D}$-representing at $F(x)$, and strictly $\precsim^l_{\text{int } D}$-monotone on $\mathcal{F}$, for all $x \in S$. Then part (*a*) is a consequence of part (*a*) of Theorem 4.1 and parts (*a*) and (*b*) of Theorem 4.2.

(*b*) As $F(x_0)$ is D-proper, by Theorem 3.5(*d*) we have that $\Phi^l_{q,x_0} : \mathcal{F} \to \mathbb{R} \cup \{\pm\infty\}$ is $\precsim^l_D$-monotone on $\mathcal{F}$, and using Theorem 3.5(*a*), (*e*) it is easy to check that Φ^l_{q,x_0} is order $\precsim^l_D$-representing at $F(x_0)$, since F is D-closed valued in S. Then the result follows by Theorem 4.1(*b*), statement (11) and Theorem 4.2(*c*). □

Remark 4.5 (*a*) Corollary 4.4(*a*) improves [19, Corollary 4.11], since it generalizes the sufficient condition of [19, Corollary 4.11] to the whole solution set of the scalarized problem and so the scalar representation (12) holds. Analogously, Corollary 4.4(*b*) reduces to [19, Corollary 4.8] by assuming that $F(x_0)$ is D-bounded instead of D-proper.

Let us observe that Corollaries 4.8 and 4.11 of [19] characterize also weak l-maximal and l-maximal solutions of problem (P).

On the other hand, the sufficient condition of Corollary 4.4(*b*) reduces to [34, Theorem 4.3] by assuming that $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = \{x_0\}$ (i.e., x_0 is a strict solution), since in this case it follows that $E(x_0) = \{x_0\}$ and so $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = E(x_0)$. Indeed, suppose that x_0 is a strict solution and consider $x \in E(x_0)$. As Φ^l_{q,x_0} is $\precsim^l_D$-monotone on $\mathcal{Y}$ we deduce that $\Phi^l_{q,x_0}(F(x)) = \Phi^l_{q,x_0}(F(x_0))$. Then $x = x_0$, since $\mathrm{S}(\Phi^l_{q,x_0} \circ F, S) = \{x_0\}$ and we have that $E(x_0) = \{x_0\}$.

(*b*) Corollary 4.4(*a*) has been stated in [1, Theorem 5.2] and [34, Theorem 4.2] by assuming that F is D-bounded and D-closed in S. The following data show that these assumptions do not guarantee both results. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $q = (1, 1)$, $F(x) = \mathbb{R}^2_+$, for all $x \in \mathbb{R}$, $x \neq 0$, and $F(0) = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_2 = 1/y_1\}$. It is easy to check that F is D-closed and D-bounded in X and $\Phi^l_{q,0}(F(x)) = 0$ for all $x \in \mathbb{R}$, but $0 \notin \mathrm{WM}^l(F, X)$, since $F(x) \prec^l_{\text{int } D} F(0)$, for all $x \in X \backslash \{0\}$.

Analogously, Corollary 4.4(*b*) clarifies Theorem 5.2 of [1], which could not be true. For example, if F is constant in the feasible set S then $\mathrm{M}^l(F, S) = S$ and for each $x_0 \in S$ it is clear that $h^l_{\inf}(F(x); F(x_0)) = 0$, for all $x \in S$.

5 Conclusions

In this paper, a detailed study on the scalarization of set optimization problems has been carried out. Several concepts of solution based on set relations have been characterized via generic scalarization mappings that satisfy suitable properties. Then these characterizations have been specified by using well-known scalarization mappings, whose properties have previously been studied. An useful research direction motivated by the results of this paper is to derive from them numerical procedures to solve this kind of optimization problems.

Recently, another approach called vectorization has been proposed to characterize solutions of set optimization problems based on set relations (see [22]). In this approach, a suitable vector optimization problem is defined whose solutions are related with the solutions of the set optimization problem. It would be interesting to relate this approach with the results of this paper. In particular, one can analyze if some of these results can be derived by combining the vectorization approach with some of the well-known scalarization processes used in vector optimization.

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