Chapter 2
Equilibrium

Abstract Tensegrity structures are classified as prestressed pin-jointed structures, and they have distinct properties compared to other pin-jointed structures: (1) they are free-standing, without any support; and (2) they have both tensile and compressive members. Prior to further studies on tensegrity structures in the following chapters, this chapter presents the formulations of (self-)equilibrium for general prestressed pin-jointed structures. The equilibrium equations are formulated in two ways: (1) using the *equilibrium matrix* associated with prestresses or axial forces, and (2) using the *force density matrix* associated with nodal coordinates. Conditions for static as well as kinematic determinacy of a prestressed pin-jointed structure are then given in terms of rank of the equilibrium matrix. Furthermore, the *non-degeneracy condition* for a prestressed free-standing pin-jointed structure is presented in terms of rank deficiency of the force density matrix.

Keywords Equilibrium equations · Static determinacy · Kinematic determinacy · Force density matrix · Non-degeneracy condition

2.1 Definition of Configuration

This section first introduces the basic mechanical assumptions for a general prestressed[1] pin-jointed structure. Configuration of a prestressed pin-jointed structure is described by its connectivity and geometry: *connectivity* defines how the nodes are connected by the members; and *geometry* (realization) of the structure is described by coordinates of the nodes.

In the category of general prestressed pin-jointed structures, the following structures are included:

- *Truss*, which carries no prestress;
- *Cable-net*, which is attached to supports, and consists of only tensile members;
- *Tensegrity-dome*, which is also attached to supports, and consists of both tensile and compressive members;
- *Tensegrity*, which has *NO* support, and consists of both tensile and compressive members.

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[1] 'Prestressed' means that prestresses are introduced to the structure a priori. Prestresses are the internal forces in the members when no external load is applied.

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2.1.1 Basic Mechanical Assumptions

There are two types of structural elements in a general prestressed pin-jointed structure:

- *Members*, which are straight; and
- *Nodes*, or joints, that connect the members.

Moreover, there are two types of nodes:

- *Fixed nodes*, or supports, which cannot have any displacement even subjected to external loads; and
- *Free nodes*, displacements of which are not constrained.

Note that there exist only free nodes in a tensegrity structure, such that it is free-standing. This makes its (self-)equilibrium distinct from other prestressed pin-jointed structures, and leads to the necessary non-degeneracy condition presented in Sect. 2.5.

Furthermore, in order to study the mechanical properties of a prestressed pin-jointed structure making use of the existing powerful mathematical tools, we adopt the following mechanical assumptions for its members and nodes (joints).

Mechanical assumptions for a prestressed pin-jointed structure:

1. Members are connected to the nodes (joints) at their two ends. The joints are *pin-joints*, also called hinge joints, allowing the members to rotate freely about them.
2. Self-weight of the structure is neglected. External loads, if exist, are applied at the nodes.
3. Member failure, such as yielding or buckling, is not considered.

From the first two assumptions, it is obvious that the members carry only axial forces, either compression or tension. The axial forces in the members are called *prestresses* or self-stresses, when no external load is applied. Some pin-jointed structures do not carry prestress, e.g., trusses. Based on the type of prestress, there are also two types of members:

Two types of members in a prestressed pin-jointed structure:

- *Struts*, which carry compressive prestresses; and
- *Cables*, which carry tensile prestresses.

A tensegrity structure consists of both struts and cables. It will be shown in Chap. 4 that this fact makes its stability properties much different from, and more complicated than, the cable-nets which consist of only cables.

In the following, we consider a prestressed pin-jointed structure, which consists of m members, n free nodes, and n^{f} fixed nodes in d-dimensional space. For simplicity, displacements of the fixed nodes are constrained in every direction, and we do not consider any other types of supports, such as roller supports, displacements of which are constrained in specified directions. Such constraints can indeed be incorporated into the formulations presented hereafter without any difficulty.

2.1.2 Connectivity

Connectivity, or *topology*, of a structure defines how its members connect the nodes. Since the members are assumed to be straight, the structure can be modeled as a directed graph in graph theory [5, 6]. The vertices and edges of the directed graph respectively represent the nodes and members of the structure.

The connectivity and directions of the members can be described by the so-called *connectivity matrix*, denoted by $\mathbf{C}^{\mathrm{s}}$. There are only two non-zero entries, 1 and -1, in each row of the connectivity matrix, corresponding to the two nodes connected by the specific member; and all other entries in the same row are zero.

Suppose that a member numbered as k $(k = 1, 2, \dots, m)$ connects node i and node j $(i, j = 1, 2, \dots, n + n^{\mathrm{f}})$. The components of the kth row $C^{\mathrm{s}}_{(k,p)}$ of the connectivity matrix $\mathbf{C}^{\mathrm{s}} \in \mathbb{R}^{m\times(n+n^{\mathrm{f}})}$ is defined as

$$C^{\mathrm{s}}_{(k,p)} = \begin{cases} \mathrm{sign}(j-p), & \text{if } p = i; \\ \mathrm{sign}(i-p), & \text{if } p = j; \\ \quad 0, & \text{otherwise}; \end{cases} \qquad (p = 1, 2, \dots, n + n^{\mathrm{f}}), \tag{2.1}$$

where

$$\mathrm{sign}(j-i) = \begin{cases} 1, & \text{if } j > i; \\ -1, & \text{if } j < i. \end{cases} \tag{2.2}$$

Hence, for the two nodes i and j, where $j < i$, the direction[2] of member k connecting these two nodes is defined as pointing from node j toward node i.

For convenience, the fixed nodes are preceded by the free nodes in the numbering sequence. Thus, the connectivity matrix $\mathbf{C}^{\mathrm{s}}$ can be partitioned into two parts as follows:

$$\mathbf{C}^{\mathrm{s}} = \left(\mathbf{C}, \ \ \mathbf{C}^{\mathrm{f}}\right), \tag{2.3}$$

where $\mathbf{C} \in \mathbb{R}^{m\times n}$ and $\mathbf{C}^{\mathrm{f}} \in \mathbb{R}^{m\times n^{\mathrm{f}}}$ respectively correspond to connectivity of the free nodes and that of the fixed nodes.

[2] Definition of member directions is not unique. Using opposite definition necessarily leads to the same equilibrium equation.

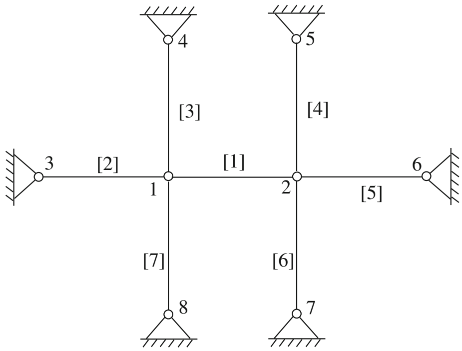

Fig. 2.1 A prestressed pin-jointed structure (cable-net) with fixed nodes considered in Example 2.1. The structure consists of two free nodes 1 and 2, six fixed nodes (supports) 3–8, and seven members [1]–[7]

Example 2.1 Connectivity matrix of a two-dimensional prestressed pin-jointed structure (cable-net) as shown in Fig. 2.1.

The prestressed pin-jointed structure as shown in Fig. 2.1 consists of eight nodes and seven members; i.e., $n + n^{\mathrm{f}} = 8$ and $m = 7$. The structure consists of both free nodes and fixed nodes (supports), and it is usually used as a simple example of cable-nets. Nodes 1 and 2 are free nodes, and nodes 3–8 are fixed nodes, where the free nodes are numbered preceding the fixed nodes. Thus, we have

$$n = 2, \;\; n^{\mathrm{f}} = 6, \;\; \text{and } m = 7. \tag{2.4}$$

The connectivity matrices corresponding to the entire structure $\mathbf{C}^{\mathrm{s}} \in \mathbb{R}^{7\times 8}$, the free nodes $\mathbf{C} \in \mathbb{R}^{7\times 2}$, and the fixed nodes $\mathbf{C}^{\mathrm{f}} \in \mathbb{R}^{7\times 6}$ are written as follows according to Eqs. (2.1) and (2.3):

$$\mathbf{C}^{\mathrm{s}} = \begin{array}{c} \begin{array}{cc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left(\begin{array}{rr|rrrrrr} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) \\ \begin{array}{c|c} \mathbf{C} & \mathbf{C}^{\mathrm{f}} \end{array} \end{array} \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \\ [5] \\ [6] \\ [7] \\ \\ \end{array}. \tag{2.5}$$

A structure is said to be *free-standing*, if it has no fixed node. A free-standing structure can freely transform while preserving distance between any pair of nodes. Moreover, its connectivity matrix becomes

$$\mathbf{C} = \mathbf{C}^{\mathrm{s}}, \tag{2.6}$$

with the vanishing sub-matrix $\mathbf{C}^{\mathrm{f}}$ corresponding to fixed nodes. From the definition of the connectivity matrix $\mathbf{C}$ for free-standing structures in Eq. (2.1), we have an important property:

$$\mathbf{C}\mathbf{i}_n = \mathbf{C}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}, \tag{2.7}$$

where all entries in the vector $\mathbf{i}_n \in \mathbb{R}^n$ are one. This comes from the fact that each row of $\mathbf{C}$ has only two non-zero entries, 1 and -1, such that sum of the entries in each row by means of multiplying the vector $\mathbf{i}_n$ turns out to be zero.

Example 2.2 Connectivity matrix of the two-dimensional free-standing structure as shown in Fig. 2.2.

The two-dimensional free-standing structure as shown in Fig. 2.2 has no fixed node. There are in total five free nodes and eight members in the structure; i.e., $n = 5$, $n^{\mathrm{f}} = 0$, and $m = 8$.

According to Eq. (2.1), the connectivity matrix $\mathbf{C}$ $(=\mathbf{C}^{\mathrm{s}} \in \mathbb{R}^{8\times 5})$ of the structure is

$$\mathbf{C} = \mathbf{C}^{\mathrm{s}} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \\ [5] \\ [6] \\ [7] \\ [8] \end{array}. \tag{2.8}$$

It is obvious that sum of the entries in each row of $\mathbf{C}$ is zero.

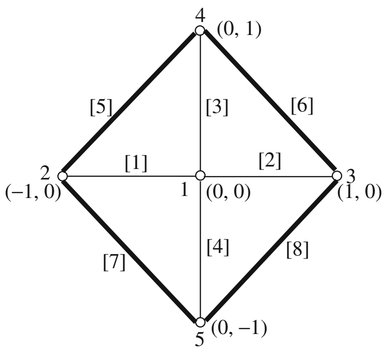

Fig. 2.2 A two-dimensional free-standing structure considered in Example 2.2. The structure consists of five free nodes 1–5 and eight members [1]–[8]. It is not attached to any fixed node or support

2.1.3 Geometry Realization

Geometry realization of a pin-jointed structure is described by coordinates of the nodes, or nodal coordinates. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ ($\in \mathbb{R}^n$), and $\mathbf{x}^{\mathrm{f}}$, $\mathbf{y}^{\mathrm{f}}$, $\mathbf{z}^{\mathrm{f}}$ ($\in \mathbb{R}^{n^{\mathrm{f}}}$) denote the vectors of coordinates of the free nodes and the fixed nodes in x-, y-, and z-directions, respectively.

The *coordinate differences* u_k, v_k, and w_k in x-, y-, and z-directions of member k connecting node i and node j can be respectively calculated as follows:

$$
\begin{aligned}
u_k &= \mathrm{sign}(j-i)\cdot x_i + \mathrm{sign}(i-j)\cdot x_j = -\mathrm{sign}(j-i)\cdot(x_j - x_i),\\
v_k &= \mathrm{sign}(j-i)\cdot y_i + \mathrm{sign}(i-j)\cdot y_j = -\mathrm{sign}(j-i)\cdot(y_j - y_i),\\
w_k &= \mathrm{sign}(j-i)\cdot z_i + \mathrm{sign}(i-j)\cdot z_j = -\mathrm{sign}(j-i)\cdot(z_j - z_i).
\end{aligned}
\tag{2.9}
$$

From the definition of connectivity matrix in Eq. (2.1), we know that $\mathrm{sign}(j-i)$ and $\mathrm{sign}(i-j)$ with $i \neq j$ in the kth row $\mathbf{C}^{\mathrm{s}}_k$ are the only two non-zero entries, which are respectively equal to 1 and -1, or -1 and 1. Thus, Eq. (2.9) can be rewritten in a matrix-vector form as follows:

$$
\begin{aligned}
u_k &= \mathbf{C}^{\mathrm{s}}_k \begin{pmatrix} \mathbf{x} \\ \mathbf{x}^{\mathrm{f}} \end{pmatrix} = \mathbf{C}_k \mathbf{x} + \mathbf{C}^{\mathrm{f}}_k \mathbf{x}^{\mathrm{f}},\\
v_k &= \mathbf{C}^{\mathrm{s}}_k \begin{pmatrix} \mathbf{y} \\ \mathbf{y}^{\mathrm{f}} \end{pmatrix} = \mathbf{C}_k \mathbf{y} + \mathbf{C}^{\mathrm{f}}_k \mathbf{y}^{\mathrm{f}},\\
w_k &= \mathbf{C}^{\mathrm{s}}_k \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^{\mathrm{f}} \end{pmatrix} = \mathbf{C}_k \mathbf{z} + \mathbf{C}^{\mathrm{f}}_k \mathbf{z}^{\mathrm{f}},
\end{aligned}
\tag{2.10}
$$

where $\mathbf{C}_k$ and $\mathbf{C}^{\mathrm{f}}_k$ are the kth rows of $\mathbf{C}$ and $\mathbf{C}^{\mathrm{f}}$, respectively.

Furthermore, *coordinate difference vectors* $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ ($\in \mathbb{R}^m$) are expressed in a matrix-vector form as follows:

$$
\begin{aligned}
\mathbf{u} &= \mathbf{C}\mathbf{x} + \mathbf{C}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}},\\
\mathbf{v} &= \mathbf{C}\mathbf{y} + \mathbf{C}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}},\\
\mathbf{w} &= \mathbf{C}\mathbf{z} + \mathbf{C}^{\mathrm{f}}\mathbf{z}^{\mathrm{f}}.
\end{aligned}
\tag{2.11}
$$

Example 2.3 Coordinate difference vectors of the two-dimensional structure as shown in Fig. 2.1, which has fixed nodes.

Consider again the the two-dimensional structure as shown in Fig. 2.1, which was studied in Example 2.1. The nodes 1 and 2 are free nodes, and the nodes 3–8 are fixed nodes.

Using the connectivity matrices $\mathbf{C} \in \mathbb{R}^{7\times 2}$ and $\mathbf{C}^{\mathrm{f}} \in \mathbb{R}^{7\times 6}$ given in Eq. (2.5), its coordinate difference vector $\mathbf{u} \in \mathbb{R}^7$ in x-direction is calculated as follows according to Eq. (2.11):

$$\mathbf{u} = \mathbf{C}\mathbf{x} + \mathbf{C}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}}$$
$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$
$$= \begin{pmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \\ x_2 - x_5 \\ x_2 - x_6 \\ x_2 - x_7 \\ x_1 - x_8 \end{pmatrix}. \tag{2.12}$$

In a similar manner, the coordinate difference vector $\mathbf{v} \in \mathbb{R}^7$ in y-direction is calculated as

$$\mathbf{v} = \mathbf{C}\mathbf{y} + \mathbf{C}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}} = \begin{pmatrix} y_1 - y_2 \\ y_1 - y_3 \\ y_1 - y_4 \\ y_2 - y_5 \\ y_2 - y_6 \\ y_2 - y_7 \\ y_1 - y_8 \end{pmatrix}. \tag{2.13}$$

For free-standing structures, the terms $\mathbf{C}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}}$ and $\mathbf{C}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}}$ corresponding to fixed nodes in Eq. (2.11) vanish. Therefore, the coordinate difference vectors are

$$\begin{aligned} \mathbf{u} &= \mathbf{C}\mathbf{x}, \\ \mathbf{v} &= \mathbf{C}\mathbf{y}, \\ \mathbf{w} &= \mathbf{C}\mathbf{z}. \end{aligned} \tag{2.14}$$

Example 2.4 Coordinate difference vectors of the two-dimensional free-standing structure as shown in Fig. 2.2.

The two-dimensional free-standing structure as shown in Fig. 2.2 was studied in Example 2.2. All of its five nodes are free nodes.

The coordinate difference vector $\mathbf{u} \in \mathbb{R}^8$ in x-direction of the two-dimensional free-standing structure as shown in Fig. 2.2 is calculated as follows, by using the connectivity matrix $\mathbf{C} \in \mathbb{R}^{8\times 5}$ in Eq. (2.8):

$$\mathbf{u} = \mathbf{C}\mathbf{x} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \\ x_1 - x_5 \\ x_2 - x_4 \\ x_3 - x_4 \\ x_2 - x_5 \\ x_3 - x_5 \end{pmatrix}. \tag{2.15}$$

Similarly, the coordinate difference vector $\mathbf{v} \in \mathbb{R}^8$ in y-direction is calculated as

$$\mathbf{v} = \mathbf{C}\mathbf{y} = \begin{pmatrix} y_1 - y_2 \\ y_1 - y_3 \\ y_1 - y_4 \\ y_1 - y_5 \\ y_2 - y_4 \\ y_3 - y_4 \\ y_2 - y_5 \\ y_3 - y_5 \end{pmatrix}. \tag{2.16}$$

By using the coordinate differences u_k, v_k, and w_k, the following relation holds for the square of length l_k of member k:

$$l_k^2 = u_k^2 + v_k^2 + w_k^2. \tag{2.17}$$

Let $\mathbf{l} \in \mathbb{R}^m$ denote the vector of member lengths, the kth entry of which is the length of member k. Let $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{L}$ $(\in \mathbb{R}^{m\times m})$ denote the diagonal versions of the coordinate difference vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and the member length vector $\mathbf{l}$; i.e.,

$$\begin{aligned} \mathbf{U} &= \mathrm{diag}(\mathbf{u}), & \mathbf{V} &= \mathrm{diag}(\mathbf{v}), \\ \mathbf{W} &= \mathrm{diag}(\mathbf{w}), & \mathbf{L} &= \mathrm{diag}(\mathbf{l}). \end{aligned} \tag{2.18}$$

Hence, square of member length matrix $\mathbf{L}$ is expressed as

$$\mathbf{L}^2 = \mathbf{U}^2 + \mathbf{V}^2 + \mathbf{W}^2, \tag{2.19}$$

where the diagonal entries of the diagonal matrix $\mathbf{L}^2$ are l_k^2. Alternatively, we can write l_k^2 as entries of the vector $\mathbf{L}\mathbf{l}$ expressed by the following equation

$$\mathbf{L}\mathbf{l} = \mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v} + \mathbf{W}\mathbf{w}. \tag{2.20}$$

2.2 Equilibrium Matrix

In this section, we obtain the (self-)equilibrium equations, in the form of equilibrium matrix associated with axial forces (prestresses) of the members, in two different ways: (1) those by assembling force balance of each node; and (2) those directly derived by applying the principle of virtual work. These equations are essentially consistent with each other. It will be shown later in Sect. 2.3 that the equilibrium matrix and its transpose, called the compatibility matrix, are the key matrices to understand static and kinematic determinacy of a pin-jointed structure. Furthermore, they are directly related to the linear stiffness matrix as will be presented in Sect. 4.2 in Chap. 4. An equivalent formulation of the equilibrium equations, using the force density matrix associated with the nodal coordinates, will be given in Sect. 2.4.

2.2.1 Equilibrium Equations by Balance of Forces

Consider a single node, for instance numbered as i, as a reference node. For simplicity, we consider only equilibrium of free nodes in the following, while equilibrium of fixed nodes can be derived in a similar manner.

Suppose the reference node i is connected to node i_j by member k_j, and there are in total m_i such members. The axial force (or prestress when no external load) carried in member k_j is denoted by s_{k_j} ($j = 1, 2, \dots, m_i$). External loads applied at the free nodes (or reaction forces at the fixed nodes) in x-, y-, and z-directions are denoted by load vectors $\mathbf{p}^x$, $\mathbf{p}^y$, and $\mathbf{p}^z$ ($\in \mathbb{R}^{n+n^{\mathrm{f}}}$), respectively. The ith entries of $\mathbf{p}^x$, $\mathbf{p}^y$, $\mathbf{p}^z$ are the loads p_i^x, p_i^y, p_i^z applied at node i in each direction. If node i is a fixed node, then p_i^x, p_i^y, p_i^z refer to the reaction forces.

Equilibrium equation of the reference node i in x-direction is

$$\begin{aligned} p_i^x + \sum_{j=1}^{m_i} s_{k_j} \frac{x_{i_j} - x_i}{l_{k_j}} &= p_i^x - \sum_{j=1}^{m_i} \mathrm{sign}(i_j - i) \frac{u_{k_j} s_{k_j}}{l_{k_j}} \\ &= p_i^x - \sum_{j=1}^{m_i} C_{(k_j,i)} \frac{u_{k_j} s_{k_j}}{l_{k_j}} \\ &= 0, \end{aligned} \tag{2.21}$$

where $x_{i_j} - x_i = -\mathrm{sign}(i_j - i) \cdot u_{k_j}$ and the (k_j, i)th entry $C_{(k_j,i)}$ of the connectivity matrix $\mathbf{C}$ of free nodes have been used.

Example 2.5 Equilibrium equation of a single node of a two-dimensional structure as shown in Fig. 2.3.

As shown in Fig. 2.3, the free node i of a two-dimensional structure is connected to three other nodes i_1, i_2, and i_3 by members k_1, k_2, and k_3, respectively. Thus, we have $m_i = 3$ for this case. Moreover, p_i^x and p_i^y are the external loads applied at node i in x- and y-directions, respectively.

Using Eq. (2.21), equilibrium equation of free node i in x-direction is written as

$$\begin{aligned} & p_i^x + s_{k_1}\frac{x_{i_1}-x_i}{l_{k_1}} + s_{k_2}\frac{x_{i_2}-x_i}{l_{k_2}} + s_{k_3}\frac{x_{i_3}-x_i}{l_{k_3}} \\ = \; & p_i^x - \mathrm{sign}(i_1-i)\frac{s_{k_1}u_{k_1}}{l_{k_1}} - \mathrm{sign}(i_2-i)\frac{s_{k_2}u_{k_2}}{l_{k_2}} - \mathrm{sign}(i_3-i)\frac{s_{k_3}u_{k_3}}{l_{k_3}} \\ = \; & p_i^x - \sum_{j=1}^{3}\mathrm{sign}(i_j-i)\frac{u_{k_j}s_{k_j}}{l_{k_j}} \\ = \; & 0. \end{aligned} \tag{2.22}$$

Similar equilibrium equation can be written for node i in y-direction.

Let $\mathbf{s} \in \mathbb{R}^m$ denote *member force vector*, the kth element s_k of which is the axial force in member k. Because the non-zero entries in the ith column of $\mathbf{C}$ correspond to the nodes that are connected to node i by the corresponding members, the equilibrium equation of the free node i in x-direction in Eq. (2.21) can be written in a matrix form as

$$(\mathbf{C}^\top)_i \mathbf{U}\mathbf{L}^{-1}\mathbf{s} = p_i^x, \tag{2.23}$$

where $(\mathbf{C}^\top)_i$ denotes the ith row of $\mathbf{C}^\top$; i.e., the transpose of the ith column of $\mathbf{C}$; moreover, $\mathbf{L}^{-1}$ denotes inverse matrix of the member length matrix $\mathbf{L}$, and the kth entry of $\mathbf{L}^{-1}$ is $1/l_k$.

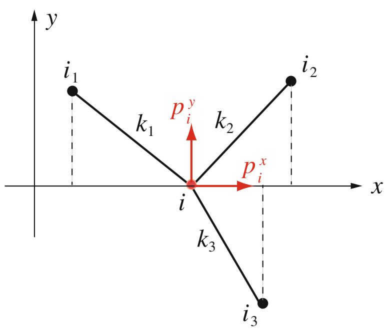

Fig. 2.3 Equilibrium of a reference node i of a two-dimensional structure subjected to external loads p_i^x and p_i^y. Nodes i_1, i_2, and i_3 are connected to node i by members k_1, k_2, and k_3, respectively

Example 2.6 Equilibrium equation of node 3 of the two-dimensional free-standing structure as shown in Fig. 2.2 in a matrix form.

Consider the equilibrium equation of the free node 3 of the two-dimensional free-standing structure as shown in Fig. 2.2. Using the third row $(\mathbf{C}^\top)_3$ of the transpose of the connectivity matrix $\mathbf{C}$ given in Eq. (2.8), we have

$$
\begin{aligned}
&(\mathbf{C}^\top)_3\mathbf{U}\mathbf{L}^{-1}\mathbf{s} \\
&= \left(0,\ -\frac{u_2}{l_2},\ 0,\ 0,\ 0,\ \frac{u_6}{l_6},\ 0,\ \frac{u_8}{l_8}\right)(s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ s_8)^\top \\
&= -\frac{u_2}{l_2}s_2 + \frac{u_6}{l_6}s_6 + \frac{u_8}{l_8}s_8 \\
&= \frac{x_3 - x_1}{l_2}s_2 + \frac{x_3 - x_4}{l_6}s_6 + \frac{x_3 - x_5}{l_8}s_8,
\end{aligned}
\tag{2.24}
$$

where

$$
\begin{aligned}
(\mathbf{C}^\top)_3 &= \left(0,\ -1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1\right), \\
\mathbf{U} &= \mathrm{diag}\left(u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ u_7,\ u_8\right), \\
\mathbf{L}^{-1} &= \mathrm{diag}\left(\frac{1}{l_1},\ \frac{1}{l_2},\ \frac{1}{l_3},\ \frac{1}{l_4},\ \frac{1}{l_5},\ \frac{1}{l_6},\ \frac{1}{l_7},\ \frac{1}{l_8}\right), \\
\mathbf{s}^\top &= \left(s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ s_8\right),
\end{aligned}
\tag{2.25}
$$

and moreover, the following coordinate differences of the members connected to node 3 have been used:

$$
\begin{aligned}
u_2 &= x_1 - x_3, \\
u_6 &= x_3 - x_4, \\
u_8 &= x_3 - x_5.
\end{aligned}
\tag{2.26}
$$

From the equilibrium equation of a single node in x-direction as defined in Eq. (2.21), we have

$$
p_3^x - \left(\frac{x_3 - x_1}{l_2}s_2 + \frac{x_3 - x_4}{l_6}s_6 + \frac{x_3 - x_5}{l_8}s_8\right) = 0, \tag{2.27}
$$

where p_3^x denotes the external load applied at node 3 in x-direction. From Eqs. (2.24) and (2.27), we have

$$
(\mathbf{C}^\top)_3\mathbf{U}\mathbf{L}^{-1}\mathbf{s} = p_3^x, \tag{2.28}
$$

which validates the equilibrium equation (2.23) in the matrix form.

The equilibrium equations of all free nodes of the structure in x-direction are summarized in a matrix form as

$$\mathbf{D}^x\mathbf{s} = \mathbf{p}^x, \tag{2.29}$$

where

$$\mathbf{D}^x = \mathbf{C}^\top\mathbf{U}\mathbf{L}^{-1}. \tag{2.30}$$

In a similar manner, the equilibrium equations in y- and z-directions are

$$\mathbf{D}^y\mathbf{s} = \mathbf{p}^y \quad \text{and} \quad \mathbf{D}^z\mathbf{s} = \mathbf{p}^z, \tag{2.31}$$

with

$$\mathbf{D}^y = \mathbf{C}^\top\mathbf{V}\mathbf{L}^{-1} \quad \text{and} \quad \mathbf{D}^z = \mathbf{C}^\top\mathbf{W}\mathbf{L}^{-1}. \tag{2.32}$$

The equilibrium equations of free nodes of a pin-jointed structure with respect to member force vector $\mathbf{s}$ can then be combined as

$$\mathbf{D}\mathbf{s} = \mathbf{p}, \tag{2.33}$$

where

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}^x \\ \mathbf{D}^y \\ \mathbf{D}^z \end{pmatrix} \quad \text{and} \quad \mathbf{p} = \begin{pmatrix} \mathbf{p}^x \\ \mathbf{p}^y \\ \mathbf{p}^z \end{pmatrix}. \tag{2.34}$$

In the above equations, $\mathbf{D} \in \mathbb{R}^{3n\times m}$ is called the *equilibrium matrix*. For a two-dimensional structure, the size of $\mathbf{D}$ is $2n \times m$, since it becomes

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}^x \\ \mathbf{D}^y \end{pmatrix}. \tag{2.35}$$

Define the reaction force vectors in x-, y-, and z-directions as $\mathbf{f}^x, \mathbf{f}^y, \mathbf{f}^z$ $(\in\mathbb{R}^{n^{\mathrm{f}}})$, respectively. Equilibrium equations of the fixed nodes can be written in a similar way as the free nodes as

$$\mathbf{D}^{\mathrm{f}}\mathbf{s} = \mathbf{f}, \tag{2.36}$$

where

$$\mathbf{D}^{\mathrm{f}} = \begin{pmatrix} (\mathbf{C}^{\mathrm{f}})^\top\mathbf{U}\mathbf{L}^{-1} \\ (\mathbf{C}^{\mathrm{f}})^\top\mathbf{V}\mathbf{L}^{-1} \\ (\mathbf{C}^{\mathrm{f}})^\top\mathbf{W}\mathbf{L}^{-1} \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}^x \\ \mathbf{f}^y \\ \mathbf{f}^z \end{pmatrix}. \tag{2.37}$$

2.2.2 Equilibrium Equations by the Principle of Virtual Work

In this subsection, the equilibrium equations are derived in a more systematic way than those obtained by considering force balance in the previous subsection.

Suppose that the structure is subjected to virtual displacements δx_i, δy_i, and δz_i $(i = 1, 2, \ldots, 3n)$, which cause virtual member length extensions δl_k $(k = 1, 2, \ldots, m)$. Thus, the total virtual work $\delta \Pi$, due to the virtual displacements and member length extensions, can be written as follows:

$$\delta \Pi = \sum_{k=1}^{m} s_k \delta l_k - \sum_{i=1}^{n} p_i^x \delta x_i - \sum_{i=1}^{n} p_i^y \delta y_i - \sum_{i=1}^{n} p_i^z \delta z_i, \tag{2.38}$$

which should be zero, when the structure is in equilibrium, according to the principle of virtual work [7]; i.e.,

$$\delta \Pi = 0, \tag{2.39}$$

for arbitrary δx_i, δy_i, δz_i, and δl_k satisfying compatibility conditions.

Using the coordinate differences u_k, v_k, and w_k in each direction of member k, the following relation holds:

$$\delta(l_k^2) = \delta(u_k^2) + \delta(v_k^2) + \delta(w_k^2), \tag{2.40}$$

which comes from the definition of member length. Equation (2.40) results in

$$l_k \delta l_k = u_k \delta u_k + v_k \delta v_k + w_k \delta w_k. \tag{2.41}$$

Substituting Eq. (2.41) into Eqs. (2.38) and (2.39) gives

$$\begin{aligned} \delta \Pi = & \left(\sum_{k=1}^{m} \frac{s_k u_k}{l_k} \delta u_k - \sum_{i=1}^{n} p_i^x \delta x_i \right) \\ & + \left(\sum_{k=1}^{m} \frac{s_k v_k}{l_k} \delta v_k - \sum_{i=1}^{n} p_i^y \delta y_i \right) \\ & + \left(\sum_{k=1}^{m} \frac{s_k w_k}{l_k} \delta w_k - \sum_{i=1}^{n} p_i^z \delta z_i \right) \\ = & \ 0. \end{aligned} \tag{2.42}$$

Because u_k, v_k, and w_k are functions of coordinates in x-, y-, and z-directions, respectively, and moreover, δx_i, δy_i, and δz_i are arbitrary (virtual) values, the following three independent equations should be true at the same time so that Eq. (2.42) holds:

$$
\begin{aligned}
x\text{-direction: } & \sum_{k=1}^{m} \frac{s_k u_k}{l_k} \delta u_k - \sum_{i=1}^{n} p_i^x \delta x_i = 0, \\
y\text{-direction: } & \sum_{k=1}^{m} \frac{s_k v_k}{l_k} \delta v_k - \sum_{i=1}^{n} p_i^y \delta y_i = 0, \\
z\text{-direction: } & \sum_{k=1}^{m} \frac{s_k w_k}{l_k} \delta w_k - \sum_{i=1}^{n} p_i^z \delta z_i = 0.
\end{aligned} \tag{2.43}
$$

In the following, we consider the equation only in x-direction, for clarity. Those in y- and z-directions can be obtained in a similar way.

If member k is connected by nodes i and j, then its coordinate difference u_k can be calculated by using the components $C_{(k,i)}$ and $C_{(k,j)}$ in the kth row of the connectivity matrix $\mathbf{C}$ defined in Eq. (2.1):

$$
u_k = C_{(k,i)} x_i + C_{(k,j)} x_j, \tag{2.44}
$$

and therefore, its variation is

$$
\delta u_k = C_{(k,i)} \delta x_i + C_{(k,j)} \delta x_j. \tag{2.45}
$$

Because all entries in the kth row of $\mathbf{C}$, except for $C_{(k,i)}$ and $C_{(k,j)}$, are zero, Eq. (2.45) can be rewritten as

$$
\delta u_k = \sum_{i=1}^{n} C_{(k,i)} \delta x_i. \tag{2.46}
$$

Substituting Eq. (2.46) into the first equation in Eq. (2.43) for node i in x-direction, we have

$$
\begin{aligned}
& \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{s_k u_k}{l_k} C_{(k,i)} \delta x_i - \sum_{i=1}^{n} p_i^x \delta x_i \\
= & \sum_{i=1}^{n} \left(\sum_{k=1}^{m} \frac{s_k u_k}{l_k} C_{(k,i)} - p_i^x \right) \delta x_i \\
= & \, 0.
\end{aligned} \tag{2.47}
$$

Because δx_i are arbitrary values, we then have n equilibrium equations

$$
\sum_{k=1}^{m} \frac{s_k u_k}{l_k} C_{(k,i)} - p_i^x = 0, \quad (i = 1, 2, \ldots, n), \tag{2.48}
$$

which can be further summarized in a matrix form as

$$\begin{aligned}\sum_{k=1}^{m} \frac{s_k u_k}{l_k} C_{(k,i)} &= (\mathbf{C}^\top)_i \mathbf{U}\mathbf{L}^{-1}\mathbf{s} \\ &= p_i^x, \qquad\qquad (i = 1, 2, \ldots, n).\end{aligned} \tag{2.49}$$

Assembling the equilibrium equations in x-direction for all nodes, we have the equilibrium equation in a matrix form as

$$\mathbf{D}^x \mathbf{s} = \mathbf{p}^x, \tag{2.50}$$

which coincides with the equilibrium equation derived in Eq. (2.29). In a similar way, we can derive the equilibrium equations in y- and z-directions, which will not be repeated here.

Since the discussions on equilibrium and stability do not depend on the units, we will omit the units in the following examples.

Example 2.7 Equilibrium matrix of the two-dimensional free-standing structure as shown in Fig. 2.2.

For simplicity, we consider a symmetric geometry realization of the two-dimensional free-standing structure as shown in Fig. 2.2. Its nodal coordinates are given in vector forms as

$$\begin{aligned}\mathbf{x} &= (0,\ -1,\ 1,\ 0,\ 0)^\top, \\ \mathbf{y} &= (0,\ 0,\ 0,\ 1,\ -1)^\top.\end{aligned} \tag{2.51}$$

From Eq. (2.15), the coordinate difference vectors $\mathbf{u}$ and $\mathbf{v}$ of the structure are

$$\mathbf{u} = \mathbf{C}\mathbf{x} = \begin{pmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \\ x_1 - x_5 \\ x_2 - x_4 \\ x_3 - x_4 \\ x_2 - x_5 \\ x_3 - x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{v} = \mathbf{C}\mathbf{y} = \begin{pmatrix} y_1 - y_2 \\ y_1 - y_3 \\ y_1 - y_4 \\ y_1 - y_5 \\ y_2 - y_4 \\ y_3 - y_4 \\ y_2 - y_5 \\ y_3 - y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.52}$$

Note that we have $\mathbf{C} = \mathbf{C}^{\mathrm{s}}$ for free-standing structures. Moreover, the member length matrix $\mathbf{L} \in \mathbb{R}^{8\times 8}$ is

$$\mathbf{L} = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2}). \tag{2.53}$$

Therefore, the equilibrium matrix $\mathbf{D} \in \mathbb{R}^{10\times 8}$ is calculated as follows by using Eq. (2.34):

$$\mathbf{D} = \begin{pmatrix} \mathbf{C}^{\top}\mathbf{U}\mathbf{L}^{-1} \\ \mathbf{C}^{\top}\mathbf{V}\mathbf{L}^{-1} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -\sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & 2 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 2 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -\sqrt{2} & -\sqrt{2} \end{pmatrix}. \tag{2.54}$$

2.3 Static and Kinematic Determinacy

Tensegrity structures are always statically indeterminate, so that their members can carry prestresses, when no external load is applied. Moreover, tensegrity structures are usually kinematically indeterminate, such that they are unstable in the absence of prestresses. Kinematically indeterminate structures can be stabilized by proper prestresses in the self-equilibrium state. The criteria and conditions for stability of tensegrity structures will be given in the next chapter. In this section, we will focus on conditions for static and kinematic determinacy of general (prestressed) pin-jointed structures.

A structure is said to be *statically indeterminate*, if the member forces and reaction forces of the structure cannot be uniquely determined by using only (static) equilibrium equations, while subjected to external loads. By contrast, member forces and reaction forces of the structure can be uniquely determined by considering only equilibrium equations, if it is *statically determinate*. Static indeterminacy of a structure enables it possess member forces (prestresses) in the members, even though no external load is involved.

A structures is said to be *kinematically indeterminate*, if there exists a nodal motion, except for the rigid-body motions, keeping all member lengths unchanged. Such motion is called *mechanism*, including infinitesimal mechanism and finite mechanism. *Infinitesimal mechanism* implies that the motion (nodal displacement) is sufficiently small, while *finite mechanism* can have large displacements. Two simple example structures illustrating the difference between infinitesimal and finite mechanisms are shown in Fig. 2.4. Finite mechanisms are widely used for machinery or deployable structures that can largely change their shapes. In this book, we will focus only on infinitesimal mechanisms.

By contrast to mechanisms, *rigid-body motions* refer to the motions that do no change the distance between any pair of nodes. Note that a pair of nodes does not limit to the two nodes connected by a member. Rigid-body motions include translation in each direction, rotation about an arbitrary axis, and the combination of these motions. Figure 2.5 shows the three rigid-body motions of a structure in the two-dimensional space, while there exist six rigid-body motions for a structure in the three-dimensional space.

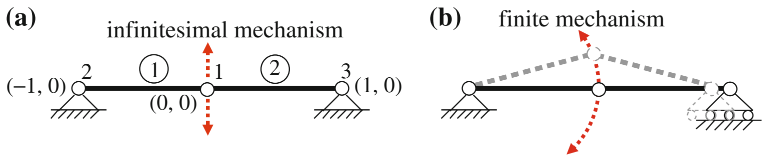

Fig. 2.4 Infinitesimal mechanism in (**a**) and finite mechanism in (**b**). Without changing the member lengths of a structure, infinitesimal mechanisms can only have sufficiently small deformations, while finite mechanisms allow large deformations. The *dashed lines* indicate possible deformations of the structures

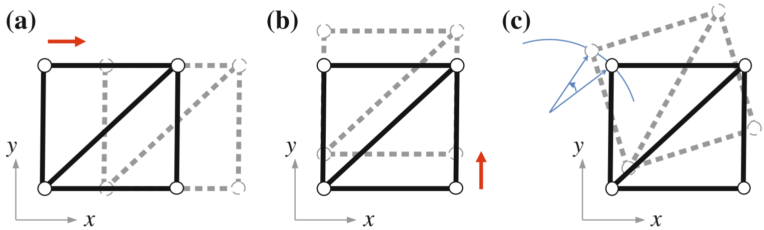

Fig. 2.5 Rigid-body motions of a structure in two-dimensional space. **a** Translation in x-direction, **b** translation in y-direction, **c** rotation about any point

2.3.1 Maxwell's Rule

There is a simple rule, called *Maxwell's rule*, to identify degrees of static and kinematic determinacy of a pin-jointed structure. The rule was proposed by Maxwell [10] in the 19th century. Using only the number m of member, the numbers n and n^{f} of nodes, the number n^{r} of reaction forces, static as well as kinematic determinacy of a d-dimensional structure consisting of fixed nodes can be identified as follows by observing the sign of the number n^{d}:

Maxwell's Rule:

$$n^{\mathrm{d}} = m + n^{\mathrm{r}} - d(n + n^{\mathrm{f}}) \begin{cases} < 0, & \text{kinematically indeterminate,} \\ = 0, & \text{statically and kinematically determinate,} \\ > 0, & \text{statically indeterminate.} \end{cases} \tag{2.55}$$

The principle of Maxwell's rule lies on whether the linear equilibrium equations can be uniquely solved: if the number of unknown parameters, including m member forces and n^{r} reaction forces, is equal to the total number $d(n + n^{\mathrm{f}})$ of (equilibrium) equations, with d equations at each of the $n + n^{\mathrm{f}}$ nodes, then the unknown parameters can be 'uniquely'[3] determined; and therefore, the structure is statically determinate. This in fact corresponds to $n^{\mathrm{d}} = 0$.

Moreover, there are two other cases concerning the sign of n^{d}:

1. When n^{d} is positive, the structure is statically indeterminate, and the degree n^{s} of static indeterminacy is n^{d}. This comes from the fact that $n^{\mathrm{s}} (= n^{\mathrm{d}})$ more equations, in addition to the existing equilibrium equations, are necessary to uniquely determine the member forces as well as the reaction forces.
2. When n^{d} is negative, the structure is kinematically indeterminate, and the degree n^{m} of kinematic indeterminacy is $-n^{\mathrm{d}}$; i.e., there exist $n^{\mathrm{m}} (= -n^{\mathrm{d}})$ independent infinitesimal mechanisms in the structure.

Example 2.8 Maxwell's rule for static and kinematic determinacy of the two-dimensional pin-jointed structures with supports as shown in Fig. 2.6.

Figure 2.6 shows four structures, all of which are two-dimensional ($d = 2$). They consist of the same number of nodes ($n + n^{\mathrm{f}} = 5$), but different numbers of members and reaction forces. According to Maxwell's rule in Eq. (2.55), we have

[3] The existence of unique solution is subjected to independence of the equilibrium equations.

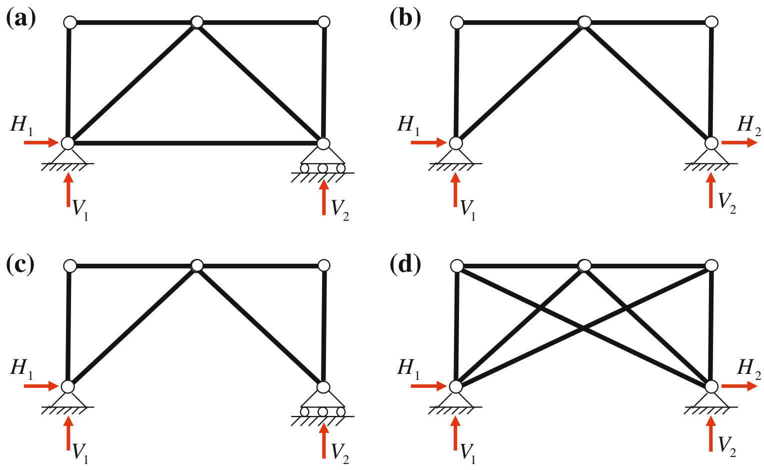

Fig. 2.6 Static and kinematic determinacy of the two-dimensional pin-jointed structures in Example 2.8. H_1, H_2 and V_1, V_2 are the reaction forces in horizontal and vertical directions, respectively. **a, b** Statically and kinematically determinate structures, **c** kinematically indeterminate structure, **d** statically indeterminate structure

$$
\begin{aligned}
&\text{(a) } m = 7,\ n^{\mathrm{r}} = 3: \quad n^{\mathrm{d}} = 7 + 3 - 2 \times 5 = 0 \\
&\qquad \Longrightarrow \text{statically and kinematically determinate;} \\
&\text{(b) } m = 6,\ n^{\mathrm{r}} = 4: \quad n^{\mathrm{d}} = 6 + 4 - 2 \times 5 = 0 \\
&\qquad \Longrightarrow \text{statically and kinematically determinate;} \\
&\text{(c) } m = 6,\ n^{\mathrm{r}} = 3: \quad n^{\mathrm{d}} = 6 + 3 - 2 \times 5 = -1 \\
&\qquad \Longrightarrow \text{kinematically indeterminate with degree of one; i.e., } n^{\mathrm{m}} = 1; \\
&\text{(d) } m = 8,\ n^{\mathrm{r}} = 4: \quad n^{\mathrm{d}} = 8 + 4 - 2 \times 5 = 2 \\
&\qquad \Longrightarrow \text{statically indeterminate with degree of two; i.e., } n^{\mathrm{s}} = 2.
\end{aligned}
\tag{2.56}
$$

In the case of free-standing structures without any fixed node; i.e., $n^{\mathrm{f}} = 0$, or reaction force; i.e., $n^r = 0$, rigid-body motions should be considered in the Maxwell's rule to verify internal static indeterminacy. The number of rigid-body motions n^{b} is given by

$$
n^{\mathrm{b}} = \frac{d^2 + d}{2}, \tag{2.57}
$$

from which n^{b} is equal to 3 for a two-dimensional case, and it is equal to 6 for a three-dimensional case.

Maxwell's Rule for Free-standing Structures:

$$n^{\mathrm{d}} = m - dn + n^{\mathrm{b}} \begin{cases} < 0, & \text{kinematically indeterminate,} \\ = 0, & \text{statically and kinematically determinate,} \\ > 0, & \text{statically indeterminate.} \end{cases} \tag{2.58}$$

Example 2.9 Maxwell's rule for the two-dimensional free-standing structure as shown in Fig. 2.2.

The number of rigid-body motions is (n^{b}=)3 for the two-dimensional structures. The two-dimensional free-standing structure as shown in Fig. 2.2 has eight members and five free nodes; i.e., $m = 8$ and $n = 5$.

According to Maxwell's rule for free-standing structures in Eq. (2.58), we have

$$\begin{aligned} n^{\mathrm{d}} &= m - dn + n^{\mathrm{b}} \\ &= 8 - 2 \times 5 + 3 \\ &= 1; \end{aligned} \tag{2.59}$$

i.e., the degree n^{d} of static indeterminacy of the structure is one. This means that there exists only one prestress mode in the structure.

It should be noted that Maxwell's rule is *NOT* a sufficient condition for identification of static or kinematic determinacy of a pin-jointed structure, because neither connectivity nor geometry realization of the structure has been taken into consideration. It has been well recognized that there are many exceptions while applying Maxwell's rule. Hence, Maxwell's rule can only be used for preliminary study; exact investigation can be conducted by checking rank of the equilibrium matrix, details on which will be given in the next two subsections.

Example 2.10 The two-dimensional pin-jointed structure in Fig. 2.7 as an exceptional example of Maxwell's rule.

The structure as shown in Fig. 2.7 consists of six members, four reaction forces, and five nodes; i.e., $m = 6$, $n^{\mathrm{r}} = 4$, and $n + n^{\mathrm{f}} = 5$. It is statically and kinematically determinate, according to Maxwell's rule:

$$\begin{aligned} & n^{\mathrm{d}} = 6 + 4 - 2 \times 5 = 0 \\ \Longrightarrow \quad & \text{statically and kinematically determinate?} \end{aligned} \tag{2.60}$$

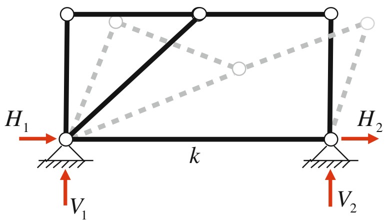

Fig. 2.7 Exceptional example of Maxwell's rule in Example 2.10. According to Maxwell's rule, it is statically and kinematically determinate. However, the structure is in fact statically and kinematically indeterminate, because there exists one (finite) mechanism as indicated by *dashed lines* in the figure, and it has one prestress mode

However, the structure is actually statically and kinematically indeterminate. It can deform as indicated by grey dashed lines in the figure, keeping lengths of all members unchanged. Moreover, member k directly connecting the two supports can have arbitrary prestress.

2.3.2 *Modified Maxwell's Rule*

To incorporate the exceptions in Maxwell's rule, a modified version of the rule was presented by Calladine [1] as follows:

Modified Maxwell's Rule for Free-standing Structures:

The following relation holds for the number n^{s} of independent prestress modes and the number n^{m} of independent infinitesimal mechanisms:

$$n^{\mathrm{s}} - n^{\mathrm{m}} = m - dn + n^{\mathrm{b}}. \tag{2.61}$$

Remember that n^{s} is the number of independent prestress modes or degree of static indeterminacy, and n^{m} is the number of independent infinitesimal mechanisms or degree of kinematic indeterminacy.

Example 2.11 Modified Maxwell's rule for the two-dimensional free-standing structure as shown in Fig. 2.2.

The two-dimensional free-standing structure as shown in Fig. 2.2 consists of five free nodes and eight members; i.e., $n = 5$ and $m = 8$. According to the modified Maxwell's rule in Eq. (2.61), we have

$$\begin{aligned} n^{\mathrm{s}} - n^{\mathrm{m}} &= m - dn + n^{\mathrm{b}} \\ &= 8 - 2 \times 5 + 3 \\ &= 1, \end{aligned} \tag{2.62}$$

which indicates that there exists at least one prestress mode, because the number n^{m} of (infinitesimal) mechanisms has to be non-negative; i.e., $n^{\mathrm{m}} \geq 0$.

Using the above relation, we can derive the number of prestress modes or infinitesimal mechanisms, if either of them is known. From Example 2.12 studied in the next subsection, we know that the structure has only one prestress mode; i.e., $n^{\mathrm{s}} = 1$. Therefore, from Eq. (2.62), the number n^{m} of independent (infinitesimal) mechanisms is

$$n^{\mathrm{m}} = n^{\mathrm{s}} - 1 = 0; \tag{2.63}$$

i.e., the structure in Fig. 2.2 consists of only one prestress mode and no mechanism. Calculation of number of mechanisms of the structure will be revisited in a formal way in Example 2.13.

Readers are encouraged to use the modified Maxwell's rule to revisit the exceptional example of Maxwell's rule presented in Example 2.10.

2.3.3 Static Determinacy

To correctly evaluate static and kinematic determinacy of a pin-jointed structure, it is necessary to investigate rank of the equilibrium matrix, or equivalently, that of the compatibility matrix. We will first prove that the compatibility matrix, which relates member extensions to nodal displacements, is the transpose of the equilibrium matrix. It will be further shown that rank of the equilibrium matrix reveals whether there are enough number of linear self-equilibrium and compatibility equations, so as to uniquely derive the non-trivial solutions for member forces and nodal displacements, respectively. In the following, we will concentrate on free-standing structures, because tensegrity structures belong to this category.

When there is no external load applied to the structure; i.e., $\mathbf{p} = \mathbf{0}$, the *self-equilibrium equation* with respect to the vector of self-equilibrium prestresses (or axial member forces) $\mathbf{s}$ of a free-standing structure is written as follows from Eq. (2.33):

$$\mathbf{D}\mathbf{s} = \mathbf{0}. \tag{2.64}$$

Denote rank of the equilibrium matrix $\mathbf{D} \in \mathbb{R}^{dn \times m}$ by r^{D}; i.e.,

$$r^{\mathrm{D}} = \mathrm{rank}(\mathbf{D}). \tag{2.65}$$

It is obvious that the rank of a matrix cannot be larger than its dimensions; i.e.,

$$r^{\mathrm{D}} \leq \min(dn, m), \tag{2.66}$$

where $\min(dn, m)$ refers to the smaller value of dn and m.

Rank r^{D} of the equilibrium matrix is indeed the number of independent equations. Thus, the number n^{s} of independent prestress modes, or degree of static indeterminacy, is calculated as

$$n^{\mathrm{s}} = m - r^{\mathrm{D}}. \tag{2.67}$$

Moreover, there are two cases concerning number of possible solutions for prestresses, by considering the value of r^{D} in comparison to the number m of prestresses:

- $r^{\mathrm{D}} < m$ or $n^{\mathrm{s}} > 0$ (Statically indeterminate):
  In this case, there exist $m - r^{\mathrm{D}}$ independent non-trivial solutions (prestress modes in members); i.e., $\mathbf{s} \neq \mathbf{0}$, satisfying the self-equilibrium equation (2.64). The possible solutions lie in the null-space of $\mathbf{D}$ spanned by the independent prestress modes $\bar{\mathbf{s}}_i$ $(i = 1, 2, \dots, m - r^{\mathrm{D}})$ as follows:

$$\mathbf{s} = \sum_{i=1}^{n^{\mathrm{s}}} \alpha_i \bar{\mathbf{s}}_i, \tag{2.68}$$

  where α_i are arbitrary coefficients, and the independent prestress modes $\bar{\mathbf{s}}_i$ satisfy

$$\mathbf{D}\bar{\mathbf{s}}_i = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{s}}_i^{\top} \bar{\mathbf{s}}_j = \delta_{ij}, \tag{2.69}$$

  with δ_{ij} referring to Kronecker's delta:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.70}$$

  Hence, the structure is *statically indeterminate* in this case, because we cannot uniquely determine the prestresses in the structure without additional information (equations).
- $r^{\mathrm{D}} = m$ or $n^{\mathrm{s}} = 0$ (Statically determinate):
  On the other hand, for the case with $r^{\mathrm{D}} = m$, the structure cannot contain any non-zero prestresses while no external load is applied. Therefore, the structure is *statically determinate* in this case.

Example 2.12 Static determinacy of the two-dimensional free-standing structure as shown in Fig. 2.2 and its single independent prestress mode.

For the two-dimensional free-standing structure in Fig. 2.2, the geometry realization of the structure in Example 2.7 is adopted. The structure consists of eight members; i.e., $m = 8$. Investigation of rank[4] r^{D} of its equilibrium matrix $\mathbf{D} \in \mathbb{R}^{10\times 8}$ in Eq. (2.54) gives

$$r^{\mathrm{D}} = 7. \tag{2.71}$$

Hence, the number n^{s} of independent prestress modes can be calculated as follows by using Eq. (2.67):

$$\begin{aligned} n^{\mathrm{s}} &= m - r^{\mathrm{D}} = 8 - 7 \\ &= 1, \end{aligned} \tag{2.72}$$

which means that there exists only one prestress mode. The normalized prestress mode $\bar{\mathbf{s}}$ of the structure is obtained from the null-space of $\mathbf{D}$ as

$$\bar{\mathbf{s}} = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}. \tag{2.73}$$

It is easy to verify that the following equation holds

$$\mathbf{D}\bar{\mathbf{s}} = \mathbf{0}. \tag{2.74}$$

Hence, prestresses of the structure in proportion to $\bar{\mathbf{s}}$ would necessarily satisfy the self-equilibrium equations.

2.3.4 Kinematic Determinacy

Let $\mathbf{d} \in \mathbb{R}^{dn}$ denote the vector of (infinitesimal) nodal displacements due to the external loads $\mathbf{p}$ applied to the structure, and let $\mathbf{e} \in \mathbb{R}^{m}$ denote the vector of member extensions, which are related to the small displacements $\mathbf{d}$ by the kinematic relations

[4] Rank and null-space of a matrix $\mathbf{D}$ can be found by using, for example, the commands *rank*($\mathbf{D}$) and *null*($\mathbf{D}$) in Octave or Matlab, respectively.

in terms of the *compatibility matrix* $\mathbf{H} \in \mathbb{R}^{m \times dn}$ [2, 3]:

$$\mathbf{Hd} = \mathbf{e}. \tag{2.75}$$

From the principle of virtual work, the virtual work done by the external loads is equal to virtual internal work done by member forces. Hence, for the virtual displacements $\delta\mathbf{d}$, which can have arbitrary values, and their corresponding virtual member extensions $\delta\mathbf{e}$, we have

$$\mathbf{p}^\top(\delta\mathbf{d}) = \mathbf{s}^\top(\delta\mathbf{e}). \tag{2.76}$$

Substituting Eq. (2.75) into Eq. (2.76), we have

$$\mathbf{p}^\top(\delta\mathbf{d}) = \mathbf{s}^\top\mathbf{H}(\delta\mathbf{d}), \tag{2.77}$$

since $\mathbf{H}$ is a constant matrix subjected to virtual displacements. From the relationship between the external loads $\mathbf{p}$ and the prestresses $\mathbf{s}$ in Eq. (2.33), we obtain that

$$\mathbf{p}^\top = \mathbf{s}^\top\mathbf{D}^\top. \tag{2.78}$$

Because the virtual displacements $\delta\mathbf{d}$ are arbitrary, Eqs. (2.77) and (2.78) lead to

$$\mathbf{s}^\top\mathbf{D}^\top = \mathbf{s}^\top\mathbf{H}. \tag{2.79}$$

Equation (2.79) is always true only if the compatibility matrix $\mathbf{H}$ is equal to the transpose $\mathbf{D}^\top$ of the equilibrium matrix $\mathbf{D}$ [1, 8]:

$$\mathbf{H} = \mathbf{D}^\top. \tag{2.80}$$

Hence, the transpose $\mathbf{D}^\top$ of the equilibrium matrix $\mathbf{D}$ is exactly the compatibility matrix, and the kinematic relation of the structure in Eq. (2.75) can be rewritten as

$$\mathbf{D}^\top\mathbf{d} = \mathbf{e}. \tag{2.81}$$

The rigid-body motions of a free-standing structure obviously do not lead to member extensions, therefore, they should be excluded in the discussions of kinematic indeterminacy. Accordingly, the total degree of freedom of a free-standing structure is $dn - n^{\mathrm{b}}$, when the number n^{b} of rigid-body motions have been excluded.

If a non-rigid-body motion $\mathbf{d}$ leads to no member extension; i.e.,

$$\mathbf{D}^\top\mathbf{d} = \mathbf{0}, \tag{2.82}$$

then $\mathbf{d}$ is a mechanism, which is usually denoted by $\mathbf{d}_{\mathrm{m}}$.

Moreover, it is easy to see that the rank of the compatibility matrix $\mathbf{D}^\top$ is equal to that of the equilibrium matrix $\mathbf{D}$:

$$\mathrm{rank}(\mathbf{D}^\top) = \mathrm{rank}(\mathbf{D}) = r^{\mathrm{D}}. \tag{2.83}$$

Therefore, the number of independent infinitesimal mechanisms, or degree of kinematic indeterminacy, n^{m} of a free-standing structure is computed by

$$n^{\mathrm{m}} = dn - n^{\mathrm{b}} - r^{\mathrm{D}}. \tag{2.84}$$

Note that for a structure, of which the rigid-body motions are constrained by the fixed nodes, n^{b} in the equation vanishes. Furthermore, we have the following two cases for the value of n^{m}:

- $r^{\mathrm{D}} < dn - n^{\mathrm{b}}$ or $n^{\mathrm{m}} > 0$ (Kinematically indeterminate):
  There exists n^{m} independent non-trivial displacements $\mathbf{m}_i$ ($i = 1, 2, \ldots, n^{\mathrm{m}}$), which are not rigid-body motions, preserving the member lengths; i.e.,

  $$\mathbf{D}^\top \mathbf{m}_i = \mathbf{0}. \tag{2.85}$$

  Therefore, the structure is *kinematically indeterminate*. Moreover, a mechanism $\mathbf{d}_{\mathrm{m}}$ can be written as a linear combination of the independent mechanisms $\mathbf{m}_i$ as follows:

  $$\mathbf{d}_{\mathrm{m}} = \sum_{i=1}^{n^{\mathrm{m}}} \beta_i \mathbf{m}_i, \tag{2.86}$$

  where β_i are arbitrary coefficients, and the independent mechanisms $\mathbf{m}_i$ are normalized as

  $$\mathbf{m}_i^\top \mathbf{m}_j = \delta_{ij}. \tag{2.87}$$

- $r^{\mathrm{D}} = dn - n^{\mathrm{b}}$ or $n^{\mathrm{m}} = 0$ (Kinematically determinate):
  The structure is *kinematically determinate*, because there exists no non-trivial displacement vector preserving the member lengths, except for the rigid-body motions.

Example 2.13 Kinematic determinacy of the two-dimensional free-standing structure as shown in Fig. 2.2.

The two-dimensional free-standing structure as shown in Fig. 2.2 consists of five free nodes; i.e., $n = 5$. The same geometry realization as in Example 2.7 is adopted for the structure. Rank r^{D} of the compatibility matrix $\mathbf{D}^\top$, or equivalently that of the equilibrium matrix as in Example 2.12, is ($r^{\mathrm{D}}=$) 7.

Therefore, degree n^{m} of kinematic indeterminacy, or number of infinitesimal mechanisms, is calculated as follows according to Eq. (2.84) for free-standing structures:

$$\begin{aligned} n^{\mathrm{m}} &= dn - n^{\mathrm{b}} - r^{\mathrm{D}} \\ &= 2 \times 5 - 3 - 7 \\ &= 0, \end{aligned} \tag{2.88}$$

which means that there exists no mechanism in the structure.

Example 2.14 Kinematic indeterminacy of the two-dimensional structure with fixed nodes as shown in Fig. 2.4a and its infinitesimal mechanism.

The two-dimensional structure in Fig. 2.4a consists of two members, one free node, and two fixed nodes; i.e., $m = 2$, $n = 1$, and $n^{\mathrm{f}} = 2$. The connectivity matrices $\mathbf{C}^{\mathrm{s}} \in \mathbb{R}^{2\times 3}$, $\mathbf{C} \in \mathbb{R}^{2\times 1}$, and $\mathbf{C}^{\mathrm{f}} \in \mathbb{R}^{2\times 2}$ of the structure respectively are

$$\mathbf{C}^{\mathrm{s}} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left(\begin{array}{c|cc} 1 & -1 & 0 \\ 1 & 0 & -1 \end{array}\right) \end{array} \begin{array}{c} \\ [1] \\ [2] \end{array}, \quad \mathbf{C} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{C}^{\mathrm{f}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.89}$$

Geometry realization of the structure is adopted as follows:

$$\mathbf{x} = (0)\,, \quad \mathbf{x}^{\mathrm{f}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \mathbf{y} = (0)\,, \quad \mathbf{y}^{\mathrm{f}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.90}$$

Thus, the coordinate difference vectors $\mathbf{u}$ and $\mathbf{v}$ ($\in \mathbb{R}^2$) are

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (0) + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ \mathbf{v} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \end{aligned} \tag{2.91}$$

and the member length vector $\mathbf{l} \in \mathbb{R}^2$ is

$$\mathbf{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.92}$$

From Eq. (2.34), the equilibrium matrix $\mathbf{D} \in \mathbb{R}^{2\times 2}$ corresponding to the only one free node is

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}^x \\ \mathbf{D}^y \end{pmatrix} = \begin{pmatrix} \mathbf{C}^\top \mathbf{U}\mathbf{L}^{-1} \\ \mathbf{C}^\top \mathbf{V}\mathbf{L}^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \tag{2.93}$$

It is obvious that the rank r^{D} of $\mathbf{D}$ is equal to one:

$$r^{\mathrm{D}} = \mathrm{rank}(\mathbf{D}) = 1. \tag{2.94}$$

According to Eqs. (2.67) and (2.84), there exist one prestress mode and one infinitesimal mechanism in the structure; i.e., $n^{\mathrm{s}} = n^{\mathrm{m}} = 1$, since we have

$$\begin{aligned} n^{\mathrm{s}} &= m - r^{\mathrm{D}} = 2 - 1 \\ &= 1, \\ n^{\mathrm{m}} &= dn - r^{\mathrm{D}} = 1 \times 2 - 1 \\ &= 1, \end{aligned} \tag{2.95}$$

where n^{b} is not included because rigid-body motions of the structure have been constrained by the two fixed nodes.

The normalized prestress mode $\bar{\mathbf{s}} \in \mathbb{R}^2$ and infinitesimal mechanism $\mathbf{d}_{\mathrm{m}} \in \mathbb{R}^2$, lying in the null-space of $\mathbf{D}$ and $\mathbf{D}^\top$, respectively, are calculated as

$$\bar{\mathbf{s}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.96}$$

Therefore, the structure is in equilibrium, if its two members carry the same prestresses, either tension or compression; and its only infinitesimal mechanism is illustrated in Fig. 2.4a.

2.3.5 Remarks

Section 2.3 presented existing methodologies for identification of static as well as kinematic determinacy of a (prestressed) pin-jointed structure. The original Maxwell's rule is neither a sufficient nor a necessary condition, but it is in a very simple form such that it can be used for preliminary studies. The modified Maxwell's rule by Calladine provides deeper understanding of static and kinematic determinacy of a structure, through the number n^{s} of independent prestress modes as well as the number n^{m} of independent mechanisms.

Degrees of static and kinematic determinacy of a d-dimensional free-standing pin-jointed structure can be identified by using the numbers of nodes n and members m, along with the rank r^{D} of the equilibrium matrix.

Static and kinematic determinacy of a free-standing pin-jointed structure:

	Statically		Kinematically	
	Determinate	Indeterminate	Determinate	Indeterminate
$n^{\mathrm{s}} = m - r^{\mathrm{D}}$	=0	>0		
$n^{\mathrm{m}} = dn - n^{\mathrm{b}} - r^{\mathrm{D}}$			=0	>0

2.4 Force Density Matrix

In this section, equilibrium equations of a prestressed pin-jointed structure are formulated with respect to nodal coordinates associated with the force density matrix. The characteristics of force density matrix are critical for understanding self-equilibrium and stability of tensegrity structures, as will be extensively used throughout the remaining of this book.

2.4.1 Definition of Force Density Matrix

Force density q_k of member k is defined as the ratio of its member force s_k to its member length l_k; i.e.,

$$q_k = \frac{s_k}{l_k}. \tag{2.97}$$

Moreover, the *force density vector* $\mathbf{q} \in \mathbb{R}^m$, consisting of force densities of all members, is calculated by

$$\mathbf{q} = \mathbf{L}^{-1}\mathbf{s}, \tag{2.98}$$

the kth entry of which is q_k.

To rewrite the equilibrium equation in Eq. (2.29) with respect to member forces $\mathbf{s}$ in x-direction into those with respect to nodal coordinates $\mathbf{x}$ and $\mathbf{x}^{\mathrm{f}}$, we have

$$\begin{aligned}\mathbf{D}^x\mathbf{s} &= \mathbf{C}^\top\mathbf{U}\mathbf{L}^{-1}\mathbf{s} = \mathbf{C}^\top\mathbf{U}\mathbf{q} = \mathbf{C}^\top\mathbf{Q}\mathbf{u} \\ &= \mathbf{C}^\top\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{C}^\top\mathbf{Q}\mathbf{C}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}},\end{aligned} \tag{2.99}$$

where $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$ is the diagonal version of the force density vector $\mathbf{q}$. Moreover, the above equations hold because we have the following relations for any vectors $\mathbf{a}$, $\mathbf{b}$ of the same size and their diagonal versions $\mathbf{A}$, $\mathbf{B}$

$$\mathbf{Ab} = \mathbf{Ba}, \text{ with } \mathbf{A} = \mathrm{diag}(\mathbf{a}) \text{ and } \mathbf{B} = \mathrm{diag}(\mathbf{b}). \tag{2.100}$$

Let K_i denote the set of members connected to the free node i. Equation (2.99) can also be derived from Eq. (2.21) as follows:

$$\begin{aligned}\sum_{k\in K_i} C_{(k,i)} \frac{u_k s_k}{l_k} &= \sum_{k\in K_i} C_{(k,i)} q_k u_k = (\mathbf{C}^\top)_i \mathbf{Qu} \\ &= (\mathbf{C}^\top)_i \mathbf{QCx},\end{aligned} \tag{2.101}$$

because the (k, i)th entry of $C_{(k,i)}$ is non-zero only if member k is connected to node i.

In a similar way to Eq. (2.99), Eq. (2.31) can be rewritten as

$$\begin{aligned}\mathbf{D}^y\mathbf{s} &= \mathbf{C}^\top\mathbf{VL}^{-1}\mathbf{s} = \mathbf{C}^\top\mathbf{Vq} = \mathbf{C}^\top\mathbf{Qv} \\ &= \mathbf{C}^\top\mathbf{QCy} + \mathbf{C}^\top\mathbf{QC}^\mathrm{f}\mathbf{y}^\mathrm{f}, \\ \mathbf{D}^z\mathbf{s} &= \mathbf{C}^\top\mathbf{WL}^{-1}\mathbf{s} = \mathbf{C}^\top\mathbf{Wq} = \mathbf{C}^\top\mathbf{Qw} \\ &= \mathbf{C}^\top\mathbf{QCz} + \mathbf{C}^\top\mathbf{QC}^\mathrm{f}\mathbf{z}^\mathrm{f}.\end{aligned} \tag{2.102}$$

Example 2.15 Equilibrium equation of free node 3 of the two-dimensional free-standing structure as shown in Fig. 2.2.

Consider equilibrium of free node 3 of the two-dimensional free-standing structure as shown in Fig. 2.2. Using the third row $(\mathbf{C}^\top)_3$ of transpose of the connectivity matrix $\mathbf{C}$ defined in Eq. (2.89), and according to Eq. (2.101), its equilibrium equation in x-direction is

$$\begin{aligned}(\mathbf{C}_3)^\top\mathbf{Qu} &= \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} q_1 &&&&&&& \\ & q_2 &&&&&& \\ && q_3 &&&&& \\ &&& q_4 &&&& \\ &&&& q_5 &&& \\ &&&&& q_6 && \\ &&&&&& q_7 & \\ &&&&&&& q_8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} \\ &= -q_2u_2 + q_6u_6 + q_8u_8\end{aligned}$$

$$= (x_3 - x_1)\frac{s_2}{l_2} + (x_3 - x_4)\frac{s_6}{l_6} + (x_3 - x_5)\frac{s_6}{l_8}$$
$$= \frac{x_3 - x_1}{l_2}s_2 + \frac{x_3 - x_4}{l_6}s_6 + \frac{x_3 - x_5}{l_8}s_8, \tag{2.103}$$

which coincides with Eq. (2.24) derived in a different way.

Define $\mathbf{E} \in \mathbb{R}^{n \times n}$ and $\mathbf{E}^{\mathrm{f}} \in \mathbb{R}^{n \times n^{\mathrm{f}}}$ as

$$\begin{aligned} \mathbf{E} &= \mathbf{C}^{\top}\mathbf{Q}\mathbf{C}, \\ \mathbf{E}^{\mathrm{f}} &= \mathbf{C}^{\top}\mathbf{Q}\mathbf{C}^{\mathrm{f}}, \end{aligned} \tag{2.104}$$

where the matrix $\mathbf{E}$ is called *force density matrix* of the free nodes, and $\mathbf{E}^{\mathrm{f}}$ is that of the fixed nodes. $\mathbf{E}$ is also called small stress matrix, for example in paper [4]. Using $\mathbf{E}$ and $\mathbf{E}^{\mathrm{f}}$, the equations in Eqs. (2.99) and (2.102) are simplified as

$$\begin{aligned} \mathbf{D}^x\mathbf{s} &= \mathbf{E}\mathbf{x} + \mathbf{E}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}}, \\ \mathbf{D}^y\mathbf{s} &= \mathbf{E}\mathbf{y} + \mathbf{E}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}}, \\ \mathbf{D}^z\mathbf{s} &= \mathbf{E}\mathbf{z} + \mathbf{E}^{\mathrm{f}}\mathbf{z}^{\mathrm{f}}. \end{aligned} \tag{2.105}$$

Therefore, the equilibrium equations in Eqs. (2.29) and (2.31) are rewritten as follows by using the force density matrices $\mathbf{E}$ and $\mathbf{E}^{\mathrm{f}}$ associated with the nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{x}^{\mathrm{f}}$, $\mathbf{y}^{\mathrm{f}}$, $\mathbf{z}^{\mathrm{f}}$:

$$\begin{aligned} \mathbf{E}\mathbf{x} + \mathbf{E}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}} &= \mathbf{p}^x, \\ \mathbf{E}\mathbf{y} + \mathbf{E}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}} &= \mathbf{p}^y, \\ \mathbf{E}\mathbf{z} + \mathbf{E}^{\mathrm{f}}\mathbf{z}^{\mathrm{f}} &= \mathbf{p}^z. \end{aligned} \tag{2.106}$$

2.4.2 Direct Definition of Force Density Matrix

Instead of formulating the force density matrix $\mathbf{E}$ through Eq. (2.104), it can also be assembled directly using force densities of the members. The (i, j)th entry $E_{(i,j)}$ of the force density matrix $\mathbf{E}$ is defined as

$$E_{(i,j)} = \begin{cases} \sum\limits_{k \in K_i} q_k & \text{for } i = j, \\ -q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k, \\ 0 & \text{for other cases.} \end{cases} \tag{2.107}$$

Example 2.16 Force density matrices of the two-dimensional cable-net as shown in Fig. 2.1 and the free-standing structure as shown in Fig. 2.2.

First, consider the two-dimensional structure (cable-net) with fixed nodes as shown in Fig. 2.1. There are in total two free nodes 1 and 2 in the structure. Node 1 is connected by members [1], [2], [3], and [7], thus, the (1, 1)th entry $E_{(1,1)}$ of the force density matrix $\mathbf{E}$ is $q_1 + q_2 + q_3 + q_7$; moreover, node 1 is connected to the other free node 2 by member [1], hence, the (1, 2)th entry $E_{(1,2)}$ of $\mathbf{E}$ is $-q_1$.

Similarly, the (2, 2)th entry $E_{(2,2)}$ of $\mathbf{E}$ is $q_1 + q_4 + q_5 + q_6$, because node 2 is connected by members [1], [4], [5], and [6]; and the (2, 1)th entry $E_{(2,1)}$ is $-q_1$ because free node 2 is connected to the other free node 1 by member [1]. Therefore, the force density matrix $\mathbf{E} \in \mathbb{R}^{2\times 2}$ corresponding to the free nodes of the structure is

$$\mathbf{E} = \begin{array}{cc} \text{Node 1} & \text{Node 2} \\ \left(\begin{array}{c} q_1 + q_2 + q_3 + q_7 \\ -q_1 \end{array}\right. & \left.\begin{array}{c} -q_1 \\ q_1 + q_4 + q_5 + q_6 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Node 1} \\ \text{Node 2} \end{array} \tag{2.108}$$

In comparison, the force density matrix $\mathbf{E} \in \mathbb{R}^{5\times 5}$ of the two-dimensional free-standing structure as shown in Fig. 2.2 is

$$\mathbf{E} = \begin{array}{c} \begin{array}{ccccc} \text{Node 1} & \text{Node 2} & \text{Node 3} & \text{Node 4} & \text{Node 5} \end{array} \\ \begin{pmatrix} q_1+q_2+q_3+q_4 & -q_1 & -q_2 & -q_3 & -q_4 \\ -q_1 & q_1+q_5+q_7 & 0 & -q_5 & -q_7 \\ -q_2 & 0 & q_2+q_6+q_8 & -q_6 & -q_8 \\ -q_3 & -q_5 & -q_6 & q_3+q_5+q_6 & 0 \\ -q_4 & -q_7 & -q_8 & 0 & q_4+q_7+q_8 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \\ \text{Node 4} \\ \text{Node 5} \end{array} \tag{2.109}$$

It is obvious from the examples that the force density matrix $\mathbf{E}$ is square and symmetric. Furthermore, it should be noted that sum of the entries in each row or column of the force density matrix in Eq. (2.109) is zero, which is always true for free-standing structures.

2.4.3 Self-equilibrium of the Structures with Supports

When the external loads are absent, the structure is said to be in a state of self-equilibrium—the nodes are equilibrated only by the prestresses in the members. Hence, self-equilibrium equations of the structure with respect to nodal coordinates can be written as follows, by setting the external loads $\mathbf{p}^x, \mathbf{p}^y, \mathbf{p}^z$ in Eq. (2.106) to zeros:

$$\begin{aligned}\mathbf{E}\mathbf{x} + \mathbf{E}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}} &= \mathbf{0},\\ \mathbf{E}\mathbf{y} + \mathbf{E}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}} &= \mathbf{0},\\ \mathbf{E}\mathbf{z} + \mathbf{E}^{\mathrm{f}}\mathbf{z}^{\mathrm{f}} &= \mathbf{0}.\end{aligned} \tag{2.110}$$

The above self-equilibrium equations are actually *non-linear* with respect to the nodal coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, because $\mathbf{E}$ and $\mathbf{E}^{\mathrm{f}}$ depend on the member lengths, which are non-linear functions of nodal coordinates. This dependency is apparent if we revisit the definitions of the coordinate differences in Eq. (2.9), the member lengths in Eq. (2.17), the force densities in Eq. (2.97), and the force density matrix in Eq. (2.104).

However, if the force densities are assigned or determined a priori, the force density matrices $\mathbf{E}$ and $\mathbf{E}^{\mathrm{f}}$ become constant. In this way, the self-equilibrium equations in Eq. (2.110) turn out to be *linear* with respect to the nodal coordinates. The only unknown parameters in Eq. (2.110) are the nodal coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ of the free nodes, while those $\mathbf{x}^{\mathrm{f}}$, $\mathbf{y}^{\mathrm{f}}$, $\mathbf{z}^{\mathrm{f}}$ of the supports are given.

The process of finding appropriate nodal coordinates as well as distribution of prestresses of a prestressed structure, satisfying the self-equilibrium equations, is generally called *form-finding* or *shape-finding*. Form-finding is a common design problem for prestressed structures, because their (self-equilibrated) configurations cannot be arbitrarily assigned in contrast to trusses carrying no prestress.

For the structures with fixed nodes, such as cable-nets and tensegrity-domes, the unknown coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ of the free nodes can be simply obtained as

$$\begin{aligned}\mathbf{x} &= -\mathbf{E}^{-1}\mathbf{E}^{\mathrm{f}}\mathbf{x}^{\mathrm{f}},\\ \mathbf{y} &= -\mathbf{E}^{-1}\mathbf{E}^{\mathrm{f}}\mathbf{y}^{\mathrm{f}},\\ \mathbf{z} &= -\mathbf{E}^{-1}\mathbf{E}^{\mathrm{f}}\mathbf{z}^{\mathrm{f}},\end{aligned} \tag{2.111}$$

if the force density matrix $\mathbf{E}$ is full-rank, or equivalently, if $\mathbf{E}$ is invertible. Hence, geometry realization of the structure, which is described in terms of nodal coordinates, can be uniquely determined.

This is the original idea of the *force density method* for the form-finding problem of cable-nets, where the non-linear self-equilibrium equations with respect to nodal coordinates are transformed into linear equations by introducing the concept of force density. Moreover, the force density matrix $\mathbf{E}$ of a cable-net is always positive definite, because it consists of only tensile members; i.e., cables, which carry positive (tensile) prestresses. Therefore, the self-equilibrium equations in Eq. (2.110) always have unique solutions as given in Eq. (2.111) [11].

This idea of force density method for cable-nets has been successfully extended to the form-finding problems of tensile membrane structures [9], by modelling them as cable-nets. This can be done because these two types of (tensile) structures share the following two common mechanical properties:

1. Both of cable-nets and membrane structures are attached to fixed nodes;
2. There exist only (positive) tensile forces in cable-nets or tensile stresses in membranes.

However, force density method cannot be directly applied to the form-finding problem of tensegrity structures. This comes from the fact that tensegrity structures are free-standing, without any fixed node, such that the corresponding force density matrix can never be full-rank. In fact, the force density matrix of a tensegrity structure should have certain rank deficiency; i.e., it has certain number of zero eigenvalues, in order to construct a structure in the space with desired dimensions (usually dimension of three). This is called the *non-degeneracy condition*, which is a necessary condition for geometry realization of tensegrity structures, and it will be presented in the next section.

Furthermore, we will discuss in Chap. 5 on how to use the idea of force density method for the form-finding of tensegrity structures, in which appropriate force densities are searched in order to satisfy this non-degeneracy condition.

2.5 Non-degeneracy Condition for Free-standing Structures

This section presents the non-degeneracy condition for a free-standing prestressed pin-jointed structure, in order to guarantee its geometry realization in the space with desired dimensions. The condition is described as an inequality with respect to the rank deficiency of the force density matrix.

Rank deficiency of a square symmetric matrix is the number of its zero eigenvalues. From the definition of force density matrix $\mathbf{E}$ in Eqs. (2.104) or (2.107), $\mathbf{E}$ of a free-standing structure has rank deficiency of at least one, because the sum of its entries in each row or column is always equal to zero. Hence, the vector $\mathbf{i}_n \in \mathbb{R}^n$ with all entries equal to one is obviously the eigenvector corresponding to this zero eigenvalue:

$$\mathbf{E}\mathbf{i}_n = \mathbf{C}^\top\mathbf{Q}\mathbf{C}\mathbf{i}_n = \mathbf{0} = 0\mathbf{i}_n, \tag{2.112}$$

because we have $\mathbf{C}\mathbf{i}_n = \mathbf{0}$ from Eq. (2.7). This property comes from the fact that free-standing structures do not have any fixed node.

Moreover, for a free-standing prestressed pin-jointed structure, the self-equilibrium equations in each direction in Eq. (2.110) become

$$\begin{aligned} \mathbf{E}\mathbf{x} &= \mathbf{0}, \\ \mathbf{E}\mathbf{y} &= \mathbf{0}, \\ \mathbf{E}\mathbf{z} &= \mathbf{0}, \end{aligned} \tag{2.113}$$

with the absence of fixed nodes $\mathbf{x}^{\mathrm{f}}$, $\mathbf{y}^{\mathrm{f}}$, $\mathbf{z}^{\mathrm{f}}$. The nodal coordinates $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ cannot be uniquely determined by solving Eq. (2.113), because the force density matrix of a free-standing structure is always rank deficient, or singular, and therefore, it is not invertible.

Example 2.17 Verification of self-equilibrium equations of the two-dimensional free-standing structure as shown in Fig. 2.2 associated with force density matrix.

In this example, we adopt the geometry realization as given in Eq. (2.51) in Example 2.7 for the two-dimensional free-standing structure as shown in Fig. 2.2. Using the prestress mode $\bar{\mathbf{s}}$ as presented in Eq. (2.73) in Example 2.12, the force densities $\mathbf{q}$ are computed from the prestresses $\mathbf{s}$ as

$$\mathbf{q} = \mathbf{L}^{-1}\mathbf{s} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & \mathbf{O} & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \sqrt{2} & & & \\ & & & & & \sqrt{2} & & \\ & & \mathbf{O} & & & & \sqrt{2} & \\ & & & & & & & \sqrt{2} \end{pmatrix}^{-1} \frac{t}{\sqrt{12}} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \frac{t}{\sqrt{24}} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \tag{2.114}$$

where $t(\neq 0)$ is an arbitrary value, the absolute value of which indicates level of the prestresses; moreover, $\mathbf{O}$ indicates zero entries in the matrix, and the member lengths in Eq. (2.53) have been used.

The force density matrix $\mathbf{E} \in \mathbb{R}^{5\times 5}$ is calculated as follows, by substituting the force densities in Eq. (2.114) into Eq. (2.109):

$$\mathbf{E} = \frac{t}{\sqrt{24}} \begin{pmatrix} 8 & -2 & -2 & -2 & -2 \\ -2 & 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{2.115}$$

Thus, the self-equilibrium equations in terms of force density matrix $\mathbf{E}$ associated with nodal coordinates $\mathbf{x}$ and $\mathbf{y}$ in x- and y-directions, respectively, are

$$\mathbf{E}\mathbf{x} = \frac{t}{\sqrt{24}} \begin{pmatrix} 8 & -2 & -2 & -2 & -2 \\ -2 & 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{E}\mathbf{y} = \mathbf{0}, \tag{2.116}$$

which show that the self-equilibrium equations are satisfied with the given geometry realization (nodal coordinates) as well as prestresses.

Define rank deficiency $\bar{r}^{\mathrm{E}}$ of force density matrix $\mathbf{E}$ as

$$\bar{r}^{\mathrm{E}} = n - \mathrm{rank}(\mathbf{E}), \tag{2.117}$$

i.e., there are in total $\bar{r}^{\mathrm{E}}$ zero eigenvalues in $\mathbf{E}$. The eigenvectors corresponding to these zero eigenvalues are denoted by $\boldsymbol{\phi}_i$ ($i = 1, 2, \ldots, \bar{r}^{\mathrm{E}}$), with

$$\boldsymbol{\phi}_i^{\top} \boldsymbol{\phi}_j = \delta_{ij}. \tag{2.118}$$

As discussed previously, the vector ($\boldsymbol{\phi}_1 =$)$\mathbf{i}_n$ in Eq. (2.112) is obviously one of these eigenvectors.

The nodal coordinates $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ satisfying Eq. (2.113) can be generally written as a linear combination of the linearly independent eigenvectors $\boldsymbol{\phi}_i$ corresponding to the zero eigenvalues of $\mathbf{E}$:

$$\mathbf{x} = \alpha_0^x \mathbf{i}_n + \sum_{i=1}^{\bar{r}^{\mathrm{E}}-1} \alpha_i^x \boldsymbol{\phi}_i,$$
$$\mathbf{y} = \alpha_0^y \mathbf{i}_n + \sum_{i=1}^{\bar{r}^{\mathrm{E}}-1} \alpha_i^y \boldsymbol{\phi}_i,$$
$$\mathbf{z} = \alpha_0^z \mathbf{i}_n + \sum_{i=1}^{\bar{r}^{\mathrm{E}}-1} \alpha_i^z \boldsymbol{\phi}_i, \tag{2.119}$$

where α_i^x, α_i^y, and α_i^z ($i = 0, 1, \ldots, \bar{r}^{\mathrm{E}} - 1$) are arbitrary coefficients. To construct a structure in the space with desired dimensions (two or three in practice), certain conditions should be satisfied for nodal coordinates $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Degeneracy of a Tensegrity Structure:

If a structure lies in a space with less dimensions than the specific dimensions d, then the structure is said to be *degenerate* in the d-dimensional space; otherwise, it is *non-degenerate*.

For example, the structure in Fig. 2.2 is non-degenerate in the two-dimensional space, and it is degenerate in the three-dimensional space, because it lies in a two-dimensional space (the plane parallel to the paper).

From the definition of degeneracy, we have the following lemma, on linear independence of the nodal coordinates, for a non-degenerate structure in d-dimensional space.

Lemma 2.1 *If a structure is non-degenerate in a d-dimensional space, its nodal coordinate vectors in each direction are linearly independent.*

Proof Consider the three-dimensional case, with $d = 3$, for instance. Suppose that the coordinate vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ of a three-dimensional structure are linearly dependent. Thus, the following equation holds:

$$\beta^x \mathbf{x} + \beta^y \mathbf{y} + \beta^z \mathbf{z} = \mathbf{0}, \tag{2.120}$$

where the coefficients β^x, β^y, and β^z are not equal to zero simultaneously. Equation (2.120) defines a plane such that the structure is degenerate in the three-dimensional space.

Therefore, the nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ have to be linearly independent, if the structure is non-degenerate in three-dimensional space.

Two-dimensional case can be proved in the same way, which completes the proof. □

Lemma 2.2 ***Non-degeneracy condition for free-standing structures***:

To guarantee that a free-standing prestressed pin-jointed structure is non-degenerate in d-dimensional space, the following relation should hold for the rank deficiency $\bar{r}^{\mathrm{E}}$ of its force density matrix:

$$\bar{r}^{\mathrm{E}} \geq d + 1. \tag{2.121}$$

Proof From Lemma 2.1, the nodal coordinate vectors of a structure should be linearly independent to ensure its non-degeneracy in the space with specific dimensions.

Because the coordinate vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ of a three-dimensional structure are in the same format as given in Eq. (2.119), they can be linearly independent only if there are no less than three independent vectors $\boldsymbol{\phi}_i$, except for the common eigenvector $\mathbf{i}_n$, in the null-space of $\mathbf{E}$. Hence, rank deficiency of the force density matrix should be equal to or greater than four for a three-dimensional structure.

For two-dimensional cases, there should be no less than two independent vectors $\boldsymbol{\phi}_i$, except for the common eigenvector $\mathbf{i}_n$, to ensure linear independency of nodal coordinate vectors $\mathbf{x}$ and $\mathbf{y}$. Hence, rank deficiency of the force density matrix should be equal to or greater than three for a two-dimensional structure.

In summary, the lemma has been proved for both two- and three-dimensional free-standing structures. □

Lemma 2.2 can be explained by geometry realization of a structure by using Eq. (2.119). Define $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$ as

$$\begin{aligned}\mathbf{x}_0 &= \alpha_0^x \mathbf{i}_n,\\ \mathbf{y}_0 &= \alpha_0^y \mathbf{i}_n,\\ \mathbf{z}_0 &= \alpha_0^z \mathbf{i}_n.\end{aligned} \tag{2.122}$$

The solutions of Eq. (2.113) can be written in a general form as follows:

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\end{pmatrix} = \begin{pmatrix}\mathbf{x}_0\\ \mathbf{y}_0\\ \mathbf{z}_0\end{pmatrix} + \sum_{i=1}^{\bar{r}^{\mathrm{E}}-1} \begin{pmatrix}\alpha_i^x & 0 & 0\\ 0 & \alpha_i^y & 0\\ 0 & 0 & \alpha_i^z\end{pmatrix} \begin{pmatrix}\boldsymbol{\phi}_i\\ \boldsymbol{\phi}_i\\ \boldsymbol{\phi}_i\end{pmatrix}, \tag{2.123}$$

where $\boldsymbol{\phi}_i$ is in the null-space of $\mathbf{E}$ such that $\mathbf{E}\boldsymbol{\phi}_i = \mathbf{0}$.

For a tensegrity structure with different rank deficiency $\bar{r}^{\mathrm{E}}$ in its force density matrix $\mathbf{E}$, we have the following discussions:

1. If $\bar{r}^{\mathrm{E}} = 1$, there exists only one non-zero solution, $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$, in each direction, hence, all nodes degenerate into the node $(\alpha_0^x, \alpha_0^y, \alpha_0^z)$. This node is called *base node*.
2. If $\bar{r}^{\mathrm{E}} = 2$, Eq. (2.123) defines a line that passes through the base node.
3. Equation (2.123) defines a two-dimensional space (plane) in the case of $\bar{r}^{\mathrm{E}} = 3$, and a three-dimensional space in the case of $\bar{r}^{\mathrm{E}} = 4$. Both of these solution spaces contain the base node.

Therefore, in order to ensure a non-degenerate tensegrity structure in d-dimensional space, rank deficiency $\bar{r}^{\mathrm{E}}$ of its force density matrix $\mathbf{E}$ should be equal to or larger than $d + 1$.

The condition in Lemma 2.2 is called the *non-degeneracy condition* for free-standing prestressed pin-jointed structures. However, it should be noted that this condition is only a necessary condition, but not a sufficient condition. As have been mentioned in Lemma 2.1, linear independence of the coordinate vectors should also be satisfied to guarantee a non-degenerate configuration.

Example 2.18 Non-degeneracy of the two-dimensional free-standing structure as shown in Fig. 2.2.

The two-dimensional free-standing structure as shown in Fig. 2.2 consists of five (free) nodes; i.e., $d = 2$ and $n = 5$.

Following the assignments of geometry and prestresses given in Example 2.16, numerical calculation shows that the rank of $\mathbf{E}$ is 2, and therefore, its rank deficiency $\bar{r}^{\mathrm{E}}$ is 3, since we have

$$\begin{aligned} \bar{r}^{\mathrm{E}} &= n - \mathrm{rank}(\mathbf{E}) = 5 - 2 = 3 \\ &= d + 1 = 3. \end{aligned} \tag{2.124}$$

This satisfies the non-degeneracy condition for a free-standing prestressed pin-jointed structure in two-dimensional space, which obviously coincides with the fact that it is a two-dimensional structure.

2.6 Remarks

In this chapter, the (self-)equilibrium equations of a (prestressed) pin-jointed structure have been presented in two different ways: those with respect to axial forces (prestresses) associated with the equilibrium matrix, and those with respect to nodal coordinates associated with the force density matrix.

Using rank (deficiency) of the equilibrium matrix, or equivalently the compatibility matrix, detailed information about degrees of static indeterminacy and kinematic indeterminacy of the structure can be achieved. Tensegrity structures are always statically indeterminate, so that they can carry prestresses in the absence of external loads; and they are usually kinematically indeterminate, such that they are unstable in the absence of prestresses. Stability criteria and conditions of tensegrity structures will be discussed in detail in Chap. 4.

Tensegrity structures are free-standing and prestressed pin-jointed structures, which are different from other types of structures, such as trusses carrying no prestress or cable-nets attached to supports. Configurations of the structures carrying prestresses cannot be arbitrarily determined, because the nodes and members have to be in the balance of prestresses. Hence, form-finding is a basic and important problem for design of tensegrity structures.

Moreover, the concept of force density is very useful for form-finding of cable-nets, but it cannot be directly utilized for tensegrity structures, because the force density matrix is singular, and therefore, non-invertible due to the fact that they are free-standing.

The non-degeneracy condition, in terms of rank deficiency of the force density matrix, has to be satisfied for a free-standing structure. The rank deficiency of a three-dimensional structure should be larger than three, while that of a two-dimensional structure should be larger than two. This condition will be used to present a strategy, making use of the idea of force density, for the form-finding of tensegrity structures in Chap. 5.

References

1. Calladine, C. R. (1978). Buckminster Fuller's "Tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frames. *International Journal of Solids and Structures*, *14*(2), 161–172.
2. Calladine, C. R., & Pellegrino, S. (1991). First-order infinitesimal mechanisms. *International Journal of Solids and Structures*, *27*(4), 505–515.
3. Calladine, C. R., & Pellegrino, S. (1992). Further remarks on first-order infinitesimal mechanisms. *International Journal of Solids and Structures*, *29*(17), 2119–2122.
4. Connelly, R. (1982). Rigidity and energy. *Inventiones Mathematicae*, *66*(1), 11–33.
5. Harary, F. (1969). *Graph theory*. Reading, MA: Addison-Wesley.
6. Kaveh, A. (1995). *Structural mechanics: graph and matrix methods*. New York: Research Studies Press.
7. Lanczos, C. (1986). *The variational principles of mechanics* (4th ed.). New York: Dover Publications.
8. Livesley, R. K. (1975). *Matrix methods of structural analysis* (2nd ed.). Oxford: Pergamon Press.
9. Maurin, B., & Motro, R. (1998). The surface stress density method as a form-finding tool for tensile membranes. *Engineering Structures*, *20*(8), 712–719.
10. Maxwell, J. C. (1864). On the calculation of the equilibrium and stiffness of frames. *Philosophical Magazine*, *27*(182), 294–299.
11. Schek, H.-J. (1974). The force density method for form finding and computation of general networks. *Computer Methods in Applied Mechanics and Engineering*, *3*(1), 115–134.

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