Thus we obtain the following

***Π*-theorem of dimension theory** *If the quantities* $x_1, \dots, x_k$ *in relation* (*) *are independent, this relation can be reduced to the function* (****) *of* $n-k$ *dimensionless parameters.*

f) Verify that if $k = n$, the function f in relation (*) can be determined up to a numerical multiple by using the Π-theorem. Use this method to find the expression $c(\varphi_0)\sqrt{l/g}$ for the period of oscillation of a pendulum (that is, a mass m suspended by a thread of length l and oscillating near the surface of the earth, where φ_0 is the initial displacement angle).

g) Find a formula $P = c\sqrt{mr/F}$ for the period of revolution of a body of mass m held in a circular orbit by a central force of magnitude F.

h) Use Kepler's law $(P_1/P_2)^2 = (r_1/r_2)^3$, which establishes for circular orbits a connection between the ratio of the periods of revolution of planets (or satellites) and the ratio of the radii of their orbits, to find, as Newton did, the exponent α in the law of universal gravitation $F = G\frac{m_1 m_2}{r^\alpha}$.

8.4 The Basic Facts of Differential Calculus of Real-Valued Functions of Several Variables

8.4.1 The Mean-Value Theorem

Theorem 1 *Let* $f : G \to \mathbb{R}$ *be a real-valued function defined in a region* $G \subset \mathbb{R}^m$, *and let the closed line segment* $[x, x+h]$ *with endpoints* x *and* $x+h$ *be contained in* G. *If the function* f *is continuous at the points of the closed line segment* $[x, x+h]$ *and differentiable at points of the open interval* $]x, x+h[$, *then there exists a point* $\xi \in]x, x+h[$ *such that the following equality holds*:

$$\boxed{f(x+h) - f(x) = f'(\xi)h.} \tag{8.53}$$

Proof Consider the auxiliary function

$$F(t) = f(x+th)$$

defined on the closed interval $0 \le t \le 1$. This function satisfies all the hypotheses of Lagrange's theorem: it is continuous on $[0, 1]$, being the composition of continuous mappings, and differentiable on the open interval $]0, 1[$, being the composition of differentiable mappings. Consequently, there exists a point $\theta \in]0, 1[$ such that

$$F(1) - F(0) = F'(\theta) \cdot 1.$$

But $F(1) = f(x+h)$, $F(0) = f(x)$, $F'(\theta) = f'(x+\theta h)h$, and hence the equality just written is the same as the assertion of the theorem. □

We now give the coordinate form of relation (8.53).

If $x = (x^1, \dots, x^m)$, $h = (h^1, \dots, h^m)$, and $\xi = (x^1 + \theta h^1, \dots, x^m + \theta h^m)$, Eq. (8.53) means that

$$
\begin{aligned}
f(x+h) - f(x) = f\bigl(x^1 + h^1, \dots, x^m + h^m\bigr) - f\bigl(x^1, \dots, x^m\bigr) &= \\
= f'(\xi)h = \left(\frac{\partial f}{\partial x^1}(\xi), \dots, \frac{\partial f}{\partial x^m}(\xi)\right)\begin{pmatrix} h^1 \\ \cdots \\ h^m \end{pmatrix} &= \\
= \partial_1 f(\xi)h^1 + \cdots + \partial_m f(\xi)h^m &= \\
= \sum_{i=1}^{m} \partial_i f\bigl(x^1 + \theta h^1, \dots, x^m + \theta h^m\bigr)h^i&.
\end{aligned}
$$

Using the convention of summation on an index that appears as both superscript and subscript, we can finally write

$$
\begin{aligned}
&f\bigl(x^1 + h^1, \dots, x^m + h^m\bigr) - f\bigl(x^1, \dots, x^m\bigr) = \\
&\quad = \partial_i f\bigl(x^1 + \theta h^1, \dots, x^m + \theta h^m\bigr)h^i,
\end{aligned}
\tag{8.54}
$$

where $0 < \theta < 1$ and θ depends on both x and h.

Remark Theorem 1 is called the mean-value theorem because there exists a certain "average" point $\xi \in]x, x+h[$ at which Eq. (8.53) holds. We have already noted in our discussion of Lagrange's theorem (Sect. 5.3.1) that the mean-value theorem is specific to real-valued functions. A general finite-increment theorem for mappings will be proved in Chap. 10 (Part 2).

The following proposition is a useful corollary of Theorem 1.

Corollary *If the function $f : G \to \mathbb{R}$ is differentiable in the domain $G \subset \mathbb{R}^m$ and its differential equals zero at every point $x \in G$, then f is constant in the domain G.*

Proof The vanishing of a linear transformation is equivalent to the vanishing of all the elements of the matrix corresponding to it. In the present case

$$
\mathrm{d}f(x)h = (\partial_1 f, \dots, \partial_m f)(x)h,
$$

and therefore $\partial_1 f(x) = \cdots = \partial_m f(x) = 0$ at every point $x \in G$.

By definition, a domain is an open connected set. We shall make use of this fact.

We first show that if $x \in G$, then the function f is constant in a ball $B(x; r) \subset G$. Indeed, if $(x+h) \in B(x; r)$, then $[x, x+h] \subset B(x; r) \subset G$. Applying relation (8.53) or (8.54), we obtain

$$
f(x+h) - f(x) = f'(\xi)h = 0 \cdot h = 0,
$$

that is, $f(x+h) = f(x)$, and the values of f in the ball $B(x;r)$ are all equal to the value at the center of the ball.

Now let $x_0, x_1 \in G$ be arbitrary points of the domain G. By the connectedness of G, there exists a path $t \mapsto x(t) \in G$ such that $x(0) = x_0$ and $x(1) = x_1$. We assume that the continuous mapping $t \mapsto x(t)$ is defined on the closed interval $0 \le t \le 1$. Let $B(x_0;r)$ be a ball with center at x_0 contained in G. Since $x(0) = x_0$ and the mapping $t \mapsto x(t)$ is continuous, there is a positive number δ such that $x(t) \in B(x_0;r) \subset G$ for $0 \le t \le \delta$. Then, by what has been proved, $(f \circ x)(t) \equiv f(x_0)$ on the interval $[0, \delta]$.

Let $l = \sup \delta$, where the upper bound is taken over all numbers $\delta \in [0, 1]$ such that $(f \circ x)(t) \equiv f(x_0)$ on the interval $[0, \delta]$. By the continuity of the function $f(x(t))$ we have $f(x(l)) = f(x_0)$. But then $l = 1$. Indeed, if that were not so, we could take a ball $B(x(l);r) \subset G$, in which $f(x) = f(x(l)) = f(x_0)$, and then by the continuity of the mapping $t \mapsto x(t)$ find $\Delta > 0$ such that $x(t) \in B(x(l);r)$ for $l \le t \le l + \Delta$. But then $(f \circ x)(t) = f(x(l)) = f(x_0)$ for $0 \le t \le l + \Delta$, and so $l \ne \sup \delta$.

Thus we have shown that $(f \circ x)(t) = f(x_0)$ for any $t \in [0, 1]$. In particular $(f \circ x)(1) = f(x_1) = f(x_0)$, and we have verified that the values of the function $f : G \to \mathbb{R}$ are the same at any two points $x_0, x_1 \in G$. □

8.4.2 A Sufficient Condition for Differentiability of a Function of Several Variables

Theorem 2 *Let $f : U(x) \to \mathbb{R}$ be a function defined in a neighborhood $U(x) \subset \mathbb{R}^m$ of the point $x = (x^1, \dots, x^m)$.*

If the function f has all partial derivatives $\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^m}$ at each point of the neighborhood $U(x)$ and they are continuous at x, then f is differentiable at x.

Proof Without loss of generality we shall assume that $U(x)$ is a ball $B(x;r)$. Then, together with the points $x = (x^1, \dots, x^m)$ and $x + h = (x^1 + h^1, \dots, x^m + h^m)$, the points $(x^1, x^2 + h^2, \dots, x^m + h^m), \dots, (x^1, x^2, \dots, x^{m-1}, x^m + h^m)$ and the lines connecting them must also belong to the domain $U(x)$. We shall use this fact, applying the Lagrange theorem for functions of one variable in the following computation:

$$\begin{aligned}
f(x+h) - f(x) &= f\bigl(x^1 + h^1, \dots, x^m + h^m\bigr) - f\bigl(x^1, \dots, x^m\bigr) = \\
&= f\bigl(x^1 + h^1, \dots, x^m + h^m\bigr) - f\bigl(x^1, x^2 + h^2, \dots, x^m + h^m\bigr) + \\
&\quad + f\bigl(x^1, x^2 + h^2, \dots, x^m + h^m\bigr) - \\
&\quad - f\bigl(x^1, x^2, x^3 + h^3, \dots, x^m + h^m\bigr) + \dots + \\
&\quad + f\bigl(x^1, x^2, \dots, x^{m-1}, x^m + h^m\bigr) - f\bigl(x^1, \dots, x^m\bigr) =
\end{aligned}$$

$$
\begin{aligned}
&= \partial_1 f\bigl(x^1+\theta^1 h^1, x^2+h^2,\dots,x^m+h^m\bigr)h^1 + \\
&\quad + \partial_2 f\bigl(x^1, x^2+\theta^2 h^2, x^3+h^3,\dots,x^m+h^m\bigr)h^2 + \dots + \\
&\quad + \partial_m f\bigl(x^1, x^2,\dots,x^{m-1}, x^m+\theta^m h^m\bigr)h^m.
\end{aligned}
$$

So far we have used only the fact that the function f has partial derivatives with respect to each of its variables in the domain $U(x)$.

We now use the fact that these partial derivatives are continuous at x. Continuing the preceding computation, we obtain

$$
\begin{aligned}
f(x+h)-f(x) &= \partial_1 f\bigl(x^1,\dots,x^m\bigr)h^1 + \alpha^1 h^1 + \\
&\quad + \partial_2 f\bigl(x^1,\dots,x^m\bigr)h^2 + \alpha^2 h^2 + \dots + \\
&\quad + \partial_m f\bigl(x^1,\dots,x^m\bigr)h^m + \alpha^m h^m,
\end{aligned}
$$

where the quantities $\alpha_1,\dots,\alpha_m$ tend to zero as $h \to 0$ by virtue of the continuity of the partial derivatives at the point x.

But this means that

$$f(x+h)-f(x) = L(x)h + o(h) \quad \text{as } h \to 0,$$

where $L(x)h = \partial_1 f(x^1,\dots,x^m)h^1 + \dots + \partial_m f(x^1,\dots,x^m)h^m$. □

It follows from Theorem 2 that if the partial derivatives of a function $f : G \to \mathbb{R}$ are continuous in the domain $G \subset \mathbb{R}^m$, then the function is differentiable at that point of the domain.

Let us agree from now on to use the symbol $C^{(1)}(G;\mathbb{R})$, or, more simply, $C^{(1)}(G)$ to denote the set of functions having continuous partial derivatives in the domain G.

8.4.3 Higher-Order Partial Derivatives

If a function $f : G \to \mathbb{R}$ defined in a domain $G \subset \mathbb{R}^m$ has a partial derivative $\frac{\partial f}{\partial x^i}(x)$ with respect to one of the variables $x^1,\dots,x^m$, this partial derivative is a function $\partial_i f : G \to \mathbb{R}$, which in turn may have a partial derivative $\partial_j(\partial_i f)(x)$ with respect to a variable x^j.

The function $\partial_j(\partial_i f) : G \to \mathbb{R}$ is called the *second partial derivative of f with respect to the variables x^i and x^j* and is denoted by one of the following symbols:

$$\partial_{ji} f(x), \qquad \frac{\partial^2 f}{\partial x^j \partial x^i}(x).$$

The order of the indices indicates the order in which the differentiation is carried out with respect to the corresponding variables.

We have now defined partial derivatives of second order.

If a partial derivative of order k

$$\partial_{i_1\cdots i_k} f(x) = \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}(x)$$

has been defined, we define by induction the partial derivative of order $k+1$ by the relation

$$\partial_{i i_1\cdots i_k} f(x) := \partial_i (\partial_{i_1\cdots i_k} f)(x).$$

At this point a question arises that is specific for functions of several variables: Does the order of differentiation affect the partial derivative computed?

Theorem 3 *If the function* $f : G \to \mathbb{R}$ *has partial derivatives*

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(x), \quad \frac{\partial^2 f}{\partial x^j \partial x^i}(x)$$

in a domain G, *then at every point* $x \in G$ *at which both partial derivatives are continuous*, *their values are the same*.

Proof Let $x \in G$ be a point at which both functions $\partial_{ij} f : G \to \mathbb{R}$ and $\partial_{ji} f : G \to \mathbb{R}$ are continuous. From this point on all of our arguments are carried out in the context of a ball $B(x; r) \subset G$, $r > 0$, which is a convex neighborhood of the point x. We wish to verify that

$$\frac{\partial^2 f}{\partial x^i \partial x^j}\bigl(x^1, \dots, x^m\bigr) = \frac{\partial^2 f}{\partial x^j \partial x^i}\bigl(x^1, \dots, x^m\bigr).$$

Since only the variables x^i and x^j will be changing in the computations to follow, we shall assume for the sake of brevity that f is a function of two variables $f(x^1, x^2)$, and we need to verify that

$$\frac{\partial^2 f}{\partial x^1 \partial x^2}\bigl(x^1, x^2\bigr) = \frac{\partial^2 f}{\partial x^2 \partial x^1}\bigl(x^1, x^2\bigr),$$

if the two functions are both continuous at the point (x^1, x^2).

Consider the auxiliary function

$$F\bigl(h^1, h^2\bigr) = f\bigl(x^1 + h^1, x^2 + h^2\bigr) - f\bigl(x^1 + h^1, x^2\bigr) - f\bigl(x^1, x^2 + h^2\bigr) + f\bigl(x^1, x^2\bigr),$$

where the displacement $h = (h^1, h^2)$ is assumed to be sufficiently small, namely so small that $x + h \in B(x; r)$.

If we regard $F(h^1, h^2)$ as the difference

$$F\bigl(h^1, h^2\bigr) = \varphi(1) - \varphi(0),$$

where $\varphi(t) = f(x^1 + th^1, x^2 + h^2) - f(x^1 + th^1, x^2)$, we find by Lagrange's theorem that

$$F(h^1, h^2) = \varphi'(\theta_1) = \big(\partial_1 f\big(x^1 + \theta_1 h^1, x^2 + h^2\big) - \partial f\big(x^1 + \theta_1 h^1, x^2\big)\big)h^1.$$

Again applying Lagrange's theorem to this last difference, we find that

$$F(h^1, h^2) = \partial_{21} f\big(x^1 + \theta_1 h^1, x^2 + \theta_2 h^2\big)h^2 h^1. \tag{8.55}$$

If we now represent $F(h^1, h^2)$ as the difference

$$F(h^1, h^2) = \tilde{\varphi}(1) - \tilde{\varphi}(0),$$

where $\tilde{\varphi}(t) = f(x^1 + h^1, x^2 + th^2) - f(x^1, x^2 + th^2)$, we find similarly that

$$F(h^1, h^2) = \partial_{12} f\big(x^1 + \tilde{\theta}_1 h^1, x^2 + \tilde{\theta}_2 h^2\big)h^1 h^2. \tag{8.56}$$

Comparing (8.55) and (8.56), we conclude that

$$\partial_{21} f\big(x^1 + \theta_1 h^1, x^2 + \theta_2 h^2\big) = \partial_{12} f\big(x^1 + \tilde{\theta}_1 h^1, x^2 + \tilde{\theta}_2 h^2\big), \tag{8.57}$$

where $\theta_1, \theta_2, \tilde{\theta}_1, \tilde{\theta}_2 \in \,]0, 1[$. Using the continuity of the partial derivatives at the point (x^1, x^2), as $h \to 0$, we get the equality we need as a consequence of (8.57).

$$\partial_{21} f\big(x^1, x^2\big) = \partial_{12} f\big(x^1, x^2\big). \qquad \square$$

We remark that without additional assumptions we cannot say in general that $\partial_{ij} f(x) = \partial_{ji} f(x)$ if both of the partial derivatives are defined at the point x (see Problem 2 at the end of this section).

Let us agree to denote the set of functions $f : G \to \mathbb{R}$ all of whose partial derivatives up to order k inclusive are defined and continuous in the domain $G \subset \mathbb{R}^m$ by the symbol $C^{(k)}(G; \mathbb{R})$ or $C^{(k)}(G)$.

As a corollary of Theorem 3, we obtain the following.

Proposition 1 *If $f \in C^{(k)}(G; \mathbb{R})$, the value $\partial_{i_1 \ldots i_k} f(x)$ of the partial derivative is independent of the order $i_1, \ldots, i_k$ of differentiation, that is, remains the same for any permutation of the indices $i_1, \ldots, i_k$.*

Proof In the case $k = 2$ this proposition is contained in Theorem 3.

Let us assume that the proposition holds up to order n inclusive. We shall show that then it also holds for order $n + 1$.

But $\partial_{i_1 i_2 \cdots i_{n+1}} f(x) = \partial_{i_1}(\partial_{i_2 \cdots i_{n+1}} f)(x)$. By the induction assumption the indices $i_2, \ldots, i_{n+1}$ can be permuted without changing the function $\partial_{i_2 \cdots i_{n+1}} f(x)$, and hence without changing $\partial_{i_1 \cdots i_{n+1}} f(x)$. For that reason it suffices to verify that one can also permute, for example, the indices i_1 and i_2 without changing the value of the derivative $\partial_{i_1 i_2 \cdots i_{n+1}} f(x)$.

Since

$$\partial_{i_1 i_2 \cdots i_{n+1}} f(x) = \partial_{i_1 i_2}(\partial_{i_3 \cdots i_{n+1}} f)(x),$$

the possibility of this permutation follows immediately from Theorem 3. By the induction principle Proposition 1 is proved. □

Example 1 Let $f(x) = f(x^1, x^2)$ be a function of class $C^{(k)}(G; \mathbb{R})$.

Let $h = (h^1, h^2)$ be such that the closed interval $[x, x+h]$ is contained in the domain G. We shall show that the function

$$\varphi(t) = f(x + th),$$

which is defined on the closed interval [0, 1], belongs to class $C^{(k)}[0, 1]$ and find its derivative of order k with respect to t.

We have

$$\varphi'(t) = \partial_1 f\bigl(x^1 + th^1, x^2 + th^2\bigr)h^1 + \partial_2 f\bigl(x^1 + th^1,\ x^2 + th^2\bigr)h^2,$$

$$\begin{aligned}\varphi''(t) = {} & \partial_{11} f(x+th)h^1h^1 + \partial_{21} f(x+th)h^2h^1 + \\ & + \partial_{12} f(x+th)h^1h^2 + \partial_{22} f(x+th)h^2h^2 = \\ = {} & \partial_{11} f(x+th)\bigl(h^1\bigr)^2 + 2\partial_{12} f(x+th)h^1h^2 + \partial_{22} f(x+th)\bigl(h^2\bigr)^2.\end{aligned}$$

These relations can be written as the action of the operator $(h^1\partial_1 + h^2\partial_2)$:

$$\varphi'(t) = \bigl(h^1\partial_1 + h^2\partial_2\bigr) f(x+th) = h^i \partial_i f(x+th),$$

$$\varphi''(t) = \bigl(h^1\partial_1 + h^2\partial_2\bigr)^2 f(x+th) = h^{i_1}h^{i_2}\partial_{i_1 i_2} f(x+th).$$

By induction we obtain

$$\varphi^{(k)}(t) = \bigl(h^1\partial_1 + h^2\partial_2\bigr)^k f(x+th) = h^{i_1}\cdots h^{i_k}\partial_{i_1\cdots i_k} f(x+th)$$

(summation over all sets $i_1, \dots, i_k$ of k indices, each assuming the values 1 and 2, is meant).

Example 2 If $f(x) = f(x^1, \dots, x^m)$ and $f \in C^{(k)}(G; \mathbb{R})$, then, under the assumption that $[x, x+h] \subset G$, for the function $\varphi(t) = f(x+th)$ defined on the closed interval [0, 1] we obtain

$$\varphi^{(k)}(t) = h^{i_1}\cdots h^{i_k}\partial_{i_1\cdots i_k} f(x+th), \tag{8.58}$$

where summation over all sets of indices $i_1, \dots, i_k$, each assuming all values from 1 to m inclusive, is meant on the right.

We can also write formula (8.58) as

$$\varphi^{(k)}(t) = \bigl(h^1\partial_1 + \cdots + h^m\partial_m\bigr)^k f(x+th). \tag{8.59}$$

8.4.4 Taylor's Formula

Theorem 4 *If the function $f : U(x) \to \mathbb{R}$ is defined and belongs to class $C^{(n)}(U(x); \mathbb{R})$ in a neighborhood $U(x) \subset \mathbb{R}^m$ of the point $x \in \mathbb{R}^m$, and the closed interval $[x, x+h]$ is completely contained in $U(x)$, then the following equality holds*:

$$f\left(x^1+h^1, \dots, x^m+h^m\right) - f\left(x^1, \dots, x^m\right) = \\ = \sum_{k=1}^{n-1} \frac{1}{k!}\left(h^1 \partial_1 + \cdots + h^m \partial_m\right)^k f(x) + r_{n-1}(x; h), \tag{8.60}$$

where

$$r_{n-1}(x; h) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}\left(h^1 \partial_1 + \cdots + h^m \partial_m\right)^n f(x+th)\, \mathrm{d}t. \tag{8.61}$$

Equality (8.60), together with (8.61), is called *Taylor's formula with integral form of the remainder.*

Proof Taylor's formula follows immediately from the corresponding Taylor formula for a function of one variable. In fact, consider the auxiliary function

$$\varphi(t) = f(x+th),$$

which, by the hypotheses of Theorem 4, is defined on the closed interval $0 \le t \le 1$ and (as we have verified above) belongs to the class $C^{(n)}[0, 1]$.

Then for $\tau \in [0, 1]$, by Taylor's formula for functions of one variable, we can write that

$$\varphi(\tau) = \varphi(0) + \frac{1}{1!}\varphi'(0)\tau + \cdots + \frac{1}{(n-1)!}\varphi^{(n-1)}(0)\tau^{n-1} + \\ + \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}\varphi^{(n)}(t\tau)\tau^n \, \mathrm{d}t.$$

Setting $\tau = 1$ here, we obtain

$$\varphi(1) = \varphi(0) + \frac{1}{1!}\varphi'(0) + \cdots + \frac{1}{(n-1)!}\varphi^{(n-1)}(0) + \\ + \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}\varphi^{(n)}(t)\, \mathrm{d}t. \tag{8.62}$$

Substituting the values

$$\varphi^{(k)}(0) = \left(h^1 \partial_1 + \cdots + h^m \partial_m\right)^k f(x) \quad (k = 0, \dots, n-1),$$

$$\varphi^{(n)}(t) = \left(h^1\partial_1 + \cdots + h^m\partial_m\right)^n f(x+th),$$

into this equality in accordance with formula (8.59), we find what Theorem 4 asserts. □

Remark If we write the remainder term in relation (8.62) in the Lagrange form rather than the integral form, then the equality

$$\varphi(1) = \varphi(0) + \frac{1}{1!}\varphi'(0) + \cdots + \frac{1}{(n-1)!}\varphi^{(n-1)}(0) + \frac{1}{n!}\varphi^{(n)}(\theta),$$

where $0 < \theta < 1$, implies Taylor's formula (8.60) with remainder term

$$r_{n-1}(x;h) = \frac{1}{n!}\left(h^1\partial_1 + \cdots + h^m\partial_m\right)^n f(x+\theta h). \tag{8.63}$$

This form of the remainder term, as in the case of functions of one variable, is called the *Lagrange form of the remainder term in Taylor's formula*.

Since $f \in C^{(n)}(U(x);\mathbb{R})$, it follows from (8.63) that

$$r_{n-1}(x;h) = \frac{1}{n!}\left(h^1\partial_1 + \cdots + h^m\partial_m\right)^n f(x) + o\left(\|h\|^n\right) \quad \text{as } h \to 0,$$

and so we have the equality

$$\begin{aligned} &f\left(x^1+h^1,\dots,x^m+h^m\right) - f\left(x^1,\dots,x^m\right) = \\ &= \sum_{k=1}^{n} \frac{1}{k!}\left(h^1\partial_1 + \cdots + h^m\partial_m\right)^k f(x) + o\left(\|h\|^n\right) \quad \text{as } h \to 0, \end{aligned} \tag{8.64}$$

called *Taylor's formula with the remainder term in Peano form*.

8.4.5 Extrema of Functions of Several Variables

One of the most important applications of differential calculus is its use in finding extrema of functions.

Definition 1 A function $f : E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}^m$ has a *local maximum* (resp. *local minimum*) at an interior point x_0 of E if there exists a neighborhood $U(x_0) \subset E$ of the point x_0 such that $f(x) \le f(x_0)$ (resp. $f(x) \ge f(x_0)$) for all $x \in U(x_0)$.

If the strict inequality $f(x) < f(x_0)$ holds for $x \in U(x_0) \setminus x_0$ (or, respectively, $f(x) > f(x_0)$), the function has a *strict local maximum* (resp. *strict local minimum*) at x_0.

Definition 2 The local minima and maxima of a function are called its *local extrema*.

Theorem 5 *Suppose a function $f : U(x_0) \to \mathbb{R}$ defined in a neighborhood $U(x_0) \subset \mathbb{R}^m$ of the point $x_0 = (x_0^1, \dots, x_0^m)$ has partial derivatives with respect to each of the variables $x^1, \dots, x^m$ at the point x_0.*

Then a necessary condition for the function to have a local extremum at x_0 is that the following equalities hold at that point:

$$\frac{\partial f}{\partial x^1}(x_0) = 0, \quad \dots, \quad \frac{\partial f}{\partial x^m}(x_0) = 0. \tag{8.65}$$

Proof Consider the function $\varphi(x^1) = f(x^1, x_0^2, \dots, x_0^m)$ of one variable defined, according to the hypotheses of the theorem, in some neighborhood of the point x_0^1 on the real line. At x_0^1 the function $\varphi(x^1)$ has a local extremum, and since

$$\varphi'(x_0^1) = \frac{\partial f}{\partial x^1}(x_0^1, x_0^2, \dots, x_0^m),$$

it follows that $\frac{\partial f}{\partial x^1}(x_0) = 0$.

The other equalities in (8.65) are proved similarly. □

We call attention to the fact that relations (8.65) give only necessary but not sufficient conditions for an extremum of a function of several variables. An example that confirms this is any example constructed for this purpose for functions of one variable. Thus, where previously we spoke of the function $x \mapsto x^3$, whose derivative is zero at zero, but has no extremum there, we can now consider the function

$$f(x^1, \dots, x^m) = (x^1)^3,$$

all of whose partial derivatives are zero at $x_0 = (0, \dots, 0)$, while the function obviously has no extremum at that point.

Theorem 5 shows that if the function $f : G \to \mathbb{R}$ is defined on an open set $G \subset \mathbb{R}^m$, its local extrema are found either among the points at which f is not differentiable or at the points where the differential $df(x_0)$ or, what is the same, the tangent mapping $f'(x_0)$, vanishes.

We know that if a mapping $f : U(x_0) \to \mathbb{R}^n$ defined in a neighborhood $U(x_0) \subset \mathbb{R}^m$ of the point $x_0 \in \mathbb{R}^m$ is differentiable at x_0, then the matrix of the tangent mapping $f'(x_0) : \mathbb{R}^m \to \mathbb{R}^n$ has the form

$$\begin{pmatrix} \partial^1 f^1(x_0) & \cdots & \partial_m f^1(x_0) \\ \vdots & \ddots & \vdots \\ \partial_1 f^n(x_0) & \cdots & \partial_m f^n(x_0) \end{pmatrix}. \tag{8.66}$$

Definition 3 The point x_0 is a *critical point of the mapping* $f : U(x_0) \to \mathbb{R}^n$ if the rank of the Jacobi matrix (8.66) of the mapping at that point is less than $\min\{m, n\}$, that is, smaller than the maximum possible value it can have.

In particular, if $n = 1$, the point x_0 is critical if condition (8.65) holds, that is, all the partial derivatives of the function $f : U(x_0) \to \mathbb{R}$ vanish.

The critical points of real-valued functions are also called the *stationary* points of these functions.

After the critical points of a function have been found by solving the system (8.65), the subsequent analysis to determine whether they are extrema or not can often be carried out using Taylor's formula and the following sufficient conditions for the presence or absence of an extremum provided by that formula.

Theorem 6 *Let $f : U(x_0) \to \mathbb{R}$ be a function of class $C^{(2)}(U(x_0); \mathbb{R})$ defined in a neighborhood $U(x_0) \subset \mathbb{R}^m$ of the point $x_0 = (x_0^1, \dots, x_0^m) \in \mathbb{R}^m$, and let x_0 be a critical point of the function f.*

If, in the Taylor expansion of the function at the point x_0

$$f\bigl(x_0^1 + h^1, \dots, x_0^m + h^m\bigr) = \\ = f\bigl(x_0^1, \dots, x_0^m\bigr) + \frac{1}{2!}\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)h^i h^j + o\bigl(\|h\|^2\bigr) \tag{8.67}$$

the quadratic form

$$\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)h^i h^j \equiv \partial_{ij} f(x_0)h^i h^j \tag{8.68}$$

a) *is positive-definite or negative-definite, then the point x_0 has a local extremum at x_0, which is a strict local minimum if the quadratic form* (8.68) *is positive-definite and a strict local maximum if it is negative-definite*;

b) *assumes both positive and negative values, then the function does not have an extremum at x_0.*

Proof Let $h \neq 0$ and $x_0 + h \in U(x_0)$. Let us represent (8.67) in the form

$$f(x_0 + h) - f(x_0) = \frac{1}{2!}\|h\|^2 \Biggl[\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)\frac{h^i}{\|h\|}\frac{h^j}{\|h\|} + o(1)\Biggr], \tag{8.69}$$

where $o(1)$ is infinitesimal as $h \to 0$.

It is clear from (8.69) that the sign of the difference $f(x_0 + h) - f(x_0)$ is completely determined by the sign of the quantity in brackets. We now undertake to study this quantity.

The vector $e = (h^1/\|h\|, \dots, h^m/\|h\|)$ obviously has norm 1. The quadratic form (8.68) is continuous as a function $h \in \mathbb{R}^m$, and therefore its restriction to the unit sphere $S(0;1) = \{x \in \mathbb{R}^m \mid \|x\| = 1\}$ is also continuous on $S(0;1)$. But the sphere S is a closed bounded subset in $\mathbb{R}^m$, that is, it is compact. Consequently, the form (8.68) has both a minimum point and a maximum point on S, at which it assumes respectively the values m and M.

If the form (8.68) is positive-definite, then $0 < m \le M$, and there is a number $\delta > 0$ such that $|o(1)| < m$ for $\|h\| < \delta$. Then for $\|h\| < \delta$ the bracket on the right-hand side of (8.69) is positive, and consequently $f(x_0 + h) - f(x_0) > 0$ for $0 < \|h\| < \delta$. Thus, in this case the point x_0 is a strict local minimum of the function.

One can verify similarly that when the form (8.68) is negative-definite, the function has a strict local maximum at the point x_0.

Thus a) is now proved.

We now prove b).

Let e_m and e_M be points of the unit sphere at which the form (8.68) assumes the values m and M respectively, and let $m < 0 < M$.

Setting $h = te_m$, where t is a sufficiently small positive number (so small that $x_0 + te_m \in U(x_0)$), we find by (8.69) that

$$f(x_0 + te_m) - f(x_0) = \frac{1}{2!}t^2\big(m + o(1)\big),$$

where $o(1) \to 0$ as $t \to 0$. Starting at some time (that is, for all sufficiently small values of t), the quantity $m + o(1)$ on the right-hand side of this equality will have the sign of m, that is, it will be negative. Consequently, the left-hand side will also be negative.

Similarly, setting $h = te_M$, we obtain

$$f(x_0 + te_M) - f(x_0) = \frac{1}{2!}t^2\big(M + o(1)\big),$$

and consequently for all sufficiently small t the difference $f(x_0 + te_M) - f(x_0)$ is positive.

Thus, if the quadratic form (8.68) assumes both positive and negative values on the unit sphere, or, what is obviously equivalent, in $\mathbb{R}^m$, then in any neighborhood of the point x_0 there are both points where the value of the function is larger than $f(x_0)$ and points where the value is smaller than $f(x_0)$. Hence, in that case x_0 is not a local extremum of the function. $\square$

We now make a number of remarks in connection with this theorem.

Remark 1 Theorem 6 says nothing about the case when the form (8.68) is semi-definite, that is, nonpositive or nonnegative. It turns out that in this case the point may be an extremum, or it may not. This can be seen, in particular from the following example.

Example 3 Let us find the extrema of the function $f(x, y) = x^4 + y^4 - 2x^2$, which is defined in $\mathbb{R}^2$.

In accordance with the necessary conditions (8.65) we write the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 4x^3 - 4x = 0, \\ \dfrac{\partial f}{\partial y}(x,y) = 4y^3 = 0, \end{cases}$$

from which we find three critical points: $(-1,0)$, $(0,0)$, $(1,0)$.

Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 4, \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) \equiv 0, \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2,$$

at the three critical points the quadratic form (8.68) has respectively the form

$$8\bigl(h^1\bigr)^2, \quad -4\bigl(h^1\bigr)^2, \quad 8\bigl(h^1\bigr)^2.$$

That is, in all cases it is positive semi-definite or negative semi-definite. Theorem 6 is not applicable, but since $f(x,y) = (x^2-1)^2 + y^4 - 1$, it is obvious that the function $f(x,y)$ has a strict minimum -1 (even a global minimum) at the points $(-1,0)$, and $(1,0)$, while there is no extremum at $(0,0)$, since for $x=0$ and $y \neq 0$, we have $f(0,y) = y^4 > 0$, and for $y = 0$ and sufficiently small $x \neq 0$ we have $f(x,0) = x^4 - 2x^2 < 0$.

Remark 2 After the quadratic form (8.68) has been obtained, the study of its definiteness can be carried out using the Sylvester[7] criterion. We recall that by the Sylvester criterion, a quadratic form $\sum_{ij=1}^{m} a_{ij}x^i x^j$ with symmetric matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$$

is positive-definite if and only if all its principal minors are positive; the form is negative-definite if and only if $a_{11} < 0$ and the sign of the principal minor reverses each time its order increases by one.

Example 4 Let us find the extrema of the function

$$f(x,y) = xy \ln\bigl(x^2 + y^2\bigr),$$

which is defined everywhere in the plane $\mathbb{R}^2$ except at the origin.

[7] J.J. Sylvester (1814–1897) – British mathematician. His best-known works were on algebra.

Solving the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = y \ln\bigl(x^2 + y^2\bigr) + \dfrac{2x^2 y}{x^2 + y^2} = 0, \\ \dfrac{\partial f}{\partial y}(x, y) = x \ln\bigl(x^2 + y^2\bigr) + \dfrac{2xy^2}{x^2 + y^2} = 0, \end{cases}$$

we find all the critical points of the function

$$(0, \pm 1); \quad (\pm 1, 0); \quad \left(\pm \frac{1}{\sqrt{2e}}, \pm \frac{1}{\sqrt{2e}}\right); \quad \left(\pm \frac{1}{\sqrt{2e}}, \mp \frac{1}{\sqrt{2e}}\right).$$

Since the function is odd with respect to each of its arguments individually, the points $(0, \pm 1)$ and $(\pm 1, 0)$ are obviously not extrema of the function.

It is also clear that this function does not change its value when the signs of both variables x and y are changed. Thus by studying only one of the remaining critical points, for example, $(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}})$ we will be able to draw conclusions on the nature of the others.

Since

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{6xy}{x^2 + y^2} - \frac{4x^3 y}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \ln\bigl(x^2 + h^2\bigr) + 2 - \frac{4x^2 y^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{6xy}{x^2 + y^2} - \frac{4xy^3}{(x^2 + y^2)^2},$$

at the point $(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}})$ the quadratic form $\partial_{ij} f(x_0) h^i h^j$ has the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

that is, it is positive-definite, and consequently at that point the function has a local minimum

$$f\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) = -\frac{1}{2e}.$$

By the observations made above on the properties of this function, one can conclude immediately that

$$f\left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right) = -\frac{1}{2e}$$

is also a local minimum and

$$f\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right) = f\left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) = \frac{1}{2e}$$

are local maxima of the function. This, however, could have been verified directly, by checking the definiteness of the corresponding quadratic form. For example, at the point $(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}})$ the matrix of the quadratic form (8.68) has the form

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

from which it is clear that it is negative-definite.

Remark 3 It should be kept in mind that we have given necessary conditions (Theorem 5) and sufficient conditions (Theorem 6) for an extremum of a function only at an interior point of its domain of definition. Thus in seeking the absolute maximum or minimum of a function, it is necessary to examine the boundary points of the domain of definition along with the critical interior points, since the function may assume its maximal or minimal value at one of these boundary points.

The general principles of studying noninterior extrema will be considered in more detail later (see the section devoted to extrema with constraint). It is useful to keep in mind that in searching for minima and maxima one may use certain simple considerations connected with the nature of the problem along with the formal techniques, and sometimes even instead of them. For example, if a differentiable function being studied in $\mathbb{R}^m$ must have a minimum because of the nature of the problem and turns out to be unbounded above, then if the function has only one critical point, one can assert without further investigation that that point is the minimum.

Example 5 (Huygens' problem) On the basis of the laws of conservation of energy and momentum of a closed mechanical system one can show by a simple computation that when two perfectly elastic balls having mass m_1 and m_2 and initial velocities v_1 and v_2 collide, their velocities after a central collision (when the velocities are directed along the line joining the centers) are determined by the relations

$$\tilde{v}_1 = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2},$$
$$\tilde{v}_2 = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}.$$

In particular, if a ball of mass M moving with velocity V strikes a motionless ball of mass m, then the velocity v acquired by the latter can be found from the formula

$$v = \frac{2M}{m + M}V, \tag{8.70}$$

from which one can see that if $0 \le m \le M$, then $V \le v \le 2V$.

How can a significant part of the kinetic energy of a larger mass be communicated to a body of small mass? To do this, for example, one can insert balls with intermediate masses between the balls of small and large mass: $m < m_1 < m_2 < \cdots < m_n < M$. Let us compute (after Huygens) how the masses $m_1, m_2, \dots, m_n$

should be chosen to that the body m will acquire maximum velocity after successive central collisions.

In accordance with formula (8.70) we obtain the following expression for the required velocity as a function of the variables $m_1, m_2, \ldots, m_n$:

$$v = \frac{m_1}{m+m_1} \cdot \frac{m_2}{m_1+m_2} \cdot \cdots \cdot \frac{m_n}{m_{n-1}+m_n} \cdot \frac{M}{m_n+M} \cdot 2^{n+1} V. \tag{8.71}$$

Thus Huygens' problem reduces to finding the maximum of the function

$$f(m_1, \ldots, m_n) = \frac{m_1}{m+m_1} \cdot \cdots \cdot \frac{m_n}{m_{n-1}+m_n} \cdot \frac{M}{m_n+M}.$$

The system of equations (8.65), which gives the necessary conditions for an interior extremum, reduces to the following system in the present case:

$$\begin{cases} m \cdot m_2 - m_1^2 = 0, \\ m_1 \cdot m_3 - m_2^2 = 0, \\ \qquad \vdots \\ m_{n-1} \cdot M - m_n^2 = 0, \end{cases}$$

from which it follows that the numbers $m, m_1, \ldots, m_n, M$ form a geometric progression with ratio q equal to $\sqrt[n+1]{M/m}$.

The value of the velocity (8.71) that results from this choice of masses is given by

$$v = \left(\frac{2q}{1+q}\right)^{n+1} V, \tag{8.72}$$

which agrees with (8.70) if $n = 0$.

It is clear from physical considerations that formula (8.72) gives the maximal value of the function (8.71). However, this can also be verified formally (without invoking the cumbersome second derivatives. See Problem 9 at the end of this section).

We remark that it is clear from (8.72) that if $m \to 0$, then $v \to 2^{n+1} V$. Thus the intermediate masses do indeed significantly increase the portion of the kinetic energy of the mass M that is transmitted to the small mass m.

8.4.6 Some Geometric Images Connected with Functions of Several Variables

a. The Graph of a Function and Curvilinear Coordinates

Let x, y, and z be Cartesian coordinates of a point in $\mathbb{R}^3$ and let $z = f(x, y)$ be a continuous function defined in some domain G of the plane $\mathbb{R}^2$ of the variables x and y.

By the general definition of the graph of a function, the graph of the function $f : G \to \mathbb{R}$ in our case is the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in G, z = f(x, y)\}$ in the space $\mathbb{R}^3$.

It is obvious that the mapping $G \xrightarrow{F} S$ defined by the relation $(x, y) \mapsto (x, y, f(x, y))$ is a continuous one-to-one mapping of G onto S, by which one can determine every point of S by exhibiting the point of G corresponding to it, or, what is the same, giving the coordinates (x, y) of this point of G.

Thus the pairs of numbers $(x, y) \in G$ can be regarded as certain coordinates of the points of a set S – the graph of the function $z = f(x, y)$. Since the points of S are given by pairs of numbers, we shall conditionally agree to call S a *two-dimensional surface in* $\mathbb{R}^3$. (The general definition of a surface will be given later.)

If we define a path $\Gamma : I \to G$ in G, then a path $F \circ \Gamma : I \to S$ automatically appears on the surface S. If $x = x(t)$ and $y = y(t)$ is a parametric definition of the path Γ, then the path $F \circ \Gamma$ on S is given by the three functions $x = x(t)$, $y = y(t)$, $z = z(t) = f(x(t), y(t))$. In particular, if we set $x = x_0 + t$, $y = y_0$, we obtain a curve $x = x_0 + t$, $y = y_0$, $z = f(x_0 + t, y_0)$ on the surface S along which the coordinate $y = y_0$ of the points of S does not change. Similarly one can exhibit a curve $x = x_0$, $y = y_0 + t$, $z = f(x_0, y_0 + t)$ on S along which the first coordinate x_0 of the points of S does not change. By analogy with the planar case these curves on S are naturally called *coordinate lines* on the surface S. However, in contrast to the coordinate lines in $G \subset \mathbb{R}^2$, which are pieces of straight lines, the coordinate lines on S are in general curves in $\mathbb{R}^3$. For that reason, the coordinates (x, y) of points of the surface S are often called *curvilinear coordinates* on S.

Thus the graph of a continuous function $z = f(x, y)$, defined in a domain $G \subset R^2$ is a two-dimensional surface S in $\mathbb{R}^3$ whose points can be defined by curvilinear coordinates $(x, y) \in G$.

At this point we shall not go into detail on the general definition of a surface, since we are interested only in a special case of a surface – the graph of a function. However, we assume that from the course in analytic geometry the reader is well acquainted with some important particular surfaces in $\mathbb{R}^3$ (such as a plane, an ellipsoid, paraboloids, and hyperboloids).

b. The Tangent Plane to the Graph of a Function

Differentiability of a function $z = f(x, y)$ at the point $(x_0, y_0) \in G$ means that

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + \\ + o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right) \quad \text{as } (x, y) \to (x_0, y_0), \qquad (8.73)$$

where A and B are certain constants.

In $\mathbb{R}^3$ let us consider the plane

$$z = z_0 + A(x - x_0) + B(y - y_0), \qquad (8.74)$$

where $z_0 = f(x_0, y_0)$. Comparing equalities (8.73) and (8.74), we see that the graph of the function is well approximated by the plane (8.74) in a neighborhood of the point (x_0, y_0, z_0). More precisely, the point $(x, y, f(x, y))$ of the graph of the function differs from the point $(x, y, z(x, y))$ of the plane (8.74) by an amount that is infinitesimal in comparison with the magnitude $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ of the displacement of its curvilinear coordinates (x, y) from the coordinates (x_0, y_0) of the point (x_0, y_0, z_0).

By the uniqueness of the differential of a function, the plane (8.74) possessing this property is unique and has the form

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0). \tag{8.75}$$

This plane is called the *tangent plane to the graph of the function* $z = f(x, y)$ *at the point* $(x_0, y_0, f(x_0, y_0))$.

Thus, the differentiability of a function $z = f(x, y)$ at the point (x_0, y_0) and the existence of a tangent plane to the graph of this function at the point $(x_0, y_0, f(x_0, y_0))$ are equivalent conditions.

c. The Normal Vector

Writing Eq. (8.75) for the tangent plane in the canonical form

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - \big(z - f(x_0, y_0)\big) = 0,$$

we conclude that the vector

$$\left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1\right) \tag{8.76}$$

is the *normal vector to the tangent plane*. Its direction is considered to be the direction normal or orthogonal to the surface S (the graph of the function) at the point $(x_0, y_0, f(x_0, y_0))$.

In particular, if (x_0, y_0) is a critical point of the function $f(x, y)$, then the normal vector to the graph at the point $(x_0, y_0, f(x_0, y_0))$ has the form $(0, 0, -1)$ and consequently, the tangent plane to the graph of the function at such a point is horizontal (parallel to the xy-plane).

The three graphs in Fig. 8.1 illustrate what has just been said.

Figures 8.1a and c depict the location of the graph of a function with respect to the tangent plane in a neighborhood of a local extremum (minimum and maximum respectively), while Fig. 8.1b shows the graph in the neighborhood of a so-called *saddle point*.

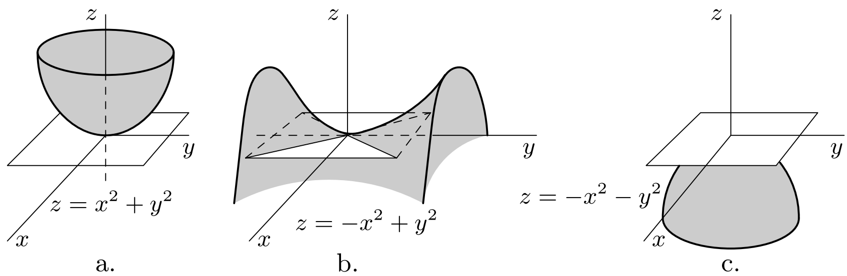


Fig. 8.1

d. Tangent Planes and Tangent Vectors

We know that if a path $\Gamma : I \to \mathbb{R}^3$ in $\mathbb{R}^3$ is given by differentiable functions $x = x(t)$, $y = y(t)$, $z = z(t)$, then the vector $(\dot{x}(0), \dot{y}(0), \dot{z}(0))$ is the velocity vector at time $t = 0$. It is a direction vector of the tangent at the point $x_0 = x(0)$, $y_0 = y(0)$, $z_0 = z(0)$ to the curve in $\mathbb{R}^3$ that is the support of the path Γ.

Now let us consider a path $\Gamma : I \to S$ on the graph of a function $z = f(x, y)$ given in the form $x = x(t)$, $y = y(t)$, $z = f(x(t), y(t))$. In this particular case we find that

$$\big(\dot{x}(0), \dot{y}(0), \dot{z}(0)\big) = \left(\dot{x}(0), \dot{y}(0), \frac{\partial f}{\partial x}(x_0, y_0)\dot{x}(0) + \frac{\partial f}{\partial y}(x_0, y_0)\dot{y}(0)\right),$$

from which it can be seen that this vector is orthogonal to the vector (8.76) normal to the graph S of the function at the point $(x_0, y_0, f(x_0, y_0))$. Thus we have shown that if a vector (ξ, η, ζ) is tangent to a curve on the surface S at the point $(x_0, y_0, f(x_0, y_0))$, then it is orthogonal to the vector (8.76) and (in this sense) lies in the plane (8.75) tangent to the surface S at the point in question. More precisely we could say that the whole line $x = x_0 + \xi t$, $y = y_0 + \eta t$, $z = f(x_0, y_0) + \zeta t$ lies in the tangent plane (8.75).

Let us now show that the converse is also true, that is, if a line $x = x_0 + \xi t$, $y = y_0 + \eta t$, $z = f(x_0, y_0) + \zeta t$, or what is the same, the vector (ξ, η, ζ), lies in the plane (8.75), then there is a path on S for which the vector (ξ, η, ζ) is the velocity vector at the point $(x_0, y_0, f(x_0, y_0))$.

The path can be taken, for example, to be

$$x = x_0 + \xi t, \qquad y = y_0 + \eta t, \qquad z = f(x_0 + \xi t, y_0 + \eta t).$$

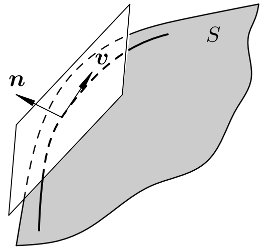
In fact, for this path,

$$\dot{x}(0) = \xi, \qquad \dot{y}(0) = \eta, \qquad \dot{z}(0) = \frac{\partial f}{\partial x}(x_0, y_0)\xi + \frac{\partial f}{\partial y}(x_0, y_0)\eta.$$

In view of the equality

$$\frac{\partial f}{\partial x}(x_0, y_0)\dot{x}(0) + \frac{\partial f}{\partial y}(x_0, y_0)\dot{y}(0) - \dot{z}(0) = 0$$

Fig. 8.2



and the hypothesis that

$$\frac{\partial f}{\partial x}(x_0, y_0)\xi + \frac{\partial f}{\partial y}(x_0, y_0)\eta - \zeta = 0.$$

We conclude that

$$\big(\dot{x}(0), \dot{y}(0), \dot{z}(0)\big) = (\xi, \eta, \zeta).$$

Hence the tangent plane to the surface S at the point (x_0, y_0, z_0) is formed by the vectors that are tangents at the point (x_0, y_0, z_0) to curves on the surface S passing through the point (see Fig. 8.2).

This is a more geometric description of the tangent plane. In any case, one can see from it that if the tangent to a curve is invariantly defined (with respect to the choice of coordinates), then the tangent plane is also invariantly defined.

We have been considering functions of two variables for the sake of visualizability, but everything that was said obviously carries over to the general case of a function

$$y = f\big(x^1, \dots, x^m\big) \tag{8.77}$$

of m variables, where $m \in \mathbb{N}$.

At the point $(x_0^1, \dots, x_0^m, f(x_0^1, \dots, x_0^m))$ the plane tangent to the graph of such a function can be written in the form

$$y = f\big(x_0^1, \dots, x_0^m\big) + \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}\big(x_0^1, \dots, x_0^m\big)\big(x^i - x_0^i\big); \tag{8.78}$$

the vector

$$\left(\frac{\partial f}{\partial x^1}(x_0), \dots, \frac{\partial f}{\partial x^m}(x_0), -1\right)$$

is the normal vector to the plane (8.78). This plane itself, like the graph of the function (8.77), has dimension m, that is, any point is now given by a set $(x^1, \dots, x^m)$ of m coordinates.

Thus, Eq. (8.78) defines a hyperplane in $\mathbb{R}^{m+1}$.

Repeating verbatim the reasoning above, one can verify that the tangent plane (8.78) consists of vectors that are tangent to curves passing through the point

$(x_0^1, \dots, x_0^m, f(x_0^1, \dots, x_0^m))$ and lying on the m-dimensional surface S – the graph of the function (8.77).

8.4.7 Problems and Exercises

1. Let $z = f(x, y)$ be a function of class $C^{(1)}(G; \mathbb{R})$.

a) If $\frac{\partial f}{\partial y}(x, y) \equiv 0$ in G, can one assert that f is independent of y in G?

b) Under what condition on the domain G does the preceding question have an affirmative answer?

2. a) Verify that for the function

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } x^2 + y^2 \neq 0, \\ 0, & \text{if } x^2 + y^2 = 0, \end{cases}$$

the following relations hold:

$$\frac{\partial f}{\partial x \partial y}(0, 0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

b) Prove that if the function $f(x, y)$ has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in some neighborhood U of the point (x_0, y_0), and if the mixed derivative $\frac{\partial^2 f}{\partial x \partial y}$ (or $\frac{\partial^2 f}{\partial y \partial x}$) exists in U and is continuous at (x_0, y_0), then the mixed derivative $\frac{\partial^2 f}{\partial y \partial x}$ (resp. $\frac{\partial^2 f}{\partial x \theta y}$) also exists at that point and the following equality holds:

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0,\ y_0).$$

3. Let $x^1, \dots, x^m$ be Cartesian coordinates in $\mathbb{R}^m$. The differential operator

$$\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x^{i^2}},$$

acting on functions $f \in C^{(2)}(G; \mathbb{R})$ according to the rule

$$\Delta f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i^2}}\big(x^1, \dots, x^m\big),$$

is called the *Laplacian*.

The equation $\Delta f = 0$ for the function f in the domain $G \subset \mathbb{R}^m$ is called *Laplace's equation*, and its solutions are called *harmonic functions in the domain G*.

a) Show that if $x = (x^1, \dots, x^m)$ and

$$\|x\| = \sqrt{\sum_{i=1}^{m} (x^i)^2},$$

then for $m > 2$ the function

$$f(x) = \|x\|^{-\frac{2-m}{2}}$$

is harmonic in the domain $\mathbb{R}^m \setminus 0$, where $0 = (0, \dots, 0)$.

b) Verify that the function

$$f\left(x^1, \dots, x^m, t\right) = \frac{1}{(2a\sqrt{\pi t})^m} \cdot \exp\left(-\frac{\|x\|^2}{4a^2 t}\right),$$

which is defined for $t > 0$ and $x = (x^1, \dots, x^m) \in \mathbb{R}^m$, satisfies the *heat equation*

$$\frac{\partial f}{\partial t} = a^2 \Delta f,$$

that is, verify that $\frac{\partial f}{\partial t} = a^2 \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i2}}$ at each point of the domain of definition of the function.

4. *Taylor's formula in multi-index notation*. The symbol $\alpha := (\alpha_1, \dots, \alpha_m)$ consisting of nonnegative integers α_i, $i = 1, \dots, m$, is called the *multi-index* α.

The following notation is conventional:

$$|\alpha| := \alpha_1 + \cdots + \alpha_m,$$

$$\alpha! := \alpha_1! \cdots \alpha_m!;$$

finally, if $a = (a_1, \dots, a_m)$, then

$$a^\alpha := a_1^{\alpha_1} \cdots a_m^{\alpha_m}.$$

a) Verify that if $k \in \mathbb{N}$, then

$$(a_1 + \cdots + a_m)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_m!} a_1^{\alpha_1} \cdots a_m^{\alpha_m},$$

or

$$(a_1 + \cdots + a_m)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha,$$

where the summation extends over all sets $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers such that $\sum_{i=1}^{m} \alpha_i = k$.

b) Let

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha 1} \cdots (\partial x^m)^{\alpha_m}}(x).$$

Show that if $f \in C^{(k)}(G; \mathbb{R})$, then the equality

$$\sum_{i_1 + \cdots + i_m = k} \partial_{i_1 \cdots i_k} f(x) h^{i_1} \cdots h^{i_k} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} D^\alpha f(x) h^\alpha,$$

where $h = (h^1, \dots, h^m)$, holds at any point $x \in G$.

c) Verify that in multi-index notation Taylor's theorem with the Lagrange form of the remainder, for example, can be written as

$$f(x+h) = \sum_{|\alpha|=0}^{n-1} \frac{1}{\alpha!} D^\alpha f(x) h^\alpha + \sum_{|\alpha|=n} \frac{1}{\alpha!} D^\alpha f(x + \theta h) h^\alpha.$$

d) Write Taylor's formula in multi-index notation with the integral form of the remainder (Theorem 4).

5. a) Let $I^m = \{x = (x^1, \dots, x^m) \in \mathbb{R}^m \mid |x^i| \le c^i, i = 1, \dots, m\}$ be an m-dimensional closed interval and I a closed interval $[a, b] \subset \mathbb{R}$. Show that if the function $f(x, y) = f(x^1, \dots, x^m, y)$ is defined and continuous on the set $I^m \times I$, then for any positive number $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x, y_1) - f(x, y_2)| < \varepsilon$ if $x \in I^m$, $y_1, y_2 \in I$, and $|y_1 - y_2| < \delta$.

b) Show that the function

$$F(x) = \int_a^b f(x, y)\, \mathrm{d}y$$

is defined and continuous on the closed interval I^m.

c) Show that if $f \in C(I^m; \mathbb{R})$, then the function

$$\mathcal{F}(x, t) = f(tx)$$

is defined and continuous on $I^m \times I^1$, where $I^1 = \{t \in \mathbb{R} \mid |t| \le 1\}$.

d) Prove *Hadamard's lemma*:

If $f \in C^{(1)}(I^m; \mathbb{R})$ *and* $f(0) = 0$, *there exist functions* $g_1, \dots, g_m \in C(I^m; \mathbb{R})$ *such that*

$$f\big(x^1, \dots, x^m\big) = \sum_{i=1}^m x^i g_i\big(x^1, \dots, x^m\big)$$

in I^m, *and in addition*

$$g_i(0) = \frac{\partial f}{\partial x^i}(0), \quad i = 1, \dots, m.$$

6. Prove the following *generalization of Rolle's theorem for functions of several variables*.

If the function f is continuous in a closed ball $\overline{B}(0; r)$, equal to zero on the boundary of the ball, and differentiable in the open ball $B(0; r)$, then at least one of the points of the open ball is a critical point of the function.

7. Verify that the function

$$f(x, y) = \left(y - x^2\right)\left(y - 3x^2\right)$$

does not have an extremum at the origin, even though its restriction to each line passing through the origin has a strict local minimum at that point.

8. *The method of least squares*. This is one of the commonest methods of processing the results of observations. It consists of the following. Suppose it is known that the physical quantities x and y are linearly related:

$$y = ax + b \tag{8.79}$$

or suppose an empirical formula of this type has been constructed on the basis of experimental data.

Let us assume that n observations have been made, in each of which both x and y were measured, resulting in n pairs of values $x_1, y_1; \dots; x_n, y_n$. Since the measurements have errors, even if the relation (8.79) is exact, the equalities

$$y_k = ax_k + b$$

may fail to hold for some of the values of $k \in \{1, \dots, n\}$, no matter what the coefficients a and b are.

The problem is to determine the unknown coefficients a and b in a reasonable way from these observational results.

Basing his argument on analysis of the probability distribution of the magnitude of observational errors, Gauss established that the most probable values for the coefficients a and b with a given set of observational results should be sought by use of the following *least-squares principle*:

If $\delta_k = (ax_k + b) - y_k$ is the discrepancy in the kth observation, then a and b should be chosen so that the quantity

$$\Delta = \sum_{k=1}^{n} \delta_k^2,$$

that is, the sum of the squares of the discrepancies, has a minimum.

a) Show that the least-squares principle for relation (8.79) leads to the following system of linear equations

$$\begin{cases} [x_k, x_k]a + [x_k, 1]b = [x_k, y_k], \\ [1, x_k]a + [1, 1]b = [1, y_k], \end{cases}$$

Table 8.1

Temperature, °C	Frequency, %	Temperature, °C	Frequency, %
0	39	20	136
5	54	25	182
10	74	30	254
15	100		

for determining the coefficients a and b. Here, following Gauss, we write $[x_k, x_k] := x_1x_1 + \cdots + x_nx_n$, $[x_k, 1] := x_1 \cdot 1 + \cdots + x_n \cdot 1$, $[x_k, y_k] := x_1y_1 + \cdots + x_ny_n$, and so forth.

b) Write the system of equations for the numbers $a_1, \dots, a_m, b$ to which the least-squares principle leads when Eq. (8.79) is replaced by the relation

$$y = \sum_{i=1}^{m} a_i x^i + b,$$

(or, more briefly, $y = a_i x^i + b$) between the quantities $x^1, \dots, x^m$ and y.

c) How can the method of least squares be used to find empirical formulas of the form

$$y = c x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

connecting physical quantities $x_1, \dots, x_m$ with the quantity y?

d) (M. Germain) The frequency R of heart contractions was measured at different temperatures T in several dozen specimens of *Nereis diversicolor*. The frequencies were expressed in percents relative to the contraction frequency at 15 °C. The results are given in Table 8.1.

The dependence of R on T appears to be exponential. Assuming $R = Ae^{bT}$, find the values of the constants A and b that best fit the experimental results.

9. a) Show that in Huygens' problem, studied in Example 5, the function (8.71) tends to zero if at least one of the variables $m_1, \dots, m_n$ tends to infinity.

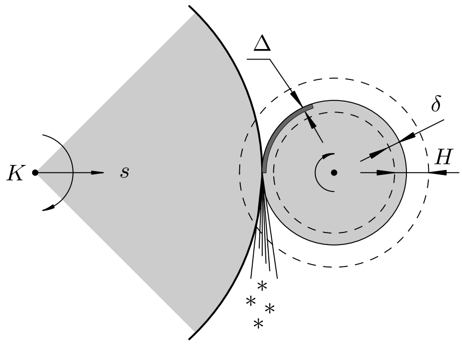
b) Show that the function (8.71) has a maximum point in $\mathbb{R}^n$ and hence the unique critical point of that function in $\mathbb{R}^n$ must be its maximum.

c) Show that the quantity v defined by formula (8.72) is monotonically increasing as n increases and find its limit as $n \to \infty$.

10. a) During so-called exterior disk grinding the grinding tool – a rapidly rotating grinding disk (with an abrasive rim) that acts as a file – is brought into contact with the surface of a circular machine part that is rotating slowly compared with the disk (see Fig. 8.3).

The disk K is gradually pressed against the machine part D, causing a layer H of metal to be removed, reducing the part to the required size and producing a smooth working surface for the device. In the machine where it will be placed this surface will usually be a working surface. In order to extend its working life, the metal of the machine part is subjected to a preliminary annealing to harden the steel. However,

Fig. 8.3



because of the high temperature in the contact zone between the machine part and the grinding disk, structural changes can (and frequently do) occur in a certain layer Δ of metal in the machine part, resulting in decreased hardness of the steel in that layer. The quantity Δ is a monotonic function of the rate s at which the disk is applied to the machine part, that is, $\Delta = \varphi(s)$. It is known that there is a certain critical rate $s_0 > 0$ at which the relation $\Delta = 0$ still holds, while $\Delta > 0$ whenever $s > s_0$. For the following discussion it is convenient to introduce the relation

$$s = \psi(\Delta)$$

inverse to the one just given. This new relation is defined for $\Delta > 0$.

Here ψ is a monotonically increasing function known experimentally, defined for $\Delta \geq 0$, and $\psi(0) = s_0 > 0$.

The grinding process must be carried out in such a way that there are no structural changes in the metal on the surface eventually produced.

In terms of rapidity, the optimal grinding mode under these conditions would obviously be a set of variations in the rate s of application of the grinding disk for which

$$s = \psi(\delta),$$

where $\delta = \delta(t)$ is the thickness of the layer of metal not yet removed up to time t, or, what is the same, the distance from the rim of the disk at time t to the final surface of the device being produced. Explain this.

b) Find the time needed to remove a layer of thickness H when the rate of application of the disk is optimally adjusted.

c) Find the dependence $s = s(t)$ of the rate of application of the disk on time in the optimal mode under the condition that the function $\Delta \overset{\psi}{\longmapsto} s$ is linear: $s = s_0 + \lambda\Delta$.

Due to the structural properties of certain kinds of grinding lathes, the rate s can undergo only discrete changes. This poses the problem of optimizing the productivity of the process under the additional condition that only a fixed number n of switches in the rate s are allowed. The answers to the following questions give a picture of the optimal mode.

d) What is the geometric interpretation of the grinding time $t(H) = \int_0^H \frac{\mathrm{d}\delta}{\psi(\delta)}$ that you found in part b) for the optimal continuous variation of the rate s?

e) What is the geometric interpretation of the time lost in switching from the optimal continuous mode of variation of s to the time-optimal stepwise mode of variation of s?

f) Show that the points $0 = x_{n+1} < x_n < \cdots < x_1 < x_0 = H$ of the closed interval $[0, H]$ at which the rate should be switched must satisfy the conditions

$$\frac{1}{\psi(x_{i+1})} - \frac{1}{\psi(x_i)} = -\left(\frac{1}{\psi}\right)'(x_i)(x_i - x_{i-1}) \quad (i = 1, \dots, n)$$

and consequently, on the portion from x_i to x_{i+1}, the rate of application of the disk has the form $s = \psi(x_{i+1})$ $(i = 0, \dots, n)$.

g) Show that in the linear case, when $\psi(\Delta) = s_0 + \lambda\Delta$, the points x_i (in part f)) on the closed interval $[0, H]$ are distributed so that the numbers

$$\frac{s_0}{\lambda} < \frac{s_0}{\lambda} + x_n < \cdots < \frac{s_0}{\lambda} + x_1 < \frac{s_0}{\lambda} + H$$

form a geometric progression.

11. a) Verify that the tangent to a curve $\Gamma : I \to \mathbb{R}^m$ is defined invariantly relative to the choice of coordinate system in $\mathbb{R}^m$.

b) Verify that the tangent plane to the graph S of a function $y = f(x^1, \dots, x^m)$ is defined invariantly relative to the choice of coordinate system in $\mathbb{R}^m$.

c) Suppose the set $S \subset \mathbb{R}^m \times \mathbb{R}^1$ is the graph of a function $y = f(x^1, \dots, x^m)$ in coordinates $(x^1, \dots, x^m, y)$ in $\mathbb{R}^m \times \mathbb{R}^1$ and the graph of a function $\tilde{y} = \tilde{f}(\tilde{x}^1, \dots, \tilde{x}^m)$ in coordinates $(\tilde{x}^1, \dots, \tilde{x}^m, \tilde{y})$ in $\mathbb{R}^m \times \mathbb{R}^1$. Verify that the tangent plane to S is invariant relative to a linear change of coordinates in $\mathbb{R}^m \times \mathbb{R}^1$.

d) Verify that the Laplacian $\Delta f = \sum_{i=1}^m \frac{\partial^2 f}{\partial x^{i2}}(x)$ is defined invariantly relative to orthogonal coordinate transformations in $\mathbb{R}^m$.

8.5 The Implicit Function Theorem

8.5.1 Statement of the Problem and Preliminary Considerations

In this section we shall prove the implicit function theorem, which is important both intrinsically and because of its numerous applications.

Let us begin by explaining the problem. Suppose, for example, we have the relation

$$x^2 + y^2 - 1 = 0 \tag{8.80}$$

between the coordinates x, y of points in the plane $\mathbb{R}^2$. The set of all points of $\mathbb{R}^2$ satisfying this condition is the unit circle (Fig. 8.4).

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