Chapter 8
AdS/CFT—Adding Probes

In real experiments, one often adds "probes" to a system to examine its response. Or one adds impurities to a system to see how they change the properties of the system. In this chapter, we discuss how to add probes in AdS/CFT. As a typical example, we add "quarks" to gauge theories as probes and see the behavior of quark potentials.

Coupling new degrees of freedom to the original system often arises new phenomena. Adding some new degrees of freedom to AdS/CFT should be also interesting. This is practically important as well. The $\mathscr{N} = 4$ SYM is clearly insufficient to mimic real worlds completely since, e.g., it does not have quarks.

In string theory, there are various fields and branes, so one may would like to add them. The resulting geometries or solutions have been known for some cases, but it is in general very difficult to solve the Einstein equation when there are multiple number of fields and branes.

So, one often adds them as "probes." This is just like the particle motion analysis in curved spacetime (Sects. 2.3 and 6.2). One fixes the background geometry and considers the case where the backreaction of the probe onto the geometry is negligible.

In this chapter, as a typical example, we add "quarks" to large-N_c gauge theories as a probe and analyze quark potentials. In Sect. 14.3, we see another example of a probe system, holographic superconductors.

8.1 Basics of Wilson Loop

The *Wilson loop* is an important observable in gauge theory, and it represents the quark-antiquark potential physically. As an example, consider a $U(1)$ gauge theory with gauge transformation given by

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \tag{8.1}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x). \tag{8.2}$$

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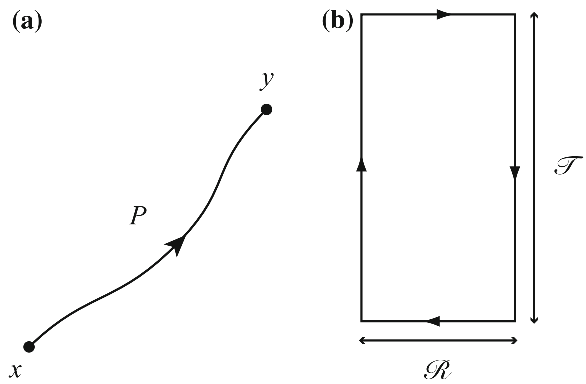

Fig. 8.1 **a** Path P. **b** The Wilson loop represents a quark-antiquark pair

A nonlocal operator such as $\phi(x)\phi^*(y)$ is not gauge invariant in general and is not an observable. But the following quantity is gauge-invariant:

$$\phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y), \tag{8.3}$$

where P is an arbitrary path from point x to y (Fig. 8.1a). It transforms as

$$\phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y) \to \phi(x)e^{i\alpha(x)}e^{i\int_P dx^\mu(A_\mu+\partial_\mu\alpha)}e^{-i\alpha(y)}\phi^*(y) \tag{8.4}$$
$$= \phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y). \tag{8.5}$$

Or if one takes a closed path P, W_P itself is gauge invariant. Thus, we define the following operator:

$$W_P(x,y) = e^{i\int_P dx^\mu A_\mu} \qquad \text{(Wilson line)}, \tag{8.6}$$
$$W_P(x,x) = e^{i\oint dx^\mu A_\mu} \qquad \text{(Wilson loop)}. \tag{8.7}$$

The Wilson loop represents the coupling of the gauge field to a test charge. Consider a charged particle with world-line $y^\mu(\lambda)$. The current is given by

$$J^\mu(x) = \oint d\lambda\, \frac{dy^\mu}{d\lambda}\delta\left(x^\mu - y^\mu(\lambda)\right). \tag{8.8}$$

The sign of the charge depends on the sign of $dy/d\lambda$. Here, we take $dy/d\lambda > 0$ for a positive charge. For a given closed path, $dy/d\lambda$ can be both positive and negative, so we have both a positive charge and a negative charge (Fig. 8.1b). Namely, the closed path describes the process of creating a "quark-antiquark pair" from the vacuum, pulling them a distance $\mathscr{R}$ apart, interacting for time $\mathscr{T}$, and annihilating them. If one uses J^μ, the coupling of the gauge field to the point particle action is written as $\delta\mathsf{S} = \int d^4x\, A_\mu J^\mu$. This perturbed action $\delta\mathsf{S}$ can be rewritten as the exponent of the

Wilson loop:

$$\delta \mathsf{S} = \int d^4x\, A_\mu(x) J^\mu(x) = \oint d\lambda \frac{dy^\mu}{d\lambda} A_\mu(y(\lambda)) = \oint dy^\mu A_\mu(y). \tag{8.9}$$

Therefore, the Wilson loop represents a partition function in the presence of a test charge:

$$\langle W_P \rangle = \frac{Z[J]}{Z[0]}. \tag{8.10}$$

Such a partition function gives the quark-antiquark potential. Let us write the Euclidean partition function formally as

$$Z = \langle f | e^{-H\mathscr{T}} | i \rangle \tag{8.11}$$

($|i\rangle$ and $|f\rangle$ are the initial state and the final state, respectively). If one uses a complete set of energy eigenstates $H|n\rangle = E_n|n\rangle$,

$$Z = \sum_n e^{-E_n\mathscr{T}} \langle f|n\rangle\langle n|i\rangle \xrightarrow{\mathscr{T}\to\infty} e^{-E_0\mathscr{T}}. \tag{8.12}$$

Thus, in the $\mathscr{T} \to \infty$ limit, the Euclidean partition function is dominated by the ground state and gives the ground state energy. When the kinetic energy is negligible, it gives the quark-antiquark potential energy.[1] Consequently,

$$\langle W_P \rangle \simeq e^{-V(\mathscr{R})\mathscr{T}}. \tag{8.13}$$

One can show that the horizontal parts of Fig. 8.1b are negligible in the large $\mathscr{T} \to \infty$ limit.

When the quark is confined like QCD, the potential grows with the separation $\mathscr{R}$, so $V(\mathscr{R}) \simeq \sigma\mathscr{R} (\mathscr{R} \gg 1)$, where σ is called the string tension. Then,

$$\langle W_P \rangle \simeq e^{-O(\mathscr{R}\mathscr{T})} = e^{-\sigma A}. \tag{8.14}$$

The exponent is proportional to the area of the Wilson loop $A = \mathscr{R}\mathscr{T}$. This behavior is known as the *area law*. An unconfined potential behaves differently. The Coulomb potential decays with the separation, and one can show that

$$\langle W_P \rangle \simeq e^{-O(\mathscr{R})} \quad (\text{when } \mathscr{R} = \mathscr{T} \gg 1). \tag{8.15}$$

This is known as the perimeter law. In this way, the Wilson loop provides a criterion for the confinement.

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[1] From $t_\mathrm{E} = it$, the Lorentzian action S_L, the Euclidean action S_E, and the potential V are related to each other by $i\mathsf{S}_L = i\int dt(-V) = -\int dt_\mathrm{E} V = -\mathsf{S}_E$.

Here, we consider only the $U(1)$ gauge theory, but a similar discussion can be done for a Yang-Mills theory.

8.2 Wilson Loops in AdS/CFT: Intuitive Approach

Let us consider the Wilson loop in AdS/CFT. The Wilson loop in AdS/CFT gives a typical example of adding a probe system to the original system. The AdS/CFT results can be understood intuitively. So, before we go through an actual computation, we first explain what kind of results one can expect in various situations. Then, we confirm our intuitive explanation via an actual computation.

The matter fields in the $\mathcal{N} = 4$ SYM are all in the adjoint representation. So, one first has to understand how to realize the fundamental representation such as a quark in AdS/CFT. Below, we describe one simple way to add such matter.

To do so, recall how the adjoint representation appeared for the D-brane (Fig. 8.2a). The open strings can have endpoints on a D-brane, but when there are multiple number of D-branes, an open string can have endpoints in various ways; there are N_c^2 possibilities. This means that the string transforms as the adjoint representation of $SU(N_c)$ gauge theory.

Now, consider an infinitely long string (Fig. 8.2b). In this case, the string can have endpoints in N_c different ways. This means that the string transforms as the fundamental representation of $SU(N_c)$ gauge theory. In this sense, such a long string represents a "quark." Such a string has an extension and tension, so the string has a large mass, which means that the long string represents a heavy quark. We discuss the Wilson loop in AdS/CFT using such a string.

We saw earlier that the string model of QCD does not describe potentials other than the confining potential (Problem 2 of Sect. 5.1). However, one can avoid this problem in AdS/CFT, and one can get the Coulomb potential which appears at short distances in QCD. The AdS/CFT result differs from the simple string model one essentially because of the curved spacetime effect as discussed below. Note that we avoided Problem 1 of Sect. 5.1 by the same trick.

First, we discuss the simplest case, the pure AdS case, to understand the basic idea of the AdS/CFT quark potential. In this case, one gets only the Coulomb potential. We then consider a more generic AdS spacetime and get a confining potential as well. Also, if we consider a black hole, we can recover behaviors in plasma phase.

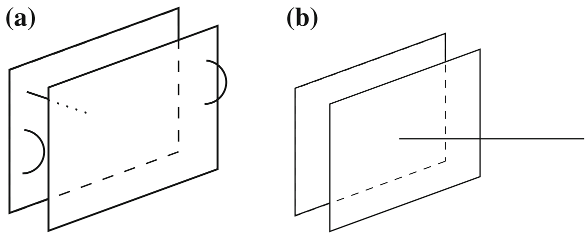

Fig. 8.2 **a** An open string can have endpoints on a D-brane. The N_c coincident D-branes represent a $SU(N_c)$ gauge theory. **b** A long string represents a massive "quark"

The pure AdS spacetime The AdS metric in Poincaré coordinates is written as

$$ds^2 = \left(\frac{r}{L}\right)^2 (-dt^2 + dx^2 + \cdots). \tag{8.16}$$

The line element has the factor r^2. We measure the gauge theory time and distance using t and x, but they differ from the proper time and distance of the AdS spacetime. This is the important point, and the qualitative behavior of the quark potential can be understood using this fact.

Figure 8.3 shows the AdS spacetime schematically. Denote the quark-antiquark separation as $\Delta x = \mathscr{R}$. The quark-antiquark pair is represented by a string which connects the pair. The string has the tension, so the tension tends to minimizes the string length. At first glance, one would connect the pair by a straight string at $r = \infty$ (Fig. 8.4). But this does not minimize the string length. This is because the coordinate distance does not represent a true distance (proper distance) in a curved spacetime. The figure does not show the proper length properly, so one needs a care. For the AdS spacetime, the proper length of the string actually gets shorter if the string goes inside the AdS spacetime ($r \neq \infty$). The line element has the factor r^2, so the proper length $r\Delta x$ gets shorter near the origin.

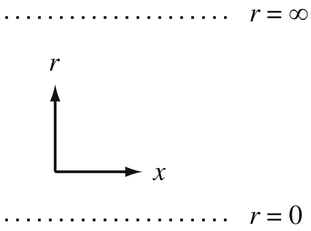

Fig. 8.3 Schematically drawn AdS spacetime. The horizontal direction represents one of three-dimensional space the gauge theory lives. The vertical direction represents the AdS radial coordinate. The radial coordinate extends from $r = 0$ to $r = \infty$, but we draw in a compact region for illustration

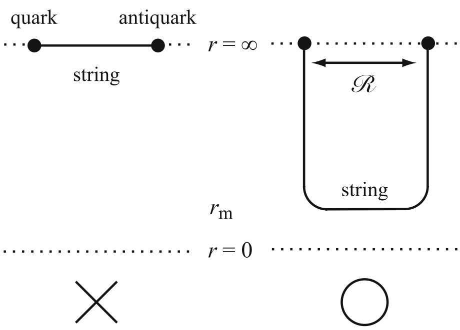

Fig. 8.4 The straight string is not the lowest energy state (left), and the string which goes inside the AdS spacetime is the lowest energy state (right)

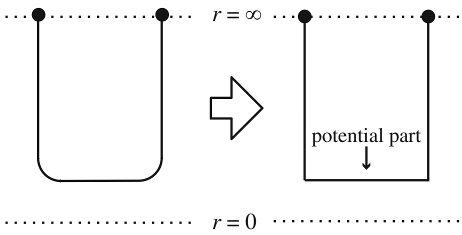

Fig. 8.5 The string which connects the quark pair (left) is approximated by a rectangular string. The energy of the horizontal string gives the quark potential

According to the analysis of Sect. 8.4, this string is roughly divided into two parts: the part the string extends vertically, and the part the string extends horizontally. So, for simplicity, let us approximate the configuration by a rectangular string (Fig. 8.5). Only the horizontal string contributes to the quark potential. This part varies as we vary the quark separation $\mathscr{R}$. On the other hand, the vertical string does not vary much. This part simply describes the quark mass.

We need a little more information to compute the potential. The explicit computation shows that the string turning point $r = r_m$ behaves as

$$r_m \propto L^2/\mathscr{R} \tag{8.17}$$

(Fig. 8.6). Also, the line element (8.16) gives two consequences. First, the proper length of the horizontal string is $(r/L)\mathscr{R}$, so the string energy $E(r)$ is given by

$$E(r) \propto \left(\frac{r}{L}\right)\mathscr{R}. \tag{8.18}$$

Second, this energy is the proper energy and not the gauge theory energy. The timelike direction also has the factor r^2 as in Eq. (8.16). The gauge theory time is the coordinate time t not the proper time. As a result, the gauge theory energy differs from the proper energy $E(r)$. From Eq. (8.16), the proper time τ_r is related to the gauge theory time by $\tau_r = (r/L)t$, so the proper energy is related to the gauge theory energy E_t by

$$E_t = \left(\frac{r}{L}\right)E(r). \tag{8.19}$$

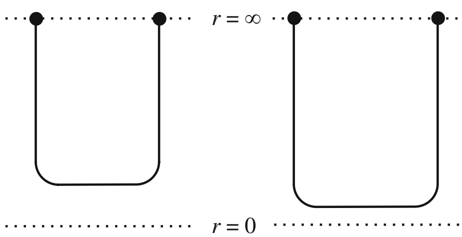

Fig. 8.6 The behavior of the string as we vary the quark separation. The larger $\mathscr{R}$ lowers the string turning point r_m as $r_m \propto 1/\mathscr{R}$

This is the UV/IR relation in Sect. 6.3 [1, 2]. Thus, the potential is given by

$$E_t = \left(\frac{r_m}{L}\right) E(r) \propto \left(\frac{r_m}{L}\right)^2 \mathscr{R} \tag{8.20}$$
$$\propto \frac{L^2}{\mathscr{R}}, \tag{8.21}$$

where we also used Eq. (8.17). This result [3, 4] has two important points:

1. First, we obtained the Coulomb potential $E \propto 1/\mathscr{R}$ not the confining potential $E \propto \mathscr{R}$. Namely, the string connecting the quark-antiquark pair does not necessarily implies a confining potential, but it can describe an unconfining potential using the curved spacetime. *In this way, we resolved Problem 2 of the string model in* Sect. 5.1. But then, how can we describe the confining potential in AdS/CFT? We will discuss this point below.
2. Second, the potential is proportional to L^2. According to the AdS/CFT dictionary, $L^2 \propto \lambda^{1/2}$, so the potential is proportional to $(g_{\mathrm{YM}}^2 N_c)^{1/2}$. But perturbatively, the potential is proportional to $g_{\mathrm{YM}}^2 N_c$. This is because the AdS/CFT result corresponds to the large-N_c limit and represents a nonperturbative effect.[2]

Let us evaluate the potential for a generic metric for later use. By repeating the above argument, the potential energy becomes

$$\boxed{E_t = \sqrt{-g_{00}}|_{r_m} E(r) = \frac{1}{2\pi l_s^2}\sqrt{-g_{00}g_{xx}}|_{r_m}\mathscr{R},} \tag{8.22}$$

where the metric is evaluated at $r = r_m$. We also included the factor of the string tension $T = 1/(2\pi l_s^2)$ which we ignored in Eq. (8.18).

The confining phase AdS/CFT can also describe the confining potential which the old string model can describe qualitatively well. The pure AdS spacetime corresponds to the $\mathscr{N} = 4$ SYM not to QCD. The $\mathscr{N} = 4$ SYM is scale invariant and the confining phase does not exist even at zero temperature. We need to modify the simple AdS geometry to describe a theory which is closer to QCD.

Many examples are known about how the AdS spacetime is deformed if one deforms the $\mathscr{N} = 4$ SYM. But we use a simple model to simplify our analysis here [7]. The AdS spacetime extends from $r = \infty$ to $r = 0$, but in this model, we cut off the AdS spacetime at $r = r_c$ (Fig. 8.7). Let us suppose that the confinement happens at a low-energy scale Λ. In AdS/CFT, the r-coordinate has the interpretation as the gauge theory energy scale. So, the confinement means that the AdS spacetime is

[2] For the $\mathscr{N} = 4$ SYM at zero temperature, the potential is evaluated nonperturbatively from the field theory point of view, and it indeed behaves as $\lambda^{1/2}$ at strong coupling [5, 6].

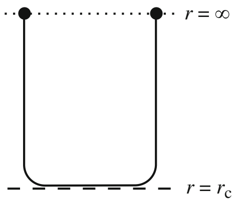

Fig. 8.7 In the cutoff AdS spacetime, the string reaches the end of the space, $r = r_c$, when the quark separation is large enough

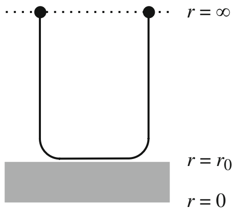

Fig. 8.8 The plasma phase case. The shaded region represents the black hole

modified deep inside the AdS spacetime $r \propto \Lambda$. The cutoff AdS roughly represents this effect.[3]

Even though we modify the spacetime, there is little difference if the string is far enough from the cutoff $r = r_c$. One gets the Coulomb potential like the pure AdS spacetime. But if the quark separation $\mathscr{R}$ is large enough, there is a new effect.

In the AdS spacetime, the turning point of the string behaves as $r_m \propto 1/\mathscr{R}$. But in the cutoff AdS spacetime, the string reaches at $r = r_c$ for a large enough $\mathscr{R}$. Once the string reaches there, the string cannot go further. Thus, from Eq. (8.22), the energy of the horizontal string is given by

$$E_t \propto r_c^2 \mathscr{R} \simeq O(\mathscr{R}), \tag{8.23}$$

which is indeed the confining potential.

After all, what contributes to the potential energy is the string at the cutoff $r = r_c$, so the AdS/CFT computation *essentially reduces to the old string model one.* AdS/CFT takes the advantage of the old string model and at the same time overcomes the difficulty of the model.

The plasma phase We now consider the finite temperature case or the plasma phase. According to AdS/CFT, the $\mathscr{N} = 4$ SYM at finite temperature corresponds to the AdS black hole (Fig. 8.8).

At finite temperature, there is a black hole horizon at $r = r_0$. But if the string is far enough from the black hole, the geometry is approximately the AdS spacetime, so one approximately has the Coulomb potential. But if the string reaches the horizon, there is a new effect.

[3] The cutoff AdS is a toy model for the confinement, but we discuss an explicit example in Appendix.

For a black hole, the line element in the timelike direction has the unique behavior, and the relation (8.19) between $E(r)$ and E_t is modified. For the Schwarzschild-AdS$_5$ (SAdS$_5$) black hole, the line element is given by

$$ds^2 = -\left(\frac{r}{L}\right)^2 \left\{1 - \left(\frac{r_0}{r}\right)^4\right\} dt^2 + \cdots , \tag{8.24}$$

so $g_{00} = 0$ at the horizon $r = r_0$. Thus, Eq. (8.22) gives

$$E_t = 0. \tag{8.25}$$

Namely, the horizontal string has no contribution to the energy. Thus, there is no force when the quark separation is large enough. This is the Debye screening in AdS/CFT [8–10].

Return of Wilson loops The Wilson loop argument here was proposed in less than two weeks after the systematic AdS/CFT researches started in 1998. Various extensions were made within a month. But people started to come back to such simple analysis since 2006.

What changed the situation? In the past, such a computation was made to find circumstantial evidences of AdS/CFT. Namely, one would like to check whether AdS/CFT correctly reproduces the behavior of gauge theories or not. People do not really have real applications in mind. This is understandable since supersymmetric gauge theories are different from QCD, so probably one was reluctant to apply them to the "real world." But in recent years, people revisits such analysis and compute various effects by taking into account the real experimental situations.

As discussed in Sect. 4.1.2, the perturbative QCD is not very effective even in the plasma phase. Thus, heavy-ion physicists try to identify the typical "fingerprints" of QGP. Some of the fingerprints discussed to date are

1. Small shear viscosity (Chaps. 4 and 12)
2. Jet quenching
3. J/Ψ-suppression

In the parton hard-scattering, jets are often formed. A jet is a collection of hadrons which travel roughly in the same direction. If jets are formed in the plasma medium, the energy of the jets are absorbed by the medium, so the number of observed hadrons are suppressed. This is the *jet quenching* (Fig. 8.9).

Another fingerprint is the *J/Ψ-suppression* [11]. J/Ψ is a "charmonium" which consists of $c\bar{c}$. Since a charm quark is heavy ($\approx$4.2 GeV), the charm pair production occurs only at the early stage of heavy-ion collisions. Now, if the production occurs in the plasma medium, the interaction between $c\bar{c}$ is screened by the light quarks and gluons in between, which is the Debye screening. Then, the charm quark is more likely to bind with the plasma constituents rather than the charm antiquark. The result is the suppression of J/Ψ production.

These phenomena have been discussed in AdS/CFT. For example, consider the jet quenching [12–17]. So far, we considered the static quark to obtain the potential.

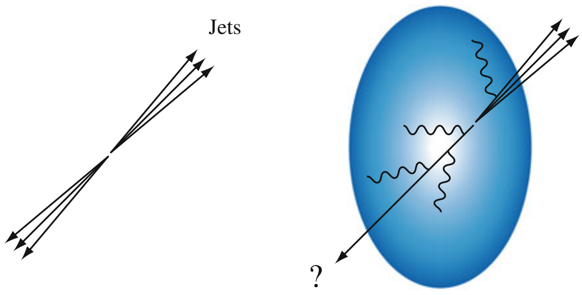

Fig. 8.9 *Left* The collision of nuclei in vacuum. *Right* Jet quenching in the plasma. The ellipsoid represents the plasma

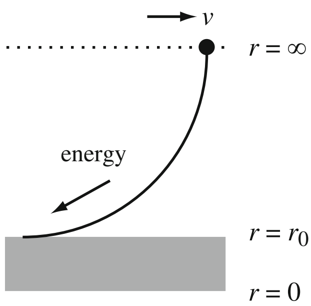

Fig. 8.10 Jet quenching in AdS/CFT

But in this case, one is interested in how the quark loses its energy. So, move the quark (string) with velocity v along the x-direction. Then, the string is dragged as in Fig. 8.10. The string is dragged because the energy of the string flows towards the horizon. This energy loss is interpreted as the energy loss of the quark in the plasma medium.

8.3 String Action

In order to confirm the intuitive explanation in the last section, let us first consider the string action. The string action is obtained using the similar argument as the particle action in Sect. 2.1. A particle draws a world-line in spacetime. Similarly, a string sweeps a two-dimensional surface, a world-sheet, in spacetime (Fig. 8.11). We write the particle action by the proper length of the world-line. Similarly, it is natural to write the string action by the area A of the world-sheet:

$$\mathsf{S} = -\mathsf{T} \int dA. \tag{8.26}$$

The parameter T has the dimensions $[\mathsf{T}] = \mathrm{L}^{-2}$, which makes the action dimensionless. Physically, it represents the string tension. It is convenient to introduce a

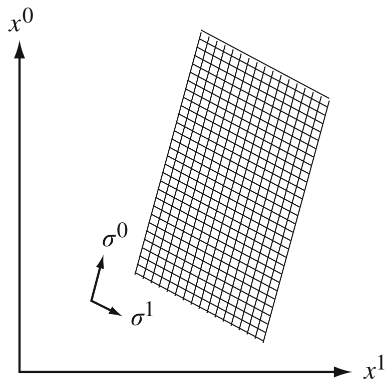

Fig. 8.11 A string sweeps a world-sheet in spacetime

parameter l_s with the dimension of length and to write the tension as

$$\mathsf{T} = \frac{1}{2\pi l_s^2}. \tag{8.27}$$

The parameter l_s represents the characteristic length scale of the string (string length).

Just as the particle action, introduce coordinates $\sigma^a = (\sigma^0, \sigma^1)$ on the world-sheet. Then, the world-sheet is described by $x^M(\sigma^a)$. Using the world-sheet coordinates σ^a, the spacetime metric is written as

$$ds^2 = \eta_{MN} dx^M dx^N = \eta_{MN} \frac{\partial x^M}{\partial \sigma^a} \frac{\partial x^N}{\partial \sigma^b} d\sigma^a d\sigma^b \tag{8.28}$$

$$=: h_{ab}\, d\sigma^a d\sigma^b, \tag{8.29}$$

where h_{ab} is known as the *induced metric*. What we are doing here is essentially the same as the embedding of a hypersurface into a higher-dimensional spacetime in Chap. 6. For example, embed S^2 into $\mathbb{R}^3$:

$$ds^2 = dX^2 + dY^2 + dZ^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{8.30}$$

In this case, we take S^2 coordinates as $\sigma^a = (\theta, \varphi)$, and the induced metric is given by

$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{8.31}$$

Using the world-sheet coordinates, one can write the area element as

$$dA = d^2\sigma \sqrt{-\det h_{ab}}. \tag{8.32}$$

For S^2, $dA = \sin\theta\, d\theta\, d\varphi$, which is the familiar area element for S^2. The coordinates σ^a are just parameterizations on the world-sheet, so the area element is invariant under

$$\sigma^{a'} = \sigma^{a'}(\sigma^b). \tag{8.33}$$

The situation is similar to general relativity. In general relativity, one writes the volume element as $d^d x\sqrt{-g}$, and the volume element is invariant under coordinate transformations. The only difference is whether one considers a spacetime or a world-sheet. In general relativity, one considers the volume element in spacetime and the coordinate transformation in spacetime, whereas Eq. (8.32) is the area element on the world-sheet and Eq. (8.33) is the coordinate transformation on the world-sheet.

Using Eq. (8.32), one gets the *Nambu-Goto action*:

$$\boxed{\mathsf{S}_{\text{NG}} = -\mathsf{T}\int d^2\sigma\sqrt{-\det h_{ab}}.} \tag{8.34}$$

From Eq. (8.28), the induced metric is written as

$$h_{ab} = \begin{pmatrix} \dot{x}\cdot\dot{x} & \dot{x}\cdot x' \\ \dot{x}\cdot x' & x'\cdot x' \end{pmatrix} \quad \left(\dot{} := \partial_{\sigma^0},\ {}' := \partial_{\sigma^1}\right). \tag{8.35}$$

Just as in Eq. (8.31), this is a matrix on (a, b) indices. Using this, we can write the Nambu-Goto action as

$$\mathsf{S}_{\text{NG}} = -\mathsf{T}\int d^2\sigma\sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2}. \tag{8.36}$$

One can consider a few extensions of the action:

1. Here, we used the Minkowski spacetime as the ambient spacetime. But one can get the curved spacetime case by replacing η_{MN} by $g_{MN}(x)$ like the particle action case in Sect. 2.1.
2. A brane action is obtained similarly. For the Dp-brane, with the $(p+1)$-dimensional induced metric h_{ab}, one writes the action as[4]

$$\mathsf{S}_{\text{D}p} = -\mathsf{T}_p\int d^{p+1}\sigma\, e^{-\phi}\sqrt{-\det h_{ab}}\,. \tag{8.37}$$

Such a brane can be added as a probe just like the string.

[4] Note the factor of the dilation $e^{-\phi}$. The dilaton ϕ and the string coupling constant g_s are related by $g_s \simeq e^{\phi}$, so this factor means that the mass density of the D-brane is proportional to $1/g_s$ [Eq. (5.58)].

8.4 Wilson Loops in AdS/CFT: Actual Computation

In this section, we confirm our intuitive explanation in Sect. 8.2 by an actual computation. As an example, we compute the Wilson loop in the pure AdS_5 spacetime.

The quark potential is given by the energy of the string in AdS/CFT. So, the starting point is the Nambu-Goto action (8.34). The action has the reparameterization invariance on the world-sheet, so we can choose convenient world-sheet coordinates by coordinate transformations (gauge fixing). Here, we take the *static gauge*[5] (Fig. 8.12):

$$\sigma^0 = t, \quad \sigma^1 = r, \quad x = x(r). \tag{8.38}$$

The induced metric on the AdS_5 spacetime is given by

$$ds_5^2 = \left(\frac{r}{L}\right)^2 (-dt^2 + d\boldsymbol{x}_3^2) + L^2 \frac{dr^2}{r^2} \tag{8.39}$$

$$= -\left(\frac{r}{L}\right)^2 dt^2 + \left\{ \left(\frac{L}{r}\right)^2 + \left(\frac{r}{L}\right)^2 x'^2 \right\} dr^2 \quad ({}' := \partial_r), \tag{8.40}$$

so the determinant of the induced metric becomes

$$-\det h_{ab} = 1 + \left(\frac{r}{L}\right)^4 x'^2. \tag{8.41}$$

Then, the action is given by

$$\mathsf{S} = -\frac{1}{2\pi l_s^2} \int d^2\sigma \sqrt{-\det h_{ab}} = -\frac{\mathscr{T}}{2\pi l_s^2} \int dr \sqrt{1 + \left(\frac{r}{L}\right)^4 x'^2}, \tag{8.42}$$

where $\mathscr{T}$ is the time duration in t. The Lagrangian does not contain x, so there is a conserved momentum p_x which is conjugate to x:

$$p_x = \frac{\partial L}{\partial x'} \propto \frac{\left(\frac{r}{L}\right)^4 x'}{\sqrt{1 + \left(\frac{r}{L}\right)^4 x'^2}} = (\text{constant}). \tag{8.43}$$

Let us determine the constant. The string has the turning point at $r = r_m$. At the turning point, $\partial_r x|_{r=r_m} = \infty$, so the constant is given by

[5] The string has the turning point at $r = r_m$, so our gauge is not well-defined in reality. But this is no problem because it is enough to consider only the half of the string by symmetry. One normally takes the gauge $\sigma^0 = t$, $\sigma^1 = x$, and $r = r(x)$ instead of Eq. (8.38). The computation is slightly easier in our gauge.

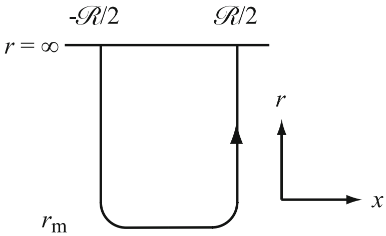

Fig. 8.12 The configuration to compute the Wilson loop

$$\text{(constant)} = \left.\frac{\left(\frac{r}{L}\right)^4 x'}{\sqrt{1+\left(\frac{r}{L}\right)^4 x'^2}}\right|_{r=r_m} = \left(\frac{r_m}{L}\right)^2. \tag{8.44}$$

Solving Eq. (8.43) in terms of x', one gets

$$x'^2 = \left(\frac{L}{r}\right)^4 \frac{1}{\left(\frac{r}{r_m}\right)^4 - 1}\,. \tag{8.45}$$

One can determine the string configuration $x(r)$ by solving Eq. (8.45). We take $x=0$ at $r=r_m$, so $x(r)$ is given by the integral

$$\int_0^x dx = \int_{r_m}^r \left(\frac{L}{r}\right)^2 \frac{dr}{\sqrt{\left(\frac{r}{r_m}\right)^4 - 1}}\,. \tag{8.46}$$

In particular, $x = \mathscr{R}/2$ at $r\to\infty$, so Eq. (8.46) gives

$$\frac{\mathscr{R}}{2} = \frac{L^2}{r_m}\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} \quad (y := r/r_m) \tag{8.47}$$

$$= \frac{L^2}{r_m}\frac{\sqrt{2}\pi^{3/2}}{\Gamma(\frac{1}{4})^2}\,. \tag{8.48}$$

From Eq. (8.48),

$$r_m \simeq \frac{L^2}{\mathscr{R}}\,, \tag{8.49}$$

which justifies Eq. (8.17). Also, when $r \gg r_m$, Eq. (8.46) gives

$$\frac{\mathscr{R}}{2} - x = \frac{L^2}{r_m}\int_{r/r_m}^\infty \frac{dy}{y^2\sqrt{y^4-1}} \simeq r^{-3}. \tag{8.50}$$

The string quickly approaches $x = \mathscr{R}/2$ for large r, which confirms the Fig. 8.5 behavior.

We determined the string configuration. We now evaluate the action (8.42) to compute the quark potential. Substituting Eq. (8.45) into Eq. (8.42), one gets

$$\mathsf{S} = -\frac{\mathscr{T}}{2\pi l_s^2} \int_{r_m}^{\infty} dr \, \frac{\left(\frac{r}{r_m}\right)^2}{\sqrt{\left(\frac{r}{r_m}\right)^4 - 1}} \,. \tag{8.51}$$

Then, the potential energy is given by

$$E = -\frac{2\mathsf{S}}{\mathscr{T}} = \frac{2}{2\pi l_s^2} r_m \int_1^{\infty} \frac{y^2 dy}{\sqrt{y^4 - 1}} \,. \tag{8.52}$$

The integral actually diverges, but this reflects the fact that the quark is infinitely heavy. We must subtract the quark mass contribution.[6] The isolated string configuration is given by $x' = 0$. By substituting $x' = 0$ into Eq. (8.42), one obtains the quark mass contribution:

$$\mathsf{S}_0 = -\frac{\mathscr{T}}{2\pi l_s^2} \int_0^{\infty} dr, \tag{8.53}$$

$$E_0 = \frac{2}{2\pi l_s^2} \int_0^{\infty} dr. \tag{8.54}$$

Thus,

$$E - E_0 = \frac{2}{2\pi l_s^2} r_m \left\{ \int_1^{\infty} \left(\frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) dy - 1 \right\} . \tag{8.55}$$

The expression is proportional to r_m, but $r_m \simeq L^2/\mathscr{R}$ from Eq. (8.49), so we get the Coulomb potential $E \simeq 1/\mathscr{R}$. The evaluation of the integral in Eq. (8.55) gives

$$E - E_0 = -\frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \frac{\lambda^{1/2}}{\mathscr{R}}, \tag{8.56}$$

which agrees with our intuitive explanation (8.21).

[6] See Ref. [18] for a more appropriate procedure.

8.5 Summary

- Adding probes to the original system is a simple but useful way to explore the system further.
- As an example, we add Wilson loops to various asymptotically AdS spacetimes. The Wilson loop is an important nonlocal observable in a gauge theory, and it represents the quark-antiquark potential.
- In AdS/CFT, the Wilson loop corresponds to adding an infinitely long string extending from the AdS boundary.
- In the pure AdS spacetime, the holographic Wilson loop gives the Coulomb potential which is a curved spacetime effect. The potential is proportional to $(g_{\mathrm{YM}}^2 N_c)^{1/2}$ which represents a strong coupling effect.
- If one changes background geometries, one gets various quark potentials such as the confining potential and the Debye screening. Also, if one considers dynamical strings, one can discuss dynamical problems such as the jet quenching in the plasma phase.

New keywords

Wilson loop	induced metric
cutoff AdS spacetime	Nambu-Goto action
jet quenching	static gauge
J/Ψ-suppression	[AdS soliton]

Appendix: A Simple Example of the Confining Phase

In the text, we discussed the cutoff AdS spacetime as a toy model of the confining phase. Here, as an explicit example, we discuss the S^1-compactified $\mathcal{N} = 4$ SYM and its dual geometry.

AdS soliton The SAdS$_5$ black hole is given by

$$ds_5^2 = \left(\frac{r}{L}\right)^2 \left(-h dt^2 + dx^2 + dy^2 + dz^2\right) + L^2 \frac{dr^2}{hr^2}, \tag{8.57}$$

$$h = 1 - \left(\frac{r_0}{r}\right)^4 . \tag{8.58}$$

We now compactify the z-direction as $0 \le z < l$.

However, the compactified SAdS$_5$ black hole is not the only solution whose asymptotic geometry is $\mathbb{R}^{1,2} \times S^1$. The "double Wick rotation"

$$z' = it, \quad z = it' \tag{8.59}$$

of the black hole gives the metric

$$ds_5^2 = \left(\frac{r}{L}\right)^2 \left(-dt'^2 + dx^2 + dy^2 + hdz'^2\right) + L^2 \frac{dr^2}{hr^2}, \tag{8.60}$$

which has the same asymptotic structure $\mathbb{R}^{1,2} \times S^1$. The geometry (8.60) is known as the *AdS soliton* [19].

As Euclidean geometries, they are the same, but they have different Lorentzian interpretations. The AdS soliton is not a black hole. Rather, because of the factor h in front of dz'^2, the spacetime ends at $r = r_0$ just like the Euclidean black hole. From the discussion in the text, this geometry describes a confining phase.

For the SAdS black hole, the imaginary time direction has the periodicity $\beta = \pi L^2/r_0$ to avoid a conical singularity. Similarly, for the AdS soliton, z' has the periodicity l given by

$$l = \frac{\pi L^2}{r_0}. \tag{8.61}$$

Wilson loop Let us consider the quark potential in this geometry. Take the quark separation as $\mathscr{R}$ in the x-direction. This corresponds to a Wilson loop on the $t' - x$ plane. Since the geometry ends at $r = r_0$, the formula (8.22) gives

$$E_t \propto \sqrt{-g_{t't'}g_{xx}}|_{r_0}\mathscr{R} = \left(\frac{r_0}{L}\right)^2 \mathscr{R}, \tag{8.62}$$

which is a confining potential.

In Sect. 8.2, we considered the Wilson loop in the SAdS black hole and discussed the Debye screening. Here, we consider a Wilson loop in the same Euclidean geometry, but the Wilson loop here is different from the one in Sect. 8.2:

- For the AdS soliton, we consider the Wilson loop on the t'-x plane (temporal Wilson loop), but as the black hole, this is a Wilson loop on the z-x plane or a spatial Wilson loop.
- For the black hole, we considered the temporal Wilson loop on the t-x plane, but as the AdS soliton, this is a spatial Wilson loop on the z'-x plane.

At high temperature $Tl > 1$, the AdS soliton undergoes a first-order phase transition to the SAdS black hole (Sect. 14.2.1).

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