Static and Free Vibration Analysis of Functionally Graded Skew Plates Using a Four Node Quadrilateral Element

S.D. Kulkarni, C.J. Trivedi and R.G. Ishi

Abstract In this work, analysis of functionally graded skew plates with different boundary conditions is performed using Modified Improved Discrete Kirchhoff Quadrilateral (MIDKQ) element. The element has seven degrees-of-freedom namely, three displacements, two rotations and two shear strains per node. The plate considered in the study has isotropic, two-constituent material distribution through the thickness. The modulus of elasticity is assumed to vary according to a power-law distribution in terms of the volume fractions of the constituents. Poisson's ratio is assumed to be constant. The finite element results for various values of the power-law index for non-dimensionalized deflection, stresses and natural frequencies for functionally graded skew plates with different boundary conditions obtained using IDKQ element are compared with the available results in literature and with the results obtained using 20 node solid element of ABAQUS. It is observed that the performance of the MIDKQ element is quite satisfactory for this case also.

Keywords Skew · Discrete · Kirchhoff theory · Finite element · Quadrilateral

1 Introduction

In recent years, functionally graded materials (FGMs) have gained considerable attention in many engineering applications. FGMs are new materials, microscopically inhomogeneous, in which the mechanical properties vary smoothly and

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continuously from one surface to the other. Many researchers have investigated the static and dynamic behavior of functionally graded plates (FGP) using analytical as well as finite element models based on different plate theories. Zenkour [1] presented generalized shear deformation theory for bending analysis of functionally graded plates. Reddy [2] presented the non-linear static analysis of FGP using third order shear deformation theory which does not require shear correction factor. The same theory was used Kulkarni [3] for the static analysis of FGP wherein he used the improved discrete Kirchhoff element by modeling the plate as a layered plate. But as it is difficult to decide the exact number of layers this approach is not suitable for all the cases. In this work instead of modeling the plate as a layered plate actual variation of material properties is considered by developing MIDKQ, which has the same number of degrees-of-freedom (DOF) as that of Improved Discrete Kirchhoff Quadrilateral (IDKQ). Using MIDKQ, functionally graded skew plates with various boundary conditions are analyzed for static and free vibration analysis. It is assumed that plate has isotropic, two constituent material distribution through the thickness. The modulus of elasticity is assumed to vary as per power law distribution in terms of volume fraction of constituents. The Poisson's ratio is assumed to be constant. The finite element results for deflection, stresses and natural frequencies for various values of power law index and skew angles are compared with available results in literature as well as with 3D finite element results of ABAQUS using twenty node solid elements. It is observed that the performance of the MIDKQ element is quite satisfactory for both static and free vibration analysis.

2 Displacements Field Approximation for Reddy's Third Order Shear Deformation Theory

The plate considered is shown in Fig. 1. The mid-plane of the plate is chosen as the reference plane (z = 0). Thus, z coordinate of top (ceramic) and bottom (metal) surfaces are h/2 and −h/2, respectively. The functional relationship between E and z for ceramic and metal FGP is assumed as

$$E(z) = E(m) + [E(m) - E(c)]\left(\frac{2z + h}{2h}\right)^{K} \tag{1}$$

where, $E(m)$, $E(c)$ are Young's modulus of metal, and ceramic respectively and k is the volume fraction index.

The normal strain components ε_x, ε_y, ε_z and shear strain components γ_{xy}, γ_{yx}, γ_{xz} are related to displacement by,

$$\begin{aligned} &\varepsilon_x = u_{x,x}, \quad \varepsilon_y = u_{y,y}, \quad \varepsilon_z = w_{,z} \\ &\gamma_{xy} = u_{x,y} + u_{y,x}, \quad \gamma_{yz} = u_{y,z} + w_{,y}, \quad \gamma_{zx} = w_{,x} + u_{x,z} \end{aligned} \tag{2}$$

where subscript comma denotes differentiation.

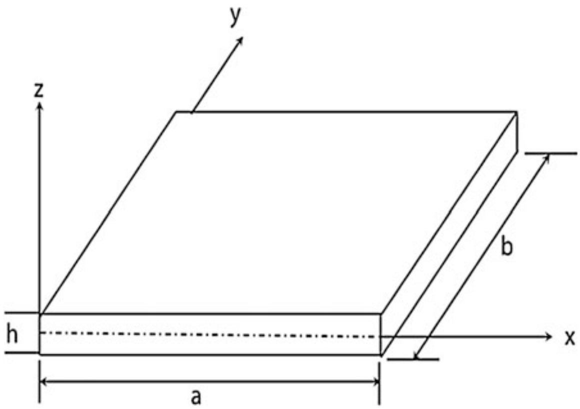

Fig. 1 Geometry of functionally graded plate

The linear constitutive equation for the stresses σ, τ are expressed using the assumption of $\sigma_z = 0$, as

$$\sigma = \bar{Q}\varepsilon, \quad \tau = \hat{Q}\gamma \tag{3}$$

where,

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_{zx} \\ \tau_{yz} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_{zx} \\ \gamma_{yz} \end{bmatrix}$$

$$\bar{Q} = \frac{E_z}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}, \quad \hat{Q} = \frac{E_z}{1-v^2}\begin{bmatrix} \frac{1-v}{2} & 0 \\ 0 & \frac{1-v}{2} \end{bmatrix}$$

In Reddy's third order theory, the in-plane displacements are expressed by imposing shear traction free condition at top and bottom surface as,

$$u(x,y,z,t) = u_0(x,y,t) - zw_{0d}(x,y) + R(z)\psi_0(x,y) \tag{4}$$

where,

$$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad u_0 = \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix}, \quad w_{0d} = \begin{bmatrix} w_{0,x} \\ w_{0,y} \end{bmatrix}, \quad \psi_0 = \begin{bmatrix} \psi_{0x} \\ \psi_{0y} \end{bmatrix}$$

In Eq. (4), ψ_{0x} and ψ_{0y} are shear strains and $R(z)$ is a global function in z given as:

$$R(z) = [z - 4z^3/(3h^2)] \tag{5}$$

Transverse displacement is considered as:

$$w_0(x, y, z, t) = w_0(x, y, t) \tag{6}$$

3 Finite Element Formulation

The variation form of 2D theory is obtained using Hamilton principle and is given in Eq. (7) using the notation $\langle .. \rangle = \int_{-h/2}^{h/2} (\ldots) dz$ notation for integration over the thickness as

$$\int_A [\left\langle \rho \partial u^T \ddot{u} + \rho \partial w_0 \partial \ddot{w}_0 \right\rangle] dA + \int_A [\langle \partial \varepsilon^T \sigma + \partial \gamma^T \tau \rangle - p_z \partial w_0(x, y, t)] dA \\ - \int_{\Gamma_L} \langle \sigma_n \partial u_n + \tau_{nz} \partial u_s + \tau_{nz} \partial w \rangle ds = 0 \tag{7}$$

where A denotes the mid-plane surface area of the plate and Γ_L denotes the boundary surface at the mid-plane of plate with normal n and tangent s. $\sigma, \varepsilon, \tau$ and γ denote in-plane stress, in-plane strain, transverse shear stress and transverse shear strain components, respectively. p_z is the force per unit area applied on the mid surface of the plate in z direction.

A four node quadrilateral element (Fig. 2) having seven degrees of freedom per node namely three translations, two rotations and two transverse shear strains at mid plane is developed based on the Reddy's third order shear deformation theory presented above.

As the highest derivatives of $u_{0x}, u_{oy}, \psi_{0x}$ and ψ_{0y} appearing in variational equation are of first order, the convergence criteria requires their interpolation function to be C^0 continuous at the element boundary.

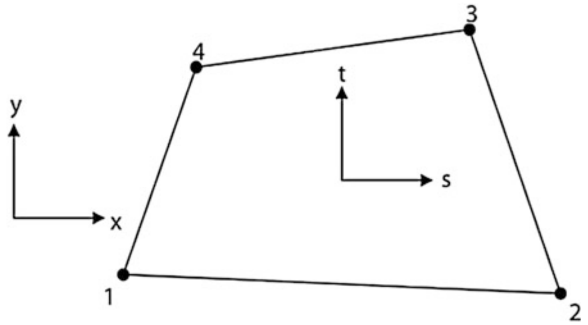

Fig. 2 Geometry of a four-node quadrilateral element

Accordingly, these variables are interpolated using bilinear Lagrange interpolation function $N_i (i = 1, 2, 3, 4)$, as

$$u_{0x} = Nu^e_{0x}, \quad u_{0y} = Nu^e_{0y}, \quad \psi_{0y} = N\psi^e_{0x}, \quad \psi_{0y} = N\psi^e_{0y} \tag{8}$$

where,

$$u^e_{0x} = \begin{bmatrix} u^1_{0x} & u^2_{0x} & u^3_{0x} & u^4_{0x} \end{bmatrix} \quad u^e_{0y} = \begin{bmatrix} u^1_{0y} & u^2_{0y} & u^3_{0y} & u^4_{0y} \end{bmatrix}$$
$$\psi^e_{0x} = \begin{bmatrix} \psi^1_{0x} & \psi^2_{0x} & \psi^3_{0x} & \psi^4_{0x} \end{bmatrix} \quad \psi^e_{0y} = \begin{bmatrix} \psi^1_{0y} & \psi^2_{0y} & \psi^3_{0y} & \psi^4_{0y} \end{bmatrix}$$
$$N = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

The presence of second derivative of w_0 in variational equation indicates that its interpolation function should have C^1 continuity at the element boundary, which is difficult to achieve for a quadrilateral element. The need for C^1 continuity requirement is circumvented by using discrete Kirchhoff constraint approach, which was proposed by Jaychandrabose et al. [4]. In this approach w_{0x}, w_{0y} are replaced by rotation variables θ_{0x}, θ_{0y}, which then requires only C^0 continuity. w_0 and θ_{0x}, θ_{0y} are interpolated independently, but the two are subsequently related by imposing the constrains $\theta_{0i} = w_{0,i}$ at discrete points on the element boundary and at the interior of the element. After applying this procedure θ_{0x}, θ_{0y} are interpolated as

$$\theta_{0x} = Gw^e_0, \quad \theta_{0x} = Hw^e_0, \tag{9}$$

where

$$w^e_0 = \begin{bmatrix} w^1_0 & w^1_{0,x} & w^1_{0,y} & w^2_0 & w^2_{0,x} & w^2_{0,y} & w^3_0 & w^3_{0,x} & w^3_{0,y} & w^4_0 & w^4_{0,x} & w^4_{0,y} \end{bmatrix}$$

And $G = \begin{bmatrix} G_1 & G_2 & \dots & G_{12} \end{bmatrix}$, $H = \begin{bmatrix} H_1 & H_2 & \dots & H_{12} \end{bmatrix}$.

G_i and H_i are the interpolation function with close form expression as given in [4]. Since no interpolation function is defined for w_0 in the interior of the element for computing strains, a bi-cubic expression for w_0 is assumed for the calculation of load vector. w_0 is expressed as,

$$\begin{aligned} w_0 &= \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2 + \alpha_7\xi^3 + \alpha_8\xi^2\eta \\ &\quad + \alpha_9\xi\eta^2 + \alpha_{10}\eta^3 + \alpha_{11}\xi\eta^3 + \alpha_{12}\xi^3\eta \\ &= \bar{N}w^e_0 \end{aligned} \tag{10}$$

where, $\bar{N} = \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \dots & \bar{N}_{12} \end{bmatrix}$.

The element stiffness matrix is obtained by substituting Eqs. (8) and (9) into Eq. (7). The element load vector is obtained similarly by substituting Eq. (10) into Eq. (7). Considering the contribution of all the element to area integral of Eq. (7) it can be expressed as

$$P = M\ddot{U} + KU \tag{11}$$

where P, M and K are assembled load vector, mass matrix and stiffness matrix respectively. For free vibration analysis P is made equal to zero in Eq. (11). Sub-space iteration technique is used to obtain natural frequencies.

4 Numerical Results and Discussion

The present formulation based on Reddy's third order shear deformation theory is assessed for static and free vibration analysis of skew FGP shown in Fig. 3 by comparing the present results with those available in literature and 3D FE results of ABAQUS.

4.1 Static Analysis

Full plate is considered for the analysis. The material properties of the plate considered for static analysis are

E_m = 70 GPa, E_c = 380 GPa and ν = 0.3.

The deflection and stresses are non-dimensionalised as in [3].

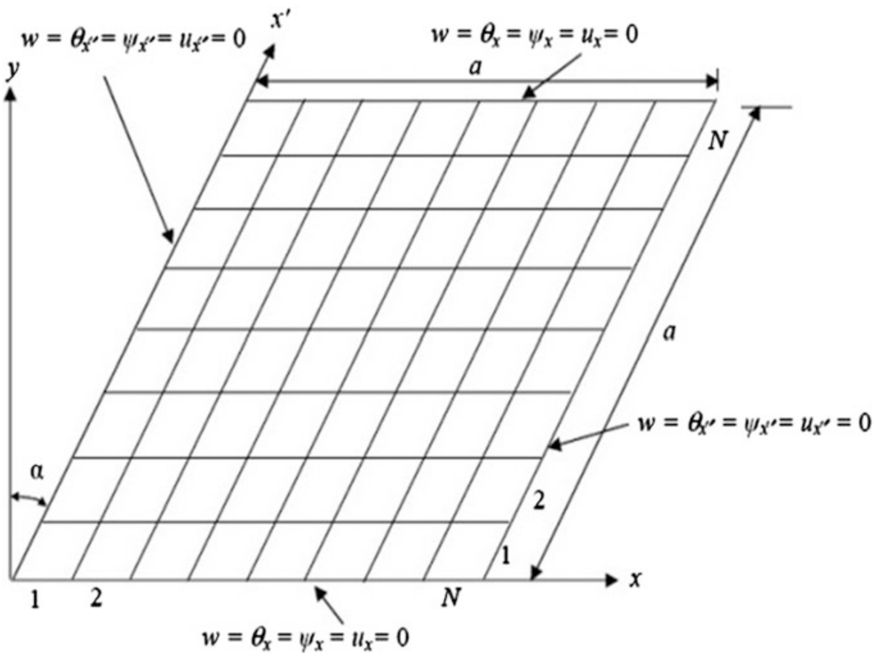

Fig. 3 Skew plate

4.1.1 All Edges Simply Supported Skew Plate with *b/a* = 1

The present results for fully ceramic plate (k = 0) for various values of skew angle are presented in Tables 1 and 2. The present results of central deflection, normal stresses and transverse shear stresses obtained using a mesh 48 × 48 are compared with the 3D FE results obtained using a mesh 48 × 48 × 4 of 20 node solid elements. For both plates results are quite close to the 3D FE results, indicating the accuracy of MIDKQ element.

Additional results of central deflection, normal stresses and transverse shear stresses for various values of skew angle and volume fraction index for a plate with S = 10 are presented in Table 3. It is observed that as volume fraction index increases central deflection also increases and as skew angle increases the central deflection decreases.

Table 1 Deflection and stresses for all edge simply-supported skew plate (SSSS) for S = 5

α	Entity	Present	ABAQUS
15°	w	4.0260	4.0289
	σ_x	0.2680	0.2728
	τ_{zx}	0.4841	0.4906
30°	w	2.8279	2.8372
	σ_x	0.2134	0.2194
	τ_{zx}	0.4189	0.4348
45°	w	1.4305	1.4401
	σ_x	0.1388	0.1393
	τ_{zx}	0.3199	0.3237

Table 2 Deflection and stresses for all edge simply-supported skew plate (SSSS) for S = 10

α	Entity	Present	ABAQUS
15°	w	4.189	4.1558
	σ_x	0.2694	0.2666
	τ_{xz}	0.4764	0.4501
30°	w	2.9687	2.9315
	σ_x	0.2147	0.2172
	τ_{xz}	0.4141	0.4156
45°	w	1.5337	1.5377
	σ_x	0.1397	0.1467
	τ_{xz}	0.3165	0.3219

Table 3 Deflection and stresses for all edges simply-supported skew plate (SSSS)

α	Entity	w	σx	τzx
15°	1	8.3351	0.4170	0.4803
	2	10.7268	0.4874	0.4488
	3	11.8744	0.5231	0.4231
	5	12.9216	0.5734	0.4019
	7	13.5266	0.6203	0.4041
30°	1	5.8821	0.3317	0.4159
	2	7.5747	0.3877	0.3885
	3	8.3971	0.4162	0.3662
	5	9.1573	0.4564	0.3478
	7	9.8573	0.4464	0.3499
45°	1	3.0144	0.2129	0.3182
	2	3.8875	0.2489	0.2970
	3	4.3254	0.2674	0.2798
	5	4.7436	0.2937	0.2656
	7	4.9805	0.3179	0.2671

4.2 Free Vibration Analysis

The material properties of the plate considered for free vibration analysis are

$E_m = 70$ GPa, $E_c = 151$ GPa, $\nu = 0.3$,
$\rho_m = 2{,}702$ kg/m^3, $\rho_c = 3{,}000$ kg/m^3.

For this analysis, also full plate is considered. The frequency is non-dimensionalised as,

$$\bar{\omega} = \omega h \sqrt{\frac{\rho_c}{E_t}}$$

4.2.1 All Edge Simply Supported Skew Plate with $S = 10$

Present results of natural frequencies using mesh of 20 × 20 for various values of skew angle and volume fraction index $k = 0$ are compared with the results of [5] and with the 3D FE element results of ABAQUS obtained using mesh of size 30 × 30 × 4 in Table 4. In Table 5 the present results for different values of skew angle and k are compared with the results of Valizadeh et al. [5]. From both the tables it is observed that the present results are quite satisfactory.

Table 4 Frequencies for SSSS plate

Mode	α	Present	Valizadeh et al. [5]	ABAQUS
1	15	6.0869	6.100	6.283
	30	7.246	7.300	7.485
	45	10.112	10.105	10.335
	60	16.785	16.800	17.016
2	15	13.314	13.500	13.674
	30	14.598	14.600	14.778
	45	18.185	18.100	18.554
	60	27.088	27.100	27.212

Table 5 Fundamental frequency for SSSS plate

k	α	Present	Valizadeh et al. [5]
0	15	6.0869	6.100
	30	7.246	7.300
	45	10.112	10.105
	60	16.785	16.800
0.5	15	5.495	5.500
	30	6.326	6.300
	45	8.881	8.850
	60	15.10	15.050
1.0	15	5.225	5.200
	30	5.960	5.900
	40	8.690	8.600
	60	14.12	14.100

Table 6 Frequency parameter for CCCC plate with S = 10

Mode	α	Present	Valizadeh et al. [5]	ABAQUS
1	15	10.505	10.500	10.732
	30	12.221	12.200	12.489
	45	16.932	16.950	17.359
	60	28.050	28.050	28.948
2	15	18.935	18.900	19.212
	30	20.531	20.500	21.044
	45	25.512	25.500	26.022
	60	38.101	37.821	37.700

4.3 All Edge Clamped Skew Plate

Similar results of natural frequencies using a mesh of 20 × 20 for various values of skew angle and $k = 0$ are compared with the results of [5] and with the 3D FE element results of ABAQUS obtained using mesh of size 30 × 30 × 4 in Table 6.

Table 7 Fundamental frequency parameter for CCCC plate with S = 10

k	α	Present	Valizadeh et al. [5]
0	15	10.505	10.500
	30	12.110	12.200
	45	16.911	16.950
	60	28.105	28.050
0.5	15	9.484	9.500
	30	11.204	11.200
	45	15.188	15.200
	60	25.221	25.200
1.0	15	9.021	9.050
	30	10.214	10.200
	40	14.389	14.500
	60	24.105	24.050

The results closely match with those of [5] and of ABAQUS. In Table 7 the present results for different values of skew angle and *k* are compared with the results of Valizadeh et al. [5].

5 Conclusion

The developed MIDKQ element has been used for the analysis of functionally graded skew plate by considering actual variation of material properties over the thickness. The results for central deflection and stresses are observed to be quite close to the 3D FE results of ABAQUS. The results of natural frequencies for all round simply supported plate and for all round clamped skew plates are compared with available literature and with 3D FE results of ABAQUS obtained using 20 node solid element observed to be quite close, indicating the satisfactory performance of MIDKQ element.

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