Chapter 2
Application to the Noumi-Yamada System with a Large Parameter

2.1 Introduction

It is known that a traditional Painlevé equation (of the variable t) is obtained by the compatibility condition of a system of second order linear differential equations of the variables x and t. Here, when we focus upon the underlying linear system, the latter variable t is often called a deformation parameter. We can consider, with the appropriate introduction of a large parameter η into these systems, the Stokes geometry for both the linear and non-linear systems in the same way as that described in the previous chapter.

It is highly expected to have geometrical correspondence between the Stokes geometry of the t-space for the Painlevé equation and that of the x-space for the underlying linear differential equation. As a matter of fact, Kawai and Takei [KT1] have shown that, in the Stokes geometry of the x-space, i.e., the one for the underlying linear differential equation, a pair of ordinary turning points is directly connected by a Stokes curve if the deformation parameter t of the linear differential equation is located at a point in a Stokes curve of the t-space for the Painlevé equation. In other words, when t belongs to the Stokes curve of the non-linear differential equation, the Stokes geometry of the x-space becomes degenerate in the sense that two different Stokes curves emanating from each turning point of the pair accidentally coincide.

Several families of non-linear equations are recently found as a higher order extension of a traditional Painlevé equation. The Noumi-Yamada system $(\mathrm{NY})_m$ $(m = 2, 3, \ldots)$ is one of such a family, and like a traditional Painlevé equation it can be obtained by the compatibility condition of a system of higher order linear differential equations. Therefore, by introducing a large parameter into these systems, the similar correspondence between the Stokes geometry of the t-space for $(\mathrm{NY})_m$ and that of the x-space for the underlying linear system is expected as that for a traditional Painlevé equation. In fact, Takei [T4] shows that, in the x-space, a pair of ordinary turning points is directly connected by a Stokes curve if the deformation parameter t is located in a Stokes curve $\mathscr{T}$ of the t-space and furthermore if t is sufficiently close to a turning point from which the Stokes curve $\mathscr{T}$ emanates.

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Since the underlying linear system is a higher order one unlike that of a traditional Painlevé equation, if $t \in \mathscr{T}$ is located far from the turning point, one cannot necessarily observe degeneration of the Stokes geometry of the x-space. That is, we often encounter a configuration of the Stokes geometry of the x-space which has no pair of turning points directly connected by a Stokes curve even if t is located at a point in a Stokes curve of the non-linear system.

The bifurcation phenomenon of a Stokes curve (Sect. 1.7) is one of the origins of such an unexpected situation; in some particular case of $(\mathrm{NYL})_{2m}$, a linear system that underlies $(\mathrm{NY})_{2m}$ (cf. (2.2.6) below), when a simple turning point s_0 hits, at $t = t^*$, a Stokes curve $\mathscr{S}$ which connects two ordinary turning points s and d, $\mathscr{S}$ then bifurcates in general (depending on the type of s_0 and that of $\mathscr{S}$) and after the bifurcation no pair of ordinary turning points is connected by a Stokes curve even if the parameter t lies in the Stokes curve of $(\mathrm{NY})_{2m}$ but each of the triplet $\{s_0,\ s,\ d\}$ of ordinary turning points is connected by a Stokes curve with one of the triplet $\{v_j\}_{j=1,2,3}$ of virtual turning points when the parameter t lies in the Stokes curve of $(\mathrm{NY})_{2m}$ [AHKKoNSShT, Fig. 10].

In order to systematically understand the repeated bifurcation phenomena and their relevance to virtual turning points, we introduce the notion of what we call a **bidirectional binary tree**, which connects several ordinary turning points in a manner specified later (Definition 2.4.3). The appearance of such a tree in the Stokes geometry of $(\mathrm{NYL})_{2m}$ is a counterpart of the appearance of a pair of ordinary turning points connected by a Stokes curve in the Stokes geometry of the underlying Schrödinger equation of a traditional Painlevé equation [KT2, Chap. 4]. Note that a bidirectional binary tree of degree other than 2 always contains, by definition, a part of a new Stokes curve as its edges, and hence, a virtual turning point is indispensable in finding a bidirectional binary tree. As we want to explain the core part of the problem in a concise manner, we do not discuss in this article the procedure to find finitely many virtual turning points needed for the description of the Stokes geometry; that is, we start with the model of the Stokes geometry in the terminology of [H3]. As a practical problem, finding the model of the Stokes geometry is an important step in obtaining a concrete figure, and it requires much computational effort, as is seen in [H1].

2.2 $(\mathrm{NY})_\ell$ and $(\mathrm{NYL})_\ell$ with a Large Parameter

The Noumi-Yamada system $(\mathrm{NY})_\ell$ ($\ell = 2, 3, \dots$) is a system of non-linear differential equations of unknown $(\ell + 1)$-functions $u(t) = (u_0(t), \dots, u_\ell(t))$ of the variable t, which was first introduced by Noumi and Yamada [NY]. It is well-known that the first member $(\mathrm{NY})_2$ and the second member $(\mathrm{NY})_3$ of $(\mathrm{NY})_\ell$ are equivalent to the traditional Painlevé equations (P_{IV}) and (P_{V}), respectively. The corresponding system with a large parameter was introduced in [T1] and intensively studied from the viewpoint of the exact WKB analysis. Let us first recall the explicit form of the Noumi-Yamada system with a large parameter η. As the structure of $(\mathrm{NY})_\ell$ depends on the parity of ℓ, we concentrate our attention, in most cases, on the case where $\ell = 2m$ ($m \in \mathbb{N}$), i.e., ℓ is even. The system $(\mathrm{NY})_{2m}$ with a large parameter η is of the following form:

$$\eta^{-1}\frac{du_j}{dt} = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \widehat{\alpha}_j \quad (j = 0, 1, \ldots, 2m), \quad (2.2.1)$$

where each index j of u_j is considered to be an element of $\mathbb{Z}/(2m+1)\mathbb{Z}$, i.e., $u_{j+2m+1} = u_j$ and $\widehat{\alpha}_j$ is a formal power series of η^{-1} with constant coefficients, that is, $\widehat{\alpha}_j$ has the form

$$\widehat{\alpha}_j = \alpha_j^{(0)} + \eta^{-1}\alpha_j^{(1)} + \eta^{-2}\alpha_j^{(2)} + \cdots \qquad (j = 0, 1, \ldots, 2m)$$

with $\alpha_j^{(k)} \in \mathbb{C}$. We sometimes denote by α_j the leading term $\alpha_j^{(0)} \in \mathbb{C}$ of $\widehat{\alpha}_j$. In addition, we assume that these $\widehat{\alpha}_j$'s satisfy the condition

$$\widehat{\alpha}_0 + \widehat{\alpha}_1 + \cdots + \widehat{\alpha}_{2m} = \eta^{-1}, \quad (2.2.2)$$

which entails that the leading terms α_j's satisfy

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = 0. \quad (2.2.3)$$

Note that it follows from the condition (2.2.2) that, by summing up all the equations of $(\mathrm{NY})_{2m}$, we have

$$\frac{d}{dt}(u_0 + \cdots + u_{2m}) = 1. \quad (2.2.4)$$

Hence we also put the following additional equation into those of $(\mathrm{NY})_{2m}$ as a normalization condition:

$$u_0 + u_1 + \cdots + u_{2m} = t. \quad (2.2.5)$$

Summing up, the system $(\mathrm{NY})_{2m}$ consists of $(2m+2)$ equations, that is, Eqs. (2.2.1) and (2.2.5).

As it is well-known, the non-linear equation $(\mathrm{NY})_\ell$ describes the compatibility condition of a system of linear partial differential equations. In our case it consists of a linear differential equation $(\mathrm{NYL})_\ell$ in x-variable that depends on a parameter t (a deformation parameter) and another linear differential equation in t-variable that controls the isomonodromic deformation of $(\mathrm{NYL})_\ell$; the explicit form of $(\mathrm{NYL})_\ell$ is as follows.

$$\frac{d\psi}{dx} = \eta A_t(x)\psi, \quad (2.2.6)$$

where $\psi = {}^t(\psi_0(x), \ldots, \psi_\ell(x))$ and $A_t(x)$ is a square matrix of the size $\ell + 1$ with a parameter t defined by

$$A_t(x) = -x^{-1} \begin{pmatrix} \widehat{e}_0 & u_1(t) & 1 & & & & \\ & \widehat{e}_1 & u_2(t) & 1 & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & \widehat{e}_{\ell-2} & u_{\ell-1}(t) & 1 \\ x & & & & & \widehat{e}_{\ell-1} & u_\ell(t) \\ xu_0(t) & x & & & & & \widehat{e}_\ell \end{pmatrix}. \tag{2.2.7}$$

Here $u(t) = (u_0(t), u_1(t), \ldots, u_\ell(t))$ is a solution of $(\mathrm{NY})_\ell$ and $\widehat{e}_j$ $(j = 0, 1, \ldots, \ell)$ is a formal power series of η^{-1} with constant coefficients determined by the relations

$$\widehat{e}_0 + \cdots + \widehat{e}_\ell = 0, \quad \widehat{\alpha}_j = \widehat{e}_j - \widehat{e}_{j+1} + \eta^{-1}\delta_{j,0} \quad (j = 0, 1, \ldots, \ell). \tag{2.2.8}$$

Here $\delta_{j,0}$ denotes the Kronecker's symbol.

2.3 Stokes Geometry of $(\mathrm{NY})_{2m}$

Now we define the Stokes geometry of the non-linear system $(\mathrm{NY})_{2m}$. For this purpose, we first construct a formal solution $\widehat{u}(t) = (\widehat{u}_0(t), \widehat{u}_1(t), \ldots, \widehat{u}_{2m}(t))$ of $(\mathrm{NY})_{2m}$ in the form

$$\widehat{u}(t) = u^{(0)}(t) + u^{(1)}(t)\eta^{-1} + u^{(2)}(t)\eta^{-2} + u^{(3)}(t)\eta^{-3} + \cdots. \tag{2.3.1}$$

Here $u^{(k)}(t) = (u_0^{(k)}(t), \ldots, u_{2m}^{(k)}(t))$ and each $u_j^{(k)}(t)$ is a multi-valued holomorphic function over $\mathbb{C}$ except for a finite number of exceptional points. We say that $\widehat{u}(t)$ is a 0-**parameter formal solution** of $(\mathrm{NY})_{2m}$ if it satisfies $(\mathrm{NY})_{2m}$ as a formal power series of η^{-1}.

We briefly explain how to construct such a 0-parameter formal solution, which does not necessarily exist for an arbitrary parameter of $(\mathrm{NY})_{2m}$. We introduce some subsets of the space of parameters $(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in \mathbb{C}^{2m+1}$ to describe a condition which assures the existence of a 0-parameter solution. By taking (2.2.3) into account, let $A^{2m} \subset \mathbb{C}^{2m+1}$ denote the space of allowable parameters

$$\{(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in \mathbb{C}^{2m+1};\ \alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = 0\}. \tag{2.3.2}$$

Then we define

$$E^{2m}_{\mathrm{cup}} := \bigcup_{\substack{0 \le i \le 2m,\\ 0 \le k \le 2m-1}} \{(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in A^{2m};\ \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k} = 0\}. \tag{2.3.3}$$

Note that the set E^{2m}_{cup} consists of a finite number of hypersurfaces in A^{2m}, and hence, $A^{2m}\backslash E^{2m}_{\mathrm{cup}}$ is an open dense subset in A^{2m}.

By putting (2.3.1) into (2.2.1) and (2.2.5), we find that the leading term $u^{(0)}(t)$ of (2.3.1) satisfies the system of algebraic equations

$$h_j(u^{(0)}(t)) = 0 \quad (j = 0, \dots, 2m) \quad \text{and} \quad g(u^{(0)}(t)) = 0,$$

where $h_j(u)$ and $g(u)$ are the polynomials of $u = (u_0, \dots, u_{2m})$ respectively defined by

$$\begin{aligned} h_j(u) &:= u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j \quad (j = 0, 1, \dots, 2m), \\ g(u) &:= \sum_{j=0}^{2m} u_j. \end{aligned}$$

Then it follows from Theorem 6 in [AH] that, if $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{2m}) \notin E^{2m}_{\mathrm{cup}}$, then $u^{(0)}(t)$ can be solved as a multi-valued holomorphic function of the variable t with a finite number of branching points of finite degree. Furthermore, it is bounded near the branching points and it is the unique solution as a multi-valued holomorphic function. Thus we can get the leading term $u^{(0)}(t)$ when $\alpha \notin E^{2m}_{\mathrm{cup}}$.

Next let $H(u)$ denote the Jacobian matrix

$$\frac{\partial(h_0, \dots, h_{2m-1}, g)}{\partial(u_0, \dots, u_{2m})} \tag{2.3.4}$$

of polynomials $h_0, h_1, \dots, h_{2m-1}$ and g of the variables u_j's. Note that, since the sum of h_j's are identically zero, and thus, their Jacobian matrix is degenerate, we substitute g for the last h_{2m} of h_j's in the above definition. Then the same theorem says that $\det H(u^{(0)}(t))$ has an only finite number of zero points as a function of t, that is, $\det H(u^{(0)}(t))$ never vanishes identically if $\alpha \notin E^{2m}_{\mathrm{cup}}$.

Now we construct the lower order term $u^{(k)}(t)$ $(k \geq 1)$. By the normalization condition (2.2.5) and the differential equations except for one corresponding to $j = 2m$ in (2.2.1), we can obtain the following recursive relations:

$$H(u^{(0)}(t))u^{(k+1)} = R^{(k)}\left(t,\ u^{(0)}(t),\ \dots,\ u^{(k)}(t),\ \frac{du^{(k)}}{dt}(t)\right) \quad (k = 0, 1, 2, \dots). \tag{2.3.5}$$

Here $R^{(k)}$ consists of polynomials of the variables $t, u^{(0)}, \dots, u^{(k)}$ and $\dfrac{du^{(k)}}{dt}$. Since $\det H(u^{(0)}(t))$ does not vanish except for a finite number of points as we have already noted, we can successively determine $u^{(k)}(t)$ by (2.3.5). Hence, for a generic parameter $\widehat{\alpha}$, we have obtained a 0-parameter formal solution $\widehat{u}(t)$ of $(\mathrm{NY})_{2m}$.

Let us now consider the linearized system of $(\mathrm{NY})_{2m}$ at the 0-parameter solution $u = \widehat{u}(t)$ thus obtained. By putting $u = \widehat{u}(t) + \widehat{U}(t)$ into the system (2.2.1) where $\widehat{U}(t) = (\widehat{U}_0(t), \widehat{U}_1(t), \ldots, \widehat{U}_{2m}(t))$ are new unknown functions and by taking its linear part with respect to $\widehat{U}$, we obtain the system of linear differential equations

$$\eta^{-1}\frac{d\widehat{U}}{dt} = \widehat{C}\,(t,\ \eta)\ \widehat{U}, \tag{2.3.6}$$

where $\widehat{C}\,(t, \eta)$ is the square matrix of size $2m+1$ of a formal power series of η^{-1} with coefficients in possibly multi-valued holomorphic functions of t, that is, $\widehat{C}\,(t,\ \eta)$ has the form

$$\widehat{C}\,(t,\ \eta) = C_0(t) + \eta^{-1}C_1(t) + \eta^{-2}C_2(t) + \cdots$$

with $C_k(t)$ being a matrix of multi-valued holomorphic functions. It is easy to see that the leading matrix $C_0(t)$ is given by the Jacobian matrix of the polynomials h_j's at $u = u^{(0)}(t)$, i.e.,

$$C_0(t) = \frac{\partial(h_0, \ldots, h_{2m})}{\partial(u_0, \ldots, u_{2m})}(u^{(0)}(t)). \tag{2.3.7}$$

Definition 2.3.1 A turning point and a Stokes curve of $(\mathrm{NY})_{2m}$ are, by definition, those of the linearized system (2.3.6) of $(\mathrm{NY})_{2m}$ at the 0-parameter solution $u = \widehat{u}(t)$.

Remark 2.3.1 In this section we use the words "a turning point" and "a Stokes curve" in the traditional sense. Since the linearized system (2.3.6) of $(\mathrm{NY})_{2m}$ is of size $(2m + 1) \times (2m + 1)$, the complete description of its Stokes geometry requires the introduction of "virtual turning points" and "new Stokes curves emanating from virtual turning points". Although no satisfactory study in this direction has yet been done for $(\mathrm{NY})_{2m}$, we believe that the study of the Stokes geometry of $(\mathrm{NYL})_{2m}$, particularly the introduction of the function $\Phi(T)$, which we will do in the subsequent sections, should play a basic role in such study. See [S2, H1] for supporting evidences of the belief. We also note that the study of the Stokes geometry of higher order Painlevé equations $(P_{\mathrm{I}})_m$ etc. also supports such a belief, although the underlying linear equations are of size 2×2 (cf. [KKoNT1, KKoNT2]).

Let $N(\nu,\ t)$ be a characteristic polynomial of the matrix $C_0(t)$, i.e.,

$$N(\nu,\ t) = \det\,(\nu I_{2m+1} - C_0(t))\,. \tag{2.3.8}$$

Then it follows from the definition of a turning point that a turning point of $(\mathrm{NY})_{2m}$ is a point $t^* \in \mathbb{C}$ at which a pair of roots of the polynomial $N(\nu,\ t)$ of ν merges. The characteristic polynomial $N(\nu,\ t)$ of $(\mathrm{NY})_{2m}$ has the following specific feature:

Lemma 2.3.1 ([T4]) *The* $\widetilde{N}(\nu,\ t) := \nu^{-1}N(\nu,\ t)$ *is a polynomial of* ν^2.

By the lemma, the roots of $N(\nu,\ t)$ consist of m-pairs $(\nu_k(t),\ -\nu_k(t))$ $(k = 0, 1, \ldots, m-1)$ and the extra root $\nu = 0$. The extra root comes from the fact that polynomials h_j's are linearly dependent, and hence, it is almost irrelevant to the

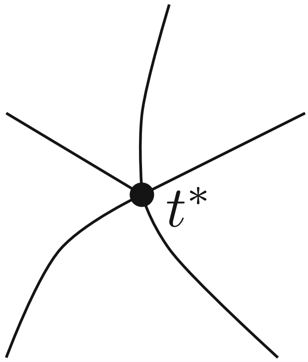

Fig. 2.1 Stokes curves emanating from a turning point of the first kind

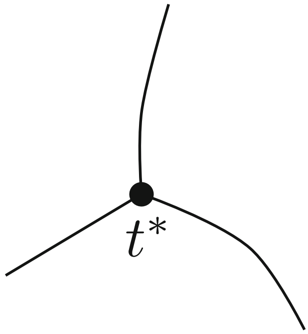

Fig. 2.2 Stokes curves emanating from a turning point of the second kind

Stokes geometry of $(\mathrm{NY})_{2m}$. In fact, if we eliminate the unknown function u_{2m} from the equations of (2.2.1) with $j = 0, 1, \ldots, 2m-1$ by the normalized condition (2.2.5) and apply the same argument to the system consisting of these $2m$-equations (where we forget the last equation of (2.2.1)), then the corresponding characteristic polynomial coincides with $\widetilde{N}(\nu,\ t)$ and the extra root never appears. Hence, in what follows, we ignore the extra root of $N(\nu,\ t)$.

Let $t^* \in \mathbb{C}$ be a turning point of $(\mathrm{NY})_{2m}$. Then, by these observations, we have the following two possibilities at t^*:

1. There exists k such that $\nu_k(t^*) = -\nu_k(t^*)$, that is, $\nu_k(t^*) = 0$. In this case, t^* is said to be a **turning point of the first kind**.
2. There exist $i \neq j$ such that either $\nu_i(t^*) = \nu_j(t^*)$ or $\nu_i(t^*) = -\nu_j(t^*)$ holds. We say that t^* is a **turning point of the second kind**.

It is known (Sect. 2.4 in [AH]) that the set of the branching points of the leading term $u^{(0)}(t)$ of the 0-parameter solution exactly coincides with that of turning points of the first kind.

Let t^* be a turning point of the first kind, that is, $\nu_k(t^*) = 0$ for some k. It is also known (Sect. 5.3 in [AH]) that, for generic α, the ramification degree of $u^{(0)}(t)$ at t^* is 2, that is, $u^{(0)}(t)$ has a Puiseux expansion of $(t-t^*)^{1/2}$ at $t = t^*$. Then, by (2.3.7),

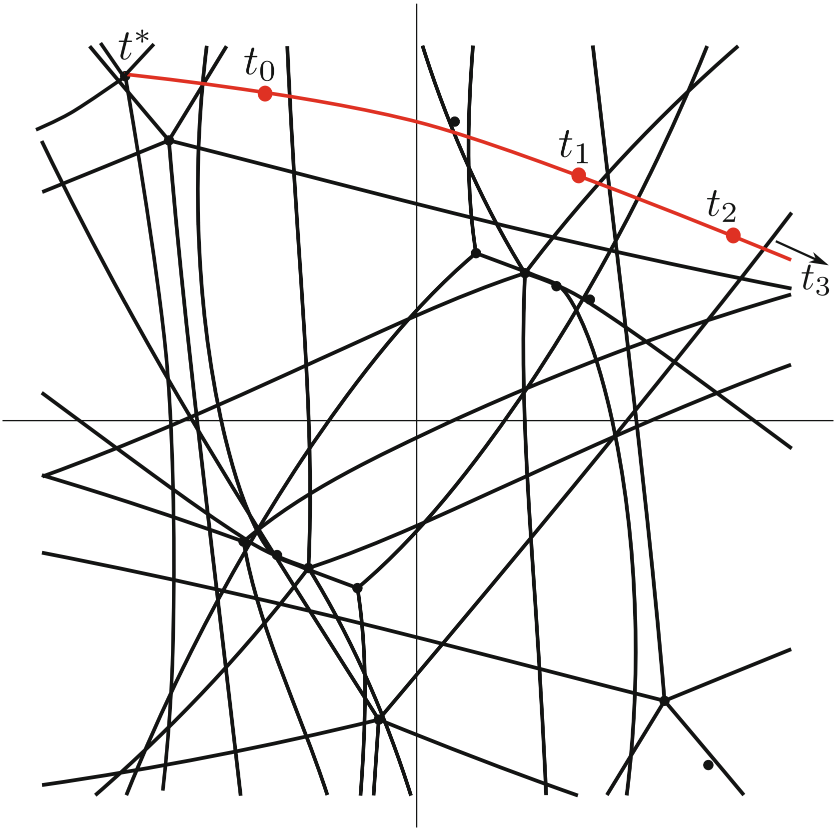

Fig. 2.3 The starting Stokes geometry of $(\mathrm{NY})_4$ (drawn with ordinary turning points and Stokes curves). The point t^* is, for example, a turning point of the first kind. For the other points t_0, t_1, t_2 and t_3 located in a Stokes curve emanating from t^*, see Example 2.4.3

the $\widetilde{N}(0,\ t)$ has also a Puiseux expansion of $(t-t^*)^{1/2}$ there, and its leading term is of order $1/2$ for generic α, i.e.,

$$\widetilde{N}(0,\ t) = c_{1/2}(t-t^*)^{1/2} + c_1(t-t^*) + \cdots \qquad (c_{1/2} \neq 0).$$

Therefore the root $\nu_k(t)$ has a Puiseux expansion of $(t-t^*)^{1/4}$ at $t=t^*$:

$$\nu_k(t) = d_{1/4}(t-t^*)^{1/4} + d_{1/2}(t-t^*)^{1/2} + \cdots \qquad (d_{1/4} \neq 0).$$

As a Stokes curve emanating from t^* is defined by

$$\mathrm{Im} \int_{t^*}^{t} (\nu_k(s) - (-\nu_k(s)))\, ds = 0 \iff \mathrm{Im} \int_{t^*}^{t} \nu_k(s) ds = 0,$$

which is equivalent to, by a Puiseux expansion of $\nu_k(t)$,

$$\mathrm{Im}\left(\frac{4d_{1/4}}{5}(t-t^*)^{5/4}+\frac{2d_{1/2}}{3}(t-t^*)^{3/2}+\cdots\right)=0,$$

we conclude that 5-Stokes curves emanate from a turning point of the first kind (cf. Fig. 2.1).

When t^* is a turning point of the second kind, then $u^{(0)}(t)$ is holomorphic near t^* and, for generic α, $\widetilde{N}(0,\ t)$ has a simple root at $t=t^*$. Hence, by the same argument as that for a turning point of the first kind, we see that 3-Stokes curves emanate from a turning point of the second kind in general (cf. Fig. 2.2).

Remark 2.3.2 The number of turning points of the first kind is less than or equal to $m2^{2m+1}$. That of the second kind is less than or equal to

$$2m(2m+1)_{2m}C_m-3m2^{2m}.$$

These estimates are strict for generic α. See [AH, AHU] for details.

2.4 A Bidirectional Binary Tree

We now study the Stokes geometry of the linear system $(\mathrm{NYL})_{2m}$. Let t_1 be a point in a Stokes curve of $(\mathrm{NY})_{2m}$ which is different from a turning point. We denote by $G:=G(t_1)$ the Stokes geometry of $(\mathrm{NYL})_{2m}$ with $t=t_1$. As stated before, when t lies in a Stokes curve of $(\mathrm{NY})_{2m}$, the corresponding Stokes geometry of $(\mathrm{NYL})_{2m}$ takes a specific configuration, in particular, there exists a so-called bidirectional binary tree in G whose definition is given now.

We first recall some conventions. Let $\mathscr{S}$ be a closed curve in $\mathbb{C}$, and let p_1, p_2, q_1 and q_2 be points in $\mathscr{S}$. We assume that, hereafter, a relevant curve does not form a loop. Then we denote by $[p_1, q_1]_{\mathscr{S}}$ or simply by $[p_1, q_1]$ the closed portion between p_1 and q_1 of the curve $\mathscr{S}$. Furthermore, we write $[p_1, q_1]\subset[p_2, q_2]$ if and only if p_2, p_1, q_1, q_2 are located in this order on $\mathscr{S}$. Hence the notation $[p_1, q_1]\subset[p_2, q_2]$ used in this article has the stronger meaning other than inclusion of sets.

Let $\mathscr{V}$ (resp. $\mathscr{W}$) be a Stokes curve emanating from a turning point v (resp. w) in G. Here, and in what follows, "a turning point" means either an ordinary one or a virtual one; when necessary, we always write so expressly. Assume that v and w are connected by both $\mathscr{V}$ and $\mathscr{W}$, that is, $[v,\ w]_{\mathscr{V}}=[v,\ w]_{\mathscr{W}}$ holds. Note that the Stokes curves $\mathscr{V}$ and $\mathscr{W}$ have the same type.

Definition 2.4.1 We say that a closed portion ℓ of $[v,\ w]$ is a **bidirectional segment** between turning points v and w in G if there exist points p and q in $[v,\ w]$ satisfying the following conditions:

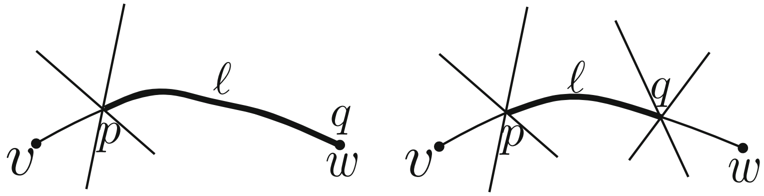

Fig. 2.4 Bidirectional segments

1. $\ell = [p, q] \subset [v, w]$.
2. The points p and q are either v or w or a point where another Stokes curve crosses.

We often write $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ to describe this situation (Fig. 2.4), and we call p and q to be the end points of ℓ.

Note that $(p, q; v, w; \mathscr{V}, \mathscr{W})$ implies that, in particular, points v, p, q, w are located in this order on the curve $\mathscr{V}$ or $\mathscr{W}$. Now let us recall the definition of a binary tree in the graph theory.

Definition 2.4.2 A **binary tree** $T = (B, E, L)$ consists of E: a set of leaf nodes, B: a set of branching nodes and L: a set of edges whose end points are in $B \cup E$, which satisfy the following conditions.

1. The degree of each leaf node is one (the degree of a node p is the number of edges with p in their end points).
2. The degree of each branching node is three.
3. For any two nodes in $B \cup E$, they are connected by a path and such a path is unique. Here a path is, by definition, a subset of edges which forms a polygonal chain.

The **degree of a binary tree** T is, by definition, the number of leaf nodes. We also define the **depth of a binary tree** T to be the number of edges of a maximal path in the tree T.

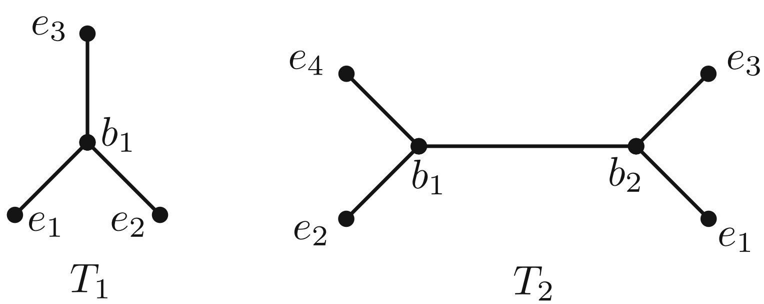

Fig. 2.5 Binary trees T_1 and T_2.

Example 2.4.1 Let us see the binary trees T_1 and T_2 in Fig. 2.5.

For examples, T_2 consists of 4-leaf nodes $\{e_1, e_2, e_3, e_4\}$, 2-branching nodes $\{b_1, b_2\}$ and 5-edges $\{e_1b_1, e_3b_1, e_2b_2, e_4b_2, b_1b_2\}$.

The degree of T_1 is 3 and that of T_2 is 4. The depth of T_1 is 2 and that of T_2 is 3.

Definition 2.4.3 A triplet $T = (B, E, L)$ is called a **bidirectional binary tree** in the Stokes geometry G if the following conditions are satisfied:

1. (B, E, L) is a binary tree where L consists of bidirectional segments in G whose end points are contained in $B \cup E$.
2. Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$ (cf. Definition 2.4.1 for the notation). Then $p \in E$ if $p = v$ and $p \in B$ otherwise, and similarly $q \in E$ if $q = w$ and $q \in B$ otherwise.
3. Let $b \in B$ and let ℓ be $(p, b; v, w; \mathscr{V}, \mathscr{W})$ in L with b in its end points. Suppose that $\ell_1 = (p_1, b; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1)$ and $\ell_2 = (p_2, b; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2)$ are the other two bidirectional segments in L with b in their end points. Then we have (cf. Fig. 2.6):

   (a) The Stokes curves $\mathscr{V}_1$ and $\mathscr{V}_2$ form an ordered crossing at b with the Stokes curve $\mathscr{W}$. That is, there exist mutually distinct indices i, j and k such that the type of $\mathscr{V}_1$ (resp. $\mathscr{V}_2$ and $\mathscr{W}$) near b is (i, j) (resp. (j, k) and (i, k)) and either

   $$\text{“}i < j \text{ on } \mathscr{V}_1 \text{ and } j < k \text{ on } \mathscr{V}_2\text{”}$$

   or

   $$\text{“}j < i \text{ on } \mathscr{V}_1 \text{ and } k < j \text{ on } \mathscr{V}_2\text{”}$$

   hold (see Definition 1.2.2 for the meaning of the labels $i < j$, etc. used here).

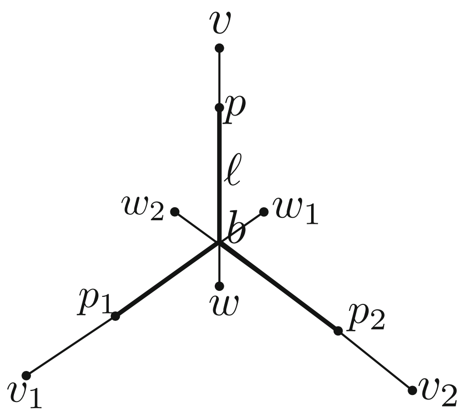

Fig. 2.6 The condition 3. of a bidirectional binary tree

(b) w is a turning point obtained from v_1 and v_2 by the method given in the proof of Proposition 1.4.1 (cf. (1.4.32)). That is, we have the integral relation

$$\int_b^{v_1} (\lambda_i(x) - \lambda_j(x))dx + \int_b^{v_2} (\lambda_j(x) - \lambda_k(x))dx = \int_b^{w} (\lambda_i(x) - \lambda_k(x))dx.$$

4. Each point in E is an ordinary turning point.

Remark 2.4.1 It follows from the conditions 3(a) and 3(b) of Definition 2.4.3 that, if $i < j$ on $\mathscr{V}_1$ and $j < k$ on $\mathscr{V}_2$ hold, we have $i < k$ on $\mathscr{W}$. Similarly, $j < i$ on $\mathscr{V}_1$ and $k < j$ on $\mathscr{V}_2$ imply $k < i$ on $\mathscr{W}$.

We will give a few examples of bidirectional binary trees.

Example 2.4.2 The simplest bidirectional binary tree is that of degree 2 (Fig. 2.7). It is nothing but a pair $(s,\ d)$ of ordinary turning points connected by a Stokes curve $\mathscr{S}$

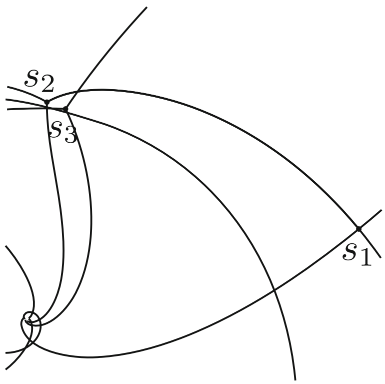

Fig. 2.7 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_0$, where only relevant Stokes curves and ordinary turning points are drawn for simplicity

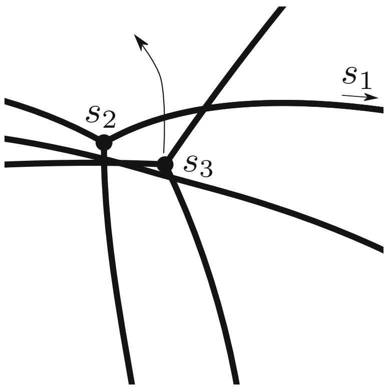

Fig. 2.8 The magnification of Fig. 2.7 near s_2

(resp. $\mathscr{D}$) emanating from s (resp. d). As a matter of fact, the bidirectional segment $(s,\ d;\ s,\ d;\ \mathscr{S},\ \mathscr{D})$ and its end points form a bidirectional binary tree of degree 2.

Example 2.4.3 We give some concrete examples of bidirectional binary trees observed in the Stokes geometry of $(\mathrm{NYL})_4$. Let t^* be a turning point of $(\mathrm{NY})_4$ and let $\mathscr{T}$ denote a Stokes curve of $(\mathrm{NY})_4$ that emanates from t^*. We take points t_0, t_1, t_2 and t_3 in $\mathscr{T}$ so that t^*, t_0, t_1, t_2 and t_3 are located in this order on $\mathscr{T}$ (cf. Fig. 2.3) .

Let us see the Stokes geometry of $(\mathrm{NYL})_4$ when $t = t_k$ $(k = 0, 1, 2, 3)$. Note that, in all figures, s_1, s_2, s_3 and s_4 are ordinary turning points and $v_1, v_2, \ldots, v_6$ are virtual turning points. Figures 2.7 and 2.8 describe the Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_0$. As t_0 in $\mathscr{T}$ is located quite near the turning point t^*, we can observe that ordinary turning points s_1 and s_2 are directly connected by a Stokes curve, and they form a bidirectional binary tree of degree 2 as it is explained in the previous example.

We also find another ordinary turning point s_3 close to the portion $[s_1,\ s_2]$. As a matter of fact, when t moves from t_0 to t_1, s_3 hits against $[s_1,\ s_2]$. As we already mentioned in Sect. 2.1, when s_3 crosses $[s_1,\ s_2]$, a bifurcation phenomenon occurs and consequently the tree of degree 2 becomes the higher one: The tree T_1 (resp. T_2) of Figs. 2.9 and 2.10 (resp. Figs. 2.11 and 2.12) is the bidirectional binary tree of degree 3. The tree T_1 consists of 3-leaf nodes $\{s_1, s_2, s_3\}$ (s_1 is a double turning point, and s_2, s_3 are simple turning points), 1-branching node $\{b_1\}$ and 3-bidirectional segments $\{s_1b_1,\ s_2b_1,\ s_3b_1\}$. For example, the bidirectional segment s_1b_1 lies in a common portion of two Stokes curves emanating from s_1 and v_1. In this way, each bidirectional segment of T_1 lies in a common portion of two Stokes curves. An important feature of T_1 is, for example, the Stokes curve emanating from s_2 and that from s_3 form an ordered crossing at b_1 and a virtual turning point v_1 is obtained from s_2 and s_3 (cf. (1.4.32)). In the same way, the Stokes curve emanating from s_1 and that from s_2 (resp. s_3) form an ordered crossing at b_1 and these turning points determine a virtual turning point v_3 (resp. v_2).

When t moves from t_1 to t_2, the tree T_1 changes its shape continuously and is deformed to the tree T_2 in Fig. 2.12. Note that, in the figure, the simple turning point s_4 is located quite near the edge s_3b_1 of T_2. Then, when t moves from t_2 to t_3 on $\mathscr{T}$, the turning point s_4 really crosses the edge s_3b_1 and the tree T_2 grows.

The degree of T_3 in Figs. 2.13 and 2.14 becomes 4 because the turning point s_4 joins in the tree as a new leaf node after s_4 hits against the edge of the tree. The tree T_3 consists of 4-leaf nodes $\{s_1, s_2, s_3, s_4\}$, 2-branching nodes $\{b_1, b_2\}$, and 5-edges $\{s_1b_1, s_2b_1, s_3b_2, s_4b_2, b_1b_2\}$. In particular, the segment b_1b_2 is in a common portion of two Stokes curves emanating from virtual turning points v_5 and v_6. Although it is not trivial, a careful study of the Stokes geometry G guarantees that branching nodes b_1 and b_2 satisfy the condition 3. of Definition 2.4.3.

For a bidirectional binary tree T in G, we can define its total integral value $\Phi(T)$ as in Definition 2.4.4 below. The value $\Phi(T)$ is closely tied up with the Stokes geometry of $(\mathrm{NY})_{2m}$, as Corollary 2.5.1 below shows. We also note that it is a counterpart of the function $\phi_J(t)$ $(J = \mathrm{I},\ \mathrm{II}, \ldots, \mathrm{VI})$ used in constructing instanton-type solutions

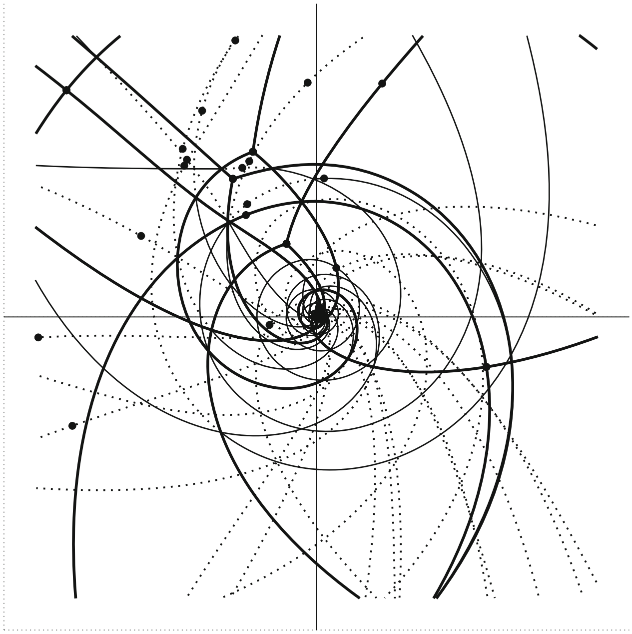

Fig. 2.9 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_1$ [H1, Fig. III-1–6]

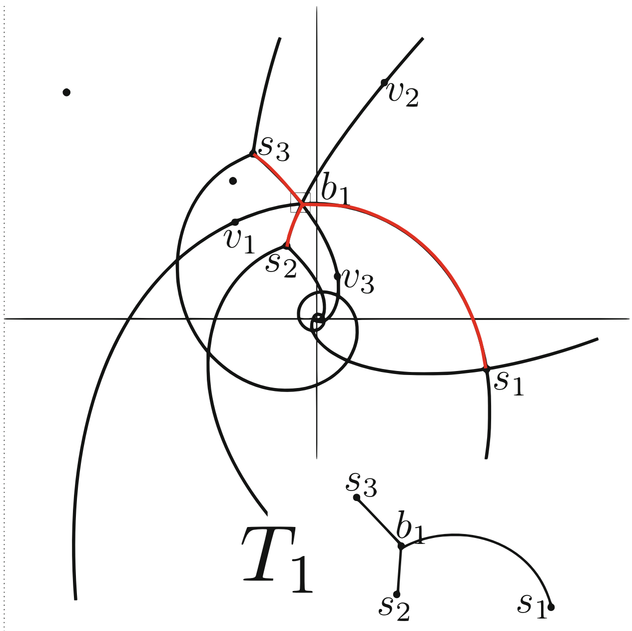

Fig. 2.10 Extract related Stokes curves and turning points of T_1 from Fig. 2.9 ($t = t_1$)

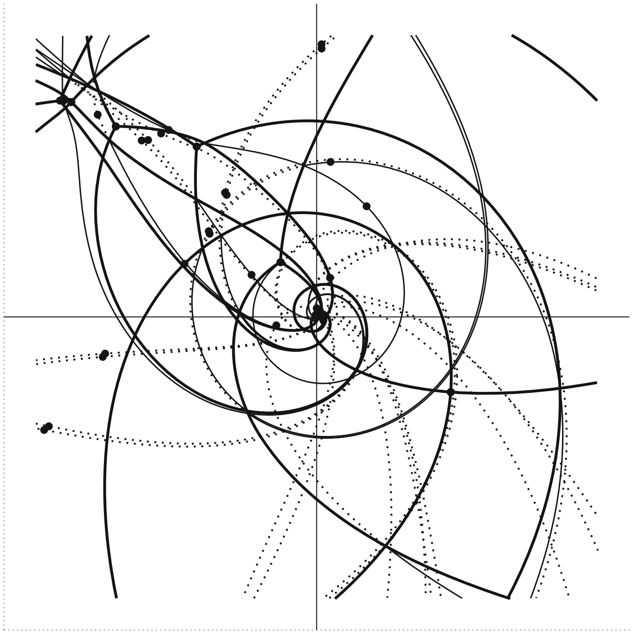

Fig. 2.11 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_2$ [H1, Fig. III-1–7]

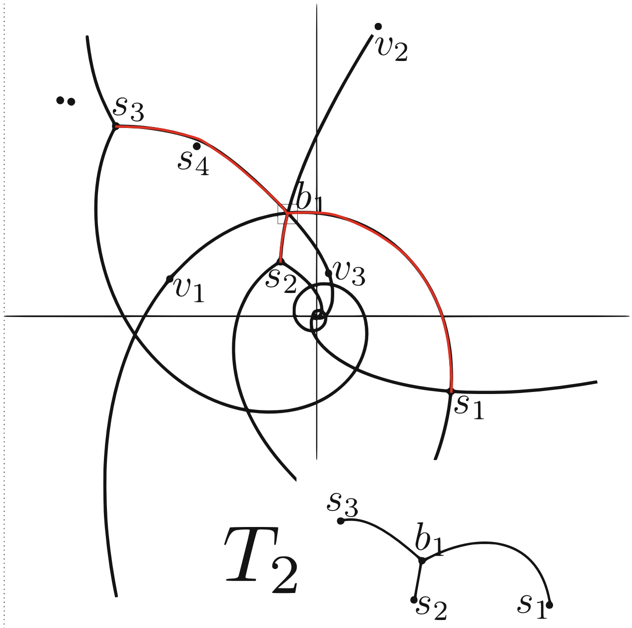

Fig. 2.12 Extract related Stokes curves and turning points of T_2 from Fig. 2.11 ($t = t_2$)

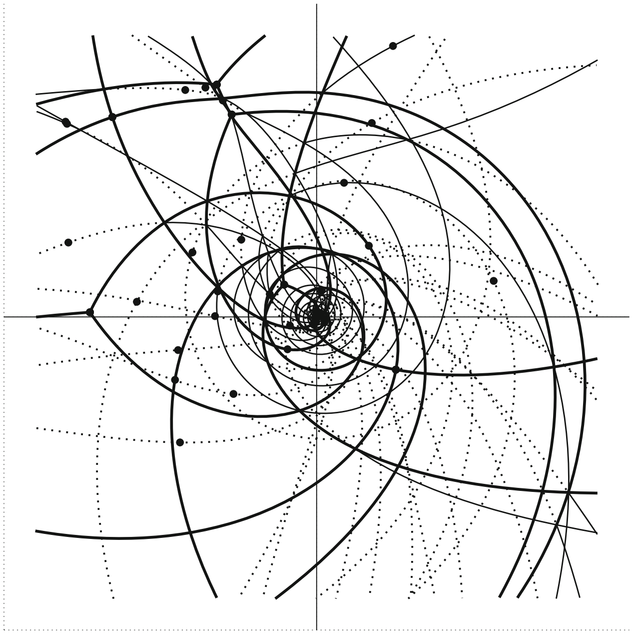

Fig. 2.13 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_3$ [H1, Fig. III-1–9]

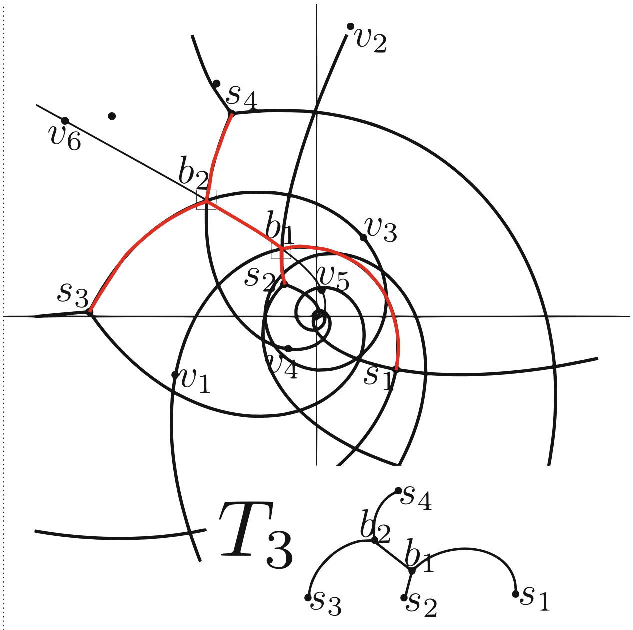

Fig. 2.14 Extract related Stokes curves and turning points of T_3 from Fig. 2.13 ($t = t_3$)

of the traditional Painlevé equation (P_J). (See [KT2, Chap. 4] for the construction of instanton-type solutions of (P_J).)

Definition 2.4.4 Let $T = (B, E, L)$ be a bidirectional binary tree in G. We define the **total integral value** $\Phi(T)$ of T as follows:

$$\Phi(T) = \sum_{\ell \in L} \left| \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|, \tag{2.4.1}$$

where, for $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$, we set $[\ell] := [p, q]$ and denote by (i_ℓ, j_ℓ) the type of the Stokes curve $\mathscr{V}$ or $\mathscr{W}$.

Remark 2.4.2 Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. Then, as $[p, q]$ is a part of the Stokes curve $\mathscr{V}$ or $\mathscr{W}$ and $p \neq q$, each integral

$$\int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$$

takes a non-zero real value. Hence, if we equip each edge in L with an appropriate orientation, we can write (2.4.1) in a much simpler form

$$\Phi(T) = \sum_{\ell \in L} \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx.$$

The following lemma gives us the most basic property of a bidirectional binary tree.

Lemma 2.4.1 *For any edge* $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$*, we have*

$$\Phi(T) = \left| \int_v^w (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|. \tag{2.4.2}$$

In particular, the integral value of the right hand side of the above equality does not depend on the choice of the edge ℓ *of the tree* T.

Proof We consider, as a bidirectional binary tree, a tree $T = (B, E, L)$ which satisfies all the conditions in Definition 2.4.2 except for the last condition 4, that is, a virtual turning point is also allowed as a leaf node. Then we show the claim for such a tree. We prove the claim by the induction on the number of edges in such a T, which is denoted by $\#T$ in this proof.

When $\#T = 1$, the only edge has the form $(v, w; v, w; \mathscr{V}, \mathscr{W})$. Hence the claim is trivial.

Now assume that the claim is true for T with $\#T \leq k$ $(k \geq 1)$. We will show the claim for T with $\#T = k + 1$. Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. We assume that $p \neq v$ and $q \neq w$, that is, p and q are branching nodes, and we will show the claim only in this case because the other cases can be proved in the same way.

Suppose that

$$\ell_1 = (p, q_1; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1), \ \ell_2 = (p, q_2; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2) \in L$$

are the other two edges with p in their end points and suppose also that

$$\ell_3 = (p_3, q; v_3, w_3; \mathscr{V}_3, \mathscr{W}_3), \ \ell_4 = (p_4, q; v_4, w_4; \mathscr{V}_4, \mathscr{W}_4) \in L$$

are the other two edges with q in their end points. Note that $T \backslash \ell$ (the tree T without the edge ℓ) consists of 4-connected components. We denote by T_1 (resp. T_2, T_3 and T_4) the connected component of $T \backslash \ell$ containing the point q_1 (resp. q_2, p_3 and p_4). Now we define the bidirectional binary tree $\widetilde{T}_1$ by T_1 where the edge ℓ_1 is replaced with

$$\widetilde{\ell}_1 = (v_1, q_1; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1),$$

i.e., $\widetilde{\ell}_1$ is obtained by substituting v_1 for p in ℓ_1. In the same way, $\widetilde{T}_2$ is the tree T_2 where ℓ_2 is replaced with

$$\widetilde{\ell}_2 = (v_2, q_2; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2).$$

Then, as $\#\widetilde{T}_1 \le k$ and $\#\widetilde{T}_2 \le k$ by the induction hypothesis, we obtain

$$\Phi(\widetilde{T}_1) = \left| \int_{v_1}^{w_1} (\lambda_{i_{\widetilde{\ell}_1}}(x) - \lambda_{j_{\widetilde{\ell}_1}}(x))dx \right|$$

and

$$\Phi(\widetilde{T}_2) = \left| \int_{v_2}^{w_2} (\lambda_{i_{\widetilde{\ell}_2}}(x) - \lambda_{j_{\widetilde{\ell}_2}}(x))dx \right|.$$

Hence, by noticing that the path $[v_1, \ p]$ (resp. $[v_2, \ p]$) appears in both sides, we have

$$\sum_{\ell' \in T_1} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_{p}^{w_1} (\lambda_{i_{\widetilde{\ell}_1}}(x) - \lambda_{j_{\widetilde{\ell}_1}}(x))dx \right|$$

and

$$\sum_{\ell' \in T_2} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_{p}^{w_2} (\lambda_{i_{\widetilde{\ell}_2}}(x) - \lambda_{j_{\widetilde{\ell}_2}}(x))dx \right|,$$

where we write $\ell' \in T_1$ (resp. T_2) when ℓ' is an edge of T_1 (resp. T_2). The condition that $\mathscr{W}_1$ and $\mathscr{W}_2$ form an ordered crossing at p entails that pairs of indices $(i_{\widetilde{\ell}_1}, \ j_{\widetilde{\ell}_1})$ and $(i_{\widetilde{\ell}_2}, \ j_{\widetilde{\ell}_2})$ share one and only one common index, and thus, we may assume $j_{\widetilde{\ell}_1} = i_{\widetilde{\ell}_2}$ (denote it by k) and $i_{\widetilde{\ell}_1} \neq j_{\widetilde{\ell}_2} \neq k$. Furthermore, the same condition implies that both $\int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx$ and $\int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx$ have the same signature. Therefore we get

$$\begin{aligned}
&\left| \int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx \right| + \left| \int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx \right| \\
&\quad = \left| \int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx + \int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx \right| \\
&\quad = \left| \int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|,
\end{aligned}$$

where the last equality comes from the fact that v is a virtual turning point obtained from w_1 and w_2. Summing up, we have

$$\sum_{\ell' \in T_1 \cup T_2} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|.$$

By applying the same argument to T_3 and T_4, we also have

$$\sum_{\ell' \in T_3 \cup T_4} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_q^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|.$$

Then, as v, p, q, w are located in this order on the Stokes curve, by noticing $\ell = [p, q]$, we have

$$\begin{aligned}
&\left| \int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right| + \left| \int_q^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right| + \left| \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right| \\
&\quad = \left| \int_v^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|,
\end{aligned}$$

and hence, we obtain

$$\begin{aligned}
\Phi(T) &= \sum_{\ell' \in T \setminus \ell} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| + \left| \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right| \\
&= \left| \int_v^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|.
\end{aligned}$$

Therefore the claim is true for T with $\#T = k + 1$, and thus, it is true for any T by the induction. This completes the proof.

2.5 Growing and Shrinking of a Bidirectional Binary Tree

Let t^* be a turning point of $(\mathrm{NY})_{2m}$ and let $\mathscr{T}$ denote a Stokes curve emanating from t^*. We take the parameterization $t = t(\theta)$ $(\theta \geq 0)$ of $\mathscr{T}$ by the length θ of the curve from t^* to $t \in \mathscr{T}$. We denote by $G(t)$ the Stokes geometry of $(\mathrm{NYL})_{2m}$ for a fixed t. Let $\theta_2 > \theta_1 > \theta_0 > 0$ and set $t_k = t(\theta_k) \in \mathscr{T}$ $(k = 0, 1, 2)$. Assume that a bidirectional binary tree $T = (B, E, L)$ exists in $G(t_1)$. We first give sufficient conditions which guarantee that T is continuously deformed when θ moves in a neighborhood of θ_1. We denote by $\mathscr{S}_T$ the set of all the Stokes curves appearing in edges of T.

C-1. An ordinary turning point of non-disjoint type never hits against a Stokes curve in $\mathscr{S}_T$. Here a turning point is said to be of non-disjoint type when the type of the turning point and that of a Stokes curve which the turning point touches share a common index.

From the condition C-1, each Stokes curve in $\mathscr{S}_T$ continuously moves when θ moves. Hence $\mathscr{S}_T$ forms a continuously moving family of Stokes curves.

C-2. Each Stokes curve in $\mathscr{S}_T$ intersects transversally with other related Stokes curves when they have some point in common.

Here, and in what follows, "a related Stokes curve" or "a related turning point" means that it appears in an element $(p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. It follows from C-1 that all the related Stokes curves and turning points continuously move with θ, and C-2 makes it sure that, in particular, a branching node also continuously moves. As a result of these conditions, each point in $B \cup E$ is regarded as a continuous function of θ, for which we also assume:

C-3. Any pair of points in $B \cup E$ never merges.

As the origin is a regular singular point for the system $(\mathrm{NYL})_{2m}$, to simplify our consideration, we prevent a point in $B \cup E$ from falling into the origin. Hence, in what follows, we always assume the following condition:

(†) Each point in $B \cup E$ stays in a compact region of $\mathbb{C}\setminus\{0\}$.

Then we obtain:

Theorem 2.5.1 *Assume that the conditions C-1–C-3 hold for every* $\theta_0 < \theta < \theta_2$. *Then, for any* $\theta \in (\theta_0, \theta_2)$, *there exists a bidirectional binary tree* T_θ *in* $G(t(\theta))$ *satisfying that* $T_{\theta_1} = T$ *and* T_θ *is continuously deformed when* θ *moves in* (θ_0, θ_2). *Furthermore the total integral value* $\Phi(T_\theta)$ *of* T_θ *is an analytic function of* $\theta \in (\theta_0, \theta_2)$.

Proof As a related Stokes curve and a point in $B \cup E$ continuously move by the conditions C-1 and C-2, it suffices to show that bidirectionality of each segment is really preserved. Let $(p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$ where p, q, v and w are known to be continuous functions of θ. By the condition C-3, we have $p \neq q$ for any

$\theta \in (\theta_0, \theta_2)$. If q is a branching node, i.e., $q \neq w$ at $\theta = \theta_1$, then we will show $q \neq w$ for any $\theta \in (\theta_0, \theta_2)$. To confirm this, let $\ell_1 = (p_1, q; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1)$ and $\ell_2 = (p_2, q; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2)$ be the other two edges with q in their end points. Since $p_1 \neq q$ and $p_2 \neq q$ hold by the condition C-3 and since $\mathscr{V}_1$ and $\mathscr{V}_2$ form an ordered crossing at q, we get

$$\left|\int_{v_1}^{q}(\lambda_i - \lambda_j)dx + \int_{v_2}^{q}(\lambda_j - \lambda_k)dx\right| \geq \left|\int_{p_1}^{q}(\lambda_i - \lambda_j)dx + \int_{p_2}^{q}(\lambda_j - \lambda_k)dx\right| \neq 0,$$

where the type of $\mathscr{V}_1$ (resp. $\mathscr{V}_2$) near q is assumed to be (i, j) (resp. (j, k)). As w is a virtual turning point obtained from v_1 and v_2, we have

$$\int_{w}^{q}(\lambda_i - \lambda_k)dx = \int_{v_1}^{q}(\lambda_i - \lambda_j)dx + \int_{v_2}^{q}(\lambda_j - \lambda_k)dx \neq 0.$$

Hence we have obtained $w \neq q$. As a conclusion, for any $\theta \in (\theta_0, \theta_2)$, the points v, p, q, w are located in this order on $\mathscr{V}$ or $\mathscr{W}$, and hence, they form a bidirectional segment in $G(t(\theta))$. Therefore we find a bidirectional binary tree T_θ at every $\theta \in (\theta_0, \theta_2)$ which is continuous deformation of T.

Let us show that $\Phi(T_\theta)$ is an analytic function of θ. Take an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. Then, by Lemma 2.4.1, we have

$$\Phi(T_\theta) = \left|\int_{v}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|,$$

and hence, it suffices to show $\int_{v}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$ to be an analytic function of θ. Since roots λ_{i_ℓ} and λ_{j_ℓ} analytically depend on θ outside ordinary turning points, we may consider the problem only near v and w, that is,

$$\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \quad \left(\text{resp. } \int_{w'}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right)$$

is analytic for a fixed v' (resp. w') sufficiently close to v (resp. w).

We only show that $\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$ is an analytic function of θ. If v is a simple turning point, then we have

$$\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx = \int_{C}\lambda_{i_\ell}(x)dx,$$

where C is a closed path which starts from v' and turns around v once with an appropriate orientation. Hence the integral is an analytic function of θ in this case.

If v is a double turning point or a virtual one, then λ_{i_ℓ} and λ_{j_ℓ} analytically depend on θ near v. Therefore it is enough to show that v itself analytically depends on θ.

When v is a double turning point, v is a simple root of the equation $\dfrac{dD}{dx}(x) = 0$ where $D(x)$ is the discriminant of the polynomial $\det(\lambda I_{2m+1} - A_t(x))$ of λ, from which analyticity of v on θ follows. When v is a virtual turning point, v is, by definition, a solution of the equation of x defined by

$$F(x) = \int_{x^*}^{x} (\lambda_{i_\ell}(z) - \lambda_{j_\ell}(z))dz + h(\theta) = 0,$$

where x^* is a fixed point near v and $h(\theta)$ is some analytic function of θ. As we have

$$\frac{dF}{dx} = \lambda_{i_\ell}(x) - \lambda_{j_\ell}(x)$$

which does not vanish near $x = v$ because v is not an ordinary turning point, we know that v also analytically depends on θ. This completes the proof.

Assume that $\theta_1 > 0$ is an exceptional point in the sense that either C-1 or C-2 or C-3 does not hold in $G(t(\theta_1))$. That is, one of the following cases happens in $G(t(\theta_1))$:

A. For an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$, an ordinary turning of non-disjoint type hits against $[v, w]$.
B. For an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$, the end points p and q of ℓ merge.
Z. At a branching node b, some edges with b in their end points become tangent at b; to be more precise, the Stokes curves containing these edges are tangent at b each other.

By taking the above theorem into account, we find that continuous deformation of T may fail at θ_1. We will investigate discontinuity of the Stokes geometry for the major cases which we often encounter in the study of the Stokes geometry of $(\mathrm{NYL})_{2m}$.

Remark 2.5.1 We have not observed Case Z alone in our concrete computations of the Stokes geometry of $(\mathrm{NYL})_{2m}$, although there is a theoretical possibility of an occurrence of such a case. Hence, in subsequent observations, we focus only on Cases A and B.

Case A: Let us consider a situation where an ordinary turning point s of non-disjoint type hits against the interior of $[p, q]$ for an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ in $G(t(\theta_1))$. We assume that s is not a leaf node of T_θ for $\theta < \theta_1$ and that s hits transversally against $\mathscr{V}$ (or $\mathscr{W}$) when θ tends to θ_1 from below. Case A is classified into the following 3 sub-cases A-1–A-3.

A-1. The type of ℓ and that of s are the same.
This is the most destructive case where the bidirectional segment ℓ is snapped by s (see Fig. 2.15). As a result, the corresponding bidirectional binary tree T_θ does not exist for $\theta > \theta_1$. The situation is a counterpart of the Nishikawa

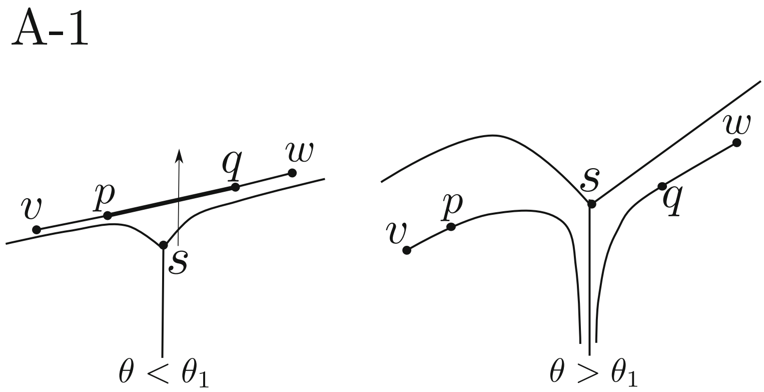

Fig. 2.15 Case A-1

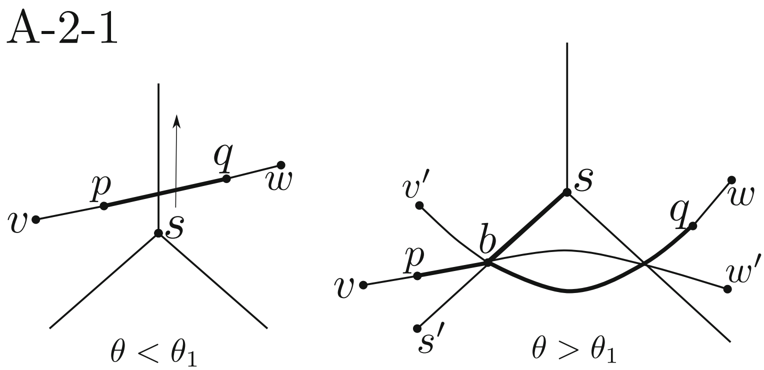

Fig. 2.16 Case A-2-1

phenomenon [KKoNT1] in the Noumi-Yamada system and it is intensively studied by Sasaki [S1, S2]. An important consequence of the vanishing of the bidirectional binary tree is that the relevant Stokes curve of $(\mathrm{NY})_{2m}$ becomes inert on the portion $\{t = t(\theta);\ \theta > \theta_1\}$; no Stokes phenomena for the solutions of $(\mathrm{NY})_{2m}$ are anticipated there, although it has not yet been proved.

A-2. The ordinary turning point s is simple, and the type of s and that of ℓ have one and only one common index.
When s touches ℓ, a bifurcation phenomenon described in Sect. 1.7 occurs, and thus, T_θ has a discontinuous change at $\theta = \theta_1$. Let us consider the case A-2 in detail. When s is sufficiently close to ℓ, the geometry near s, in a generic situation, becomes graphically equivalent to A-2-1 or A-2-2 described respectively in Fig. 2.16 or Fig. 2.17, that is, either one or two Stokes curves emanating from s intersect with ℓ in a sufficiently small neighborhood of s for $\theta < \theta_1$. (In Fig. 2.16 for $\theta < \theta_1$, we have omitted, for the sake of simplicity, a

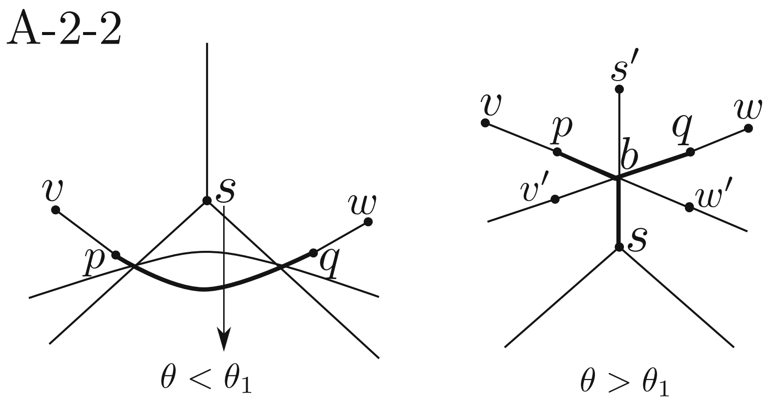

Fig. 2.17 Case A-2-2

new Stokes curve irrelevant to the structure of the tree in question. However, in Fig. 2.17 for $\theta < \theta_1$, we have included its counterpart to make the figure look well-balanced.) After s hits against ℓ, as shown in the same figure, still the bidirectional binary tree T_θ continues to exist where s becomes a new leaf node of T_θ and the degree of T_θ increases. That is, the tree T_θ $(\theta > \theta_1)$ is obtained from the tree T_θ $(\theta < \theta_1)$ in which the edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ is replaced with the following 3-edges ℓ_1, ℓ_2 and ℓ_3 having the same branching node b as their end points:

$$\ell_1 = (s, b; s, s'; \mathscr{S}, \mathscr{S}'), \quad \ell_2 = (p, b; v, w'; \mathscr{V}, \mathscr{W}'),$$
$$\ell_3 = (q, b; w, v'; \mathscr{W}, \mathscr{V}').$$

A-3. The geometric situation is similar to that of A-2 for $\theta < \theta_1$, but s is a double turning point.
The bidirectional binary tree T_θ continues to exist near $\theta = \theta_1$ where degree of T_θ is unchanged. This is because the vector field which defines the related Stokes curve of ℓ is non-degenerate near s, and hence, no bifurcation phenomenon happens and the tree T_θ is continuously deformed when θ moves near θ_1.

Case B: Let us consider a situation where end points of an edge $\ell = (v, b; v, w; \mathscr{V}, \mathscr{W})$ merge in $G(t(\theta_1))$, that is, b coincides with v. Here we only consider the case when one of the end points of ℓ is a leaf node. See [H2] for the other cases.

B-1. v is a double turning point d.
The ℓ has the form $(d, b; d, d'; \mathscr{D}, \mathscr{D}')$ (cf. Fig. 2.18). Since all the roots are holomorphic near d, related Stokes curves also continuously move when θ moves near $\theta = \theta_1$. Therefore no discontinuous phenomenon happens in this case. Note that, when $\theta = \theta_1$, that is, b and d coincide, the other two edges with

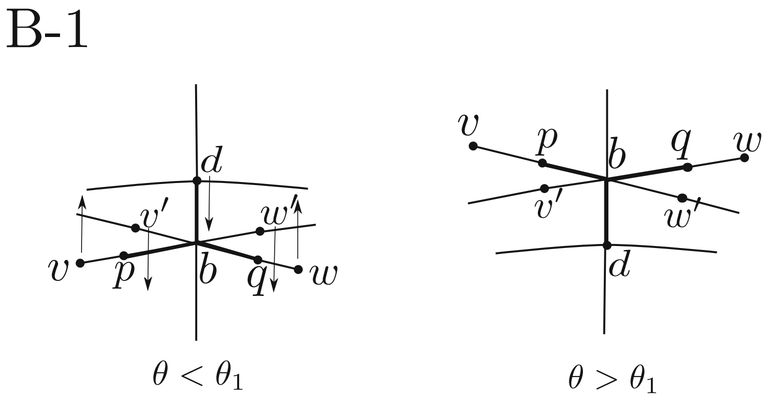

Fig. 2.18 Case B-1

b in their end points, i.e., $[p, b]$ and $[q, b]$ in Fig. 2.18, become tangent at b. To be more precise, the Stokes curves containing these two edges are tangent at b. Therefore the configuration of the relevant part of T_θ becomes the one described in Fig. 2.18, and hence, the bidirectional binary tree T_θ continues to exist after θ_1 and the degree of T_θ remains constant.

B-2. v is a simple turning point s.
This is the reverse case of A-2, that is, configurations at $\theta > \theta_1$ in Figs. 2.16 and 2.17 are the initial ones in this case, and the resulting configurations are those at $\theta < \theta_1$ in the same figures. Hence the leaf node v will be removed from the bidirectional binary tree T_θ, and the degree of T_θ decreases when $\theta > \theta_1$.

As an immediate consequence of these observations, we have the following proposition.

Proposition 2.5.1 *Assume that, for each exceptional point $\theta > 0$, only one of the cases A-2, A-3, B-1 or B-2 occurs in $G(t(\theta))$. Then there exists a bidirectional binary tree T_θ $(\theta > 0)$ satisfying that T_θ is continuously deformed when θ moves and its total integral value $\Phi(T_\theta)$ is an analytic function of θ.*

Proof When $\theta > 0$ is sufficiently small, it follows from Theorems 2.1 and 2.2 in [T4] that there exists a pair of ordinary turning points which are connected by a Stokes curve. Hence a bidirectional binary tree of degree 2 exists for a sufficiently small $\theta > 0$. Then the first part of the proposition follows from Theorem 2.5.1 and the above observations. Hence it suffices to show $\Phi(T_\theta)$ to be an analytic function of θ at $\theta = \theta_1$ where one of the cases A-2, A-3, B-1 or B-2 occurs in $G(t(\theta_1))$. Here we show the claim for A-2. The other cases can be proved by modifying the path of integration in the same way as that of A-2.

Suppose that a simple turning point s hits on an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$ at $t = t(\theta_1)$. Further we assume that the type of s is (i, j), the type of $\mathscr{V}$ is (j, k) and $j > k$ holds on $\mathscr{V}$. By Lemma 2.4.1, we have, for $\theta < \theta_1$,

$$\begin{aligned}\Phi(T_\theta) &= \left|\int_v^w (\lambda_j(x) - \lambda_k(x))dx\right| = \int_v^w (\lambda_j(x) - \lambda_k(x))dx \\ &= \int_{C_j} \lambda_j(x)dx - \int_{C_k} \lambda_k(x)dx,\end{aligned}$$

where the paths C_j and C_k are $[v, w]$ with orientation from v to w. The last expression is still valid for $\theta > \theta_1$ by modifying the path C_j so that the ordinary turning point s avoids hitting against C_j.

On the other hand, when $\theta > \theta_1$, the tree T_θ is obtained from the tree T_θ $(\theta < \theta_1)$ in which the edge ℓ is replaced with the following 3-edges having the same branching node b (cf. Figs. 2.16 and 2.17):

$$\ell_1 = (s, b; s, s'; \mathscr{S}, \mathscr{S}'), \quad \ell_2 = (p, b; v, w'; \mathscr{V}, \mathscr{W}'),$$
$$\ell_3 = (q, b; w, v'; \mathscr{W}, \mathscr{V}').$$

Here we may assume that the type of $\mathscr{S}$ (resp. $\mathscr{V}$ and $\mathscr{W}$) near b is (i, j) (resp. (j, k) and (i, k)). Then it follows from Lemma 2.4.1 and the fact $j > k$ on $\mathscr{V}$, and hence, $k > i$ on $\mathscr{W}$ and $j > i$ on $\mathscr{S}'$ ($\iff$ $i > j$ on $\mathscr{S}$) that we have

$$\begin{aligned}\Phi(T_\theta) &= \left|\int_v^b (\lambda_j - \lambda_k)dx\right| + \left|\int_s^b (\lambda_i - \lambda_j)dx\right| + \left|\int_w^b (\lambda_k - \lambda_i)dx\right| \\ &= \int_v^b (\lambda_j - \lambda_k)dx + \int_s^b (\lambda_i - \lambda_j)dx + \int_w^b (\lambda_k - \lambda_i)dx \\ &= \int_v^b (\lambda_j - \lambda_k)dx + \int_C \lambda_j dx + \int_w^b (\lambda_k - \lambda_i)dx\end{aligned}$$

for $\theta > \theta_1$, where the path C is a closed path which starts from b and turns around s once. Then it is equal to

$$\left(\int_v^b \lambda_j dx + \int_C \lambda_j dx + \int_b^w \lambda_i dx\right) - \left(\int_v^b \lambda_k dx + \int_b^w \lambda_k dx\right).$$

As λ_j is changed to λ_i after the analytic continuation along C, we find

$$\left(\int_v^b \lambda_j dx + \int_C \lambda_j dx + \int_b^w \lambda_i dx\right) = \int_{C_j} \lambda_j(x)dx$$

and

$$\left(\int_v^b \lambda_k dx + \int_b^w \lambda_k dx\right) = \int_{C_k} \lambda_k(x)dx.$$

Hence we have obtained, for $\theta > \theta_1$,

$$\Phi(T_\theta) = \int_{C_j} \lambda_j(x)dx - \int_{C_k} \lambda_k(x)dx,$$

which entails that $\Phi(T_\theta)$ is analytic at $\theta = \theta_1$. This completes the proof.

Let ν_1 and ν_2 be a pair of roots which defines the turning point t^* of $(\mathrm{NY})_{2m}$, that is, $\nu_1(t)$ and $\nu_2(t)$ merge at $t = t^*$. Then, as a corollary of Proposition 2.5.1, we have:

Corollary 2.5.1 *Assume the same conditions as those in Proposition 2.5.1 hold. Then we have*

$$\Phi(T_\theta) = \frac{1}{2}\left|\int_{t^*}^{t(\theta)} (\nu_1(s) - \nu_2(s))ds\right| \qquad (\theta > 0). \tag{2.5.1}$$

Proof When $\theta > 0$ is sufficiently small, T_θ is of degree 2, that is, two ordinary turning points are connected by a Stokes curve. In this case, the equality was shown in Theorems 2.1 and 2.2 in [T4]. Then both sides of (2.5.1) are analytic functions of θ. Hence the equality holds for any $\theta > 0$.

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