Higher Order Hybrid Invexity Frameworks and Discrete Multiobjective Fractional Programming Problems

Ram U. Verma

Abstract Based on the higher order hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities, first some parametrically generalized sufficient efficiency conditions for multiobjective fractional programming are developed and then efficient solutions to the multiobjective fractional programming problems are established. Furthermore, the obtained results on sufficient efficiency conditions are generalized to the case of the $\varepsilon-$efficient solutions. The results thus obtained generalize and unify a wide spectrum of investigations on the theory and applications to the multiobjective fractional programming based on the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity frameworks.

Keywords Higher order hybrid invexity · Multiobjective fractional programming · Efficient solutions

AMS Subject Classification 90C32 · 90C45

1 Introduction

Mangasarian [8] investigated second order duality for a conventional nonlinear programming problem, where the approach is based on constructing a second order dual problem by taking linear and quadratic approximations of the objective and constraint functions for an arbitrary but fixed point leading to the Wolfe dual model for the approximated problem, while letting the fixed point to vary. Recently, Verma [22] investigated a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency constraints for multiobjective fractional programming problems leading to weakly $\varepsilon-$efficient solutions. Motivated by these research developments, we first introduce the higher order hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities, second, introduce some parametrically sufficient efficiency conditions for multiobjective fractional programming, and finally, explore the

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efficient solutions to multiobjective fractional programming problems. The results established in this paper, not only generalize and unify the results on general sufficient efficiency conditions for multiobjective fractional programming problems based on the hybrid invexity of functions, but also generalize second order invexity results in more general settings.

We consider, based on the higher order hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities of functions, the following multiobjective fractional programming problem:
(P)

$$Minimize \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$, where X is an open convex subset of $\Re^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:
(Pλ)

$$Minimize \Big(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x)\Big)$$

subject to $x \in Q$ with

$$\lambda = \Big(\lambda_1, \lambda_2, \cdots, \lambda_p\Big) = \left(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\right),$$

where x^* is an efficient solution to (P).
General mathematical programming problems offer a wide range of applications to other fields, such as applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquake-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, and others. Recently, Pitea and Postolache [18] introduced and studied a new class of multitime multiobjective variational problems for minimizing a vector of functionals of curvilinear integral type relating to Mond-Weir-Zalmai type duality based on the notion of (ρ, b)-quasiivexity. They also established some weak duality theorems showing the value of the objective function of the primal cannot exceed the value of the dual. On the other hand, there are accelerated advances on duality models for a class of multiobjective control problems with generalized invexity, especially the work of Zhian and Qingkai [41], where they have discussed the duality models for multiobjective control problems using the generalized invexity. For more details on generalized efficiency and efficiency results and applications, we recommend the reader [1–41].

This submission is organized as follows: the introductory section deals with a brief historical development for the multiobjective fractional mathematical programming, while emphasizing the roles of the generalized invex functions. In Sect. 2, the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invex functions of higher orders are introduced, and Sect. 3 deals with sufficient efficiency conditions leading to the solvability of the problem (P) using the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities.

2 Hybrid Invexities

In this section, we introduce the notion of higher order $(\Phi, \rho, \eta, \zeta, \theta)-$invexities, which encompass most of the existing generalized invexities in the current literature. Let X be an open convex subset of $\Re^n$ (n-dimensional Euclidean space). Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $z \in \Re^n$. Suppose that $f : X \to \Re$ is a real-valued twice continuously differentiable function defined on X, and that $\triangledown f(y)$ and $\nabla^2 f(y)$ denote, respectively, the gradient and Hessian of f at y.

Definition 2.1 A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \theta, \zeta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \langle \triangledown f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x, x^*)\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2.$$

Definition 2.2 A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\langle \triangledown f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x, x^*)\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.3 A twice differentiable function $f : X \to \Re$ is said to be strictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ pseudo-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\langle \triangledown f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x, x^*)\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) > 0.$$

Definition 2.4 A twice differentiable function $f : X \to \Re$ is said to be prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function

$\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x,x^*)\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$
$$\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.5 A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re)$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\Phi\big(f(x) - f(x^*)\big) \leq 0$$
$$\Rightarrow \langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x,x^*)\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0.$$

Definition 2.6 A twice differentiable function $f : X \to \Re$ is said to be strictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\Phi\big(f(x) - f(x^*)\big) \leq 0$$
$$\Rightarrow \langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x,x^*)\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 < 0.$$

Definition 2.7 A twice differentiable function $f : X \to \Re$ is said to be prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\Phi\big(f(x) - f(x^*)\big) < 0$$
$$\Rightarrow \langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x,x^*)\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

equivalently,

$$\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, \zeta(x,x^*)\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$
$$\Rightarrow \Phi\big(f(x) - f(x^*)\big) \geq 0.$$

Definition 2.8 A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \, \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x^*)}{g_j(x^*)} \text{ for some } j \in \{1, \cdots, p\}.$$

Next to this context, we have the following auxiliary problem:
(P$\bar{\lambda}$)

$$minimize_{x \in Q}(f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,
where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, and $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)}$.

Example 2.1 Consider a twice differentiable function $f : X \to \Re$ such that there exist functions $\Phi : \Re \to \Re$, $\rho : X \times X \to \Re$, $\eta, \theta, \zeta : X \times X \to \Re^n$. Then f is hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invex at x^* of second order since for each $x \in X$, and $z \in \Re^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle + \rho(x, x^*)\|\theta(x, x^*\|^2.$$

Example 2.2 Consider a differentiable function $f : X \to \Re$ such that there exist functions $\Phi : \Re \to \Re$, $\rho : X \times X \to \Re$, $\eta, \theta, \zeta : X \times X \to \Re^n$. Then f is hybrid $(\Phi, \rho, \eta, \theta)-$invex at x^* of first order since for each $x \in X$, and $z \in \Re^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \langle \nabla f(x^*), \eta(x, x^*)\rangle + \rho(x, x^*)\|\theta(x, x^*\|^2.$$

Next, we introduce the efficiency solvability conditions for (P$\bar{\lambda}$) problem.

Definition 2.9 A point $x^* \in Q$ is an efficient solution to (P$\bar{\lambda}$) if there does not exist an $x \in Q$ such that

$$f_i(x) - \bar{\lambda}_i g_i(x) \leq f_i(x^*) - \bar{\lambda}_i g_i(x^*) \, \forall \, i = 1, \cdots, p,$$

$$f_j(x) - \bar{\lambda}_j g_j(x) < f_j(x^*) - \bar{\lambda}_j g_j(x^*) \text{ for some} j \in \{1, \cdots, p\},$$

where $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)}$ for $i = 1, \cdots, p$.
Next, we recall the following result (Verma [24]) that provides a set of necessary efficiency conditions for problem (P) for developing some sufficient efficiency conditions for the next section based on second order $(\Phi, \rho, \eta, \zeta, \theta)-$invexities.

Theorem 2.1 *[24] Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$ for each $i \in \underline{p}$, and let f_i and g_i be twice continuously differentiable at x^* for each $i \in \underline{p}$. For each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an efficient solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\{\Big(\nabla G_j(x^*,t),\langle y,\nabla^2 G_j(x^*,t)y\rangle\Big):t\in\hat{T}_j(x^*),\ j\in\underline{q}\}$$
$$+\ span\{\Big(\nabla H_k(x^*,s),\langle y,\nabla^2 H_k(x^*,s)y\rangle\Big):s\in S_k,k\in\underline{r}\},$$

where $\hat{T}_j(x^*)\equiv\{t\in T_j:G_j(x^*,t)=0\}$, *is closed, then there exist* $u^*\in U\equiv\{u\in\mathbb{R}^p:u\geq 0,\sum_{i=1}^p u_i=1\}$ *and integers* ν_0^* *and* ν^* *with* $0\leq\nu_0^*\leq\nu^*\leq n+1$ *such that there exist* ν_0^* *indices* j_m *with* $1\leq j_m\leq q$ *together with* ν_0^* *points* $t^m\in\hat{T}_{j_m}(x^*)$, $m\in\underline{\nu_0^*}$, $\nu^*-\nu_0^*$ *indices* k_m *with* $1\leq k_m\leq r$ *together with* $\nu^*-\nu_0^*$ *points* $s^m\in S_{k_m}$ *for* $m\in\underline{\nu^*}\backslash\underline{\nu_0^*}$, *and* ν^* *real numbers* v_m^* *with* $v_m^*>0$ *for* $m\in\underline{\nu_0^*}$ *with the property that*

$$\sum_{i=1}^p u_i^*[\nabla f_i(x^*)-\lambda^*(\nabla g_i(x^*)]+\sum_{m=1}^{\nu_0^*}v_m^*[\nabla G_{j_m}(x^*,t^m)$$
$$+\sum_{m=\nu_0^*+1}^{\nu^*}v_m^*\nabla H_k(x^*,s^m)=0, \tag{2.1}$$

$$\langle y,\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)-\lambda^*\nabla^2 g_i(x^*)]+\sum_{m=1}^{\nu_0^*}v_m^*\nabla^2 G_{j_m}(x^*,t^m)$$
$$+\sum_{m=\nu_0^*+1}^{\nu^*}v_m^*\nabla^2 H_k(x^*,s^m)\Big]y\rangle\geq 0, \tag{2.2}$$

$$u_i^*[f_i(x^*)-\lambda^* g_i(x^*)]=0,\quad i\in\underline{p}, \tag{2.3}$$

where $\underline{\nu}\setminus\underline{\nu_0}$ *is the complement of the set* $\underline{\nu_0}$ *relative to the set* $\underline{\nu}$.

3 Sufficient Efficiency Conditions for Problem (P)

This section deals with some parametrically sufficient efficiency conditions for problem (P) under the hybrid frameworks for $(\Phi,\rho,\eta,\zeta,\theta)-$invexities. We begin with real-valued functions $E_i(.,x^*,u^*)$ and $B_j(.,v)$ defined by

$$E_i(x,x^*,u^*)=u_i[f_i(x)-\Big(\frac{f_i(x^*)}{g_i(x^*)}\Big)g_i(x)],\ i\in\{1,\cdots,p\}$$

and

$$B_j(.,v)=v_jH_j(x),\ \ j=1,\cdots,m.$$

Theorem 3.1 *Let $x^* \in Q$, f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*) = 0, \tag{3.1}$$

$$\Big\langle \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*) \Big] z, \eta(x, x^*) \Big\rangle \geq 0, \tag{3.2}$$

$$-\frac{1}{2} \Big\langle \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*) \Big] z, \zeta(x, x^*) \Big\rangle \geq 0, \tag{3.3}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{3.4}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* with $\bar{\Phi}(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(ii) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iii) $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\Phi, \rho_1, \eta, \theta)-$invex and $-g_i$ is hybrid $(\Phi, \Psi, \rho_2, \eta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, $H_j(., v^*)$ $\forall j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \rho_3, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$, and $\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \rho^*(x, x^*) \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof If (i) holds, and if $x \in Q$, then it follows from (3.1)–(3.3) that

$$\Big\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] + \sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*) z], \eta(x, x^*) \Big\rangle$$

$$
\begin{aligned}
&-\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\
&+ \Big\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) + \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*)z, \eta(x,x^*)\Big\rangle \\
&- \frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle \geq 0. \qquad (3.5)
\end{aligned}
$$

Since $v^* \geq 0$, $x \in Q$ and (3.4) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and in light of assumptions on $\tilde{\Phi}$, we find

$$\tilde{\Phi}\Big(\Sigma_{j=1}^m v_j^* H_j(x) - \Sigma_{j=1}^m v_j^* H_j(x^*)\Big) \leq 0,$$

which applying the hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, results in

$$
\begin{aligned}
&\Big\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) + \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*)z, \eta(x,x^*)\Big\rangle \\
&- \frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle + \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \qquad (3.6)
\end{aligned}
$$

It follows from (3.5) and (3.6) that

$$
\begin{aligned}
&\Big\langle \Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\
&+ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \eta(x,x^*)\Big\rangle \\
&- \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\
&\geq \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \geq -\rho(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.7)
\end{aligned}
$$

Since $\rho(x,x^*) \geq 0$, applying the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invexity at x^* to (3.7) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned}
&\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\
&\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\
&= 0.
\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.8}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \ \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

Next, If (ii) holds, and if $x \in Q$, then it follows from (3.1)–(3.3) that

$$\begin{aligned}
&\Big\langle \Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] \\
&+ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \eta(x, x^*)\Big\rangle \\
&- \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x, x^*)\Big\rangle \\
&+ \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*) + \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \eta(x, x^*)\Big\rangle \\
&- \frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \zeta(x, x^*)\Big\rangle \geq 0.
\end{aligned} \tag{3.9}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which results (using assumptions on $\tilde{\Phi}$) in

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0.$$

Now, in light of the strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we find

$$\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*) + \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z, \eta(x, x^*)\Big\rangle \\ - \frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z, \zeta(x, x^*)\Big\rangle + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ + \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z], \eta(x, x^*)\Big\rangle \\ - \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z], \zeta(x, x^*)\Big\rangle \\ > \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 > -\rho(x, x^*)\|\theta(x, x^*)\|^2. \tag{3.11}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invexity at x^* to (3.11) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) \\ = 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.12}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

The important aspect of the proof applying (iii) is that we use the equivalent form for Definition 2.7 instead. Since $B_j(\cdot, v_{j^*})$ is strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$, we have

$$\begin{aligned}&\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*) + \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \eta(x, x^*)\Big\rangle \\ &- \frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \zeta(x, x^*)\Big\rangle + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 < 0. \qquad (3.13)\end{aligned}$$

Next, applying (3.13)–(3.15), we arrive at

$$\begin{aligned}&\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] \\ &+ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \eta(x, x^*)\Big\rangle \\ &- \frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x, x^*)\Big\rangle \\ &> \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 > -\rho(x, x^*)\|\theta(x, x^*)\|^2. \qquad (3.14)\end{aligned}$$

At this stage, since $E_i(.; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned}&\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ &\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) = 0.\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \qquad (3.15)$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \;\forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

Finally, we prove using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, which implies $\bar{\Phi}\Big(H_j(x) - H_j(x^*)\Big) \leq 0$. Then applying the hybrid $(\bar{\Phi}, \rho_3, \eta, \zeta, \theta)-$quasi-invexity of H_j at x^* and $v^* \in R^m_+$, we have

$$\begin{aligned} &\Big\langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*) + \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \eta(x, x^*)\Big\rangle \\ &- \frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \zeta(x, x^*)\Big\rangle + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \end{aligned} \tag{3.16}$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, it follows from the hybrid $(\Phi, \rho_3, \eta, \zeta, \theta)-$invexity assumptions that

$$\begin{aligned} &\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \\ &= \Phi\Big(\Sigma_{i=1}^p u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)})[g_i(x) - g_i(x^*)]\}\Big) \\ &\geq \Sigma_{i=1}^p u_i^*\{\langle \nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*) \\ &+ \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z, \eta(x, x^*)\rangle] \\ &- \frac{1}{2}\langle \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z, \zeta(x, x^*)\rangle] \\ &+ \Sigma_{i=1}^p u_i^*[\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*)]\|\theta(x, x^*)\|^2 \\ &\geq -\Big[\langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*) + \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \eta(x, x^*)\Big\rangle \\ &- \frac{1}{2}\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \zeta(x, x^*)\rangle\Big] \\ &+ \Sigma_{i=1}^p u_i^*[\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*)]\|\theta(x, x^*)\|^2 \\ &\geq (\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \Sigma_{i=1}^p u_i^*[\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*)])\|\theta(x, x^*)\|^2 \\ &= (\Sigma_{j=1}^m v_j^* \rho_3 + \rho^*(x, x^*))\|\theta(x, x^*)\|^2 \\ &\geq 0, \end{aligned}$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$ and $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*))$.
This implies that

$$\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \geq 0.$$

Next we consider the case when the functions are first-order differentiable, Theorem 3.1 reduces to the result which is similar to the results of Zalmai ([35], Theorems 3.1, 3.2), and Zalmai and Zhang [38].

Theorem 3.2 *For $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\left\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*) \right\rangle \geq 0 \quad (3.17)$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \quad (3.18)$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are first-order hybrid $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are first-order hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.
(ii) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are first-order hybridprestrictly $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are first-order strictly hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.
(iii) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ are first-order prestrictly hybrid $(\Phi, \rho, \eta, \theta)-$quasi-invex at x^* $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are first-order strictly hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.
(iv) For each $i \in \{1, \cdots, p\}$, f_i is first-order hybrid $(\Phi, \rho_1, \eta, \theta)-$invex and $-g_i$ is first-order hybrid $(\Phi, \rho_2, \eta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$. $H_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \bar{\rho_3}, \eta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \rho^*(x, x^*) \geq 0$ for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$, $\rho^*(x, x^*) = \Sigma_{i=1}^p u_i^* (\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

We observe that Theorem 3.1 can be further generalized to the case of the $\varepsilon-$efficient conditions based on the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity frameworks. As a matter of fact, we generalize the $\varepsilon-$efficient solvability conditions for problem (P) based on the work of Verma [22], and Kim, Kim and Lee [6], where they have investigated the $\varepsilon-$efficiency as well as the weak $\varepsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. We recall some auxiliary concepts (for the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity) crucial to the problem on hand.

Definition 3.1 A point $x^* \in Q$ is an $\varepsilon-$efficient solution to (P) if there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \; \forall i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f(_j x^*)}{g_j(x^*)} - \varepsilon_j \text{ for some } j \in \{1, \cdots, p\},$$

where ε_i=$(\varepsilon_1, \cdots, \varepsilon_p)$ is with $\varepsilon_i \geq 0$ for $i = 1, \cdots, p$.
For $\varepsilon = 0$, Definition 3.1 reduces to the case that $x^* \in Q$ is an efficient solution to (P).
Next, we start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x)],\ i \in \{1, \cdots, p\}$$

and

$$B_j(., v^*) = v_j^* H_j(x),\ \ j = 1, \cdots, m.$$

Theorem 3.3 *Let* $x^* \in Q$*,* f_i, g_i *for* $i \in \{1, \cdots, p\}$ *with* $f_i(x^*) \geq \varepsilon_i g_i(x^*)$*,* $g_i(x^*) > 0$ *and* H_j *for* $j \in \{1, \cdots, m\}$ *be twice continuously differentiable at* $x^* \in Q$*, and let there exist* $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$*,* $v^* \in \Re_+^m$ *and* $z \in \Re^n$ *such that*

$$\Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0, \tag{3.19}$$

$$\Big\langle\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big]z, \eta(x, x^*)\Big\rangle \geq 0, \tag{3.20}$$

$$-\frac{1}{2}\Big\langle\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big]z, \zeta(x, x^*)\Big\rangle \geq 0, \tag{3.21}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.22}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$*):*

(i) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ *are hybrid* $(\Phi, \rho, \eta, \theta)-$*pseudo-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$*, and* $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ *are hybrid* $(\tilde{\Phi}, \bar{\rho}, \eta, \theta)-$*quasi-invex at* x^* *for* $\tilde{\Phi}$ *increasing and* $\tilde{\Phi}(0) = 0$.
(ii) $E_i(.\,; x^*, u^*)$ $\forall i \in \{1, \cdots, p\}$ *are prestrictly hybrid* $(\Phi, \rho, \eta, \theta)-$*pseudo-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$*, and* $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ *are strictly hybrid* $(\Phi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\bar{\Phi}$ *increasing and* $\bar{\Phi}(0) = 0$.

(iii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are prestrictly hybrid* $(\Phi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$*, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are strictly hybrid* $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$*quasi-invex at* x^* *for* $\bar{\Phi}$ *increasing and* $\bar{\Phi}(0) = 0$.

(iv) *For each* $i \in \{1, \cdots, p\}$, f_i *is hybrid* $(\Phi, \rho_1, \eta, \theta)-$*invex and* $-g_i$ *is* $(\Phi, \rho_2, \eta, \theta)-$*invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$*, and* $H_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *is hybrid* $(\bar{\Phi}, \rho_3, \eta, \theta)-$*quasi-invex at* x^* *for* $\bar{\Phi}$ *increasing and* $\bar{\Phi}(0) = 0$ *and* $\Sigma_{j=1}^{m} v_j^* \rho_3(x, x^*) + \rho^*(x, x^*) \geq 0$ *for* $\rho^* = \Sigma_{i=1}^{p} u_i^*(\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*))$*, where* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$.

Then x^* *is an* $\varepsilon-$*efficient solution to (P).*

Proof The proofs are similar to that of Theorem 3.1.

4 Concluding Remarks

We observe that the higher order hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ invexities can effectively be applied generalizing/unifying the first-order sufficient efficiency condition results [35], first-order parametric duality model results [36] as well as second order duality model results (Zalmai [37]) on Hanson-Antczak-type generalized V-invex functions in semi-infinite multiobjective fractional programming. Based on new duality models and suitable constraint structures, the weak, strong, and strict converse duality theorems can be established using appropriate hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ invexities.

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