# Chapter 2
# Optimal Control for Diffusion Processes

**Abstract** This chapter deals with completely observable stochastic control problems for diffusion processes, described by SDEs. The decision maker chooses an optimal decision among all possible ones to achieve the goal. Namely, for a control process, its response evolves according to a (controlled) SDE and the payoff on a finite time interval is given. The controller wants to minimize (or maximize) the payoff by choosing an appropriate control process from among all possible ones. Here we consider three types of control processes:

1. $(\mathcal{F}_t)$-progressively measurable processes.
2. Brownian-adapted processes.
3. Feedback controls.

In order to analyze the problems, we mainly use the dynamic programming principle (DPP) for the value function.

The reminder of this chapter is organized as follows. Section 2.1 presents the formulation of control problems and basic properties of value functions, as preliminaries for later sections. Section 2.2 focuses on DPP. Although DPP is known as a two stage optimization method, we will formulate DPP by using a semigroup and characterize the value function via the semigroup. In Sect. 2.3, we deal with verification theorems, which give recipes for finding optimal Markovian policies. Section 2.4 considers a class of Merton-type optimal investment models, as an application of previous results.

## 2.1 Introduction

This section is devoted to formulating the time horizon stochastic control and analyze basic notions. We introduce control processes, payoffs and value functions in Sect. 2.1.1, and investigate their properties in Sect. 2.1.2.

Before we formulate the stochastic control problem, we give a typical example, called the linear quadratic control.

*Example 2.1 (Linear quadratic* (LQ) *control).* Consider a $d$-dimensional stochastic system with an external random force $\gamma(\cdot)$,

$$dX(t) = (b(t)X(t) + \gamma(t))\,dt + d\,W(t),$$



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where $W$ is a $d$-dimensional $(\mathcal{F}_t)$-Wiener process, $b : [0,T] \mapsto \mathbb{R}^d \otimes \mathbb{R}^d$ and $\gamma : [0,T] \times \Omega \mapsto \mathbb{R}^d$ are $(\mathcal{F}_t)$-progressively measurable.

Let $S_+^d$ denote the set of $d \times d$ non-negative definite matrices. Suppose that $M, N : [0,T] \mapsto S_+^d$ are given. The problem is to choose $\gamma(\cdot)$ so that the payoff:

$$J(\theta, x; \gamma(\cdot)) = E_{\theta x}\Big[\int_\theta^T (X(t)^\top M(t)X(t) + \gamma(t)^\top N(t)\gamma(t))\,dt + |X(T)|^2\Big]$$

is minimized and calculate the value function;

$$v(\theta, x) = \inf_{\gamma(\cdot)} J(\theta, x; \gamma(\cdot)).$$

In particular, when $d = 1, M = N = b = 0$ and $|\gamma(\cdot)| \le 1$ are assumed, the choice $\hat{\gamma}(t) := -\mathrm{sgn} X(t)$ is optimal (see Examples 2.4 and 2.5).

### 2.1.1 Formulations

We are going to formulate the finite time horizon stochastic control problem. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and $W$ an $m$-dimensional $(\mathcal{F}_t)$-Wiener process. We call $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ a reference probability system. Let $T > 0$ be given. A $\sigma$-compact and convex subset $\Gamma$ of $\mathbb{R}^q$ is called a control region, where $q$ is a positive integer.

**Definition 2.1.** The $\Gamma$-valued $(\mathcal{F}_t)$-progressively measurable process $(\gamma(t), t \in [0,T])$ is called a control process, if $\gamma(\cdot) \in L^\infty([0,T] \times \Omega, (\mathcal{F}_t); \Gamma)$, namely, there is a compact set $\Gamma_{\gamma(\cdot)} (\subset \Gamma)$, such that $\gamma(t,\omega) \in \Gamma_{\gamma(\cdot)}$, for almost all $(t,\omega)$.

The 6-tuple $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$ is called an admissible control. We denote by $\mathbb{A}$ the set of all admissible controls.

Let $b : [0,T] \times \mathbb{R}^d \times \Gamma \mapsto \mathbb{R}^d$ and $\alpha : [0,T] \times \mathbb{R}^d \times \Gamma \mapsto \mathbb{R}^d \otimes \mathbb{R}^m$ be given and satisfy the following condition:

$(b_1)$

$$\begin{aligned} &|b(t_1, x_1, \gamma_1) - b(t_2, x_2, \gamma_2)| + |\alpha(t_1, x_1, \gamma_1) - \alpha(t_2, x_2, \gamma_2)| \\ &\le l|x_1 - x_2| + \bar{m}(|t_1 - t_2| + |\gamma_1 - \gamma_2|) \end{aligned} \tag{2.1}$$

and

$$|b(t, 0, \gamma)| + |\alpha(t, 0, \gamma)| \le K, \quad \forall (t, \gamma) \in [0,T] \times \Gamma, \tag{2.2}$$

where $l$ and $K$ are positive constants and $\bar{m}(\cdot)$ is a bounded modulus function, say $\bar{m}(\cdot) \le \bar{M}$.

When $A(\in \mathbb{A})$ is applied, the stochastic system evolves according to the SDE (more precisely, controlled SDE)

$$dX(t) = b(t, X(t), \gamma(t))\, dt + \alpha(t, X(t), \gamma(t))\, dW(t), \quad (0 \le)\theta < t \le T, \tag{2.3}$$

with the initial condition

$$X(\theta) = x(\in \mathbb{R}^d). \tag{2.4}$$

By Theorem 1.2, there exists a unique solution of SDE (2.3)–(2.4), denoted by $X^A_{\theta x}$. This solution is sometimes called the response for $A$. $X^A_{\theta x}(t)$ is clearly $\sigma(\gamma(s), W(s') - W(\theta), s, s' \in [\theta, t])$-measurable.

We omit the indices $\theta$, $x$ and $A$, when no confusion occurs.

Let us introduce three functions $f, \phi$ and $\kappa$ with the conditions $(b_2)$–$(b_5)$.

$(b_2)$ $f : [0, T] \times \mathbb{R}^d \times \Gamma \mapsto \mathbb{R}^1$ is continuous and $f(t, x, \gamma)$ is continuous w.r.t. $x$, uniformly in $(t, \gamma)$,

$(b_3)$ $\phi : \mathbb{R}^d \mapsto \mathbb{R}^1$ is continuous,

$(b_4)$ $|f(t, x, \gamma)| + |\phi(x)| \le \hat{k}(1 + |x|^2)$, $\forall t, x, \gamma$ with a constant $\hat{k}$,

$(b_5)$ $\kappa : [0, T] \times \mathbb{R}^d \times \Gamma \mapsto [0, c_0]$, satisfies $(b_2)$.

$f$ and $\phi$ are called the running cost and the terminal cost, respectively, and $\kappa$ is the discount rate.

For the response $X = X^A_{\theta x}$, the cost function on time interval $[s, t]$ $(\subset [\theta, T])$ is given by

$$\begin{aligned} &C(t, s, A; \phi) \\ &= \int_s^t \exp\{-\int_s^\lambda \kappa(h, X(h), \gamma(h))\, dh\} f(\lambda, X(\lambda), \gamma(\lambda))\, d\lambda \\ &\quad + \exp\{-\int_s^t \kappa(h, X(h), \gamma(h))\, dh\}\phi(X(t)). \end{aligned} \tag{2.5}$$

When the response $X$ stops at time $t$, we define the payoff (or criterion) by

$$J(t, \theta, x, A; \phi) = E_{\theta x} C(t, \theta, A; \phi), \tag{2.6}$$

where $(\theta, x)$ refer to the initial condition of $X$. Clearly $J(t, \theta, x, A; \phi)$ depends on the joint probability of $(\gamma(s), W(s') - W(\theta); s, s' \in [\theta, t])$, but not on $A$ itself.

We want to minimize (or maximize) the payoff, by choosing an appropriate admissible control.

**Definition 2.2.** $v(\cdot)$, defined by

$$v(t, \theta, x; \phi) = \inf_{A \in \mathbb{A}} J(t, \theta, x, A; \phi) \quad (\text{or } \sup_{A \in \mathbb{A}} J(t, \theta, x, A; \phi)) \tag{2.7}$$

is called the value function. If $A \in \mathbb{A}$ gives the infimum (or supremum) of the RHS, $A$ is called an optimal control.

Thus we are concerned with the characterization of the value function and an optimal control.

Let us introduce different classes of admissible controls: Brownian adapted controls and feedback controls.

1. Let $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$ be an admissible control. $A$ is called Brownian adapted, if $\gamma(\cdot)$ is $(\mathcal{F}_t^W)$-progressively measurable. $\mathbb{A}^W$ denotes the set of all Brownian adapted controls. For a fixed reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$, $\boldsymbol{\Gamma}^W$ denotes the set of all $(\mathcal{F}_t^W)$-progressively measurable control processes. Since the payoff is calculated in terms of the joint probability distribution of $(W(\cdot), \gamma(\cdot))$, for any given a reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$, it holds that

$$\inf_{A \in \mathbb{A}^W} J(t, \theta, x, A; \phi) = \inf_{\gamma(\cdot) \in \boldsymbol{\Gamma}^W} J(t, \theta, x, A; \phi)$$

2. We control a system by using the data on the system, in the customary manners. A $\Gamma$-valued Borel function $\hat{\gamma}(\cdot)$, defined on $[0, T] \times \mathbb{R}^d$, is called a Markovian policy, if it is bounded. When we apply $\hat{\gamma}(\cdot)$, the system evolves according to the SDE

$$dX(t) = b(t, X(t), \hat{\gamma}(t, X(t)))\, dt + \alpha(t, X(t), \hat{\gamma}(t, X(t)))\, dW(t). \tag{2.8}$$

We can hardly expect that a strong solution will exist for (2.8), but a weak solution does exist under mild conditions (see Sect. 1.3.3). Hence there exist $W^*$ and $X^*$, on an appropriate filtered probability space $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*), P^*)$, such that (2.8) holds. Putting $\gamma^*(s) = \hat{\gamma}(s, X^*(s))$, we have $A^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*), P^*, W^*, \gamma^*(\cdot)) \in \mathbb{A}$, but not in $\mathbb{A}^W$. We call $A^*$ an admissible control associated with the Markovian policy $\hat{\gamma}(\cdot)$.

### *2.1.2 Value Functions: Basic Properties*

Recalling Theorem 1.2, we first list some basic properties of responses.

**Proposition 2.1.** *Let $p \geq 1$ be given. Then there exists a constant $K_p (> 0)$, such that the following estimates hold:*

(i) *For any $(\theta, x) \in [0, T] \times \mathbb{R}^d$,*

$$E_{\theta x}\Big[\sup_{\theta \leq t \leq T} |X^A(t)|^{2p}\Big] \leq K_p(1 + |x|^{2p}), \quad \forall A \in \mathbb{A}. \tag{2.9}$$

(ii) *For any $(\theta, x) \in [0, T] \times \mathbb{R}^d$ and $\theta \leq t_1 < t_2 \leq T$,*

$$E_{\theta x}\Big[\sup_{t_1\le s\le t_2}|X^A(s)-X^A(t_1)|^{2p}\Big]\le K_p(1+|x|^{2p})(t_2-t_1)^p,\quad \forall A\in\mathbb{A}. \tag{2.10}$$

(iii) *For any* $x, y \in \mathbb{R}^d$ *and* $\theta \in [0, T]$,

$$E\Big[\sup_{\theta\le s\le T}|X^A_{\theta x}(s)-X^A_{\theta y}(s)|^2\Big]\le K_1|x-y|^2,\quad \forall A\in\mathbb{A}. \tag{2.11}$$

(iv) *For* $0 \le \theta_1 < \theta_2 \le T$ *and* $x \in \mathbb{R}^d$,

$$E\Big[\sup_{\theta_2\le s\le T}|X^A_{\theta_1 x}(s)-X^A_{\theta_2 x}(s)|^2\Big]\le K_1(1+|x|^2)(\theta_2-\theta_1),\quad \forall A\in\mathbb{A}. \tag{2.12}$$

Regarding the dependence on the control process, we have;

**Proposition 2.2.** *There exists a constant* $K_0$ *such that, for any controls* $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$ *and* $\hat{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \hat{\gamma}(\cdot))$,

$$\begin{aligned}E_{\theta x}\Big[\sup_{\theta\le s\le t}|X^A(s)-X^{\hat{A}}(s)|^2\Big] &\le K_0\int_\theta^t E[\bar{m}(|\gamma(s)-\hat{\gamma}(s)|)^2]\,ds\\ &\le K_0\Big\{a^2(t-\theta)+\bar{M}^2\frac{E\mathbf{1}\gamma(s)-\hat{\gamma}(s)\mathbf{1}^2_{\theta t}}{(\bar{m}^{-1}(a))^2}\Big\},\\ &\forall a>0,\quad \forall 0\le\theta\le t\le T,\quad x\in\mathbb{R}^d\end{aligned} \tag{2.13}$$

*where* $\bar{M}$ *is a bound of* $\bar{m}(\cdot)$, *and* $\mathbf{1}\gamma(\cdot)-\hat{\gamma}(\cdot)\mathbf{1}_{\theta t} = \|\gamma(\cdot)-\hat{\gamma}(\cdot)\|_{L^2[\theta,t]}$. *We put* $\mathbf{1} * \mathbf{1} = \mathbf{1} * \mathbf{1}_{\theta T}$.

*Proof.* Set

$$Z(t)=X^A(t)-X^{\hat{A}}(t),\quad \triangle b(t)=b(t,X^A(t),\gamma(t))-b(t,X^{\hat{A}}(t),\hat{\gamma}(t)),$$

and similarly for $\alpha(\cdot)$. Then

$$\sup_{\theta\le s\le t}|Z(s)|\le\int_\theta^t|\triangle b(s)|\,ds+\sup_{\theta\le s\le t}\Big|\int_\theta^s\triangle\alpha(h)\,dW(h)\Big|$$

holds by (2.3). Therefore, the Burkholder–Davis–Gundy inequality yields

$$\rho(t)\le C\Big[\int_\theta^t\rho(s)\,ds+\int_\theta^t E[\bar{m}(|\gamma(s)-\hat{\gamma}(s)|)^2]\,ds\Big] \tag{2.14}$$

with a constant $C > 0$, where

$$\rho(t) = E\Big[\sup_{\theta \le s \le t} |Z(s)|^2\Big].$$

Now Gronwall's inequality and (2.14) lead to the left inequality in (2.13).

Now using the estimation

$$\begin{aligned} E\bar{m}(|\xi|)^2 &\le a^2 + E[\bar{m}(|\xi|)^2; |\xi| > \bar{m}^{-1}(a)] \\ &\le a^2 + \bar{M}^2 P(|\xi| > \bar{m}^{-1}(a)) \\ &\le a^2 + \bar{M}^2 \frac{E|\xi|^2}{(\bar{m}^{-1}(a))^2}, \end{aligned}$$

we obtain the right inequality in (2.13). This completes the proof. □

Hence, when $A, A_n, n = 1, 2, \ldots$ have the same reference probability system, we obtain

**Corollary 2.1.** *The response depends on its control process continuously in* $L^2([0,T] \times \Omega)$. *Further, if*

$$\lim_{n \to \infty} E\mathbf{1}\gamma_n(\cdot) - \gamma(\cdot)\mathbf{1}^2 = 0,$$

*then we can choose a subsequence $n'$ such that, for any $(\theta, x) \in [0,T] \times \mathbb{R}^d$,*

$$\lim_{n' \to \infty} \sup_{\theta \le s \le T} |X_{\theta x}^{A_{n'}}(s) - X_{\theta x}^{A}(s)| = 0 \quad P\text{-a.s.}$$

*and*

$$\lim_{n' \to \infty} E_{\theta x}\Big[\sup_{\theta \le s \le T} |X^{A_{n'}}(s) - X^{A}(s)|^2\Big] = 0.$$

Next we discuss continuity properties of the payoff and the value function, replying on Propositions 2.1 and 2.2.

**Theorem 2.1.** *Suppose $(b_1)$–$(b_5)$ hold. Then $J(t, \theta, x, A; \phi)$ and $v(t, \theta, x; \phi)$ have the following properties:*

(i) *There is a constant $\bar{K}$ such that, for any $0 \le \theta < t \le T$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} &|J(t, \theta, x, A; \phi)| \le \bar{K}(1 + |x|^2), \quad \forall A \in \mathbb{A}, \\ &|v(t, \theta, x; \phi)| \le \bar{K}(1 + |x|^2). \end{aligned} \tag{2.15}$$

(ii) ***Continuous dependence on initial conditions***
*Let $R$ and $\varepsilon(>0)$ be given. Then there is a constant $\delta_{\varepsilon R} > 0$ such that, for any $x_1, x_2 \in S_R$ and $0 \le \theta_1 < \theta_2 < t \le T$,*

$$|J(t, \theta_1, x_1, A; \phi) - J(t, \theta_2, x_2, A; \phi)| < \varepsilon, \quad \forall A \in \mathbb{A}, \tag{2.16}$$

*whenever $|x_1 - x_2| + |\theta_1 - \theta_2| < \delta_{\varepsilon R}$, and*

$$|v(t, \theta_1, x_1; \phi) - v(t, \theta_2, x_2; \phi)| < \varepsilon$$

*whenever $|x_1 - x_2| + |t_1 - t_2| < \delta_{\varepsilon R}$.*

(iii) ***Continuous dependence on the terminal time***
*Let $R$ and $\varepsilon(>0)$ be given. Then there is a constant $\Delta_{\varepsilon R} > 0$ such that, for any $x \in S_R$ and $0 \le \theta \le t_1 < t_2 \le T$,*

$$|J(t_1, \theta, x, A; \phi) - J(t_2, \theta, x, A; \phi)| < \varepsilon, \quad \forall A \in \mathbb{A} \tag{2.17}$$

*and*

$$|v(t_1, \theta, x; \phi) - v(t_2, \theta, x; \phi)| < \varepsilon,$$

*whenever $|t_1 - t_2| < \Delta_{\varepsilon R}$.*

*Proof.* We assume $\kappa = 0$, because the proof in the case of general $\kappa$ is similar.

(i) is immediate from $(b_4)$ and (2.9).

(ii) Fix $\theta$ and $A$ and put $X(t, x) = X^A_{\theta x}(t)$ for simplicity. For given $\varepsilon > 0$ and $R > 0$, set

$$\gamma_{\varepsilon R} = (K_1(1 + (R+1)^2)\varepsilon^{-1})^{\frac{1}{2}},$$

with $K_1$ of (2.9). Then, for any $x \in S_{R+1}$,

$$P\Big(\sup_{\theta \le t \le T} |X(t, x)| > \gamma_{\varepsilon R}\Big) \le E\Big[\sup_{\theta \le t \le T} |X(t, x)|^2\Big]\gamma_{\varepsilon R}^{-2} < \varepsilon. \tag{2.18}$$

Next choose a small $\delta_0 = \delta_0(\varepsilon, R) > 0$ such that, for any $x, y \in S_R$,

$$|\phi(x) - \phi(y)| + \sup_{\gamma, t} |f(t, x, \gamma) - f(t, y, \gamma)| < \varepsilon, \qquad \text{whenever } |x - y| < \delta_0, \tag{2.19}$$

which is possible thanks to $(b_2)$ and $(b_3)$.

Let us fix $y \in S_R$ arbitrarily. Then

$$\begin{aligned} &P\Big(\sup_{\theta \le t \le T} |X(t,x) - X(t,y)| > \delta_0\Big) \\ &< E\Big[\sup_{\theta \le t \le T} |X(t,x) - X(t,y)|^2\Big]\delta_0^{-2} < K_1|x-y|^2\delta_0^{-2} \end{aligned} \tag{2.20}$$

follows from (2.11).

Now using (2.18) and (2.20), we introduce two sets $\Omega_x$ and $\tilde{\Omega}_x$ by

$$\Omega_x = \Big\{\omega \in \Omega;\ \sup_{\theta \le t \le T} |X(t,x;\omega)| \le \gamma_{\varepsilon R}\Big\} \tag{2.21}$$

for $x$ with $|x-y| \le 1$, and

$$\tilde{\Omega}_x = \Big\{\omega \in \Omega;\ \sup_{\theta \le t \le T} |X(t,x;\omega) - X(t,y;\omega)| < \delta_0\Big\}. \tag{2.22}$$

Then (2.18) and (2.20) imply

$$P(\Omega_x) > 1-\varepsilon, \qquad P(\Omega_y) > 1-\varepsilon, \tag{2.23}$$

and

$$P(\tilde{\Omega}_x) > 1-\varepsilon \tag{2.24}$$

whenever $|x-y|^2 < \varepsilon\delta_0^2 K_1^{-1}$.

Let us estimate the quantity

$$I_1(x,y) := E\Big[\int_\theta^t |f(s,X(s,x),\gamma(s)) - f(s,X(s,y),\gamma(s))|\,ds\Big].$$

We have

$$\begin{aligned} &I_1(x,y) \\ &\le E\Big[\int_\theta^t |f(s,X(s,x),\gamma(s)) - f(s,X(s,y),\gamma(s))|\,ds;\ \Omega_x \cap \Omega_y \cap \tilde{\Omega}_x\Big] \\ &+ \hat{K}E\Big[\int_\theta^t (2 + |X(s,x)|^2 + |X(s,y)|^2)\,ds;\ \Omega_x^c \cup \Omega_y^c \cup \tilde{\Omega}_x^c\Big] \end{aligned} \tag{2.25}$$

thanks to $(b_4)$. By (2.9) and (2.22)–(2.25), we can find a constant $c_1 > 0$, independent of $\theta, t, A, \varepsilon$, and $R$, such that

$$I_1(x,y) < c_1\sqrt{\varepsilon}(1+R^2) \tag{2.26}$$

whenever $x, y \in S_R$ and $|x-y| < \sqrt{\frac{\varepsilon}{K_1}}\delta_0$.

Indeed, (2.19) and (2.22) imply that

$$\text{1st term in the RHS of (2.25)} \; < \varepsilon T. \tag{2.27}$$

For the 2nd term, (2.9), (2.23), and (2.24) yield

$$\begin{aligned}
&E\Big[\int_\theta^t |X(s,x)|^2\,ds;\ \Omega_x^{c} \cup \Omega_y^{c} \cup \tilde{\Omega}_x^{c}\Big] \\
&\le \Big(E(\int_\theta^t |X(s,x)|^2\,ds)^2\Big)^{\frac{1}{2}}\sqrt{3\varepsilon} \\
&\le TK_2(1+|x|^2)\sqrt{3\varepsilon} \le TK_2(1+(R+1)^2)\sqrt{3\varepsilon}
\end{aligned} \tag{2.28}$$

whenever $x, y \in S_R$ and $|x-y| < \sqrt{\frac{\varepsilon}{K_1}}\delta_0$.

Since for $X(s,y)$ one has the same estimate as (2.28), (2.27) and (2.28) together with (2.25) establish (2.26).

Applying the same arguments to the terminal cost and using (2.26), we can find $\delta_1 := \delta_1(\varepsilon, R) > 0$, such that

$$\begin{aligned}
&|J(t,\theta,x,A;\phi) - J(t,\theta,y,A;\phi)| < \varepsilon, \\
&\qquad\qquad \forall A \in \mathbb{A}, \quad 0 \le \theta < t \le T,
\end{aligned} \tag{2.29}$$

whenever $x, y \in S_R$ and $|x-y| < \delta_1$.

Finally, we consider the initial time. Let $\theta_1 < \theta_2 < t$ and $x \in S_R$. Set $X(s,\theta) = X_{\theta x}^A(s)$, and with the same $\gamma_{\varepsilon R}$ and $\delta_0$, define $\Omega_i$ and $\tilde{\Omega}$ by

$$\Omega_i = \Big\{\omega \in \Omega;\ \sup_{\theta_i \le s \le T} |X(s,\theta_i)| < \gamma_{\varepsilon R}\Big\}, \quad i = 1, 2,$$

and

$$\tilde{\Omega} = \Big\{\omega \in \Omega;\ \sup_{\theta_2 \le s \le T} |X(s,\theta_1) - X(s,\theta_2)| < \delta_0\Big\}.$$

By arguing as in the previous estimations, we can take $\delta_2 = \delta_2(\varepsilon, R) > 0$, so that

$$\begin{aligned}
&|J(t,\theta_1,x,A;\phi) - J(t,\theta_2,x,A;\phi)| < \varepsilon, \\
&\qquad\qquad \forall t \ge \theta_2, \quad \forall x \in S_R, \quad \forall A \in \mathbb{A},
\end{aligned} \tag{2.30}$$

whenever $\theta_2 - \theta_1 < \delta_2$.

Now (ii) the proof of follows from (2.29) and (2.30).

Since (iii) the proof of mimics of (ii), the result is established. □

Theorem 2.1 asserts that, if the terminal cost function $\phi$ has quadratic growth, then the value function also has quadratic growth. Put

$$\tilde{C} = \left\{ \psi \in C(\mathbb{R}^d);\ \tilde{\psi}(x) := \frac{\psi(x)}{1+|x|^2} \in C_b(\mathbb{R}^d) \right\}. \tag{2.31}$$

We introduce the norm $\|\psi\|_{\tilde{C}}$ by

$$\|\psi\|_{\tilde{C}} = \|\tilde{\psi}\|_{C(\mathbb{R}^d)} \tag{2.32}$$

and the order $\leq$ by

$$\varphi \leq \psi \Longleftrightarrow \varphi(x) \leq \psi(x), \quad \forall x \in \mathbb{R}^d. \tag{2.33}$$

Then $\tilde{C}$ becomes a Banach lattice.

Recalling $(b_3)$ and $(b_4)$, we define a two-parameter operator $V_{\theta t}; \tilde{C} \mapsto \tilde{C}$ $(0 \leq \theta \leq t \leq T)$ by

$$V_{\theta t}\phi(x) = v(t, \theta, x; \phi). \tag{2.34}$$

**Proposition 2.3.** *$V_{\theta t}$ has the following properties:*

(i) *$V_{\theta\theta}$ is the identity map on $\tilde{C}$.*
(ii) *Monotonicity property: $\phi \leq \psi \Rightarrow V_{\theta t}\phi \leq V_{\theta t}\psi$.*
(iii) *There is a constant $\tilde{K} > 0$, such that*

$$\|V_{\theta t}\phi - V_{\theta t}\psi\|_{\tilde{C}} \leq \tilde{K}\|\phi - \psi\|_{\tilde{C}}, \quad \forall \theta \leq t, \quad \forall \phi, \psi \in \tilde{C}, \tag{2.35}$$

*and*

$$\|V_{\theta t}0\|_{\tilde{C}} \leq \tilde{K}, \quad \forall \theta \leq t. \tag{2.36}$$

*Proof.* (i) and (ii) are clear from the definition of payoff.

(iii) From the inequality $|\phi(x) - \psi(x)| \leq \|\phi - \psi\|_{\tilde{C}}(1 + |x|^2)$ it follows that

$$|J(t, \theta, x, A; \phi) - J(t, \theta, x, A; \psi)| \leq \|\phi - \psi\|_{\tilde{C}} E_{\theta x}\Big(1 + \sup_{\theta \leq t \leq T} |X(t)|^2\Big).$$

Now (2.9) yields (iii). $\square$

## 2.2 Dynamic Programming Principle (DPP)

The dynamic programming principle (DPP), introduced by R. Bellman [Be52, Be57], gives a powerful tool for stochastic control problems and is known as a two-stages optimization method. Here we will formulate DPP as a two parameter semigroup on a suitable Banach space, constructed by using time discretization in Sects. 2.1.1–2.2.3 [N76, N81]. We characterize the semigroup as the envelope of Markovian transition semigroups and show that its generator is related to HJB equation in Sect. 2.2.5. We have two kinds of control processes on a fixed reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$: one is an $(\mathcal{F}_t)$-progressively measurable control process and the other is an $(\mathcal{F}_t^W)$-progressively measurable one. But we note that these two kind controls give the same value function (see Sect. 2.2.4).

In this section, we always assume the conditions $(b_1)$–$(b_5)$.

### 2.2.1 Discrete-Time Dynamic Programming Principle

Let $\mathcal{D}, 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$ be a division of $[0, T]$. We put $\mathcal{D} = (t_1, \ldots, t_p)$, $\mathscr{P}_{\mathcal{D}}$ = the set of division points of $\mathcal{D}(= \{t_1, \ldots, t_p\})$ and $|\mathcal{D}| = \max_{i=0,\ldots,p} |t_i - t_{i+1}|$. For divisions $\mathcal{D}$ and $\tilde{\mathcal{D}}$, we say that $\mathcal{D} \subset \tilde{\mathcal{D}}$, if $\mathscr{P}_{\mathcal{D}} \subset \mathscr{P}_{\tilde{\mathcal{D}}}$.

Let $\mathbb{A}^{\mathcal{D}}$ denote the set of all admissible controls $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$ with

$$\gamma(t, \omega) = \gamma(t_i, \omega) \quad \text{for } t \in [t_i, t_{i+1}), \quad i = 0, \ldots, p. \tag{2.37}$$

$\gamma(\cdot)$ is called a switching control at $\mathcal{D}$, or a $\mathcal{D}$-admissible control. When we restrict $\mathbb{A}$ to $\mathbb{A}^{\mathcal{D}}$, the value function $v^{\mathcal{D}}$ is given by

$$v^{\mathcal{D}}(t, \theta, x; \phi) = \inf_{A \in \mathbb{A}^{\mathcal{D}}} J(t, \theta, x, A; \phi). \tag{2.38}$$

Since $v^{\mathcal{D}}$ clearly satisfies (i)–(iii) in Theorem 2.1, $V_{\theta t}^{\mathcal{D}}$ defined by

$$V_{\theta t}^{\mathcal{D}} \phi(x) = v^{\mathcal{D}}(t, \theta, x; \phi) \tag{2.39}$$

is a mapping from $\tilde{C}$ into $\tilde{C}$.

We will show the following subsidiary theorem;

**Theorem 2.2 (Discrete-time *DPP*).** *Let $\Gamma$ be convex and compact. Then for $l < m < n$,*

$$V_{t_l t_m}^{\mathcal{D}}(V_{t_m t_n}^{\mathcal{D}} \phi) = V_{t_l t_n}^{\mathcal{D}} \phi, \tag{2.40}$$

*that is,*

$$\inf_{A \in \mathbb{A}^{\mathcal{D}}} J(t_m, t_l, x, A; v^{\mathcal{D}}(t_n, t_m, \cdot\,; \phi)) = v^{\mathcal{D}}(t_n, t_l, x; \phi). \tag{2.41}$$

Before we embark upon the proof, let us consider constant controls. Set $A_\gamma = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma)$, where $\gamma \in \Gamma$ stands for a constant control process: $\gamma(t, \omega) = \gamma$, for all $t$ and $\omega$. Then its response $X_{\theta y}^{A_\gamma}$ is the strong solution of (2.3), measurable w.r.t. $\sigma(W(\cdot) - W(\theta), y, \gamma)$, and the payoff $J(t, \theta, y, A_\gamma; \phi)$ depends only on $t, \theta, y, \gamma$ and $\phi$. Denote the payoff by $J(t, \theta, y, \gamma; \phi)$. We will seek an optimal $\gamma(\in \Gamma)$ by using the measurable selection theorem (see [SV79], Lemma 12.1.7 and Theorem 12.1.10).

Since $J(t, \theta, y, \gamma; \phi)$ is continuous in $(y, \gamma)$ by (2.11) and (2.13), the compactness of $\Gamma$ implies that, for $y \in \mathbb{R}^d$, the set

$$\Gamma_y = \{\gamma \in \Gamma; J(t, \theta, y, \gamma; \phi) = \inf_{\gamma \in \Gamma} J(t, \theta, y, \gamma; \phi)\} \tag{2.42}$$

is non-empty and compact. Hence there is a minimum selector (Borel function) $\gamma_{\theta t}(\cdot; \phi); \mathbb{R}^d \mapsto \Gamma$, such that

$$\gamma_{\theta t}(y; \phi) \in \Gamma_y, \quad \forall y \in \mathbb{R}^d \tag{2.43}$$

(see [FR75], Lemma B in Appendix).

Consequently

$$J(t, \theta, y, \gamma_{\theta t}(y; \phi); \phi) = \inf_{\gamma \in \Gamma} J(t, \theta, y, \gamma; \phi) =: \nu_{\theta t}\phi(y). \tag{2.44}$$

Further, by (2.9) and $(b_4)$, there is a constant $c > 0$, such that

$$\begin{aligned} &|f(s, x, \gamma)| + |\nu_{\theta t}\phi(x)| \leq c(1 + |x|^2), \\ &\qquad\qquad \forall x \in \mathbb{R}^d, \quad \forall \theta \leq s \leq t, \quad \forall \gamma \in \Gamma. \end{aligned} \tag{2.45}$$

Let us prove Theorem 2.2 by using the minimum selector.

*Proof.* For the proof, we assume $\kappa = 0$, because, for the general $\kappa$, the proof is entirely similar. Putting

$$\Big(\prod_{k=i}^{j} \nu_{t_k t_{k+1}}\Big)\phi = \nu_{t_i t_{i+1}} \nu_{t_{i+1} t_{i+2}} \cdots \nu_{t_j t_{j+1}}\phi, \tag{2.46}$$

we will show the inequality,

$$V^{\mathcal{D}}_{t_i t_{j+1}}\phi(x) \geq \Big(\prod_{k=i}^{j} \nu_{t_k t_{k+1}}\Big)\phi(x) \tag{2.47}$$

and its opposite,

$$V^{\mathcal{D}}_{t_i t_{j+1}}\phi(x) \leq \Big(\prod_{k=i}^{j} \nu_{t_k t_{k+1}}\Big)\phi(x). \tag{2.48}$$

Put $X^A(\cdot) = X^A_{t_i x}(\cdot)$. Since $X^A(t_j)$ and $\gamma(t_j)$ are $\mathcal{F}_{t_j}$-measurable and $W(\cdot + t_j) - W(t_j)$ is independent of $\mathcal{F}_{t_j}$,

$$\begin{aligned} E(C(t_{j+1}, t_j, A; \phi)|\mathcal{F}_{t_j}) &= J(t_{j+1}, t_j, X^A(t_j), \gamma(t_j); \phi) \\ &\geq \nu_{t_j t_{j+1}} \phi(X^A(t_j)) \quad P\text{-a.s.} \end{aligned} \tag{2.49}$$

holds for any $A \in \mathbb{A}^{\mathcal{D}}$. Hence, we have

$$J(t_{j+1}, t_i, x, A; \phi) \geq J(t_j, t_i, x, A; \nu_{t_j t_{j+1}} \phi) \geq \Big(\prod_{k=i}^{j} \nu_{t_k t_{k+1}}\Big) \phi(x). \tag{2.50}$$

Taking the infimum of LHS over $A \in \mathbb{A}^{\mathcal{D}}$, we obtain (2.47).

For (2.48), we construct a control process $\gamma^*(\cdot)$ for which the RHS of (2.47) is attained. This is done by using the minimum selector $\gamma_{\theta t}(\cdot)$ of (2.43). On a reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$, we first define $\gamma^*(t_i)$ by

$$\gamma^*(t_i) = \gamma_{t_i t_{i+1}}\Big(x; \Big(\prod_{k=i+1}^{j} \nu_{t_k t_{k+1}}\Big) \phi\Big), \tag{2.51}$$

and, consider the SDE

$$\begin{cases} dX(t) = b(t, X(t), \gamma^*(t_i))\, dt + \alpha(t, X(t), \gamma^*(t_i))\, dW(t), \quad t \in (t_i, t_{i+1}], \\ X(t_i) = x. \end{cases} \tag{2.52}$$

Then we define $\gamma^*(t_{i+1})$ by

$$\gamma^*(t_{i+1}) = \gamma_{t_{i+1} t_{i+2}}\Big(X(t_{i+1}); \Big(\prod_{k=i+2}^{j} \nu_{t_k t_{k+1}}\Big) \phi\Big). \tag{2.53}$$

Repeating this procedure, we obtain an $(\mathcal{F}_t^W)$-progressively measurable $\gamma^*(\cdot)$, such that $A^* := (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma^*(\cdot))$ is in $\mathbb{A}^{\mathcal{D}}$ and

$$J(t_{j+1}, t_i, x, A^*; \phi) = \Big(\prod_{k=i}^{j} \nu_{t_k t_{k+1}}\Big) \phi(x), \tag{2.54}$$

which yields (2.48).

Now from (2.47) and (2.48) it follows that

$$\begin{aligned} V^{\mathcal{D}}_{t_l t_n}\phi(x) &= \Big(\prod_{k=l}^{m-1} \nu_{t_k t_{k-1}}\Big)(V^{\mathcal{D}}_{t_m t_n}\phi)(x) \\ &= V^{\mathcal{D}}_{t_l t_n}(V^{\mathcal{D}}_{t_m t_n}\phi)(x). \end{aligned} \tag{2.55}$$

This completes the proof. □

Put $\boldsymbol{\Gamma}^W = L^\infty([0,T]\times\Omega,(\mathcal{F}^W_t);\boldsymbol{\Gamma})$ and $\boldsymbol{\Gamma}^{W,\mathcal{D}} = \{\gamma(\cdot)\in\boldsymbol{\Gamma}^W$; switching at $\mathcal{D}\}$. Then referring to the proof above and to the minimum selector, we have

*Remark 2.1.* For $\theta, t \in \mathscr{P}_{\mathcal{D}} \cup \{0,T\}$ and $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W)$, there is $\gamma^*(\cdot)\in \boldsymbol{\Gamma}^{W,\mathcal{D}}$, such that

$$V^{\mathcal{D}}_{\theta t}\phi(x) = J(t,\theta,x,A^*;\phi),$$

where $A^* = (\Omega,\mathcal{F},(\mathcal{F}_t),P,W,\gamma^*(\cdot))$.

Next we study the discrete-time DP property.

**Proposition 2.4.** *Let $\mathcal{D} = (t_1,\dots,t_p)$ be given.*

(i) *Let $\gamma^*(\cdot)$ be an optimal control process in $\boldsymbol{\Gamma}^{W,\mathcal{D}}$, given by the minimum selector. Then, for any $(\mathcal{F}^W_t)$-stopping time $\tau$ taking values in $\mathscr{P}_{\mathcal{D}}\cup\{0,T\}$,*

$$\begin{aligned} V^{\mathcal{D}}_{0T}\phi(x) = &E_{0x}\Big[\int_0^\tau \exp\{-\int_0^s \kappa(\lambda,X^*(\lambda),\gamma^*(\lambda))\,d\lambda\} f(s,X^*(s),\gamma^*(s))\,ds \\ &+ \exp\{-\int_0^\tau \kappa(\lambda,X^*(\lambda),\gamma^*(\lambda))\,d\lambda\} V^{\mathcal{D}}_{\tau T}\phi(X^*(\tau))\Big], \end{aligned} \tag{2.56}$$

*where $X^*$ is the response for $\gamma^*(\cdot)$, with $X^*(0)=x$,*

(ii) *Let $\gamma(\cdot)\in\boldsymbol{\Gamma}^{W,\mathcal{D}}$ and $X$ its response with $X(0)=x$. Then, for any $(\mathcal{F}^W_t)$-stopping time $\tau$ taking values in $\mathscr{P}_{\mathcal{D}}\cup\{0,T\}$,*

$$\begin{aligned} V^{\mathcal{D}}_{0T}\phi(x) \le &E_{0x}\Big[\int_0^\tau \exp\{-\int_0^s \kappa(\lambda,X(\lambda),\gamma(\lambda))\,d\lambda\} f(s,X(s),\gamma(s))\,ds \\ &+ \exp\{-\int_0^\tau \kappa(\lambda,X(\lambda),\gamma(\lambda))\,d\lambda\} V^{\mathcal{D}}_{\tau T}\phi(X(\tau))\Big]. \end{aligned} \tag{2.57}$$

*Proof.* $P(\tau=0)=1$ or 0, because $(\tau=0)\in\mathcal{F}^W_0$. When $P(\tau=0)=1$, (2.56) and (2.57) are trivial. Hence we may assume in the proof that the set of values of $\tau$ is $\{t_1,\dots,t_p,T\}$.

We again assume $\kappa = 0$.

(i) We have

$$C(T,0,\gamma^*(\cdot);\phi)$$
$$=\sum_{k=1}^{p+1}\chi(\tau = t_k)\Big(\int_0^{t_k} f(s,X^*(s),\gamma^*(s))\,ds + C(T,t_k,\gamma^*(\cdot);\phi)\Big), \tag{2.58}$$

where $t_{p+1} = T$.

From the definition of $\gamma^*(\cdot)$, it follows that

$$E(C(T,t_p,\gamma^*(\cdot);\phi)|\mathcal{F}^W_{t_p}) = \nu_{t_pT}\phi(X^*(t_p)),$$
$$E(C(T,t_{p-1},\gamma^*(\cdot);\phi)|\mathcal{F}^W_{t_{p-1}})$$
$$= E\Big((\int_{t_{p-1}}^{t_p} f(s,X^*(s),\gamma^*(s))\,ds + E(C(T,t_p,\gamma^*(\cdot);\phi)|\mathcal{F}^W_{t_p}))|\mathcal{F}^W_{t_{p-1}}\Big)$$
$$= E(C(t_p,t_{p-1},\gamma^*(\cdot);\nu_{t_pT}\phi)|\mathcal{F}^W_{t_{p-1}})$$
$$= \nu_{t_{p-1},t_p}(\nu_{t_pT}\phi)(X^*(t_{p-1}))$$
$$= V^{\mathcal{D}}_{t_{p-1}T}\phi(X^*(t_{p-1}))$$

and

$$E(C(T,t_k,\gamma^*(\cdot);\phi)|\mathcal{F}^W_{t_k}) = V^{\mathcal{D}}_{t_kT}\phi(X^*(t_k)). \tag{2.59}$$

In view of (2.58) and (2.59), we obtain

$$V^{\mathcal{D}}_{0T}\phi(x) = E_{0x}C(T,0,\gamma^*(\cdot);\phi)$$
$$= \sum_{k=1}^{p+1} E_{0x}\Big[\chi(\tau{=}t_k)\Big\{\int_0^{t_k} f(s,X^*(s),\gamma^*(s))\,ds + E(C(T,t_k,\gamma^*(\cdot);\phi)|\mathcal{F}^W_{t_p})\Big\}\Big]$$
$$= E_{0x}\Big[\int_0^{\tau} f(s,X^*(s),\gamma^*(s))\,ds + V^{\mathcal{D}}_{\tau T}\phi(X^*(\tau))\Big].$$

(ii) Let us define $\tilde{\gamma}(t)$ by

$$\tilde{\gamma}(t) = \sum_{k=1}^{p+1}\chi(\tau = t_k)(\gamma(t)\chi(t < t_k) + \gamma^*(t)\chi(t \ge t_k)).$$

Then $\tilde{\gamma}(\cdot) \in \boldsymbol{\Gamma}^{W,\mathcal{D}}$. Further (2.59) yields

$$
\begin{aligned}
&V_{0T}^{\mathcal{D}}\phi(x) \le \; J(T,0,x,\tilde{\gamma}(\cdot);\phi) \\
&= \sum_{k=1}^{p+1} E_{0x}\Big[\chi(\tau = t_k)\Big(\int_0^{t_k} f(s,X(s),\gamma(s))\,ds + E(C(T,t_k,\gamma^*(\cdot);\phi)|\mathcal{F}_{t_k}^W)\Big)\Big] \\
&= \text{RHS of (2.57)},
\end{aligned}
$$

which concludes the proof of (ii). □

## 2.2.2 Approximation Theorem

Before we prove the dynamic programming principle, we establish the following approximation result.

**Theorem 2.3.** *Let $\Gamma$ be convex and compact and* $\mathcal{D}_n = (t_{n,1},\ldots,t_{n,j(n)})$, $n = 1,2,\ldots$.

*Suppose that*

$$
\mathcal{D}_n \subset \mathcal{D}_{n+1}, \quad n = 1,2,\ldots \quad \textit{and} \quad \lim_{n\to\infty} |\mathcal{D}_n| = 0. \tag{2.60}
$$

*Let* $A = (\Omega,\mathcal{F},(\mathcal{F}_t),P,W,\gamma(\cdot))$ $(\in \mathbb{A})$ *be given. Then there exists* $A_n = (\Omega,\mathcal{F},(\mathcal{F}_t),P,W,\gamma_n(\cdot))$ *of* $\mathbb{A}^{\mathcal{D}_n}$ *such that*

$$
\lim_{n\to\infty} E\mathbf{1}\gamma_n(\cdot) - \gamma(\cdot)\mathbf{1}^2 = 0 \tag{2.61}
$$

*and*

$$
\lim_{n\to\infty} |\gamma_n(s) - \gamma(s)| = 0 \quad a.e. \quad \text{on } [0,T]\times\Omega. \tag{2.62}
$$

*Proof.* Since $\Gamma$ is convex and compact, we can apply a routine.

First fix $\gamma_0 \in \Gamma$ arbitrarily and set

$$
\gamma(s) = \gamma_0 \quad \text{for } s < 0.
$$

With this convention, we define $\bar{\gamma}_l(\cdot)$ by

$$
\bar{\gamma}_l(t) = 2^l \int_{t-2^{-l}}^{t} \gamma(s)\,ds, \quad t \ge 0.
$$

Then $\bar{\gamma}_l(\cdot)$ is a continuous control process, satisfying

$$
\lim_{l\to\infty} E\mathbf{1}\bar{\gamma}_l(\cdot) - \gamma(\cdot)\mathbf{1}^2 = 0. \tag{2.63}
$$

Second, we define $\bar{\gamma}_{l,m}(\cdot)$ by

$$\bar{\gamma}_{l,m}(t) = \bar{\gamma}_l(t_{m,j}) \quad \text{for } t \in [t_{m,j}, t_{m,j+1}), \quad j = 0, 1 \dots, j(m) - 1. \tag{2.64}$$

Then $\bar{\gamma}_{l,m}(\cdot)$ provides a switching control at $\mathcal{D}_m$, and, as $m \to \infty$, $\bar{\gamma}_{l,m}(\cdot)$ approaches $\bar{\gamma}_l(\cdot)$ uniformly on $[0, T]$, $P$-a.s.

Third, we can choose $(l_n, m_n)$ so that $m_n \geq n$ and

$$E[\mathbf{1}\gamma_{l_n,m_n}(\cdot) - \gamma(\cdot)\mathbf{1}^2] < 2^{-n}. \tag{2.65}$$

Finally we define $\gamma_p(\cdot)$ as follows:

$$\gamma_1(t) = \cdots = \gamma_{m_1-1}(t) = \gamma_0, \quad \forall t \in [0, T],$$
$$\gamma_{m_n}(t) = \cdots = \gamma_{m_{n+1}-1}(t) = \gamma_{l_n,m_n}(t), \quad \forall t \in [0, T], \quad n = 1, 2, \dots.$$

Then $\gamma_p(\cdot)$, $p = m_n, \dots, m_{n+1} - 1$ become switching controls at $\mathcal{D}_{m_n}$, and hence at $\mathcal{D}_p$, by (2.60). By (2.65), $\gamma_p(\cdot)$ satisfies (2.61).

Appealing to the Borel–Cantelli Lemma, we can choose a subsequence $\gamma_{p_j}(\cdot)$, so that (2.62) holds. This completes the proof. □

Regarding the responses $X^A$ and $X^{A_n}$, (2.13) and (2.61) lead to

$$\lim_{n\to\infty} E_{\theta x}\Big[\sup_{\theta \leq t \leq T} |X^{A_n}(t) - X^A(t)|^2\Big] = 0, \quad \forall(\theta, x). \tag{2.66}$$

Set

$$\mathbb{A}_s = \bigcup_{\mathcal{D}} \mathbb{A}^{\mathcal{D}} = \text{set of all switching controls}.$$

Using (2.66), we get

**Corollary 2.2.** *Let $\Gamma$ is convex and compact. Then*

$$\begin{aligned} V_{\theta t}\phi(x) &= \inf_{A\in\mathbb{A}_s} J(t, \theta, x, A; \phi) \\ &= \lim_{n\to\infty} \inf_{A\in\mathbb{A}^{\mathcal{D}_n}} J(t, \theta, x, A; \phi), \quad \forall \phi \in \tilde{C} \end{aligned} \tag{2.67}$$

*whenever $\mathcal{D}_n, n = 1, 2, \dots$ satisfy* (2.60).

### *2.2.3 Dynamic Programming Principle*

Now we are ready to prove DPP

**Theorem 2.4.** *Let $\Gamma$ be convex and $\sigma$-compact. We assume* $(b_1)$–$(b_5)$ *in Sect.* 2.1.1. *Then, for any $\theta < \theta_1 < \theta_2 \leq T$,*

$$V_{\theta\theta_2}\phi = V_{\theta\theta_1}(V_{\theta_1\theta_2}\phi) \quad \textit{for } \phi \in \tilde{C}. \tag{2.68}$$

*Proof.* We divide the proof into two steps.

**Step 1.** Let $\Gamma$ be convex and compact. Take a sequence $\mathcal{D}_n, n = 1, 2, \ldots$ such that $\theta, \theta_1, \theta_2 \in \mathscr{P}_{\mathcal{D}_1} \cup \{0, T\}$ and (2.60) holds.
Then Theorem 2.2 shows that

$$V_{\theta\theta_2}^{\mathcal{D}_n}\phi = V_{\theta\theta_1}^{\mathcal{D}_n} V_{\theta_1\theta_2}^{\mathcal{D}_n}\phi. \tag{2.69}$$

Since $V_{\theta_1\theta_2}^{\mathcal{D}_n}\phi \geq V_{\theta_1\theta_2}\phi$, the monotonicity property of $V_{\theta\theta_1}^{\mathcal{D}_n}$ leads to

$$V_{\theta\theta_1}^{\mathcal{D}_n}(V_{\theta_1\theta_2}^{\mathcal{D}_n}\phi) \geq V_{\theta\theta_1}^{\mathcal{D}_n}(V_{\theta_1\theta_2}\phi). \tag{2.70}$$

Hence from (2.69) and (2.70), it follows that

$$V_{\theta\theta_2}^{\mathcal{D}_n}\phi \geq V_{\theta\theta_1}^{\mathcal{D}_n}(V_{\theta_1\theta_2}\phi) \tag{2.71}$$

Letting $n \to \infty$, and using Corollary 2.2 we get

$$V_{\theta\theta_2}\phi \geq V_{\theta\theta_1}(V_{\theta_1\theta_2}\phi) \tag{2.72}$$

Next we shall show the converse of inequality (2.72). Since $V_{\theta\theta_2}^{\mathcal{D}_{n+m}}\phi(x) = V_{\theta\theta_1}^{\mathcal{D}_{n+m}}(V_{\theta_1\theta_2}^{\mathcal{D}_{n+m}}\phi) \leq V_{\theta\theta_1}^{\mathcal{D}_{n+m}}(V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi)(x)$, we obtain, letting $n \to \infty$,

$$\begin{aligned} V_{\theta\theta_2}\phi(x) &\leq V_{\theta\theta_1}(V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi)(x) \quad (m = 1, 2, \ldots) \\ &\leq J(\theta_1, \theta, x, A; V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi), \quad \forall A \in \mathbb{A}. \end{aligned} \tag{2.73}$$

Using (2.15) and (2.9), we have

$$\begin{aligned} E_{\theta x}|V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi(X^A(\theta_1))|^2 &\leq c_1 E_{\theta x}(1 + |X^A(\theta_1)|^2)^2 \\ &\leq c_2(1 + |x|^4), \end{aligned} \tag{2.74}$$

with constants $c_1$ and $c_2$ independent of $A$ and $\mathcal{D}_m$.
Since Corollary 2.2 implies

$$\lim_{m\to\infty} V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi(X(\theta_1)) = V_{\theta_1\theta_2}\phi(X(\theta_1)) \quad P\text{-a.s.}, \tag{2.75}$$

(2.74) and the convergence theorem lead to

$$\lim_{m\to\infty} J(\theta_1, \theta, x, A; V_{\theta_1\theta_2}^{\mathcal{D}_m}\phi) = J(\theta_1, \theta, x, A; V_{\theta_1\theta_2}\phi). \tag{2.76}$$

Therefore the inequality

$$V_{\theta\theta_2}\phi(x) \le J(\theta_1, \theta, x, A; V_{\theta_1\theta_2}\phi), \quad \forall A \in \mathbb{A} \tag{2.77}$$

follows from (2.73) and (2.76). Taking the infimum of the RHS of (2.77) over $A \in \mathbb{A}$, we obtain the converse of inequality (2.72).

This completes the proof of (2.68) when $\Gamma$ is convex and compact.

**Step 2.** Let $\Gamma$ be convex and $\sigma$-compact.

We can take a sequence of convex and compact sets $\Gamma_n, n = 1, 2, \cdots$ such that

$$\Gamma_n \subset \Gamma_{n+1}, \quad n = 1, 2, \ldots \quad \text{and} \quad \bigcup_n \Gamma_n = \Gamma. \tag{2.78}$$

We denote by $\mathbb{A}^n$ the set of all admissible controls with control region $\Gamma_n$. Then

$$\mathbb{A}^n \subset \mathbb{A}^{n+1} \quad \text{and} \quad \bigcup_n \mathbb{A}^n = \mathbb{A}, \tag{2.79}$$

by Definition 2.1. Observing that (2.79) implies

$$J(t, \theta, x, A; \phi) \ge \lim_{n\to\infty} V^n_{\theta t}\phi(x), \quad \forall A \in \mathbb{A}, \tag{2.80}$$

where

$$V^n_{\theta t}\phi(x) = \inf_{A \in \mathbb{A}^n} J(t, \theta, x, A; \phi), \tag{2.81}$$

and taking the infimum of (2.80) w.r.t. $A$ over $\mathbb{A}$, we have

$$V_{\theta t}\phi(x) = \lim_{n\to\infty} V^n_{\theta t}\phi(x), \quad \forall \theta \in t, \quad x \in \mathbb{R}^d, \tag{2.82}$$

because $V_{\theta t}\phi(x) \le V^n_{\theta t}\phi(x)$, $\forall n$.

Since $V^n_{\theta t}$ satisfies DPP,

$$\begin{aligned} V_{\theta\theta_2}\phi(x) &= \lim_{n\to\infty} V^n_{\theta\theta_2}\phi(x) \\ &= \lim_{n\to\infty} V^n_{\theta\theta_1}(V^n_{\theta_1\theta_2}\phi)(x) \\ &\ge \lim_{n\to\infty} V^n_{\theta\theta_1}(V_{\theta_1\theta_2}\phi)(x) \quad (\text{by } V^n_{\theta_1\theta_2}\phi \ge V_{\theta_1\theta_2}\phi) \\ &= V_{\theta\theta_1}(V_{\theta_1\theta_2}\phi)(x). \end{aligned} \tag{2.83}$$

For the converse inequality, we have

$$\begin{aligned} V^{n+m}_{\theta\theta_2}\phi(x) &= V^{n+m}_{\theta\theta_1}(V^{n+m}_{\theta_1\theta_2}\phi)(x) \\ &\le V^{n+m}_{\theta\theta_1}(V^m_{\theta_1\theta_2}\phi)(x), \quad m, n = 1, 2, \ldots. \end{aligned} \tag{2.84}$$

Letting $n \to \infty$, we get

$$\begin{aligned} V_{\theta\theta_2}\phi(x) &\le V_{\theta\theta_1}(V^m_{\theta_1\theta_2}\phi)(x), \quad m = 1, 2, \ldots \\ &\le J(\theta_1, \theta, x, A; V^m_{\theta_1\theta_2}\phi), \quad \forall A \in \mathbb{A}, \quad m = 1, 2, \ldots. \end{aligned} \tag{2.85}$$

Again (2.74) and the convergence theorem yield

$$\lim_{m\to\infty} J(\theta_1, \theta, x, A; V^m_{\theta_1\theta_2}\phi) = J(\theta_1, \theta, x, A; V_{\theta_1\theta_2}\phi). \tag{2.86}$$

Thus, the converse of inequality (2.83) follows from (2.85) and (2.86).

This completes the proof of Theorem 2.4. □

### 2.2.4 Brownian Adapted Controls

In this section, we will comment on the value function in the case where one restricts the control processes to the class of Brownian adapted ones.

We denote the set of all admissible controls, $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$ where $\gamma(\cdot)$ is $(\mathcal{F}^W_t)$-progressively measurable, by $\mathbb{A}^W$. Put

$$V^W_{\theta t}\phi(x) = \inf_{A\in\mathbb{A}^W} J(t, \theta, x, A; \phi). \tag{2.87}$$

Then we have:

**Proposition 2.5.**

$$V^W_{\theta t}\phi = V_{\theta t}\phi, \quad \forall \theta < t. \tag{2.88}$$

*Proof.* Since we can easily see that (2.82) is also valid for $V^W_{\theta t}$, we may assume for the proof that $\Gamma$ is convex and compact. Now the proposition follows from Remark 2.1 and Corollary 2.2. □

For $A \in \mathbb{A}^W$, $J(t, \theta, x, A; \phi)$ can be calculated by means of the joint probability distribution $(W, \gamma(\cdot))$. So, we fix a reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ and identify $A$ with its control process $\gamma(\cdot)$.

For a stopping time $\tau$, we again denote the payoff by $J(\cdot)$:

$$\begin{aligned} J(\tau, \theta, x, \gamma(\cdot); \psi) = &E_{\theta x}\Big[\int_\theta^\tau \exp\{-\int_\theta^s \kappa(\lambda, X(\lambda), \gamma(\lambda))\, d\lambda\} f(s, X(s), \gamma(s))\, ds \\ &+ \exp\{-\int_\theta^\tau \kappa(\lambda, X(\lambda), \gamma(\lambda))\, d\lambda\}\psi(X(\tau))\Big], \end{aligned} \tag{2.89}$$

where $X = X^{\gamma(\cdot)}$.

**Proposition 2.6 (DP Property).**
*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ and $(\theta, x) \in [0, T) \times \mathbb{R}^d$ be given.*

(i) *For a $[\theta, T]$-valued $(\mathcal{F}_t^W)$-stopping time $\tau$ and $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$,*

$$V_{\theta T}\phi(x) \leq J(\tau, \theta, x, \gamma(\cdot); V_{\tau T}\phi). \tag{2.90}$$

(ii) *For $\varepsilon > 0$, there is $\gamma^\varepsilon(\cdot) \in \boldsymbol{\Gamma}^W$, such that*

$$V_{\theta t}\phi(x) + \varepsilon \geq J(\tau, \theta, x, \gamma^\varepsilon(\cdot); V_{\tau T}\phi) \tag{2.91}$$

*for any $[\theta, T]$-valued $(\mathcal{F}_t^W)$-stopping time $\tau$.*

*Proof.* Although the DP property was given in [FS06], Theorem 7.1, we will sketch the proof by using Proposition 2.4. Assume that $\theta = 0, \kappa = 0$, and $\Gamma$ is convex and compact.

Put $\tau_n = 2^{-n}[1+2^n\tau] \wedge T$. For $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$, we are mainly interested in estimating $J(\tau_n, 0, x, \gamma(\cdot); V_{\tau_n T}\phi) - J(\tau, 0, x, \gamma(\cdot); V_{\tau T}\phi)$. We have

$$\begin{aligned}
I_n &:= E_{0x} \int_\tau^{\tau_n} |f(s, X(s), \gamma(s))|\, ds \\
&\leq \sqrt{T} 2^{-\frac{n}{2}} \Big( E_{0x} \int_0^t K(1 + |X(s)|^4)\, ds \Big)^{\frac{1}{2}} \quad (\text{use } (\mathrm{b}_4)) \\
&\leq 2^{-\frac{n}{2}} c_1 (1 + |x|^2)
\end{aligned} \tag{2.92}$$

with a constant $c_1$ independent of $n$ and $\gamma(\cdot)$.

Since $V_{tT}\phi(y)$ is continuous w.r.t. $(t, y)$,

$$|V_{\tau_n T}\phi(X(\tau_n))| \leq \bar{K}\Big(1 + \sup_{0 \leq t \leq T} |X(t)|^2\Big) \tag{2.93}$$

by (2.15), and the RHS of (2.93) is integrable, the dominated convergence theorem yields

$$\lim_{n\to\infty} E_{0x}|V_{\tau_n T}\phi(X(\tau_n)) - V_{\tau T}\phi(X(\tau))| = 0. \tag{2.94}$$

Let us fix $\tau_n$ and take a switching controls, $\gamma_l(\cdot) \in \boldsymbol{\Gamma}^W, l = 1, 2, \ldots,$ approaching $\gamma(\cdot)$, so that

$$\lim_{l\to\infty} E_{0x}\Big[\sup_t |X^{\gamma_l(\cdot)}(t) - X(t)|^2\Big] = 0 \tag{2.95}$$

and

$$\lim_{l\to\infty} \sup_t |X^{\gamma_l(\cdot)}(t) - X(t)| = 0 \quad P\text{-a.s.} \tag{2.96}$$

We take a sequence of divisions $\mathcal{D}_l, l = 1, 2, \dots,$ such that (2.60) holds and $\mathcal{D}_l$ contains all values of $\tau_n$ and switching times of $\gamma_l(\cdot)$. Then Corollary 2.2 yields

$$\lim_{m\to\infty} V_{\tau_n T}^{\mathcal{D}_{l+m}} \phi(X^{\gamma_l(\cdot)}(\tau_n)) = V_{\tau_n T}\phi(X^{\gamma_l(\cdot)}(\tau_n)) \quad P\text{-a.s. and in } L^1(\Omega). \tag{2.97}$$

On the other hand, Proposition 2.4 (ii) implies

$$V_{0T}^{\mathcal{D}_{l+m}}\phi(x) \le J(\tau_n, 0, x, \gamma_l(\cdot); V_{\tau_n T}^{\mathcal{D}_{l+m}}\phi). \tag{2.98}$$

Letting $m \to \infty$ and using (2.97), we obtain

$$V_{0T}\phi(x) \le E_{0x}\Big[\int_0^{\tau_n} f(s, X^{\gamma_l(\cdot)}(s), \gamma_l(s))\, ds + V_{\tau_n T}\phi(X^{\gamma_l(\cdot)}(\tau_n))\Big]. \tag{2.99}$$

Now taking $l \to \infty$ and $n \to \infty$ one obtains (i).

(ii) Let $\varepsilon > 0$ be given. By Corollary 2.2 and Remark 2.1, we can choose a switching control $\gamma^\varepsilon(\cdot) \in \boldsymbol{\Gamma}^W$, such that

$$V_{0T}\phi(x) + \varepsilon > J(T, 0, x, \gamma^\varepsilon(\cdot); \phi). \tag{2.100}$$

Suppose that $\tau$ is a finitely-many valued $(\mathcal{F}^W)$-stopping time, say $(t_k, k = 1, \dots, p)$. Since $\gamma^\varepsilon(t)$ is a functional of $(W(\theta), \theta \le t_k)$ and $(W(s) - W(t_k), s \in [t_k, t])$, under $P(\cdot|\mathcal{F}_{t_k}^W)$, $\gamma^\varepsilon(\cdot)$ becomes a control process by freezing $(W(\theta), \theta \le t_k)$, and similarly for its response $X^\varepsilon(t)$. Hence

$$E\Big(\int_{t_k}^T f(s, X^\varepsilon(s), \gamma^\varepsilon(s))\, ds + \phi(X^\varepsilon(T))|\mathcal{F}_{t_k}^W\Big) \ge V_{t_k T}\phi(X^\varepsilon(t_k)) \quad P\text{-a.s.}. \tag{2.101}$$

Considering

$$\begin{aligned} J(T, 0, x, \gamma^\varepsilon(\cdot); \phi) = E_{0x}\sum_{k=1}^p \chi(\tau = t_k)\Big\{\int_0^{t_k} f(s, X^\varepsilon(s), \gamma^\varepsilon(s))\, ds \\ + \int_{t_k}^T f(s, X^\varepsilon(s), \gamma^\varepsilon(s))\, ds + \phi(X^\varepsilon(T))\Big\} \end{aligned} \tag{2.102}$$

and applying (2.101) to (2.102), we have

$$\begin{aligned} &J(T, 0, x, \gamma^\varepsilon(\cdot); \phi) \\ &\ge E_{0x}\sum_{k=1}^p \chi(\tau = t_k)\Big\{\int_0^{t_k} f(s, X^\varepsilon(s), \gamma^\varepsilon(s))\, ds + V_{t_k T}\phi(X^\varepsilon(T))\Big\} \\ &= J(\tau, 0, x, \gamma^\varepsilon(\cdot); V_{\tau T}\phi). \end{aligned} \tag{2.103}$$

Since any stopping time can be approached by finitely-many valued ones, (2.100) and (2.103) complete the proof. □

*Example 2.2 (*DPP *with hitting time).* Assume $f(\cdot) = 0$ and $\kappa(\cdot) = 0$. Let $\mathcal{O}$ be an open set. For $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$, $X^{\gamma(\cdot)}$ and $\tau^{\gamma(\cdot)}$ denote its response and the hitting time of $\mathcal{O}$ by $X^{\gamma(\cdot)}$ respectively. Then DP property yields that the value function $v(\theta, x) = V_{\theta T}\phi(x)$ satisfies

$$v(\theta, x) = \inf_{\gamma(\cdot)\in\boldsymbol{\Gamma}^W} E_{\theta x} v(\tau^{\gamma(\cdot)},\ X^{\gamma(\cdot)}(\tau^{\gamma(\cdot)})) \quad \text{for } \theta \in [0, T), \quad x \notin \mathcal{O}. \tag{2.104}$$

### 2.2.5 Characterization of the Semigroup $(V_{\theta t}, \theta \le t)$

Referring to Theorems 2.1 and 2.4 we summarize the basic properties of the semigroup $(V_{\theta t}, \theta \le t), \theta, t \in [0, T]$.

**Proposition 2.7.** *$V_{\theta t} : \tilde{C} \to \tilde{C}$ satisfies the followings:*

(i) $V_{\theta\theta} =$ *identity,* $\forall \theta \in [0, T]$,
(ii) ***Semigroup property***

$$V_{\theta t} = V_{\theta s} V_{st} \quad \text{for } \theta \le s \le t,$$

(iii) ***Monotonicity***

$$\phi \le \psi \Longrightarrow V_{\theta t}\phi \le V_{\theta t}\psi,$$

(iv) $\exists \tilde{k} > 0$ *such that* $\forall \phi, \psi \in \tilde{C}$ *and* $\theta \le t$,

$$\begin{cases} \|V_{\theta t}\phi - V_{\theta t}\psi\|_{\tilde{C}} \le \tilde{k}\|\phi - \psi\|_{\tilde{C}}, \\ \|V_{\theta t}0\|_{\tilde{C}} \le \tilde{k}, \end{cases} \tag{2.105}$$

(v) ***Continuity w.r.t. the time parameters***
*Let $R > 0$ be given. For any $\hat{t}$ and $\hat{\theta}$, we have*

$$\begin{cases} \lim_{t\to\hat{t}} \sup_{|x|\le R} |V_{\theta t}\phi(x) - V_{\theta\hat{t}}\phi(x)| = 0, \ \forall \theta < \hat{t}, \\ \lim_{\theta\to\hat{\theta}} \sup_{|x|\le R} |V_{\theta t}\phi(x) - V_{\hat{\theta} t}\phi(x)| = 0, \ \forall t > \hat{\theta}. \end{cases} \tag{2.106}$$

Next we will characterize $(V_{\theta t}, \theta \le t)$ from the point of view of semigroups.

For a constant control $\gamma(t) = \gamma, \forall t \in [0, T]$, its response is a diffusion governed by the SDE

$$dX(t) = b(t, X(t), \gamma)\, dt + \alpha(t, X(t), \gamma)\, dW(t).$$

Let us define $\mathrm{H}^{\gamma}_{\theta t} : \tilde{C} \mapsto \tilde{C}$ $(\theta \le t)$, by

$$\mathrm{H}^{\gamma}_{\theta t}\phi(x) = J(t, \theta, x, \gamma; \phi), \quad \forall x \in \mathbb{R}^d. \tag{2.107}$$

Then $(\mathrm{H}^{\gamma}_{\theta t}; \theta \le t)$ is a semigroup satisfying properties (i)–(v) of Proposition 2.7 and

$$V_{\theta t}\phi \le \mathrm{H}^{\gamma}_{\theta t}\phi, \quad \forall \phi \in \tilde{C}. \tag{2.108}$$

**Theorem 2.5.** *Let* $\mathrm{U}_{\theta t}; \tilde{C} \mapsto \tilde{C}, \theta \le t$*, be a semigroup with the monotonicity property. If*

$$\mathrm{U}_{\theta t}\phi \le \mathrm{H}^{\gamma}_{\theta t}\phi, \quad \forall \gamma \in \Gamma, \quad \forall \phi \in \tilde{C}, \quad \forall \theta \le t, \tag{2.109}$$

*then*

$$\mathrm{U}_{\theta t}\phi \le V_{\theta t}\phi, \quad \forall \phi \in \tilde{C}, \quad \forall \theta \le t. \tag{2.110}$$

*In other words,* $(V_{\theta t}, \theta \le t)$ *is the maximal element in the set of monotone semigroups satisfying* (2.109).

$(V_{\theta t}, \theta \le t)$ is called the envelope of $\{(\mathrm{H}^{\gamma}_{\theta t}, \theta \le t); \gamma \in \Gamma\}$.

The proof is easy, using (2.47), (2.48), and Corollary 2.2.

Finally, using the properties of the value function, we compute the generator of $(V_{\theta t}, \theta \le t)$. Put

$$\mathcal{D} = \{\psi \in \tilde{C} \cap C^2_p(\mathbb{R}^d); \partial_x\psi \text{ and } \partial_{xx}\psi \text{ satisfy the polynomial growth condition}\}.$$

Then

$$\mathrm{G}^{\gamma}_{\theta} = \frac{1}{2}\mathrm{tr}(a(\theta, x, \gamma)\partial_{xx}) + b(\theta, x, \gamma) \cdot \partial x$$

is the generator of $X^{\gamma}$, with domain $\mathcal{D}$, and

$$\mathrm{G}^{\gamma}_{\theta}\psi(x) - \kappa(\theta, x, \gamma)\psi(x) + f(\theta, x, \gamma)$$

is the generator of $(\mathrm{H}^{\gamma}_{\theta t}, \theta \le t)$. Put

$$\mathrm{G}_{\theta}\psi(x) := \inf_{\gamma \in \Gamma}(\mathrm{G}^{\gamma}_{\theta}\psi(x) - \kappa(\theta, x, \gamma)\psi(x) + f(\theta, x, \gamma)).$$

**Proposition 2.8.** *For any* $R_0 > 0$ *and* $\psi \in \mathcal{D}$*,*

$$\lim_{t \to \theta} \sup_{|x| \le R_0} \left| \frac{1}{t - \theta}(V_{\theta t}\psi(x) - \psi(x)) - \mathrm{G}_{\theta}\psi(x) \right| = 0. \tag{2.111}$$

Straightforward computations together with Proposition 2.1 give the following lemma.

**Lemma 2.1.** *Suppose that the Borel function $h$ defined on $[0, T] \times \mathbb{R}^d \times \Gamma$ is continuous w.r.t. $(t, x)$ uniformly on $\Gamma$ and satisfies*

$$|h(t, x, \gamma)| \leq K(1 + |x|^{2p}), \quad \forall t, x, \gamma, \tag{2.112}$$

*with constants $K > 0$ and $p \geq 1$. Then, for any positive $\varepsilon$ and $R_0$, there is $\delta_{\varepsilon R_0} > 0$ such that for any $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$,*

$$\sup_{|x| \leq R_0} E_{\theta x}\Big[\sup_{\theta \leq s \leq t} |h(s, X^{\gamma(\cdot)}(s), \gamma(s)) - h(\theta, x, \gamma(s))|\Big] < \varepsilon \tag{2.113}$$

*whenever $|t - \theta| < \delta_{\varepsilon R_0}$.*

*Proof (Proposition 2.8).* Put

$$\begin{aligned} h(t, x, \gamma) =& \frac{1}{2}\mathrm{tr}(a(t, x, \gamma)\partial_{xx}\psi(x)) + b(t, x, \gamma) \cdot \partial_x \psi(x) \\ &- \kappa(t, x, \gamma)\psi(x) + f(t, x, \gamma), \quad \psi \in \mathcal{D}. \end{aligned}$$

Since $h(\cdot)$ satisfies the conditions of Lemma 2.1,

$$\begin{aligned} J_1 :=& \Big| \inf_{\gamma(\cdot) \in \boldsymbol{\Gamma}^W} E_{\theta x}\Big[\int_\theta^t h(s, X^{\gamma(\cdot)}(s), \gamma(s))\, ds\Big] \\ &- \inf_{\gamma(\cdot) \in \boldsymbol{\Gamma}^W} E_{\theta x}\Big[\int_\theta^t h(\theta, x, \gamma(s))\, ds\Big]\Big| \\ &< \varepsilon(t - \theta), \quad \forall |x| \leq R_0 \end{aligned} \tag{2.114}$$

whenever $|t - \theta| < \delta_{\varepsilon R_0}$.

On the other hand, (2.112) and ($b_5$) yield

$$\begin{aligned} J_2 &:= E_{\theta x}\Big[\int_\theta^t |e^{-\int_\theta^s \kappa(\lambda, X^{\gamma(\cdot)}(\lambda), \gamma(\lambda))\, d\lambda} - 1||h(s, X^{\gamma(\cdot)}(s), \gamma(s))|\, ds\Big] \\ &< c_3 E_{\theta x} \int_\theta^t (s - \theta)(1 + |X^{\gamma(\cdot)}(s)|^{2p})\, ds \\ &\leq \frac{c_3}{2}(t - \theta)^2 E_{\theta x}\Big[\sup_{\theta \leq s \leq t}(1 + |X^{\gamma(\cdot)}(s)|^{2p})\Big] \\ &\leq c_4(t - \theta)^2(1 + |x|^{2p}), \quad \forall \gamma(\cdot) \in \boldsymbol{\Gamma}^W, \end{aligned} \tag{2.115}$$

with constant $c_3$ and $c_4$ independent of $\gamma(\cdot), t, \theta$, and $x$.

Thus, Itô's formula together with (2.114) and (2.115) yield

$$\begin{aligned}&\Big|V_{\theta t}\psi(x)-\psi(x)-\inf_{\gamma(\cdot)\in\boldsymbol{\Gamma}^W}\Big[E_{\theta x}\int_\theta^t h(\theta,x,\gamma(s))\,ds\Big]\Big|\\&\le J_1+J_2<(t-\theta)(c_4(1+R_0^{2p})(t-\theta)+\varepsilon),\quad \forall|x|\le R_0,\end{aligned}\tag{2.116}$$

whenever $|t-\theta|<\delta_{\varepsilon R_0}$.

Using the inequalities

$$\begin{aligned}\inf_{\gamma(\cdot)\in\boldsymbol{\Gamma}^W}E_{\theta x}\int_\theta^t h(\theta,x,\gamma(s))\,ds&\ge(t-\theta)\inf_{\gamma\in\boldsymbol{\Gamma}}h(\theta,x,\gamma)\\&=\inf_{\gamma\in\Gamma}\int_\theta^t h(\theta,x,\gamma)\,ds\\&\ge\inf_{\gamma(\cdot)\in\boldsymbol{\Gamma}^W}E_{\theta x}\int_\theta^t h(\theta,x,\gamma(s))\,ds,\end{aligned}$$

we have

$$\inf_{\gamma(\cdot)\in\boldsymbol{\Gamma}^W}E_{\theta x}\int_\theta^t h(\theta,x,\gamma(s))\,ds=(t-\theta)\inf_{\gamma\in\Gamma}h(\theta,x,\gamma).\tag{2.117}$$

Inserting (2.117) into (2.114) and referring to Proposition 2.5, we complete the proof. □

Proposition 2.8 asserts that if the value function is smooth, then it satisfies the Cauchy problem for the (nonlinear) parabolic equation;

$$-\frac{\partial V}{\partial t}(t,x)+\mathrm{H}(t,x,\partial_{xx}V(t,x),\partial_x V(t,x),V(t,x))=0,\quad\forall(t,x)\in[0,T)\times\mathbb{R}^d\tag{2.118}$$

with the lateral boundary condition

$$V(T,x)=\phi(x),\quad x\in\mathbb{R}^d,\tag{2.119}$$

where $\mathrm{H}:[0,T]\times\mathbb{R}^d\times S^d\times\mathbb{R}^d\times\mathbb{R}^1\mapsto\mathbb{R}^1$, is given by

$$\begin{aligned}&\mathrm{H}(t,x,A,P,v)\\&=\sup_{\gamma\in\Gamma}\Big(-\frac{1}{2}\mathrm{tr}(a(t,x,\gamma)A)-b(t,x,\gamma)\cdot p+\kappa(t,x,\gamma)v-f(t,x,\gamma)\Big).\end{aligned}\tag{2.120}$$

The parabolic equation (2.118) is called Hamilton–Jacobi–Bellman equation (HJB equation in short), or the dynamic programming equation.

Viscosity solutions of HJB equations will be considered in Chap. 3.

*Example 2.3 (Time-homogeneous case).* Suppose that all the coefficients $\alpha, b, \kappa$ and $f$ are independent of the time variable. Here we use Brownian adapted control processes. In this case, we have

$$\nu_{\theta t} = \nu_{0(t-\theta)}. \tag{2.121}$$

Indeed, for $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$, its response is the solution of the SDE

$$\begin{cases} dX(t) = b(X(t), \gamma(t))\,dt + \alpha(X(t), \gamma(t))\,dW(t), \quad t > \theta, \\ X(\theta) = x. \end{cases} \tag{2.122}$$

If $\gamma(\cdot + \theta)$ is adapted to $W^\theta(\cdot) := W(\cdot + \theta) - W(\theta)$, then $\hat{\gamma}(t) := \gamma(t + \theta)$ can be regarded as an element of $\boldsymbol{\Gamma}^{W^\theta}$ and the solution $\hat{X}$ of the SDE

$$\begin{cases} d\hat{X}(t) = b(\hat{X}(t), \hat{\gamma}(t))\,dt + \alpha(\hat{X}(t), \hat{\gamma}(t))\,dW^\theta(t), \quad t > 0, \\ \hat{X}(0) = x, \end{cases} \tag{2.123}$$

is nothing but $X(\cdot + \theta)$. Consequently,

$$\begin{aligned} V_{\theta t}\phi(x) &\le \inf_{\gamma(\cdot) \in \boldsymbol{\Gamma}^{W^\theta}} J(t, \theta, x, \gamma(\cdot); \phi) \\ &= V_{0(t-\theta)}\phi(x). \end{aligned} \tag{2.124}$$

However, we have seen in (2.101) that for any $\gamma(\cdot) \in \boldsymbol{\Gamma}^W$,

$$\begin{aligned} E_{\theta x} C(t, \theta, \gamma(\cdot); \phi) &= E_{\theta x}\left[E(C(t, \theta, \gamma(\cdot); \phi) | \mathcal{F}_\theta^W)\right] \\ &\ge \inf_{\gamma(\cdot) \in \boldsymbol{\Gamma}^{W^\theta}} J(t, \theta, x, \gamma(\cdot); \phi). \end{aligned} \tag{2.125}$$

Now (2.124) and (2.125) yield (2.121).

Putting $V_t := V_{0t}$, we have the semigroup $(V_t, t \in [0, T])$ with the generator G given by

$$\begin{aligned} \mathrm{G}\psi(x) &= \lim_{t \to 0} \frac{1}{t}(V_t\psi(x) - \psi(x)) \\ &= \inf_{\gamma \in \Gamma}\Big(\frac{1}{2}\mathrm{tr}(a(x, \gamma)\partial_{xx}\psi) + b(x, \gamma) \cdot \partial_x\psi - \kappa(x, \gamma)\psi + f(x, \gamma)\Big) \end{aligned} \tag{2.126}$$

for $\psi \in \mathcal{D}$.

## 2.3 Verification Theorems and Optimal Controls

In this section we show how to construct optimal controls or optimal Markovian policies using the HJB equations. In the previous section, we have shown, using DPP and Itô's formula, that under appropriate regularity assumption the value function satisfies the HJB equation. However, it is rather difficult to verify the regularity of value function. But, if a classical solution of HJB equation exists, then we can take an optimal Markovian policy by using the minimum selector and obtain the value function. This assertion is called a verification theorem.

In Sect. 2.3.1, we state verification theorems, which are useful in seeking optimal controls in practical problems. In Sect. 2.3.2, we give three simple examples.

### *2.3.1 Verification Theorems*

Here we assume the following conditions $(b_1)'$ and $(b_2)'$ weaker than $(b_1)$–$(b_5)$:

$(b_1)'$

$$b : [0, T] \times \mathbb{R}^d \times \Gamma \longmapsto \mathbb{R}^d, \quad \alpha : [0, T] \times \mathbb{R}^d \times \Gamma \longmapsto \mathbb{R}^d \otimes \mathbb{R}^m$$

are uniformly continuous and satisfy

$$\begin{aligned} &|b(t_1, x_1, \gamma_1) - b(t_2, x_2, \gamma_2)| + |\alpha(t_1, x_1, \gamma_1) - \alpha(t_2, x_2, \gamma_2)| \\ \le & l|x_1 - x_2| + m(|t_1 - t_2| + |\gamma_1 - \gamma_2|), \end{aligned} \tag{2.127}$$

with a constant $l$ and $a$ modulus function $m(\cdot)$, and

$$|b(t, x, \gamma)| + |\alpha(t, x, \gamma)| \le K(1 + |x| + |\gamma|), \tag{2.128}$$

with a constant $K$.

$(b_2)'$

$$\kappa : [0, T] \times \mathbb{R}^d \times \Gamma \mapsto [0, c_0], \quad f : [0, T] \times \mathbb{R}^d \times \Gamma \mapsto \mathbb{R}^1, \quad \phi : \mathbb{R}^d \mapsto \mathbb{R}^1$$

are continuous and

$$|f(t, x, \gamma)| + |\phi(x)| \le \hat{K}(1 + |x|^2 + |\gamma|^2), \tag{2.129}$$

with a constant $\hat{K}$.

Here we admit a slightly bigger class of control processes. Specifically let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ be a reference probability system. We take $(\gamma_*(t), t \in [0, T])$ to be a $\Gamma$-valued $(\mathcal{F}_t)$-progressively measurable process, satisfying

$$E\Big[\int_0^T |\gamma_*(t)|^n \, ds\Big] < \infty \quad \text{for } n = 1, 2, \ldots. \tag{2.130}$$

This condition is sometimes convenient for establishing the existence of optimal controls (see Example 2.4).

For a control $A_* = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma_*(\cdot))$, its response evolves according to the SDE

$$dX(t) = b(t, X(t), \gamma_*(t))\, dt + \alpha(t, X(t), \gamma_*(t))\, dW(t), \quad t \in [\theta, T], \tag{2.131}$$

with the initial condition

$$X(\theta) = x. \tag{2.132}$$

We clearly have a unique solution of (2.3)–(2.4), satisfying

$$E_{\theta x}\Big[\sup_{\theta \le s \le t} |X(s)|^{2p}\Big] \le c_p\Big(1 + |x|^{2p} + E\int_\theta^t |\gamma_*(s)|^{2p}\, ds\Big), \quad \forall \theta, t, x, A_*, \tag{2.133}$$

with a constant $c_p$.

By (2.131) and $(b_2)'$, we can define the payoff $J(T, \theta, x, A_*; \phi)$ by (2.6) and the value function $v_*(\cdot)$ by

$$v_*(\theta, x) = \inf_{A_*} J(T, \theta, x, A_*; \phi), \tag{2.134}$$

respectively.

*Remark 2.2.*

$$v_*(\theta, x) = v(\theta, x).$$

*Proof.* That $v_*(\theta, x) \le v(\theta, x)$ is clear. For the opposite inequality, we fix $A_* = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma_*(\cdot))$ arbitrarily and claim that

$$J(T, \theta, x, A_*; \phi) \ge v(\theta, x). \tag{2.135}$$

Put $\gamma_N(t) = \gamma_*(t)\chi(|\gamma_*(t)| \le N)$ and $A_N = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, \gamma_N(\cdot))$. Then $A_N \in \mathbb{A}$ and

$$\lim_{N\to\infty} E\Big[\int_0^T |\gamma_*(s) - \gamma_N(s)|^n \, ds\Big] = \lim_{N\to\infty} E\Big[\int_0^T |\gamma_*(s)|^n \chi(|\gamma_*(s)| > N)\, ds\Big] = 0. \tag{2.136}$$

$X$ and $X_N$ denote the responses for $\gamma_*(\cdot)$ and $\gamma_N(\cdot)$, respectively.

Let $\rho_N(t) := E_{\theta x}[\sup_{\theta \le s \le t} |X(s) - X_N(s)|^2]$. From (2.127) and the Burkholder–Davis–Gundy inequality, we easily deduce that

$$\rho_N(t) \le cE\Big(\int_\theta^t \{\rho_N(s) + (1 + |\gamma_*(s)|^2)\chi(|\gamma_*(s)| \ge N)\}\, ds\Big), \tag{2.137}$$

with a constant $c$ independent of $\theta, x, t, N$, and $\gamma_*(\cdot)$.

Hence, (2.136) and (2.137) yield

$$\lim_{N \to \infty} E_{\theta x}\Big[\sup_{\theta \le s \le T} |X(s) - X_N(s)|^2\Big] = 0.$$

Choosing a subsequence $N_j$ so that

$$\lim_{N_j \to \infty} \sup_{\theta \le s \le T} |X(s) - X_{N_j}(s)| = 0 \quad P\text{-a.s.}$$

and using (2.133), we conclude that

$$\lim_{N_j \to \infty} J(T, \theta, x, A_{N_j}; \phi) = J(T, \theta, x, A_*; \phi),$$

which in turn yields (2.135). □

Let us consider the HJB equation;

$$-\frac{\partial V}{\partial t}(t, x) + \mathrm{H}(t, x, \partial_{xx} V(t, x), \partial_x V(t, x), V(t, x)) = 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}^d \tag{2.118$'$}$$

with the lateral boundary condition

$$v(T, x) = \phi(x), \quad x \in \mathbb{R}^d. \tag{2.119$'$}$$

The following result holds true.

**Theorem 2.6 (([FS06], p. 159) *Verification theorem*).**
*Let $\omega \in C^{12}([0, T) \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d)$ be a solution of (2.118′)–(2.119′), satisfying*

$$|\omega(t, x)| \le K_1(1 + |x|^2), \quad \forall t, x, \tag{2.138}$$

*with a constant $K_1 > 0$.*

*Then*

(i) *For any $A_* = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma_*(\cdot))$,*

$$\omega(t, x) \le J(T, t, x, A_*; \phi), \quad \forall t, x. \tag{2.139}$$

(ii) *Let* $(t, x)$ *be given. Suppose that there exists* $\hat{A}_* = (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{P}, \hat{W}, \hat{\gamma}_*(\cdot))$ *such that*

$$\begin{aligned}
&\min_{\gamma\in\Gamma}(\mathrm{G}_s^{\gamma}\omega(s, \hat{X}(s)) - \kappa(s, \hat{X}(s), \gamma)\omega(s, \hat{X}(s)) + f(s, \hat{X}(s), \gamma)) \\
=&\mathrm{G}_s^{\hat{\gamma}_*(s)}\omega(s, \hat{X}(s)) - \kappa(s, \hat{X}(s), \hat{\gamma}_*(s))\omega(s, \hat{X}(s)) + f(s, \hat{X}(s), \hat{\gamma}_*(s)), \\
&\forall s \in [t, T], \quad \hat{P}\text{-a.s.}, \qquad (2.140)
\end{aligned}$$

*where* $\hat{X}$ *is the response for* $\hat{A}_*$ *with* $\hat{X}(t) = x$.

*Then*

$$\omega(t, x) = J(T, t, x, \hat{A}_*; \phi). \qquad (2.141)$$

(iii) $\hat{A}_*$ *is optimal for* $(t, x)$, *i.e.,*

$$v_*(t, x) = \omega(t, x) = J(T, t, x, \hat{A}_*; \phi). \qquad (2.142)$$

For the case of optimal Markovian policies, we deduce from Theorem 2.6 and Proposition 1.6 the following

**Corollary 2.3.** *In addition to* $(b_1)'$ *and* $(b_2)'$, *we assume that*

(a) $\Gamma$ *is convex and compact,*
(b) $d = m$,
(c) $b(\cdot)$ *is bounded,*
(d) $\alpha(\cdot)$ *is bounded, symmetric and uniformly parabolic.*

*Suppose that the HJB equation (2.118′)–(2.119′) has a classical solution, satisfying the growth condition* (2.138). *Then the maximum selector of* $\mathrm{H}(\cdot)$ *provides an optimal Markovian policy.*

Regarding the classical solution of HJB equation, we recall the following result.

**Theorem 2.7 ([FS06], p. 163).** *Let* $\kappa = 0$ *and assume;*

(a) $\Gamma$ *is compact and* $d = m$,
(b) $\alpha(\cdot)$ *is uniformly parabolic and independent of* $\gamma$. *Moreover,* $\alpha(\cdot) \in C^{12}([0, T]\times \mathbb{R}^d)$ *and* $\alpha(\cdot), \alpha(\cdot)^{-1}$, *and* $\partial_x\alpha(\cdot)$ *are bounded,*
(c) $b(t, x, \gamma) = \tilde{b}(t, x) + \alpha(t, x)\Theta(t, x, \gamma)$, *where* $\tilde{b}(\cdot) \in C^{12}([0, T] \times \mathbb{R}^d)$ *and* $\partial_x\tilde{b}(\cdot)$ *is bounded, and* $\Theta(\cdot)$ *and* $\partial_x\Theta(\cdot)$ *are in* $C_b([0, T] \times \mathbb{R}^d \times \Gamma)$,
(d) $f(\cdot)$ *and* $\partial_x f(\cdot)$ *are in* $C_p([0, T] \times \mathbb{R}^d \times \Gamma)$,
(e) $\phi(\cdot) \in C^3(\mathbb{R}^d) \cap C_p^1(\mathbb{R}^d)$.

*Then the* HJB *equation* (2.118′)–(2.119′) *has a unique classical solution.*

### *2.3.2 Examples of Optimal Control*

We are concerned with explicit formulations of optimal controls for three simple models. The first two examples are related to Gaussian diffusions effected by linear control processes. The third one is a stochastic control with state constraint.

*Example 2.4 (Linear Gaussian quadratic regulator).* Let $\Gamma = \mathbb{R}^q$ and suppose that matrix-valued continuous functions of suitable sizes $A, B, \sigma, M, N$ and $D$ are given on $[0, T]$. For an admissible control $A_* = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma_*(\cdot))$, we have the $d$-dimensional SDE

$$dX(t) = (A(t)X(t) + B(t)\gamma_*(t))\,dt + \sigma(t)\,dW(t), \quad t \in [\theta, T], \tag{2.143}$$

with the initial condition

$$X(\theta) = x (\in \mathbb{R}^d) \tag{2.144}$$

and the payoff

$$\begin{aligned} &J(T, \theta, x, A_*) \\ =&E_{\theta x}\Big[\int_\theta^T (X(s)^\top M(s)X(s) + \gamma_*(s)^\top N(s)\gamma(s))\,ds + X(T)^\top DX(T)\Big]. \end{aligned} \tag{2.145}$$

We assume that for any $t \in [0, T]$,

(a) $D, M(t)$ are non-negative definite symmetric $d \times d$ matrices;
(b) $N(t)$ is a positive definite symmetric $q \times q$ matrix.

Since the corresponding HJB equation reads

$$\begin{aligned} 0 =&\frac{\partial \omega}{\partial t}(t, x) + \frac{1}{2}\mathrm{tr}(a(t)\partial_{xx}\omega(t, x)) + (A(t)x) \cdot \partial_x \omega(t, x) \\ &+ x^\top M(t)x + \inf_{\gamma \in \Gamma}((B(t)\gamma) \cdot \partial_x \omega(t, x) + \gamma^\top N(t)\gamma), \\ &\qquad t \in [0, T), \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.146}$$

with the lateral boundary condition

$$\omega(T, x) = x^\top Dx, \tag{2.147}$$

we are looking for the solution of (2.146)–(2.147).

Observing that the expression whose inf is taken in (2.146) is

$$\left|\frac{1}{2}N(t)^{-\frac{1}{2}}B(t)^\top \partial_x \omega(t, x) + N(t)^{\frac{1}{2}}\gamma\right|^2 - \frac{1}{4}|N(t)^{-\frac{1}{2}}B(t)^\top \partial_x \omega(t, x)|^2 \tag{2.148}$$

and assuming that $\omega(\cdot)$ is quadratic, we have

$$\omega(t,x) = x^\top P(t)x + \int_t^T \operatorname{tr}(a(s)P(s))\,ds \tag{2.149}$$

where $P(\cdot)$ is the solution of the following Riccati equation:

$$\begin{aligned} 0 = & \frac{dP}{dt}(t) + M(t) + A(t)^\top P(t) + P(t)A(t) \\ & - P(t)B(t)N(t)^{-1}B(t)^\top P(t), \quad t \in [0,T) \end{aligned} \tag{2.150}$$

with

$$P(T) = D. \tag{2.151}$$

Hence, from (2.148) and (2.149) it follows that an optimal Markovian policy is given by $-N(t)^{-1}B(t)^\top P(t)x$.

More details for LQ problems are given in [YZ99], Chapter 6.

*Example 2.5 (1-dimensional bang-bang control).* Let $\Gamma = [-1,1]$. Suppose that $g \in C_p^1(\mathbb{R}^1)$ is even, $g(0) = 0$ and, $g'(x) \geq 0$, on $[0,\infty)$.

Let us consider the following time-homogeneous simple model. For $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$, its response $X^A$ evolves according to the SDE

$$dX(t) = \gamma(t)\,dt + dW(t), \quad t \in (0,T] \tag{2.152}$$

with the initial condition

$$X(0) = x \ (\in \mathbb{R}^1), \tag{2.153}$$

and the payoff $j(\cdot)$ is given by

$$j(t,x,A) = E_x g(X^A(t)). \tag{2.154}$$

The (formal) HJB equation for the value function $v(t,x) = \inf_{A\in\mathbb{A}} j(t,x,A)$ reads

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}\partial_{xx}v(t,x) + \inf\limits_{|\gamma|\le 1}(\gamma\partial_x v(t,x)) \\ \qquad\qquad = \dfrac{1}{2}\partial_{xx}v(t,x) - |\partial_x v(t,x)|, \quad t > 0, \quad x \in \mathbb{R}^1, \\ v(0,x) \quad = g(x), \quad x \in \mathbb{R}^1. \end{cases} \tag{2.155}$$

Now we look for the explicit solution of (2.155).

**Proposition 2.9 (Communicated by F. Asakura).**

$$v(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2}} \int_{-\infty}^{\infty} g(y) \exp\Big(-\frac{|x|^2 - y}{2t} + |x| - |y|\Big)\, dy$$
$$- \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{|x|-|y|} \Big( \int_t^{\infty} s^{-\frac{3}{2}} (|x|+|y|+s) \exp\Big(-\frac{|x|+|y|+s^2}{2s}\Big)\, ds\Big)\, dy$$
$$+ \int_{-\infty}^{\infty} e^{-2|y|} g(y)\, dy \tag{2.156}$$

*and*

$$\operatorname{sgn}(\partial_x v(t,x)) = \operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{2.157}$$

*Outline of proof.* Consider the auxiliary equation

$$\frac{\partial \omega}{\partial t}(t,x) = \partial_{xx}\omega(t,x), \quad t > 0, \quad x > 0 \tag{2.158}$$

with the boundary and initial conditions

$$\begin{cases} \partial_x \omega(t,0) + \dfrac{1}{2}\omega(t,0) = 0, & t > 0, \\ \omega(0,x) = e^{-\frac{x}{2}} g\Big(\dfrac{x}{2}\Big), & x > 0. \end{cases} \tag{2.159}$$

We solve (2.158)–(2.159) using the Laplace transformation. Putting $u(t,x) = \omega(t,x) e^{\frac{2x-t}{4}}$, we have

$$\partial_x u(t,x) > 0 \quad \text{for } x > 0 \tag{2.160}$$

and

$$\lim_{t \to 0} u(t,x) = g\Big(\frac{x}{2}\Big). \tag{2.161}$$

Further, $v(t,x) := u(2t, 2|x|),\ t > 0,\ x \in \mathbb{R}^1$ is given by expression (2.156) and satisfies (2.155). From (2.156), it follows that

$$\operatorname{sgn}(\partial_x v(t,x)) = \operatorname{sgn} x. \tag{2.162}$$

This completes the proof. □

By Theorem 2.6, $v(\cdot)$ is the value function and the minimum selector $\hat{\gamma}(t,x) = -\mathrm{sgn}(\partial_x v(t,x)) = -\mathrm{sgn}\, x$ gives an optimal Markovian policy. Its response $X$ evolves according to SDE

$$dX(t) = -\mathrm{sgn}\, X(t) + dW(t). \tag{2.163}$$

Since (2.163) admits a unique strong solution, $X$ is the diffusion with generator $\frac{1}{2}\frac{d^2}{dx^2} - (\mathrm{sgn}\, x)\frac{d}{dx}$ and $e^{-2|x|}$ gives the density of the corresponding invariant probability measure. From (2.156) it follows that for any $R > 0$, there is a constant $C_R > 0$ such that

$$\sup_{|x|\le R}\left|v(t,x) - \int_{-\infty}^{\infty} e^{-2|y|} g(y)\, dy\right| \le C_R e^{-\frac{t}{2}}, \quad \forall t > 1. \tag{2.164}$$

For related topics consult [Be75, BSW80] and [IW81], Chap. 6.

Finally we will consider stochastic control with state constraint. We assume that the coefficients $\alpha, b$, and $f$ are independent of the time variable, and satisfy $(b_1)'$ and $(b_2)'$.

Let $\mathcal{O}$ be a bounded and open set of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. For an admissible control $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$, the response $X^A$ evolves according to the SDE

$$\begin{cases} dX(t) = b(x(t), \gamma(t))\, dt + \alpha(X(t), \gamma(t))\, dW(t), & t > 0, \\ X(0) = x. \end{cases} \tag{2.165}$$

By $\tau^A$ we denote the exit time of $X^A$ from $\mathcal{O}$, that is

$$\tau^A = \begin{cases} \inf\{s > 0; X^A(s) \notin \mathcal{O}\}, \\ \infty, \quad \text{if } \{\cdots\} \text{ is empty.} \end{cases} \tag{2.166}$$

For a given continuous function $g$ on $\mathcal{O}$ and a positive constant $\kappa$, we define the payoff $J(\cdot)$ by

$$J(x, A) = E_x\Big[\int_0^{\tau^A} e^{-\kappa s} f(X^A(s), \gamma(s))\, ds + e^{-\kappa\tau^A} g(X^A(\tau^A))\Big], \quad x \in \mathcal{O}; \tag{2.167}$$

here, when $\tau^A = \infty$, $e^{-\kappa\tau^A} g(X^A(\tau^A))$ stands for 0.

When $\gamma(\cdot)$ is constant $\gamma \in \Gamma$, its response $X^A$ is the diffusion with the generator $\mathrm{G}^\gamma$ given by

$$\mathrm{G}^\gamma u = \frac{1}{2}\mathrm{tr}(a(x,\gamma)\partial_{xx} u) + b(x,\gamma)\cdot\partial_x u \tag{2.168}$$

and $J(\cdot, A)$ satisfies the elliptic PDE with boundary value $g$,

$$\begin{cases} \mathrm{G}^\gamma J(x, A) - \kappa J(x, A) + f(x, \gamma) = 0, & \forall x \in \mathcal{O}, \\ \lim\limits_{y \to x} J(y, A) = g(x), & x \in \partial\mathcal{O}, \end{cases} \tag{2.169}$$

provided $J(\cdot, A)$ is smooth up to boundary. $v(x) := \inf_{A \in \mathbb{A}} J(x, A)$ is called the value function.

Let us consider the following Dirichlet problem for the HJB equation:

$$\begin{cases} \inf\limits_{\gamma \in \Gamma} (\mathrm{G}^\gamma u(x) - \kappa u(x) + f(x, \gamma)) = 0, & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \partial\mathcal{O}. \end{cases} \tag{2.170}$$

Then, the following result holds true.

**Theorem 2.8 (Verification theorem).** *Let $u(\cdot)$ be a classical solution of* (2.170). *Then, it is valid that*

(i) $u(x) \le J(x, A), \ \forall x \in \mathcal{O}, \forall A \in \mathbb{A}$,
(ii) *Let $x_0 \in \mathcal{O}$ be given. If there exists $A^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*), P^*, W^*, \gamma^*(\cdot))$ and its response $X^*$ with $X^*(0) = x_0$, such that*

$$\begin{aligned} 0 =& \min_{\gamma \in \Gamma} (\mathrm{G}^\gamma u(X^*(t)) - \kappa u(X^*(t)) + f(X^*(t), \gamma)) \\ =& \mathrm{G}^{\gamma^*(t)} u(X^*(t)) - \kappa u(X^*(t)) + f(X^*(t), \gamma^*(t)), \\ & \qquad \forall t < \tau^{A^*} \quad P\text{-a.s.}, \end{aligned} \tag{2.171}$$

*then*

$$u(x_0) = J(x_0, A^*). \tag{2.172}$$

*From* (i) *and* (2.172)*, it follows that $u(x_0) = v(x_0)$ and $A^*$ gives an optimal control for the initial state $x_0$.*

Regarding the classical solution of (2.170), we can refer to [E83], [E98], [Mo10], Sect. 5.3.

*Example 2.6 (Production planning problem).* Consider the production planning for one commodity, in the presence of random demand, fluctuating according to the SDE

$$dZ(t) = b\,dt - \sigma\,dW, \tag{2.173}$$

where the positive constant $b$ represents the expected demand rate and $\sigma$ and $W$ denote a positive constant and a real Wiener process. The firm adjusts its production rate $\gamma(\cdot)$ and the finished products are stored in a buffer of size $\bar{K}$. Since the running cost $f(\cdot)$ and the terminal cost $g(\cdot)$ are needed, the firm's objective is to minimize its expected total cost by choosing a suitable $\gamma(\cdot)$.

Let us formulate the problem in precise terms. Let $K$ be the production capacity and put $\Gamma = [0, K]$. Then, for $A = (\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, \gamma(\cdot))$, the inventory level $X(t)$ evolves according to the SDE

$$\begin{cases} dX(t) = \gamma(t)\,dt - dZ(t) = (\gamma(t) - b)\,dt + \sigma\,dW(t), \quad t > 0, \\ X(0) = x. \end{cases}$$

$X(t) > 0$ (resp. $< 0$) means a surplus (resp. backlog) of product. Since the inventory state cannot exceed the buffer size $\overline{K}$ and the firm fixes the backlog size $\underline{K}(< 0)$, because a new policy is needed for a large amount of backlog, we give the payoff by

$$J(x, A) = E_x\Big[\int_0^{\tau_A} e^{-\kappa s} f(X^A(s), \gamma(s))\,ds + e^{-\kappa\tau_A} g(X^A(\tau^A))\Big], \tag{2.174}$$

where $\tau^A$ = exit time of $X^A$ from $(\underline{K}, \overline{K}) := \mathcal{O}$.

We suppose that $f(\cdot)$ is given by

$$f(x, \gamma) = p(x \vee 0) + \gamma^2, \tag{2.175}$$

with a positive constant $p$.

The corresponding HJB equation reads

$$\frac{1}{2}\sigma^2 u''(x) - bu'(x) - \kappa u(x) + \inf_{\gamma\in\Gamma}(\gamma u'(x) + \gamma^2) + p(x \vee 0) = 0, \quad \forall x \in (\underline{K}, \overline{K}), \tag{2.176}$$

with the boundary conditions

$$u(\underline{K}) = g(\underline{K}), \quad u(\overline{K}) = g(\overline{K}).$$

Since there exists a unique classical solution of (2.176), the verification theorem asserts that the solution is equal to the value function and an optimal Markovian policy is given by the minimum selector;

$$\hat{\gamma}(x) = \begin{cases} 0 & \text{on } \{x \in \mathcal{O}; u'(x) \geq 0\}, \\ -\dfrac{u'(x)}{2} & \text{on } \{x \in \mathcal{O}; u'(x) \in (-2K, 0)\}, \\ K & \text{on } \{x \in \mathcal{O}; u'(x) \leq -2K\}. \end{cases}$$

Refer to [Mo10], Chapter 6 for related topics.

## 2.4 Optimal Investment Models

This section is devoted to finite time horizon optimal investment problems. We consider a market with one bond and $m(\geq 1)$ risky assets, where $d$ factor processes, $X^1, \ldots, X^d$, determine the performance of the market. This model, called a factor model, was introduced by Merton [Me71] (refer to [Me73, HP81, Na03]).

Suppose that the factor process $X = (X^1, \ldots, X^d)$ evolves according to the SDE

$$dX = b(t, X(t))\,dt + dW(t). \tag{2.177}$$

The prices of the bond and of the $i$-th asset are given by

$$\begin{cases} dS^0(t) = S^0(t) r(t, X(t))\,dt, & t > 0, \\ S^0(0) = s^0 > 0, \end{cases} \tag{2.178}$$

and by the SDE

$$\begin{cases} dS^i(t) = S^i(t)\Big(g^i(t, X(t))\,dt + \sum_{j=1}^{d+m} \sigma^i_j(t, X(t))\,dW^j\Big), & t > 0, \\ S^i(0) = s^i > 0, \quad i = 1, \ldots, m, \end{cases} \tag{2.179}$$

respectively, where $W = (W^1, \ldots, W^d)$ is the Wiener process of (2.177).

Let us consider an agent who invests at any $t \in (0, T)$ a proportion $\pi^i(t)$ of his/her wealth in the $i$-th risky asset ($i = 1, \ldots, m$) and $\pi^0(t) = 1 - \sum_{i=l}^m \pi^i(t)$ in the bond. The agent wants to maximize the expected utility from the terminal wealth by choosing a suitable $(\pi^i(\cdot), i = 1, \ldots, m)$.

As an application of our previous results, we study the problem by using a DPP argument. We note that the martingale approach is also powerful (refer to [HP81] and [HP83]).

### *2.4.1 Formulations*

Let

$$\begin{aligned} \sigma &: [0, T] \times \mathbb{R}^d \longmapsto \mathbb{R}^m \otimes \mathbb{R}^{d+m}, \\ b &: [0, T] \times \mathbb{R}^d \longmapsto \mathbb{R}^d, \\ g &: [0, T] \times \mathbb{R}^d \longmapsto \mathbb{R}^m, \\ r &: [0, T] \times \mathbb{R}^d \longmapsto [0, K_0] \end{aligned}$$

be continuous and Lipschitz continuous in the variable $x \in \mathbb{R}^d$, uniformly on $[0, T]$. We assume that

(a) $a(t,x) := \sigma(t,x)\sigma(t,x)^\top$ is uniformly parabolic, i.e.,

$$y^\top a(t,x)y \geq \lambda_0|y|^2, \quad \forall y \in \mathbb{R}^m, \quad \forall t,x,$$

for a positive constant $\lambda_0$,

(b) $\sigma(t,x)^\top a(t,x)^{-1}\sigma(t,x)$ is bounded,

(c) $g(t,x)^\top a(t,x)^{-1}g(t,x)$ and $\sigma(t,x)^\top a(t,x)^{-1}g(t,x)$ satisfy linear growth condition in $x$ uniformly on $[0,T]$.

Let $\tilde{W} = (W^1,\ldots,W^d,W^{d+1},\ldots,W^{d+m})$ be a $(d+m)$-dimensional Wiener process, defined on $(\Omega,\mathcal{F},P)$, and put $W = (W^1,\ldots,W^d)$ (= first $d$-component of $\tilde{W}$).

Hence, the strong solution of SDE (2.177) is a diffusion process with the generator $\mathrm{G}_t$:

$$\mathrm{G}_t\psi = \frac{1}{2}\Delta\psi + b(t,x)\cdot\partial_x\psi, \tag{2.180}$$

where $\Delta$ is the $d$-dimensional Laplacian operator.

Further,

$$\mathcal{F}_t^W = \mathcal{F}_t^X, \quad \forall t \in [0,T],$$

if $X(0)$ =constant.

Since $S^0$ and $S^i$ evolve according to (2.178) and (2.179), respectively, we have

$$S^0(t) = s^0\exp\Big(\int_0^t r(s,X(s))\,ds\Big) \tag{2.181}$$

and

$$S^i(t) = s^i\exp\Big\{\int_0^t\Big(g(s,X(s))^i - \frac{1}{2}|\sigma(s,X(s))^i|^2\Big)\,ds + \int_0^t\sigma(s,X(s))^i\,d\tilde{W}(s)\Big\}. \tag{2.182}$$

$g(t,X(t))$ and $\sigma(t,X(t))$ are called the mean return process and the volatility process, respectively.

We assume

$$g(t,x)^i > r(t,x), \quad \forall t,x, \quad i = 1,\ldots,m. \tag{2.183}$$

By using the data of $X$ and $S^i, i = 1,\ldots,m$, the agent invests at any time $t \in (0,T)$ a proportion $\pi^i(t)$ of its wealth in the $i$-th asset, $i = 1,\ldots,m$, and $\pi^0(t) = 1 - \sum_{i=1}^m\pi^i(t)$ in the bond. $(\pi^i(t) > 1)$ and $(\pi^i(t) < 0)$ stand for borrowing money and selling, respectively. We call $\pi(\cdot) = (\pi^1(\cdot),\ldots,\pi^m(\cdot))$ an investment strategy or an admissible strategy, if

$$\pi(\cdot)^\top \sigma(\cdot, X(\cdot)) \in L^\infty([0,T] \times \Omega, (\mathcal{F}_t^{\tilde{W}}); \mathbb{R}^1 \otimes \mathbb{R}^{d+m}). \tag{2.184}$$

Let $\mathscr{A}$ denote the set of all admissible strategies.

For $\pi(\cdot) \in \mathscr{A}$, the agent's wealth $Z^{\pi(\cdot)}$ evolves according to the SDE

$$\begin{aligned}
&\frac{dZ^{\pi(\cdot)}(t)}{Z^{\pi(\cdot)}(t)} \\
&= \sum_{i=1}^m \pi^i(t)(g(t,x(t))^i \, dt + \sigma(t, X(t))^i \, d\tilde{W}(t)) + \pi^0(t) r(t, X(t)) \, dt \\
&= (r(t, X(t)) + \pi(t) \cdot (g(t, X(t)) - r(t, X(t)) 1^m)) \, dt + \pi(t)^\top \sigma(t, X(t)) \, d\tilde{W}(t)
\end{aligned} \tag{2.185}$$

by (2.178) and (2.179). The quantity

$$\mu(t,x) := g(t,x) - r(t,x)1^m \tag{2.186}$$

is the excess rate of return from risky assets, where $1^m = (1, 1, \dots, 1) \in \mathbb{R}^m$.

From (2.185), we deduce that

$$\begin{aligned}
Z^{\pi(\cdot)}(t) =& z \exp\Big\{ \int_0^t (r(s, X(s)) + \pi(s) \cdot \mu(s, X(s))) \, ds \\
&+ \int_0^t \pi(s)^\top \sigma(s, X(s)) \, d\tilde{W}(s) - \frac{1}{2} \int_0^t |\pi(s)^\top \sigma(s, X(s))|^2 \, ds \Big\},
\end{aligned} \tag{2.187}$$

where $z =$ the initial wealth $> 0$.

$U \in C^2((0, \infty))$ is called a utility function, if $U' > 0$ (increasing) and $U'' < 0$ (concave).

We are interested in HARA (hyperbolic absolute risk aversion) utility functions

$$\text{power utility function: } U(x) = \frac{x^\delta}{\delta}, \quad x > 0 \quad \text{with } \delta < 1, \neq 0, \tag{2.188}$$

and

$$\text{logarithmic utility function: } U(x) = \log x, \quad x > 0. \tag{2.189}$$

The agent's objective is to maximize the expected utility from the terminal wealth, $E[U(Z^{\pi(\cdot)}(T))]$, by choosing an appropriate investment strategy.

## 2.4.2 Investment Problems for Power Utility Function

By (2.187) and (2.188), we have

$$U(Z^{\pi(\cdot)}(t)) = \frac{z^\delta}{\delta} M^{\pi(\cdot)}(t) \exp\Big(\delta \int_0^t \eta(s, X(t), \pi(s))\, ds\Big) \tag{2.190}$$

where

$$M^{\pi(\cdot)}(t) = \exp\Big(\int_0^t \delta\pi(s)^\top \sigma(s, X(s))\, d\tilde{W}(s) - \frac{1}{2}\int_0^t |\delta\pi(s)^\top \sigma(s, X(s))|^2\, ds\Big) \tag{2.191}$$

and

$$\eta(s, x, \pi(s)) = -\frac{1-\delta}{2}|\pi(s)^\top \sigma(s, x)|^2 + \pi(s)\cdot\mu(s, x) + r(s, x). \tag{2.192}$$

$M^{\pi(\cdot)}$ is an exponential $(\mathcal{F}_t^{\tilde{W}})$-martingale, because of the boundedness of $\pi(\cdot)^\top\sigma(\cdot)$.

For the computation of $EU(Z^{\pi(\cdot)}(t))$, we apply Girsanov's transformation;

$$P^{\pi(\cdot)} = M^{\pi(\cdot)}(T) \circ P \quad \text{on } \mathcal{F}_T^{\tilde{W}}. \tag{2.193}$$

The following (i)–(iii) are valid with respect to $P^{\pi(\cdot)}$

(i) $\tilde{W}^{\pi(\cdot)}$ given by (2.194) is an $(\mathcal{F}_t^{\tilde{W}})$-Wiener process, i.e.,

$$\tilde{W}^{\pi(\cdot)}(t) = \tilde{W}(t) - \delta\int_0^t \sigma(s, X(s))^\top \pi(s)\, ds, \quad t \in [0, T]. \tag{2.194}$$

(ii) The factor process $X$ is described by SDE.

$$dX(t) = \big(b(t, X(t)) + \delta\bar{\sigma}(t, X(t))^\top \pi(t)\big)\, dt + dW^{\pi(\cdot)}(t), \tag{2.195}$$

where $W^{\pi(\cdot)}$ is the first $d$-component of $\tilde{W}^{\pi(\cdot)}$ and $\bar{\sigma}(\cdot)$ is the $m \times d$-matrix consisting of the first $d$ columns of $\sigma(\cdot)$.

(iii) Since $\eta(\cdot)$ is independent of $z$,

$$E_{zx}U(Z^{\pi(\cdot)}(t)) = z^\delta E_x^{\pi(\cdot)}\Big[\frac{1}{\delta}\exp\Big(\delta\int_0^t \eta(s, X(s), \pi(s))\, ds\Big)\Big], \tag{2.196}$$

where $E^{\pi(\cdot)}$ denotes the expectation w.r.t. $P^{\pi(\cdot)}$.

Referring to (2.195), we consider the following control problem. Let $\Gamma = \mathbb{R}^m$ and fix a reference probability system $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, \tilde{B})$ with a $(d+m)$-dimensional Wiener process $\tilde{B}$. For a control process $\pi(\cdot) \in \boldsymbol{\Gamma}^{\tilde{B}}$, its response $Y^{\pi(\cdot)}$ evolves according to the SDE

$$dY(s) = \big((b(s, Y(s)) + \delta\bar{\sigma}(s, Y(s))^\top \pi(s)\big)\, ds + dB(s), \quad s \in (t, T], \tag{2.197}$$

with the initial condition

$$Y(t) = x \ (\in \mathbb{R}^d), \tag{2.198}$$

where $B$ is the first $d$-dimensional component of $\tilde{B}$. We define the payoff $j(\cdot)$ and the value function $u(\cdot)$ by

$$j(t, x, \pi(\cdot)) = E_{tx}\Big[\frac{1}{\delta}\exp\Big(\delta\int_t^T \eta(s, Y(s), \pi(s))\, ds\Big)\Big] \tag{2.199}$$

and

$$u(t, x) = \sup_{\pi(\cdot)} j(t, x, \pi(\cdot)), \tag{2.200}$$

respectively. Now let us apply the logarithmic transformation,

$$v(t, x) = \frac{1}{\delta}\log(\delta u(t, x)). \tag{2.201}$$

Calculating formally, we obtain

$$\begin{aligned} 0 =& \frac{\partial v}{\partial t} + \frac{1}{2}(\Delta v + \delta|\partial_x v|^2 + 2b(t, x)\cdot\partial_x v + 2r(t, x)) \\ &+ \sup_{\pi\in\mathbb{R}^m}\Big\{-\frac{1-\delta}{2}|\pi^\top\bar{\sigma}(t, x)|^2 + \pi^\top(\mu(t, x) + \delta\bar{\sigma}(t, x)\partial_x v)\Big\}. \end{aligned} \tag{2.202}$$

Since the supremum is attained for $\pi = \hat{\Pi}$, where

$$\hat{\Pi}(t, x) = \frac{1}{1-\delta}\{a(t, x)^{-1}(\mu(t, x) + \delta\bar{\sigma}(t, x)\partial_x v(t, x))\}, \tag{2.203}$$

we obtain

$$0 = \frac{\partial v}{\partial t} + \frac{1}{2}\Delta v + \mathrm{H}(t, x, \partial_x v(t, x)) \quad \text{on } [0, T)\times\mathbb{R}^d, \tag{2.204}$$

with the lateral boundary condition

$$v(T, x) = 0, \quad x \in \mathbb{R}^d, \tag{2.205}$$

where

$$
\begin{aligned}
\mathrm{H}(t,x,p) =& \frac{\delta}{2}\Big\{|p|^2 + \frac{\delta}{1-\delta}p(\bar{\sigma}^\top a^{-1}\bar{\sigma})(t,x)p^\top\Big\} \\
&+ b(t,x)\cdot p + \frac{\delta}{2(1-\delta)}\{(\mu^\top a^{-1}\bar{\sigma})(t,x)p^\top + p(\bar{\sigma}^\top a^{-1}\mu)(t,x)\} \\
&+ r(t,x) + \frac{1}{2(1-\delta)}(\mu^\top a^{-1}\mu)(t,x).
\end{aligned}
\tag{2.206}
$$

## 2.4.3 Optimal Investment Strategy

For the existence of optimal strategy, we need to take a broader class than $\mathscr{A}$. Put

$$\mathscr{A}_2 = \{\pi(\cdot);\pi(\cdot)^\top\sigma(\cdot,X(\cdot)) \in L^2([0,T]\times\Omega,(\mathcal{F}_t^{\tilde{W}});\mathbb{R}^1\otimes\mathbb{R}^m\}. \tag{2.207}$$

By the condition (a),

$$\pi(\cdot)\in L^2([0,T]\times\Omega,(\mathcal{F}_t^{\tilde{W}});\mathbb{R}^m) \quad \text{if } \pi(\cdot)\in\mathscr{A}_2 \tag{2.208}$$

and

$$\pi(\cdot)\in L^\infty([0,T]\times\Omega,(\mathcal{F}_t^{\tilde{W}});\mathbb{R}^m) \quad \text{if } \pi(\cdot)\in\mathscr{A}. \tag{2.209}$$

First we prepare Lemma, by using (2.208).

**Lemma 2.2 (Approximation).**

(i) *For $\pi(\cdot)\in\mathscr{A}_2$, there is a sequence $\pi_n(\cdot), n=1,2,\ldots$, in $\mathscr{A}$, such that*

$$\lim_{n\to\infty} Z^{\pi_n(\cdot)}(T) = Z^{\pi(\cdot)}(T) \quad P\text{-a.s.} \tag{2.210}$$

(ii)

$$\sup_{\mathscr{A}_2} E_{zx}[U(Z^{\pi(\cdot)}(T))] = \sup_{\mathscr{A}} E_{zx}[U(Z^{\pi(\cdot)}(T))]. \tag{2.211}$$

*Proof.* We divide the proof for (i) into three steps.

**Step 1.** For $\pi(\cdot)\in\mathscr{A}_2$, we can take a bounded $\pi_l(\cdot)$, such that

$$E\mathbf{1}\pi_l(\cdot)-\pi(\cdot)\mathbf{1}^2 < 2^{-l} \quad (l=1,2,\ldots). \tag{2.212}$$

On the other hand, for $\varepsilon>0$, there is a positive constant $k=k_\varepsilon$, such that

$$P_x\Big(\sup_{0\le s\le T}|X(s)|\ge k\Big)<\varepsilon. \tag{2.213}$$

Put

$$\pi_{l\varepsilon}(t) = \pi_l(t)\chi(\tau_k > t), \tag{2.214}$$

where

$$\tau_k = \begin{cases} \inf\{t < T; |X(t)| \geq k\}, \\ T \quad \text{if } \{\cdots\} = \text{empty}. \end{cases}$$

Then $\pi_{l,\varepsilon}(\cdot)$ is in $\mathscr{A}$.

**Step 2.** We evaluate the stochastic integral term

$$\xi := \int_0^T \pi(s)^\top \sigma(s, X(s))\, d\tilde{W}(s). \tag{2.215}$$

Replacing $\pi(\cdot)$ by $\pi_l(\cdot)$ and $\pi_{l\varepsilon}(\cdot)$, we define $\xi_l$ and $\xi_{l\varepsilon}$, respectively. Compute

$$P_x(|\xi_{l\varepsilon} - \xi| > \varepsilon) = P_x(|\xi_{l\varepsilon} - \xi| > \varepsilon; \tau_k = T) + P_x(|\xi_{l\varepsilon} - \xi| > \varepsilon; \tau_k < T). \tag{2.216}$$

Since $\pi_l(s) = \pi_{l\varepsilon}(s)$, for all $s(< T)$, for $\tau_k = T$, one has that,

$$\begin{aligned} \text{1st term in RHS } &= P_x(|\xi_l - \xi| > \varepsilon;\ \tau_k = T) \\ &\leq P_x\left(\left|\int_0^T (\pi_l(s) - \pi(s))^\top \sigma(s, X(s))\, d\tilde{W}(s)\right| > \varepsilon\right) \\ &\leq \frac{1}{\varepsilon^2} E[\mathbf{1}\pi_l(\cdot) - \pi(\cdot)\mathbf{1}^2] c_1^2 (1 + k^2), \end{aligned} \tag{2.217}$$

because $|\sigma(s,x)| \leq c_1(1 + |x|)$.

Combining (2.212), (2.213), (2.216), and (2.217), we can take a large integer $l_\varepsilon$ such that

$$P_x(|\xi_{l_\varepsilon} - \xi| > \varepsilon) < 2\varepsilon \quad \text{whenever } l > l_\varepsilon. \tag{2.218}$$

**Step 3.** By (2.212), (2.214), and (2.218), we take a sequence $\pi_n(\cdot), n = 1, 2, \ldots$ in $\mathscr{A}$, such that

$$\lim_{n\to\infty} \mathbf{1}\pi_n(\cdot) - \pi(\cdot)\mathbf{1} = 0 \quad P\text{-a.s.} \tag{2.219}$$

and

$$\lim_{n\to\infty} \int_0^T \pi_n(s)^\top \sigma(s, X(s))\, d\tilde{W}(s) = \int_0^T \pi(s)^\top \sigma(s, X(s))\, d\tilde{W}(s) \quad P\text{-a.s.} \tag{2.220}$$

From (2.187), (2.219), and (2.220), we obtain (2.210).

(ii) (2.211) is an easy consequence of (2.210) together and Fatou's lemma. This completes the proof. □

Now we are going to seek an optimal strategy.

**Theorem 2.9.** *Assume* (a)–(c). *Suppose that* (2.204)–(2.205) *has a solution* $\tilde{v} \in C^{12}([0, T] \times \mathbb{R}^d)$*, satisfying*

$$|\partial_x \tilde{v}(t, x)| \leq K(1 + |x|), \quad \forall t, x, \tag{2.221}$$

*with a positive constant* $K$*. Then*

$$\sup_{\mathscr{A}_2} E_{zx}[U(Z^{\pi(\cdot)}(T))] = z^\delta \tilde{u}(0, x), \tag{2.222}$$

*where*

$$\tilde{u}(t, x) = \frac{1}{\delta} \exp(\delta \tilde{v}(t, x)). \tag{2.223}$$

*Further,* $\hat{\Pi}(\cdot)$ *given by* (2.203) *provides an optimal strategy.*

*Proof.* We divide the proof into two steps.

**Step 1.** We show that $\hat{\pi}(t) := \hat{\Pi}(t, X(t))$ is in $\mathscr{A}_2$ and $M^{\hat{\pi}(\cdot)}$ is an exponential $(\mathcal{F}_t^{\tilde{W}})$-martingale.
Indeed, since $\hat{\Pi}(\cdot)$ is the maximum selector of (2.202), (b), (c), and (2.221) show that $\hat{\Pi}(t, x)^\top \sigma(t, x)$ is linearly growing w.r.t. $x$ uniformly in $t$. Hence, $\hat{\pi}(\cdot)$ is in $\mathscr{A}_2$. Thus, Proposition 1.5 concludes step 1.

**Step 2.** We have

$$E_{zx}[U(Z^{\hat{\pi}(\cdot)}(T))] \geq \sup_{\mathscr{A}} E_{zx}[U(Z^{\pi(\cdot)}(T))]. \tag{2.224}$$

Indeed, for $\pi(\cdot) \in \mathscr{A}$, $M^{\pi(\cdot)}$ is an exponential $(\mathcal{F}_t^{\tilde{W}})$-martingale. Put

$$\zeta^{\pi(\cdot)}(t) = \tilde{u}(t, X(t)) \exp\Big(\delta \int_0^t \eta(s, X(s), \pi(s))\, ds\Big). \tag{2.225}$$

Then Itô's formula and (2.201) yield

$$\begin{aligned} E_{zx}[U(Z^{\pi(\cdot)}(T))] &= z^\delta E_x^{\pi(\cdot)}\Big[\frac{1}{\delta} \exp\Big(\int_0^T \delta \eta(s, X(s), \pi(s))\, ds\Big)\Big] \\ &= z^\delta E_x^{\pi(\cdot)}[\zeta^{\pi(\cdot)}(T)] \quad \Big(\text{by } \tilde{u}(T, \cdot) = \frac{1}{\delta}\Big). \end{aligned} \tag{2.226}$$

Since (2.202) and (2.203) imply

$$E_x^{\pi(\cdot)}[\zeta^{\pi(\cdot)}(T)] \le E_x^{\pi(\cdot)}[\zeta^{\pi(\cdot)}(0)] = \tilde{u}(0,x) \tag{2.227}$$

and

$$E_x^{\hat{\pi}(\cdot)}[\zeta^{\hat{\pi}(\cdot)}(T)] = E_x^{\hat{\pi}(\cdot)}[\zeta^{\hat{\pi}(\cdot)}(0)] = \tilde{u}(0,x), \tag{2.228}$$

we have (2.224).

Now, using Lemma 2.2 we conclude that

$$E_{zx}[U(Z^{\hat{\pi}(\cdot)}(T))] = \sup_{\mathscr{A}_2} E_{zx}[U(Z^{\pi(\cdot)}(T))],$$

which completes the proof. □

Refer to [Na03, FSh00, FSh02] for details and related topics.

*Example 2.7 (Linear Gaussian model).* Suppose that

$$b(t,x) = b_0 + b_1 x, \quad \sigma(t,x) = \sigma,$$
$$g(t,x) = g_0 + g_1 x, \quad r(t,x) = r,$$

where $b_0, b_1, \sigma, g_0, g_1$ and r are constant and $a := \sigma\sigma^\top$ is a regular matrix. Hence the $d$-dimensional factor process $X$ and the asset prices $S^i, i = 1, \dots, m$, evolve according to the SDEs

$$\begin{cases} dX(t) = (b_0 + b_1 X(t))\,dt + dW(t), & t \in (0,T], \\ X(0) = x \ (\in \mathbb{R}^d) \end{cases} \tag{2.229}$$

and

$$\begin{cases} dS^i(t) = S^i(t)((g_0^i + g_1^i X(t))\,dt + \sigma^i\,d\tilde{W}(t)), & t \in (0,T], \\ S^i(0) = s^i > 0 \quad (i = l, \dots, m), \end{cases} \tag{2.230}$$

respectively.

The bond price $S^0(t)$ is given by

$$S^0(t) = s^0 e^{rt}. \tag{2.231}$$

Following Theorem 2.9, we will seek an optimal investment strategy. In order to solve the semilinear parabolic equation (2.204)–(2.205) for the Gauss model, we assume the quadratic from of $v(\cdot)$:

$$v(t,x) = \frac{1}{2} x^\top Q(t) x + R(t) \cdot x + S(t), \tag{2.232}$$

with $d \times d$ symmetric matrix $Q(t)$, $R(t) \in \mathbb{R}^d$ and $S(t) \in \mathbb{R}^1$. Then straightforward computations lead to the following equations:

$$\begin{cases} 0 = \dfrac{dQ}{dt}(t) + Q(t)K_0Q(t) \\ \qquad + K_1^\top Q(t) + Q(t)K_1 + \dfrac{1}{2(1-\delta)} g_1^\top a^{-1} g_1, \quad t \in [0,T), \\ Q(T) = 0, \end{cases} \tag{2.233}$$

where

$$K_0 = \delta I_d + \frac{\delta}{1-\delta}\bar{\sigma}^\top a^{-1}\bar{\sigma}, \quad K_1 = b_1 + \frac{\delta}{1-\delta}\bar{\sigma}^\top a^{-1} g_1;$$

$$\begin{cases} 0 = \dfrac{dR}{dt}(t) + (K_1 + K_0 Q(t))^\top R(t) + Q(t)K_2 + K_3, \quad t \in [0,T), \\ R(T) = 0, \end{cases} \tag{2.234}$$

where

$$K_2 = b_0 + \frac{\delta}{1-\delta}\bar{\sigma}^\top a^{-1}(g_0 - r1^m), \quad K_3 = \frac{1}{1-\delta} g_1^\top a^{-1}(g_0 - r1^m);$$

and

$$\begin{cases} 0 = \dfrac{dS}{dt}(t) + \dfrac{1}{2}\mathrm{tr}Q(t) + \dfrac{1}{2}R(t)^\top K_0 R(t) \\ \qquad + K_2 \cdot R(t) + r + \dfrac{1}{1-\delta}(g_0 - r1^m)^\top a^{-1}(g_0 - r1^m), \quad t \in [0,T), \\ S(T) = 0. \end{cases} \tag{2.235}$$

If the Riccati equation (2.233) has a solution, then we can solve (2.234) and (2.235), and $v(\cdot)$ given by (2.232) becomes a solution of (2.204)–(2.205) with linearly growing $\partial_x v(\cdot)$.

Hence

$$\hat{\pi}(t) := a^{-1}\Big(g_0 - r1^m + g_1 X(t) + \delta\bar{\sigma}\partial_x v(t, X(t))\Big) \tag{2.236}$$

gives an optimal investment strategy (refer to [KN02, W71, Wo68] for Riccati equations).

*Example 2.8 (Uniformly elliptic volatility model).* We consider a random volatility model presented in [Ph02].

Let $d = m = 1$. Suppose that $\rho \in (-1, 1)$ and $\varepsilon > 0$ are given. We assume the following forms:

$$b(t,x) = b_0 + b_1 x,\ \sigma(t,x) = (\rho\sqrt{c^2x^2+\varepsilon}, \sqrt{1-\rho^2}\sqrt{c^2x^2+\varepsilon}),$$
$$g(t,x) = g_0 + g_1 x,\ r(t,x) = r > 0$$

with constants $b_0, b_1, c, g_0, g_1$ and $r$.

$\rho$ is called the correlation between asset and the factor process. The factor process $X$ and the asset price $S$ evolve according to the SDEs (2.237) and (2.238) respectively,

$$dX(t) = (b_0 + b_1 X(t))\,dt + dW(t), \tag{2.237}$$

and

$$\begin{aligned} dS(t) =& S(t)\Big((g_0 + g_1X(t))\,dt + \rho\sqrt{c^2X(t)^2+\varepsilon}\,dW(t) \\ &+ \sqrt{1-\rho^2}\sqrt{c^2X(t)^2+\varepsilon}\,dB(t)\Big), \end{aligned} \tag{2.238}$$

where $W$ and $B$ are mutually independent real Wiener processes.

Let $v(\cdot)$ be the logarithmic transformation of the value function. Then,

$$\begin{cases} 0 = \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\delta}{2}\Big(1 + \dfrac{\rho^2}{1-\delta}\Big)\Big|\dfrac{\partial v}{\partial x}\Big|^2 \\ \quad + \Big(b_0 + b_1 x + \dfrac{\delta\rho}{1-\delta}\dfrac{g_0 - r + g_1 x}{\sqrt{c^2x^2+\varepsilon}}\Big)\dfrac{\partial v}{\partial x} \\ \quad + \dfrac{1}{2(1-\delta)}\dfrac{|g_0 - r + g_1x|^2}{c^2x^2+\varepsilon} + r, \quad t \in [0,T), \quad x \in \mathbb{R}^1, \\ v(T,x) = 0, \quad x \in \mathbb{R}^1 \end{cases} \tag{2.239}$$

holds by (2.204). Since [Ph02], Theorem 4.1 provides a solution $v(\cdot) \in C^{12}([0,T)\times\mathbb{R}^1) \cap C([0,T]\times\mathbb{R}^1)$ with $\frac{\partial v}{\partial x}$ satisfying the linear growth condition, we conclude that

$$\hat{\pi}(t) = \frac{1}{\sqrt{c^2X(t)^2+\varepsilon}}\Big(\frac{g_0 - r + g_1X(t)}{\sqrt{c^2X(t)^2+\varepsilon}} + \delta\rho\frac{\partial v}{\partial x}(t, X(t))\Big)$$

is an optimal investment strategy.

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