Chapter 2
General Relativity and Black Holes

In this book, black holes frequently appear, so we will describe the simplest black hole, the Schwarzschild black hole and its physics.

Roughly speaking, a black hole is a region of spacetime where gravity is strong so that even light cannot escape from there. The boundary of a black hole is called the *horizon*. Even light cannot escape from the horizon, so the horizon represents the boundary between the region which is causally connected to distant observers and the region which is not.

General relativity is mandatory to understand black holes properly, but a black hole-like object can be imagined in Newtonian gravity. Launch a particle from the surface of a star, but the particle will return if the velocity is too small. In Newtonian gravity, the particle velocity must exceed the escape velocity in order to escape from the star. From the energy conservation, the escape velocity is determined by

$$\frac{1}{2}v^2 = \frac{GM}{r}. \tag{2.1}$$

If the radius r becomes smaller for a fixed star mass M, the gravitational potential becomes stronger, so the escape velocity becomes larger. When the radius becomes smaller, eventually the escape velocity reaches the speed of light. Then, no object can escape from the star. Setting $v = c$ in the above equation gives the radius

$$r = \frac{2GM}{c^2}, \tag{2.2}$$

which corresponds to the horizon. For a solar mass black hole, the horizon radius is about 3 km, which is 2.4×10^5 times smaller than the solar radius.

To be precise, the above argument is false from several reasons:

1. First, the speed of light is arbitrary in Newtonian mechanics. As a result, the speed of light decreases as light goes away from the star. But in special relativity the speed of light is the absolute velocity which is independent of observers.

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2. Newtonian mechanics cannot determine how gravity affects light.
3. In the above argument, light can temporally leave from the "horizon." But in general relativity light cannot leave even temporally.

The Newtonian argument has various problems, but the horizon radius (2.2) itself remains true in general relativity, and we utilize Newtonian arguments again later.

Below we explain black holes using general relativity, but we first discuss the particle motion in a given spacetime. For the flat spacetime, this is essentially a review of special relativity. We take this approach from the following reasons: (i) We study the particle motion around black holes later in order to understand black hole physics; (ii) The main purpose of this book is not to obtain a new geometry but to study the behavior of a "probe" such as a particle in a known geometry; (iii) String theory is a natural extension of the particle case below.

2.1 Particle Action

Flat spacetime case—review of special relativity First, let us consider the particle motion in the flat spacetime. We denote the particle's coordinates as $x^\mu := (t, x, y, z)$. According to special relativity, the distance which is invariant relativistically is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.3}$$

The distance is called timelike when $ds^2 < 0$, spacelike when $ds^2 > 0$, and null when $ds^2 = 0$. For the particle, $ds^2 < 0$, so one can use the *proper time* τ given by

$$ds^2 = -d\tau^2. \tag{2.4}$$

The proper time gives the relativistically invariant quantity for the particle, so it is natural to use the proper time for the particle action:

$$\mathsf{S} = -m \int d\tau. \tag{2.5}$$

The action takes the familiar form in the nonrelativistic limit. With the velocity $v^i := dx^i/dt$, $d\tau$ is written as $d\tau = dt(1 - v^2)^{1/2}$, so

$$\mathsf{S} = -m \int dt(1 - v^2)^{1/2} \simeq -m \int dt \left(1 - \frac{1}{2}v^2 + \cdots\right), \quad (v \ll 1). \tag{2.6}$$

In the final expression, the first term represents the particle's rest mass energy, and the second term represents the nonrelativistic kinetic energy.

A particle draws a *world-line* in spacetime (Fig. 2.1). Introducing an arbitrary parametrization λ along the world-line, the particle coordinates or the particle motion

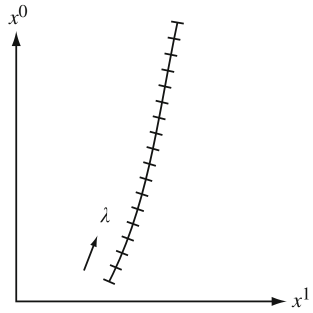

Fig. 2.1 A particle draws a world-line in spacetime

are described by $x^\mu(\lambda)$. Using the parametrization,

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu = -\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\lambda^2 \quad (\dot{} := d/d\lambda), \tag{2.7}$$

so the action is written as

$$\mathsf{S} = -m \int d\lambda \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = \int d\lambda\, L. \tag{2.8}$$

The parametrization λ is a redundant variable, so the action should not depend on λ. In fact, the action is invariant under

$$\lambda' = \lambda'(\lambda). \tag{2.9}$$

The canonical momentum of the particle is given by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} = m \frac{dx_\mu}{d\tau} \tag{2.10}$$

($\dot{x}^2 := \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$). Note that the canonical momentum satisfies

$$p^2 = m^2 \frac{\dot{x}^2}{-\dot{x}^2} = -m^2, \tag{2.11}$$

so its components are not independent:

$$\boxed{p^2 = -m^2\,.} \tag{2.12}$$

The Lagrangian does not contain x^μ itself but contains only $\dot{x}^\mu$, so p_μ is conserved. Thus, $p_\mu = m\, dx_\mu/d\tau =$ (constant), which describes the free motion.

The particle's *four-velocity* u^μ is defined as

$$u^\mu := \frac{dx^\mu}{d\tau}. \tag{2.13}$$

In terms of the ordinary velocity v^i,

$$u^\mu = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(1, v^i), \quad \left(\frac{d\tau}{dt}\right)^2 = 1 - v^2 := \gamma^{-2}. \tag{2.14}$$

Since $p_\mu = mu_\mu$ and $p^2 = -m^2$, u^μ satisfies $u^2 = -1$.

The action (2.5) is proportional to m, and one cannot use it for a massless particle. The action which is also valid for a massless particle is given by

$$\mathsf{S} = \frac{1}{2}\int d\lambda \left\{e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2\right\}. \tag{2.15}$$

From this action,

$$\text{Equation of motion for } e\text{:} \quad \dot{x}^2 + e^2m^2 = 0, \tag{2.16}$$

$$\text{Canonical momentum:} \qquad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}\dot{x}_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}. \tag{2.17}$$

Use Eq. (2.16) at the last equality of Eq. (2.17). Using Eq. (2.16), the Lagrangian reduces to the previous one (2.8):

$$\frac{1}{2}\left\{e^{-1}\dot{x}^2 - em^2\right\} = -m\sqrt{-\dot{x}^2}. \tag{2.18}$$

This action also has the reparametrization invariance: the action (2.15) is invariant under

$$\lambda' = \lambda'(\lambda), \tag{2.19}$$

$$e' = \frac{d\lambda}{d\lambda'}e. \tag{2.20}$$

Particle action (curved spacetime) Now, move from special relativity to general relativity. The invariant distance is given by replacing the flat metric $\eta_{\mu\nu}$ with a curved metric $g_{\mu\nu}(x)$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{2.21}$$

Here, we first consider the particle motion in a curved spacetime and postpone the discussion how one determines $g_{\mu\nu}$.

The action is obtained by replacing the flat metric $\eta_{\mu\nu}$ with a curved metric $g_{\mu\nu}$:

$$\mathsf{S} = -m \int d\tau = -m \int d\lambda \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \tag{2.22}$$

Just like the flat spacetime, the canonical momentum is given by

$$p_\mu = m \frac{g_{\mu\nu}(x)\dot{x}^\nu}{\sqrt{-\dot{x}^2}}, \quad \dot{x}^2 := g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu, \tag{2.23}$$

and the constraint $p^2 = -m^2$ exists. Also,[1]

$$\boxed{\text{If the metric is independent of } x^\mu \text{, its conjugate momentum } p_\mu \text{ is conserved.}}$$

The variational principle $\delta\mathsf{S} = 0$ gives the world-line which extremizes the action. For the flat spacetime, the particle has the free motion and has the "straight" world-line. For the curved spacetime, the world-line which extremizes the action is called a *geodesic*. The variation of the action with respect to x^μ gives the equation of motion for the particle:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{2.24}$$

This is known as the *geodesic equation*.[2] Here, $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}). \tag{2.25}$$

The particle motion is determined by solving the geodesic equation. However, black holes considered in this book have enough number of conserved quantities so that one does not need to solve the geodesic equation.

The massless particle action is also obtained by substituting $\eta_{\mu\nu}$ with $g_{\mu\nu}$ in Eq. (2.15). The particle action described here can be naturally extended into string and the objects called "branes" in string theory (Sect. 8.3).

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[1] What is conserved is p_μ which may not coincide with p^μ in general. In the flat spacetime, p_μ and p^μ are the same up to the sign, but in the curved spacetime, the functional forms of p_μ and p^μ differ by the metric $g_{\mu\nu}(x)$.

[2] Note that we use the proper time τ here not the arbitrary parametrization λ. The equation of motion does not take the form of the geodesic equation for a generic λ. A parameter such as τ is called an *affine parameter*. As one can see easily from the geodesic equation, the affine parameter is unique up to the linear transformation $\tau \to a\tau + b$ (a, b: constant). For the massless particle, the proper time cannot be defined, but the affine parameter is possible to define.

2.2 Einstein Equation and Schwarzschild Metric

So far we have not specified the form of the metric, but the metric is determined by the Einstein equation[3]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}, \tag{2.26}$$

where G is the Newton's constant, and $T_{\mu\nu}$ is the energy-momentum tensor of matter fields. The Einstein equation claims that the spacetime curvature is determined by the energy-momentum tensor of matter fields.

We will encounter various matter fields. Of prime importance in this book is

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}, \tag{2.27}$$

where Λ is called the *cosmological constant*. In this case, the Einstein equation becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0. \tag{2.28}$$

From Eq. (2.27), the cosmological constant acts as a constant energy density, and the positive cosmological constant, $\Lambda > 0$, has been widely discussed as a dark energy candidate. On the other hand, what appears in AdS/CFT is the negative cosmological constant, $\Lambda < 0$. The anti-de Sitter spacetime used in AdS/CFT is a solution of this case (Chap. 6).

For now, let us consider the Einstein equation with no cosmological constant and with no matter fields:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.29}$$

The simplest black hole, the *Schwarzschild black hole*, is the solution of the above equation:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2. \tag{2.30}$$

Here, $d\Omega_2^2 := d\theta^2 + \sin^2\theta d\varphi^2$ is the line element of the unit S^2. We remark several properties of this black hole:

- The metric approaches the flat spacetime $ds^2 \to -dt^2 + dr^2 + r^2 d\Omega_2^2$ as $r \to \infty$.
- As we will see in Sect. 2.3.2, M represents the black hole mass. We will also see that the behavior GM/r comes from the four-dimensional Newtonian potential.

[3] In App., we summarize the formalism of general relativity for the readers who are not familiar to it.

- The horizon is located at $r = 2GM$ where $g_{00} = 0$.
- A coordinate invariant quantity such as

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{2.31}$$

  diverges at $r = 0$. This location is called a *spacetime singularity*, where gravity is infinitely strong.

We now examine the massive and massless particle motions around the black hole to understand this spacetime more.

2.3 Physics of the Schwarzschild Black Hole

2.3.1 Gravitational Redshift

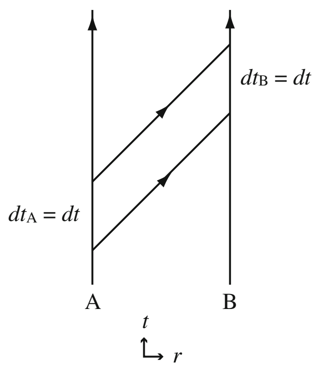
The *gravitational redshift* is one of three "classic tests" of general relativity; the other two are mentioned in Sect. 2.3.2. The discussion here is used to discuss the surface gravity (Sect. 3.1.2) and to discuss the gravitational redshift in the AdS spacetime (Sect. 6.2).

Consider two static observers at A and B (Fig. 2.2). The observer at A sends light, and the observer at B receives light. The light follows the null geodesics $ds^2 = 0$, so

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 = 0, \tag{2.32}$$

$$dt^2 = \frac{g_{rr}}{-g_{00}}dr^2 \quad \to \quad \int_A^B dt = \int_A^B \sqrt{\frac{g_{rr}(r)}{-g_{00}(r)}}dr. \tag{2.33}$$

Fig. 2.2 Exchange of light between A and B



The right-hand side of the final expression does not depend on when light is sent, so the coordinate time until light reaches from A to B is always the same. Thus, if the observer at A emits light for the interval dt, the observer at B receives light for the interval dt as well.

However, the proper time for each observer differs since $d\tau^2 = |g_{00}|dt^2$:

$$d\tau_A^2 \simeq |g_{00}(A)|dt^2, \tag{2.34}$$
$$d\tau_B^2 \simeq |g_{00}(B)|dt^2. \tag{2.35}$$

But both observers should agree to the total number of light oscillations, so

$$\omega_B d\tau_B = \omega_A d\tau_A. \tag{2.36}$$

The energy of the photon is given by $E = \hbar\omega$, so $E_B d\tau_B = E_A d\tau_A$, or

$$\boxed{\frac{E_B}{E_A} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}}.} \tag{2.37}$$

For simplicity, consider the Schwarzschild black hole and set $r_B = \infty$ and $r_A \gg GM$. Then,

$$E_\infty = \sqrt{|g_{00}(A)|}E_A \simeq E_A - \frac{GM}{r_A}E_A < E_A. \tag{2.38}$$

Here, we used $\sqrt{|g_{00}(A)|} = (1 - 2GM/r_A)^{1/2} \simeq 1 - (GM)/r_A$. Thus, the energy of the photon decreases at infinity. The energy of the photon decreases because the photon has to climb up the gravitational potential. Indeed, the second term of Eq. (2.38) takes the form of the Newtonian potential for the photon. Also, suppose that the point A is located at the horizon. Since $g_{00}(A) = 0$ at the horizon, $E_\infty \to 0$, namely light gets an infinite redshift.

2.3.2 Particle Motion

Motion far away The particle motion can be determined from the geodesic equation (2.24). However, there are enough number of conserved quantities for a static spherically symmetric solution such as the Schwarzschild black hole, which completely determines the particle motion without solving the geodesic equation.

- First, because of spherical symmetry, the motion is restricted to a single plane, and one can choose the equatorial plane ($\theta = \pi/2$) as the plane without loss of generality.

- Second, as we saw in Sect. 2.1, when the metric is independent of a coordinate x^μ, its conjugate momentum p_μ is conserved. For a static spherically symmetric solution, the metric is independent of t and φ, so the energy p_0 and the angular momentum p_φ are conserved.

Then, the particle four-momentum is given by

$$p_0 =: -mE, \tag{2.39}$$

$$p_\varphi =: mL, \tag{2.40}$$

$$p^r = m\frac{dr}{d\tau}, \tag{2.41}$$

$$p^\theta = 0. \tag{2.42}$$

(E and L are the energy and the angular momentum per unit rest mass.) Because the four-momentum satisfies the constraint $p^2 = -m^2$,

$$g^{00}(p_0)^2 + m^2 g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g^{\varphi\varphi}(p_\varphi)^2 = -m^2. \tag{2.43}$$

Substitute the metric of the Schwarzschild black hole. When the angular momentum $L = 0$,

$$\left(\frac{dr}{d\tau}\right)^2 = (E^2 - 1) + \frac{2GM}{r}. \tag{2.44}$$

Since $(dr/d\tau)^2 \simeq E^2 - 1$ as $r \to \infty$, $E = 1$ represents the energy when the particle is at rest at infinity, namely the rest mass energy of the particle. Differentiating this equation with respect to τ and using $\tau \simeq t$ in the nonrelativistic limit, one gets

$$\frac{d^2r}{dt^2} \simeq -\frac{GM}{r^2}, \tag{2.45}$$

which is nothing but the Newton's law of gravitation. Thus, M in the Schwarzschild black hole (2.30) represents the black hole mass.

Similarly, when $L \neq 0$,

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{2.46}$$

$$= (E^2 - 1) + \frac{2GM}{r} - \frac{L^2}{r^2} + \frac{2GML^2}{r^3}. \tag{2.47}$$

The third term in Eq. (2.47) represents the centrifugal force term. On the other hand, the fourth term is a new term characteristic of general relativity. General relativity has "classic tests" such as

- The perihelion shift of Mercury
- The light bending

in addition to the gravitational redshift, and both effects come from this fourth term.[4] The fourth term is comparable to the third term only when the particle approaches $r \simeq 2GM$. This distance corresponds to the horizon radius of the black hole and is about 3 km for a solar mass black hole, so the effect of this term is normally very small.

We will generalize the discussion here to a generic static metric in order to discuss the surface gravity in Sect. 3.1.2. We will also examine the particle motion in the AdS spacetime in Sect. 6.2.

Motion near horizon We now turn to the particle motion near the horizon. How long does it take until the particle reaches the horizon? For simplicity, we assume that the particle is at rest at infinity ($E = 1$) and that the particle falls radially ($L = 0$). From Eq. (2.44), the particle motion for $E = 1$ and $L = 0$ is given by

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r_0}{r} \tag{2.48}$$

($r_0 = 2GM$). Near the horizon,

$$\frac{dr}{d\tau} \simeq -1. \tag{2.49}$$

We choose the minus sign since the particle falls inward (r decreases as time passes). So, to go from $r = r_0 + R$ to $r = r_0 + \varepsilon$,

$$\tau \simeq -\int_{r_0+R}^{r_0+\varepsilon} dr = R - \varepsilon. \tag{2.50}$$

Namely, the particle reaches the horizon in a *finite proper time.*

However, the story changes from the point of view of the coordinate time t. By definition, $p^0 = m dt/d\tau$, and from the conservation law, $p^0 = g^{00} p_0 = m(1 - r_0/r)^{-1}$. Then,

$$\frac{d\tau}{dt} = \frac{r - r_0}{r}. \tag{2.51}$$

Thus,

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 \left(\frac{d\tau}{dt}\right)^2 = \frac{r_0(r - r_0)^2}{r^3} \tag{2.52}$$

or

$$\frac{dr}{dt} \simeq -\frac{r - r_0}{r_0} \tag{2.53}$$

[4] For the light bending, use the equation for the massless particle instead of Eq. (2.43).

near the horizon, and

$$t \simeq -r_0 \int_{r_0+R}^{r_0+\varepsilon} \frac{dr}{r - r_0} = r_0(\ln R - \ln \varepsilon), \tag{2.54}$$

so $t \to \infty$ as $\varepsilon \to 0$. Namely, it takes an *infinite coordinate time* until the particle reaches the horizon. Incidentally, Eq. (2.53) near the horizon takes the same form as the massless case below. Namely, the particle moves with the speed of light near the horizon.

Let us consider the massless case. For the massless particle, $p^2 = 0$ or $ds^2 = 0$, so

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 = 0 \tag{2.55}$$

$$\to \left(\frac{dr}{dt}\right)^2 = -\frac{g_{00}}{g_{rr}} = \left(1 - \frac{r_0}{r}\right)^2 \tag{2.56}$$

$$\to \frac{dr}{dt} = -\left(1 - \frac{r_0}{r}\right) \simeq -\frac{r - r_0}{r_0}. \tag{2.57}$$

Near the horizon, the expression takes the same form as the massive case (2.53) as promised. We considered the infalling photon, but one can consider the outgoing photon. In this case, $t \to \infty$ until the light from the horizon reaches the observer at finite r.

There is nothing special to the horizon from the point of view of the infalling particle. But there is a singular behavior from the point of view of the coordinate t. This is because the Schwarzschild coordinates (t, r) are not well-behaved near the horizon. Thus, we introduce the coordinate system which is easier to see the infalling particle point of view.

2.4 Kruskal Coordinates

The particle motion discussed so far can be naturally understood by using a new coordinate system, the *Kruskal coordinates*. The Kruskal coordinates (u, v) are defined by

$$r > r_0 \begin{cases} u = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/(2r_0)} \cosh\left(\frac{t}{2r_0}\right) \\ v = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/(2r_0)} \sinh\left(\frac{t}{2r_0}\right) \end{cases} \tag{2.58}$$

$$r < r_0 \begin{cases} u = \left(1 - \frac{r}{r_0}\right)^{1/2} e^{r/(2r_0)} \sinh\left(\frac{t}{2r_0}\right) \\ v = \left(1 - \frac{r}{r_0}\right)^{1/2} e^{r/(2r_0)} \cosh\left(\frac{t}{2r_0}\right) \end{cases} \tag{2.59}$$

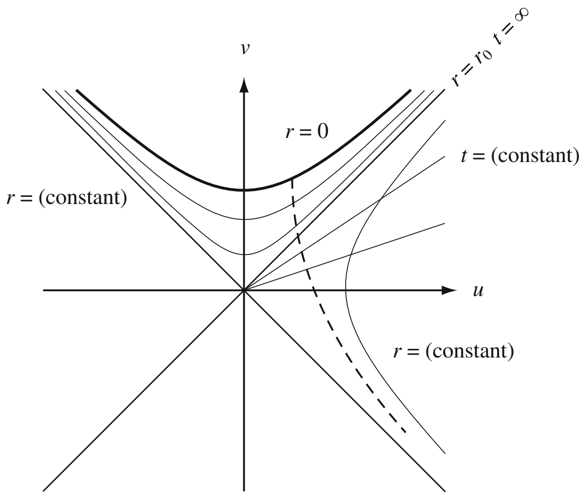


Fig. 2.3 Kruskal coordinates. The light-cones are kept at 45°, which is convenient to see the causal structure. The dashed line represents an example of the particle path. Once the particle crosses the horizon, it must reach the singularity

By the coordinate transformation, the metric (2.30) becomes

$$ds^2 = \frac{4r_0^3}{r}e^{-r/r_0}(-dv^2 + du^2) + r^2 d\Omega_2^2. \tag{2.60}$$

Here, we use not only (u, v) but also use r, but r should be regarded as $r = r(u, v)$ and is determined by

$$\left(\frac{r}{r_0} - 1\right) e^{r/r_0} = u^2 - v^2. \tag{2.61}$$

One can see the following various properties from the coordinate transformation and the metric (see also Fig. 2.3):

- The metric (2.60) is not singular at $r = r_0$. There is a singularity at $r = 0$. The transformation (2.58) is singular at $r = r_0$, but this is not a problem. Because the transformation relates the coordinates which are singular at $r = r_0$ to the coordinates which are not singular at $r = r_0$, the transformation should be singular there.
- The null world-line $ds^2 = 0$ is given by $dv = \pm du$. In this coordinate system, the lines at 45° give light-cones just like special relativity, which is convenient to see the causal structure of the spacetime.
- The $r =$ (constant) lines are hyperbolas from Eq. (2.61).
- In particular, in the limit $r = r_0$, the hyperbola becomes a null line, so *the horizon $r = r_0$ is a null surface*. Namely, the horizon is not really a spatial boundary but is a light-cone. In special relativity, the events inside light-cones cannot influence the events outside light-cones. Similarly, the events inside the horizon cannot influence the events outside the horizon. Then, even light cannot reach from $r < r_0$ to $r > r_0$.
- For $r < r_0$, the $r =$ (constant) lines become spacelike. This means that a particle cannot remain at $r =$ (constant) because the geodesics of a particle cannot be

spacelike. The singularity at $r = 0$ is spacelike as well. Namely, the singularity is not a point in spacetime, but rather it is the end of "time."
- The $t =$ (constant) lines are straight lines. In particular, the $t \to \infty$ limit is given by $u = v$. One can see that it takes an infinite coordinate time to reach the horizon.

To summarize, the particle falling into the horizon cannot escape and necessarily reaches the singularity.

New keywords

After you read each chapter, try to explain the terms in "New keywords" by yourself to check your understanding.

horizon	cosmological constant
proper time	Schwarzschild black hole
world-line	spacetime singularity
four-velocity	gravitational redshift
geodesic	Kruskal coordinates
affine parameter	

Appendix: Review of General Relativity

Consider a coordinate transformation

$$x'^\mu = x'^\mu(x). \tag{2.62}$$

Under a coordinate transformation, a quantity is called a vector if it transforms as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu, \tag{2.63}$$

and as a 1-form if it transforms "oppositely":

$$V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu. \tag{2.64}$$

The tensors with a multiple number of indices are defined similarly.

In general, the derivative of a tensor such as $\partial_\mu V^\nu$ does not transform as a tensor, but the covariant derivative ∇_μ of a tensor transforms as a tensor. The covariant derivatives of the vector and the 1-form are given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + V^\alpha \Gamma^\nu_{\alpha\mu} , \tag{2.65}$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - V_\alpha \Gamma^\alpha_{\mu\nu}. \tag{2.66}$$

As a useful relation, the covariant divergence of a vector is given by

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right), \tag{2.67}$$

where $g := \det g$. This can be shown using a formula for a matrix M:

$$\partial_\mu(\det M) = \det M \,\mathrm{tr}(M^{-1}\partial_\mu M). \tag{2.68}$$

For example, dx^μ transforms as a vector

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}dx^\nu \tag{2.69}$$

and the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x). \tag{2.70}$$

Thus, the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is invariant under coordinate transformations. Under the infinitesimal transformation $x'^\mu = x^\mu - \xi^\mu(x)$, Eq. (2.70) is rewritten as

$$g'_{\mu\nu}(x-\xi) = \left(\delta^\rho_{\ \mu} + \partial_\mu\xi^\rho\right)\left(\delta^\sigma_{\ \nu} + \partial_\nu\xi^\sigma\right)g_{\rho\sigma} \tag{2.71}$$

or

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + (\partial_\mu\xi^\rho)g_{\rho\nu} + (\partial_\nu\xi^\rho)g_{\mu\rho} + \xi^\rho\partial_\rho g_{\mu\nu} \tag{2.72}$$

$$= g_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{2.73}$$

In general relativity, an action must be a scalar which is invariant under coordinate transformations. From Eq. (2.69),

$$d^4x' = \left|\frac{\partial x'}{\partial x}\right|d^4x, \tag{2.74}$$

where $|\partial x'/\partial x|$ is the Jacobian of the transformation. On the other hand, $\sqrt{-g}$ transforms in the opposite manner:

$$\sqrt{-g'} = \left|\frac{\partial x}{\partial x'}\right|\sqrt{-g}. \tag{2.75}$$

Thus, $d^4x\,\sqrt{-g}$ is the volume element which is invariant under coordinate transformations.

The metric is determined by the Einstein-Hilbert action:

$$\mathsf{S} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \tag{2.76}$$

Here, G is the Newton's constant, and the Ricci scalar R is defined by the Riemann tensor $R^\alpha{}_{\mu\nu\rho}$ and the Ricci tensor $R_{\mu\nu}$ as follows:

$$R^\alpha{}_{\mu\nu\rho} = \partial_\nu \Gamma^\alpha_{\mu\rho} - \partial_\rho \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - \Gamma^\alpha_{\sigma\rho}\Gamma^\sigma_{\mu\nu}, \tag{2.77}$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{2.78}$$

The variation of the Einstein-Hilbert action gives

$$\begin{aligned}\delta\mathsf{S} = \frac{1}{16\pi G} \int d^4x \, \Big\{ &\sqrt{-g} R_{\mu\nu}(\delta g^{\mu\nu}) + (\delta\sqrt{-g}) R_{\mu\nu} g^{\mu\nu} \\ &+ \sqrt{-g}(\delta R_{\mu\nu}) g^{\mu\nu} \Big\}, \end{aligned} \tag{2.79}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. The second term can be rewritten by using[5]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{2.80}$$

One can show that the third term reduces to a surface term, so it does not contribute to the equation of motion.[6] Therefore,

$$\delta\mathsf{S} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}, \tag{2.81}$$

and by requiring $\delta\mathsf{S} = 0$, one gets the vacuum Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.82}$$

The contraction of Eq. (2.82) gives $R_{\mu\nu} = 0$.

When one adds the matter action $\mathsf{S}_{\text{matter}}$, the equation of motion becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{2.83}$$

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[5] Using Eq. (2.68) gives $\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}$. Then, use another matrix formula $\delta M = -M\delta(M^{-1})M$ which can be derived from $MM^{-1} = I$. Then, $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$. Note that $\delta g_{\mu\nu} \neq g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$. Namely, we do not use the metric to raise and lower indices of the metric variation.

[6] Care is necessary to the surface term in order to have a well-defined variational principle. This issue will be discussed in Chap. 7 App. and Chap. 12 App. 1.

where $T_{\mu\nu}$ is the energy-momentum tensor for matter fields:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta \mathsf{S}_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{2.84}$$

Various matter fields appear in this book, but the simplest term one can add to the Einstein-Hilbert action is given by

$$\mathsf{S}_{cc} = -\frac{1}{8\pi G} \int d^4x \sqrt{-g}\,\Lambda, \tag{2.85}$$

which is the cosmological constant term. From Eq. (2.84),

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}, \tag{2.86}$$

so the Einstein equation becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0. \tag{2.87}$$

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