# Analysis of an Eco-Epidemiological Model with Migrating and Refuging Prey

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**Abstract** This paper concerns a predator–prey system with migrating and refuging prey with disease infection. Analysis of the model regarding stability has been performed. The effect of time delay on the above system is also studied. By assuming the time delay a bifurcation parameter, the stability of the positive equilibrium, and Hopf-bifurcation is studied. Further, the directions of Hopf-bifurcation and the stability of bifurcated periodic solutions are calculated using the famous normal form theory, Riesz representation theorem and central manifold theorem. This is not a case study, hence real data is not available. However, to verify our theoretical predictions, some numerical simulations are also included.

**Keywords** Predator–prey model · Stability · Hopf-bifurcation · Migration · Refuge · Delay

## 1 Introduction

The dynamic relation between prey and predator has been studied extensively in the literature. At first sight, prey–predator dynamics may seem very simple mathematically, but they are, in fact very difficult and challenging. The classical Lotka-Volterra model is a first stepping stone in the study of prey–predator dynamics and interactions [1, 23]. In mathematical ecology, this model is extensively used and cited and proved a milestone in the progress of mathematical ecology. On the other hand, the famous work of Kermack-Mckendric [25] in epidemiological studies received much attention among applied mathematicians, scientists, and ecologists. After the work of [1, 23, 25], many mathematical models have been published for reference

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(see [2, 6, 7, 13–17, 21, 24, 27, 29], etc., and references therein). Combined and/or overlapping study of ecology and epidemiology is termed as Eco-epidemiology. Eco-epidemiological models are gaining popularity day-by-day. The present study also falls under the purview of Eco-epidemiology.

To study the environmental impact on prey–predator models, the 'time delay' has been investigated by the researchers. A good number of papers are available in the literature for instance (see [4, 8, 18]). In these papers, most of the authors investigate the 'time delay' as a game changing. Time delay may cause changes in stability, occurrence for limit cycle, bifurcation, etc.

Further, migration of species especially prey is also evolved and few references are available in the literature, for reference we can refer [11, 19, 20]. Prey-refuge in prey-predator models also play an important role in dynamical nature. Further if prey-refuge is more than outbreak of the prey population occurs. To understand a role of prey-refuge in mathematical ecology few publications are available. At this juncture we may refer readers to ([9, 22, 24, 26, 29] and references therein).

Pal and Samanta [3] proposed the following mathematical model by incorporating prey-refuge in the model proposed of Xiao and Chen [28]:

$$\begin{cases} \frac{dS}{dt} = r_1 S(1 - \frac{S+I}{k}) - SI\beta, \\ \frac{dI}{dt} = SI\beta - cI - \frac{bIY}{aY+I}, \\ \frac{dY}{dt} = -dY + \frac{pbIY}{aY+I}. \end{cases} \tag{1}$$

Motivated by the model of Samanta [18] and model in [12], Hu and Li [10] proposed the following model:

$$\begin{cases} \frac{dS}{dt} = rS\left(1 - \frac{S+I}{k}\right) - SI\beta - p_1 SY, \\ \frac{dI}{dt} = -cI + SI\beta - p_2 IY, \\ \frac{dY}{dt} = -dY + qp_1 S(t-\tau)Y(t-\tau) + qp_2 I(t-\tau)Y(t-\tau). \end{cases} \tag{2}$$

In order to study the influence of prey-refuge, migration, and disease on the Prey-predator system, in this paper, we concentrate on an eco-epidemiological prey-predator system consisting of three species as in [10]. Motivated by the models in [10] and [3], we propose a mathematical model in which prey is migrating and refuging with disease in both species. We present stability and Hopf-bifurcation analysis of the mathematical model. Detailed assumptions for model formulation are listed in the next section.

The rest of the paper is structured as follows: In the next section, we formulate our main mathematical model with the help of biological and ecological assumptions. In Sect. 3, we consider the model without delay. In Sect. 4, we discuss the stability of mathematical model with delay. In Sect. 5, we discuss the direction and stability of Hopf-bifurcation using the normal form theory, Riesz representation theorem and central manifold theorem as in [10]. Numerical simulations have been done in Sect. 6 followed by discussion in the last Sect. 7.

## 2 The Model

In this paper, we propose to study a prey-predator system by means of mathematical modeling. To formulate the model and in view of simplicity we make the following assumptions:

- In the absence of disease and predation the healthy (susceptible) prey population has logistic growth with growth rate $r$ and carrying capacity $k$, i.e.,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right). \tag{3}$$

- Disease is spreading in both populations. After disease prey population is divided into two parts susceptible prey(S) and infected prey (I). Thus total biomass of prey population is $S(t) + I(t)$.
- Due to mathematical complexity, the bifurcation of predator population and the detailed dynamics of the disease infection in the predator population is omitted. Further, it is also assumed that disease infection in predator occurs due to eating of the infected prey and not due to outside infection. In other words, it is easy to understand that the disease infection starts from prey and then carries forward to predator. For example H1N1, H5N1, etc., may be pointed out here to understand the physical phenomenon better. Thus total biomass of predator population is Y.
- Infected prey population does not become immune as well as they have no reproduction rate. However, infected prey population contributes the carrying capacity $k$.
- Predator population consumes both susceptible as well as infected prey population.
- Due to environmental and fear factors, we consider out migration in prey population, i.e., once prey migrated they will not return. Let $m_1$ and $m_2$ be the migration rates of susceptible and infected prey respectively. Further, healthy prey population is more active compared to infected one. Hence, healthy prey can migrate more easily than infected prey before their predation. Hence, by using this ecological information, we can impose the mathematical condition $m_1 > m_2$.
- Let $d_2$ and $d_3$ be natural death rates for infected prey and predator population respectively.
- Death (mortality) rate due to disease for infected prey population and predator population are denoted by c and $d_4$ respectively.
- The coefficient for S-prey and I-prey to predator are denoted by $q_1$ and $q_2$ respectively. The relationship between $q_1$ and $q_2$ is established later.
- Let a refuge protecting $m_3S$ of healthy prey and $m_4I$ that of infected prey, where $m_3, m_4 \in [0, 1)$. Hence, $(1 - m_3)S$ and $(1 - m_4)I$ of healthy and infected prey, respectively, are available to the predator for predation.

Based on these assumptions, model takes the following form:

$$\begin{cases} \frac{dS}{dt} = rS\left(1 - \frac{S+I}{k}\right) - SI\beta - p_1(1-m_3)SY - m_1S, \\ \frac{dI}{dt} = SI\beta - p_2(1-m_4)IY - d_2I - m_2I - cI, \\ \frac{dY}{dt} = q_1p_1(1-m_3)S(t-\tau)Y(t-\tau) + q_2p_2(1-m_4)I(t-\tau)Y(t-\tau) - d_3Y - d_4Y. \end{cases} \tag{4}$$

The initial conditions are

$$\begin{cases} S(t) = \phi_1(t) > 0, \\ I(t) = \phi_2(t) > 0, \\ Y(t) = \phi_3(t) > 0, \\ (\phi_1(t), \phi_2(t), \phi_3(t)) \in C = C\left([-\tau, 0], R_+^3\right), \\ R_+^3 = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0\}, \end{cases} \tag{5}$$

where,

$\beta$ : Disease Contact Rate
$p_1, p_2$ : Predation Coefficients of Susceptible (S) and Infected (I) Prey
$\tau$ : Gestation period (delay).

Ecological and biological assumptions suggests the following relationship between $q_1$ and $q_2$:

$$q_2 \neq q_1 \text{ and } 0 < q_1 \leq 1,$$

$$q_2 > q_1 \text{ and } 0 < q_2 \leq 1.$$

## 3 Analysis of the Model Without Delay

In this section, model (4) is investigated under the condition $\tau = 0$. Before going to main analysis, we state two lemmas for our model without proof.

**Lemma 1** *Each solution of the system (4) without delay with the initial conditions (5) are strictly positive for all* $t \geq 0$.

**Lemma 2** *Solutions of the system (4) without delay with the initial conditions (5) are eventually bounded, i.e., uniformity bounded in* $R_+^3$.

### 3.1 Equilibrium Points

System of ODEs under consideration has the following equilibrium points:

(i) $E_1(0, 0, 0) = (0, 0, 0)$.
(ii) $E_2(\widehat{S}, 0, 0)$. where $\widehat{S} = (r - m_1)\frac{k}{r}$.
(iii) $E_3(S^*, 0, Y^*)$, where

$$\begin{cases} S^* = & \frac{d_3 + d_4}{q_1 p_1 (1 - m_3)}, \\ Y^* = & \frac{k p_1 (1 - m_3) q_1 (r - m_1) - r(d_3 + d_4)}{k q_1 (1 - m_3) p_1^2}. \end{cases}$$

(iv) $E_4(\overline{S}, \overline{I}, 0)$, where

$$\begin{cases} \overline{S} = & \frac{c + d_2 + m_2}{\beta}, \\ \overline{I} = & \frac{\{(r - m_1)k\beta - r(c + d_2 + m_2)\}}{(\beta(r + k\beta))}. \end{cases}$$

(v) $E_5(\widetilde{S}, \widetilde{I}, \widetilde{Y})$, where

$$\begin{cases} \widetilde{S} = & \frac{(r - m_1) q_2 (1 - m_4) p_2 k + (c + d_2 + m_2) q_2 p_1 (1 - m_3)(1 - m_4) p_2 k - (r + k\beta)(d_3 + d_4)}{r q_2 p_2 (1 - m_4) - q_1 p_1 (1 - m_3)(r + k\beta) + q_2 (1 - m_4) p_2 \beta k}, \\ \widetilde{I} = & \frac{(d_3 + d_4) - q_1 p_1 (1 - m_3)\widetilde{S}}{q_2 p_2 (1 - m_4)}, \\ \widetilde{Y} = & \frac{\beta\widetilde{S} - (c + d_2 + m_2)}{p_2 (1 - m_4)}. \end{cases}$$

### 3.1.1 Existence Conditions

We have the following existence conditions:

(i) Trivial equilibrium $E_1$ always exists.
(ii) $E_2$ exists provided $(r - m_1) > 0$ or $r > m_1$. Physical meaning implies that existence of $E_2$ is independent of other parameters and depends only on growth rate and migration of S, viz., $r$ and $m_1$. $E_2$ exists if growth rate of S is greater than migration of itself.
(iii) $E_3$ exists provided $\frac{r - m_1}{d_3 + d_4} > \frac{r}{k q_1 p_1 (1 - m_3)}$. This is the case when no disease infection occurs in the prey population.
(iv) $E_4$ exists provided $\frac{r - m_1}{c + d_2 + m_2} > \frac{r}{k\beta}$. This is the case when predator does not survive.
(v) $E_5$ exists provided the following conditions are satisfied:

$$\begin{cases} d_3 + d_4 > q_1 p_1 (1 - m_3)\widetilde{S}, \\ \beta\widetilde{S} > (c + d_2 + m_2), \\ (r - m_1) q_2 p_2 (1 - m_4) k + (c + d_2 + m_2) q_2 p_2 (1 - m_4) p_1 (1 - m_3) k > (r + k\beta)(d_3 + d_4), \\ (r q_2 p_2 (1 - m_4) + q_2 p_1 (1 - m_3)\beta k) > q_1 p_1 (1 - m_3)(r + k\beta). \end{cases}$$

This equilibrium point is very important, since it provides the coexistence of all the three populations.

## 3.2 Stability Analysis

Jacobian matrix of the system is given by

$$J = \begin{pmatrix} (r - m_1 - \frac{2rS}{K} - \frac{rI}{k} - \beta I - p_1(1-m_3)Y) & \left(-\frac{rS}{k} - \beta S\right) & (-p_1(1-m_3)S) \\ (\beta I) & (\beta S - p_2(1-m_4)Y - c - d_2 - m_2) & (-p_2(1-m_4)I) \\ (q_1(1-m_3)p_1 Y) & (q_2 p_2(1-m_4)Y) & (q_1 p_1(1-m_3)S + q_2 p_2(1-m_4)I - d_3 - d_4) \end{pmatrix}, \tag{6}$$

with this matrix stability analysis is carried out. We will focus on the non zero equilibrium point.

After a little calculation we see that trivial equilibrium point is locally stable if $r < (m_1)$. Equilibrium $(E_2)$ is locally asymptotically stable provided the following conditions are satisfied:

$$\begin{cases} (\beta\widehat{S} - c - d_2 - m_2) = \frac{\beta k(r-m_1)}{r} < 0, \\ (q_1 p_1(1-m_3)\widehat{S} - d_3 - d_4) = q_1 p_1(1-m_3)\frac{k(r-m_1)}{r} - d_3 - d_4 < 0. \end{cases}$$

Equilibrium $(E_3)$ is locally asymptotically stable if

$$\begin{cases} \left[\frac{\beta(d_3+d_4)}{q_1(1-m_3)p_1} - \frac{p_2(1-m_4)(r-m_1)}{p_1(1-m_3)} + \frac{p_2(1-m_3)r(d_3+d_4)}{kq_1(1-m_3)p_1^2} - (c + d_2 + m_2)\right] < 0; \\ \text{Quadratic equation } (\lambda^2 - \xi\lambda + \zeta) \text{ have roots with negative real parts, where} \\ \xi = \frac{-(d_3+d_4)[kq_1 p_1(1-m_3)+r]}{kq_1(1-m_3)p_1}, \\ \zeta = \frac{(d_3+d_4)[kq_1 p_1(1-m_3)(r-m_1)-(d_3+d_4)r]}{kq_1 p_1(1-m_3)}. \end{cases}$$

Equilibrium $(E_4)$ is locally asymptotically stable if

$$\begin{cases} (q_1 p_1\overline{S} + q_2 p_2\overline{I} - d_3 - d_4) = -(d_3 + d_4) + \frac{q_1 p_1(c+d_2+m_2)}{\beta} + \frac{q_2 p_2[(r-m_1)k\beta - r(c+d_2+m_2)]}{\beta(r+k\beta)} < 0; \\ \text{Equation } (\lambda^2 - B\lambda + C) \text{ have roots with negative real parts, where} \\ B = \frac{-3r(c+d_2+m_2)}{k\beta}, \\ C = (c + d_2 + m_2)\left[(r - m_1) - \frac{r(c+d_2+m_2)}{k\beta}\right]. \end{cases}$$

*Remark 1* In this case no infection occurs in the system, hence ecologically mortality due to infection in predator population may be omitted. Similarly, parameter $c$ may also be deleted. Jacobian matrix at $E_3$ is now reduced to

$$J = \begin{bmatrix} \left(r - m_1 - \frac{2rS^*}{k} - p_1(1-m_3)Y^*\right) & \left(-\frac{rS^*}{k} - \beta S\right) & (-p_1(1-m_3)S^*) \\ 0 & (\beta S^* - p_2(1-m_4)Y^* - d_2 - m_2) & 0 \\ (q_1 p_1(1-m_3)Y^*) & (q_2 p_2(1-m_4)Y^*) & (q_1 p_1(1-m_3)S^* - d_3) \end{bmatrix}, \tag{7}$$

where $S^* = \frac{d_3}{q_1 p_1(1-m_3)}$ and $Y^* = \frac{kq_1 p_1(1-m_3)(r-m_1)-r(d_3)}{kq_1(1-m_3)p_1^2}$.

Characteristic equation is given by $(\lambda - \lambda_1)(\lambda^2 - \xi\lambda + \zeta) = 0$, where $\lambda_1 = (\beta S^* - p_2(1-m_4)Y^* - d_2 - m_2) = \frac{\beta(d_3)}{q_1 p_1(1-m_3)} - \frac{p_2(1-m_4)(r-m_1)}{p_1(1-m_3)} + \frac{p_2(1-m_4)r(d_3)}{kq_1(1-m_3)p_1^2} - (d_2+m_2)$,

$\xi = \frac{-(d_3)[kq_1p_1(1-m_3)+r]}{kq_1(1-m_3)p_1}$, $\zeta = \frac{(d_3)[kq_1p_1(1-m_3)(r-m_1)-(d_3)r]}{kq_{1(1-m_3)}p_1}$.

Thus $E_3$ is locally asymptotically stable if the following conditions are satisfied:

$$\begin{cases} \left[\frac{\beta(d_3)}{q_1(1-m_3)p_1} - \frac{p_2(1-m_4)(r-m_1)}{p_1(1-m_3)} + \frac{p_2(1-m_4)r(d_3)}{kq_1(1-m_3)p_1^2} - (d_2+m_2)\right] < 0, \\ \text{Equation } (\lambda^2 - \xi\lambda + \zeta) \text{ have roots with negative real parts.} \end{cases}$$

### 3.2.1 Positive Equilibrium

In this case, populations of all the three species exists simultaneously. As promised, we will furnish the detail of the stability of the positive equilibrium point. For the stability of the positive equilibrium $E_5$, we state the following theorem:

**Theorem 1** *System (4) with $\tau = 0$ is locally asymptotically stable at $E_5$ if the following conditions are satisfied:*

*(i)* $\Gamma\widetilde{S} + \Delta\widetilde{I} + \Theta\widetilde{Y} + \Lambda < 0.$
*(ii)* $A_1A_2 + A_3 > 0$,

*where* $\Gamma = (q_1p_1(1-m_3) + \beta - \frac{2r}{k})$,
$\Delta = (-(\frac{r}{k} + \beta) + q_2(1-m_4)p_2)$,
$\Theta = -(p_1(1-m_3) + p_2(1-m_4))$,
$\Lambda = (r - m_1 - m_2 - d_2 - d_3 - d_4 - c)$,
$A_1 = \Gamma\widetilde{S} + \Delta\widetilde{I} + \Theta\widetilde{Y} + \Lambda$,
$A_2 = \widetilde{S}^2[\beta q_1p_1(1-m_3) - \frac{2rq_1(1-m_3)p_1}{k} - \frac{2r\beta}{k}] + \widetilde{Y}^2[p_1(1-m_3)(1-m_4)p_2] + \widetilde{I}^2[-(\frac{r}{k}+\beta)q_2p_2(1-m_4)] + \widetilde{S}\widetilde{I}[-(\frac{r}{k}+\beta)q_1p_1(1-m_3) + \beta q_2p_2(1-m_4) - \frac{2rq_1p_1(1-m_3)}{k}] + \widetilde{Y}\widetilde{I}[-q_2p_2(1-m_4)(1-m_3)p_1 + p_2(1-m_4)(\frac{r}{k}+\beta)] + \widetilde{S}\widetilde{Y}[-q_1p_2p_1(1-m_4)(1-m_3) + \frac{2rp_2(1-m_4)}{k} - p_1(1-m_3)\beta] + \widetilde{S}[-\beta(d_3+d_4) - (c+m_2+d_2)q_1p_1(1-m_3) + (r-m_1)q_1p_1(1-m_3) + \frac{2r(d_3+d_4)}{k} + \beta(r-m_1) + \frac{2r(c+d_2+m_2)}{k}] + \widetilde{I}[-(c+d_2+m_2)(1-m_4)q_2p_2 + (r-m_1)q_2(1-m_4)p_2 + (\frac{r}{k}+\beta)(d_3+d_4) + (\frac{r}{k}+\beta)(c+d_2+m_2)] + \widetilde{Y}[p_1(1-m_3)(d_3+d_4) - p_2(r-m_1)(1-m_4) + p_1(1-m_3)(c+m_2+d_2)] + [p_2(1-m_4)(d_3+d_4) + (c+m_2+d_2)(d_3+d_4) - (d_3+d_4)(r-m_1) - (r-m_1)(c+m_2+d_2)]$,
$A_3 = \widetilde{S}^3[-\frac{2r\beta q_1p_1(1-m_3)}{k}] + \widetilde{S}^2\widetilde{Y}[\frac{2rq_1p_1p_2(1-m_4)(1-m_3)}{k}] + \widetilde{I}^2\widetilde{S}[q_2p_2(1-m_4)\beta(\frac{r}{k}+\beta)] + \widetilde{S}^2\widetilde{I}[\frac{2r\beta q_2p_2(1-m_4)}{k}] + \widetilde{S}^2[(r-m_1)\beta q_1(1-m_3)p_1 + \frac{2r\beta(d_3+d_4)}{k} + \frac{2rq_1p_1(1-m_3)(c+m_2+d_2)}{k}] + \widetilde{I}2[(\frac{r}{k}+\beta)q_2(1-m_4)p_2(c+d_2+m_2)] + \widetilde{S}\widetilde{I}\widetilde{Y}[-2\beta q_2p_1p_2(1-m_4)(1-m_3) + q_1p_1p_2(1-m_4)(1-m_3)(\frac{r}{k}+\beta)] + \widetilde{S}\widetilde{I}[(r-m_1)\beta q_2p_2(1-m_4) + (\frac{r}{k}+\beta)(c+m_2+d_2)q_1p_1(1-m_3) + \frac{2r}{k}(c+m_2+d_2)q_2p_2(1-m_4)] + \widetilde{S}\widetilde{Y}[\beta p_1(1-m_3)(d_3+d_4)] + \widetilde{I}\widetilde{Y}[(c+m_2+d_2)q_2p_1p_2(1-m_4)(1-m_3)] +$

$\widetilde{S}[(r - m_1)\{-\beta(d_3 + d_4) - (c + m_2 + d_2)q_1 p_1(1 - m_3)\} - \frac{2rp_2(1-m_4)(d_3+d_4)}{k} - \frac{2r(c+m_2+d_2)(d_3+d_4)}{k}] + \widetilde{I}[-(r - m_1)(c + m_2 + d_2)q_2 p_2(1 - m_4) - (\frac{r}{k} + \beta)(c + m_2 + d_2)(d_3 + d_4)] + [p_2(1 - m_4)(d_3 + d_4)(r - m_1) + (r - m_1)(c + m_2 + d_2)(d_3 + d_4)]$.

*Proof 1* Jacobian matrix at $E_5$ is given by

$$J = \begin{pmatrix} (r - \frac{2r\widetilde{S}}{k} - fracr\widetilde{I}k - \beta\widetilde{I} - p_1(1-m_3)\widetilde{Y}) & (-\frac{r\widetilde{S}}{k} - \beta\widetilde{S}) & (-p_1(1-m_3)\widetilde{S}) \\ (\beta\widetilde{I}) & (\beta\widetilde{S} - p_2(1-m_4)\widetilde{Y} - c - d_2 - m_2) & (-p_2(1-m_4)\widetilde{I}) \\ (q_1 p_1(1-m_3)\widetilde{Y}) & (q_2 p_2(1-m_4)\widetilde{Y}) & (q_1 p_1(1-m_3)\widetilde{S} + q_2 p_2(1-m_4)\widetilde{I} - d_3 - d_4) \end{pmatrix}. \tag{8}$$

The characteristic equation is given by $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$, where $A_1$, $A_2$ and $A_3$ are the same as defined in statement of the theorem. By Routh-Hurwitz criteria the theorem follows.

## 4 Stability Analysis of the Model with Time Delay

In this section, model (4) with $\tau \neq 0$ is considered. It is also important to mention that we will consider the positive equilibrium ($E_*$) only. At any point, jacobian matrix of system (4) is given by

$$J = \begin{bmatrix} \left(r\left(1 - \frac{S+I}{k}\right) - \frac{rS}{k} - \beta I - p_1(1-m_3)Y - m_1\right) & \left(-\frac{rS}{k} - \beta S\right) & (-p_1(1-m_3)S) \\ (\beta I) & (\beta S - p_2(1-m_4)Y - c - d_2 - m_2) & (-p_2(1-m_4)I) \\ 0 & 0 & (-d_3 - d_4) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (q_1 p_1(1-m_3)Y) & (q_2 p_2(1-m_4)Y) & (q_1 p_1(1-m_3)S + q_2 p_2(1-m_4)I) \end{bmatrix} (e^{-\lambda\tau}),$$

here $\lambda$ being a complex number. Now we state two more lemmas from [10, 27];

**Lemma 3** *Let* $\lambda = (A + iB)$ $A > 0$, $B > 0$ *then,*

- *if* $A < B$ *, all roots of the equation* $\lambda + A - Be^{-\lambda\tau} = 0$ *have positive real parts for* $\tau < \frac{1}{\sqrt{B^2-A^2}} cos^{-1}\left(\frac{A}{B}\right)$.
- *if* $A > B$*, all roots of the equation* $\lambda + A - Be^{-\lambda\tau} = 0$ *have negative real parts for any* $\tau$.

**Lemma 4** *Let the polynomial,* $h(z) = z^3 + p_0 z^2 + q_0 z + r_0 = 0$

*(i)* *if* $r_0 < 0$*, then this equation has at least one positive root;*
*(ii)* *if* $r_0 \geq 0$ *and* $\triangle = (p_0^2 - 3q_0) \leq 0$ *then this equation has no positive roots;*
*(iii)* *if* $r_0 \geq 0$ *and* $\triangle = (p_0^2 - 3q_0) > 0$*, then this equation has positive roots if and only if* $z_1^* = \frac{-p_0+\sqrt{\triangle}}{3}$ *and* $h(z_1^*) \leq 0$.

Let us study nonzero equilibrium $E_* = (S_*, I_*, Y_*)$. The jacobian matrix at $E_* = (S_*, I_*, Y_*)$ is given as

$$J(E_*) = \begin{pmatrix} \left(r\left(1-\frac{S_*+I_*}{k}\right)-\frac{rS_*}{k}-p_1(1-m_3)Y_*-\beta I_*-m_1\right) & \left(-\frac{rS_*}{k}-\beta S_*\right) & (-p_1(1-m_3)S_*) \\ \beta I_* & (\beta S_*-p_2(1-m_4)Y_*-c-d_2-m_2) & p_2(1-,m_4)I_* \\ (q_1p_1(1-m_3)Y_*e^{-\lambda\tau}) & (q_2p_2(1-m_4)Y_*e^{-\lambda\tau}) & \begin{matrix}(q_1p_1(1-m_3)S_* \\ +q_2p_2(1-m_4)I_*)e^{-\lambda\tau}-d_3-d_4)\end{matrix} \end{pmatrix}.$$

The characteristics equation is given as $(\lambda^3+M_2\lambda^2+M_1\lambda+M_0)+(n_2\lambda^2+n_1\lambda+n_0)e^{-\lambda\tau}=0$, where,

$$M_2 = \left(\frac{2r}{k}+\frac{p_1(1-m_3)\beta}{p_2(1-m_4)}-\frac{q_1p_1(1-m_3)\beta}{q_2p_2}(1-m_4)S_*\right) + \left(\frac{r(d_3+d_4)}{q_2p_2(1-m_4)K}+d_3+d_4-\frac{p_1(1-m_3)(c+d_2+m_2)}{p_2(1-m_4)}-r\right),$$

$M_1 = (\{(d_3+d_4)(c+d_2+m_2)-(r-m_1)(c+d_2+m_2)\}+S_*\{-(d_3+d_4)\beta+\frac{2r}{K}(d_3+d_4)+(r-m_1)\beta+\frac{2r}{k}(c+d_2+m_2)\}+I_*\{(\frac{r}{k}+\beta)(d_3+d_4+c+d_2+m_2)\}+Y_*\{(d_3+d_4)p_2(1-m_4)+(d_3+d_4)p_1(1-m_3)+(c+d_2+m_2)p_1(1-m_3)+(r-m_1)p_2(1-m_4)\}+S_*^2(-\frac{2r\beta}{k})+Y_*^2(p_1p_2(1-m_3)(1-m_4))+S_*Y_*(-\beta p_1(1-m_3)+\frac{2r}{k}p_2(1-m_4))+I_*Y_*(\frac{r}{k}+\beta)p_2(1-m_4))$,

$M_0 = (\{-(r-m_1)(c+d_2+m_2)(d_3+d_4)+S_*\{(d_3+d_4)\beta(r-m_1)+\frac{2r}{k}(d_3+d_4)\}(c+d_2+m_2)\}+I_*\{(d_3+d_4)(c+d_2+m_2)(\frac{r}{k}+\beta)\}+Y_*\{(d_3+d_4)(c+d_2+m_2)p_1(1-m_3)-(r-m_1)(d_3+d_4)p_2\}+S_*^2(-\frac{2r}{k}(d_3+d_4)\beta)+Y_*^2(d_3+d_4)p_2p_1(1-m_3)(1-m_4))+S_*Y_*\{-(d_3+d_4)\beta p_1(1-m_3)-\frac{2r}{k}(d_3+d_4)p_2(1-m_4)\}+I_*Y_*\{(\frac{r}{k}+\beta)p_2(1-m_4)(d_3+d_4)\})$,

$n_2 = -(d_3+d_4)$,

$n_1 = (S_*^2\{q_1p_1(1-m_3)\beta-\frac{-2r}{k}q_1p_1(1-m_3)\}+I_*^2\{-(\frac{r}{k}+\beta)q_2p_2(1-m_4)\}+S_*Y_*\{-q_1p_2p_1(1-m_3)(1-m_4)\}+S_*I_*\{q_2p_2(1-m_4)\beta+\frac{2r}{k}q_2p_2(1-m_4)\}+Y_*I_*\{-q_2p_2p_1(1-m_3)(1-m_4)\}+S_*\{\{-(c+d_2+m_2)-(r-m_1)\}q_1p_1(1-m_3)\}+I_*\{-(c+d_2+m_2)q_2p_2(1-m_4)+(r-m_1)q_2p_2(1-m_4)\})$,

$n_0 = -(S_*^2\{q_1p_1(1-m_3)\beta(r-m_1-\frac{2r}{k})+\frac{2r}{K}(c+d_2+m_1)q_1p_1(1-m_3)-q_1\{p_1(1-m_3)\}^2\beta\}+I_*^2\{(\frac{r}{k}+\beta)(c+d_2+m_2)q_2p_2(1-m_4)\}+S_*^2Y_*\{\frac{2r}{k}q_1p_1p_2(1-m_3)(1-m_3)+q_1\{p_1(1-m_3)\}^2\beta\}+S_*^2I_*\{-\frac{2r}{k}q_2p_2(1-m_4)\beta\}+S_*I_*Y_*\{(\frac{r}{k}+\beta)q_1p_1p_2(1-m_3)(1-m_4)-2\beta q_2p_1p_2(1-m_3)(1-m_4)\}+S_*Y_*\{\{-(r-m_1)+(\frac{r}{k}+\beta)\}q_1p_1p_2(1-m_3)(1-m_4)\}+S_*I_*\{\beta(r-m_1)q_2p_2(1-m_4)+\frac{2r}{k}(c+d_2+m_2)q_2p_2(1-m_4)+(c+d_2+m_2)(\frac{r}{k}+\beta)q_1p_1(1-m_3)\}+Y_*I_*\{(c+d_2+m_2)q_2p_2p_1(1-m_3)(1-m_4)\}+S_*\{(c+d_2+m_2)(r-m_1)q_1p_1(1-m_3)\}+I_*\{-(c+d_2+m_2)(r-m_1)q_2p_2(1-m_4)\})$.

Now we put $\lambda = i\omega$ $(\omega > 0)$ we get

**Real Part**: $\{n_2\omega^2+n_0\}\cos\omega\tau+\{n_1\omega\sin\omega\tau - M_2\omega^2+M_0\}$, **Imaginary Part**: $n_1\omega\cos\omega\tau-(-n_2\omega^2+n_0)\sin\omega\tau+M_1\omega-\omega^3$ $(\textbf{Real Part})^2+(\textbf{Imaginary Part})^2 = \omega^6+p_0\omega^4+q_0\omega^2+r_0$. Hence, we have $\omega^6+p_0\omega^4+q_0\omega^2+r_0=0$, where

$p_0=(M_2^2-2M_1-n_2^2)$ $q_0=(M_1^2-2M_2M_0+2n_2n_0-n_1^2)$ $r_0=(M_0^2-n_0^2)$. If we put $z=\omega^2$, then we have the equation $z^3+p_0z^2+q_0z+r_0=0$. If $M_0^2 \geq n_0^2$, then we will have $r_0 \geq 0$, we have two situations for $\Delta$ (i)$\Delta = (p_0^2 - 3q_0) \leq 0$. (ii)$\Delta = (p_0^2 - 3q_0) > 0$.

In situation (i) we have to say that $E_*$ is absolutely stable if $r_0 \geq 0$ and $\Delta = (p_0^2 - 3q_0) \leq 0$. Also, if we have and $r_0 \geq 0$ $\Delta = (p_0^2 - 3q_0) > 0$ then equation has negative roots if and only if $h(z_1^*) > 0$ where $z_1^* = \frac{-p_0+\sqrt{\Delta}}{3}$ thus we have the following theorem for the stability of $E_*$.

**Theorem 2** *$E_*(S_*, I_*, Y_*)$ is absolutely stable if one of the following conditions holds:*

*(i)* $\Delta = (p_0^2 - 3q_0) \leq 0$.
*(ii)* $\Delta = (p_0^2 - 3q_0) > 0$ *and* $z_1^* = \frac{-p_0+\sqrt{\Delta}}{3} < 0$.
*(iii)* $\Delta = (p_0^2 - 3q_0) > 0$, $z_1^* = \frac{-p_0+\sqrt{\Delta}}{3} > 0$ *and* $h(z_1^*) > 0$ *provided* $r_0 \geq 0$.

Next, if we consider the case when $r_0 < 0$ or $\{r_0 \geq 0, \Delta = (p_0^2 - 3q_0) > 0, z_1^* > 0, h(z_1^*) < 0\}$. Then, according to lemma , equation will have one positive root say $\omega_0$ that is the characteristic equation has a pair of purely imaginary roots say $\pm i\omega_0$. Now assume that $i\omega_0, \omega_0 > 0$ is a root of $h(z)$, then we have real and imaginary parts as under.

**Real Part** $= \{n_2\omega^2 + n_0\}\cos\omega\tau + \{n_1\omega\sin\omega\tau - M_2\omega^2 + M_0\} = 0$.

**Imaginary Part** $= n_1\omega\cos\omega\tau - (-n_2\omega^2 + n_0)\sin\omega\tau + M_1\omega - \omega^3 = 0$.

Solving the above equation for $\tau$, we have (by eliminating $\sin\omega\tau$ between these equations)

$$\tau = \frac{1}{\omega_0}\cos^{-1}\left(\frac{n_1\omega_0^2\{\omega_0 - M_1\} - \{M_2\omega_0^2 - M_0\}\{n_2\omega_0^2 - n_0\}}{n_1^2\omega_0^2 + n_2\omega_0^2 - n_0}\right) + \frac{2k\pi}{\omega_0}, (k = 0, 1, 2, \ldots)$$

We call it as a 'critical value' and may be denoted as $\tau_k = \frac{1}{\omega_0}\cos^{-1}\left(\frac{n_1\omega_0^2\{\omega_0 - M_1\} - \{M_2\omega_0^2 - M_0\}\{n_2\omega_0^2 - n_0\}}{n_1^2\omega_0^2 + n_2\omega_0^2 - n_0}\right) + \frac{2k\pi}{\omega_0}$, $(k = 0, 1, 2, \ldots)$. This is corresponding to the characteristic equation as it has purely imaginary roots $\pm i\omega$, which is a result similar to that of Hu et al. 2012 [10]. Differentiating the characteristics equation w.r.t. $\tau$, we get $(\frac{d\lambda}{d\tau})^{-1} = \frac{(3\lambda^2+2M_2\lambda+M_1)e^{\lambda\tau}}{(\lambda^2 n_2+\lambda n_1+n_0)\lambda} + \frac{2n_2\lambda+n_1}{(\lambda^2 n_2+\lambda n_1+n_0)\lambda} - \frac{\tau}{\lambda}$ or $(\frac{d\lambda}{d\tau})^{-1} = \frac{(3\lambda^2+2M_2\lambda+M_1)e^{\lambda\tau} + (2n_2\lambda+n_1) - \tau(2n_2\lambda+n_1)}{(\lambda^2 n_2+\lambda n_1+n_0)\lambda}$. As proved in [10], it is easy to prove the transversality condition at $\tau_k$ e.g. $\frac{d(Re\lambda)}{d\tau} \neq 0$. $\tau_k$ is used as a point for direction of Hopf Bifurcation as in the next section.

*Remark 2* The equilibrium points $E_5$ of model (4) with $\tau = 0$ and $E_*$ of model (4) with $\tau \neq 0$ are ecologically similar. Both convey the message that all the species exist simultaneously.

## 5 Direction and Stability of the Hopf-Bifurcation

With the symbols used in [10] and procedure explained in [5], we have the following system of functional differential equation, $\dot{u}(t) = L_\mu(\mu_t) + F(\mu, u_t)$, where $u_t(\theta) = u(t+\theta) \in R^3$ and $L_\mu : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^3$ and $F : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^3$ are given as

$$L_\mu\phi = (\tau_k+\mu)\begin{bmatrix} -\frac{rS_*}{k} & -\left(\frac{r}{k}+\beta\right)S_* & (-p_1(1-m_3)S_*) \\ \beta I_* & (\beta S_* - p_2(1-m_4)Y_* - c - d_2 - m_2) & -p_2(1-m_4)I_* \\ 0 & 0 & -d_3-d_4) \end{bmatrix} \times \phi(0)$$
$$+ (\tau_k+\mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1p_1(1-m_3)Y_* & q_2p_2(1-m_4)Y_* & q_1p_1(1-m_3)S_* + q_2p_{2(1-m_4)}I_* \end{bmatrix} \times \phi(-1),$$

and

$$F(\mu,\theta) = \begin{pmatrix} -\frac{r}{k}\phi_1^2(0) - \left(\frac{r}{k}+\beta\right)\phi_1(0)\phi_2(0) - p_1(1-m_3)\phi_1(0)\phi_3(0) \\ \beta\phi_1(0)\phi_2(0) - p_2(1-m_4)\phi_2(0)\phi_3(0) \\ q_1p_1(1-m_3)\phi_1(-1)\phi_3(-1) + q_2p_2(1-m_4)\phi_1(-1)\phi_2(-1) \end{pmatrix},$$

$\phi(0) \equiv (\phi_1(0), \phi_1(0), \phi_1(0))^T \in \mathbb{C}$ i.e.

$$L_\mu\phi = (\tau_k+\mu)\begin{bmatrix} -\frac{rS_*}{k} & -\left(\frac{r}{k}+\beta\right)S_* & (-p_1(1-m_3)S_*) \\ \beta I_* & (\beta S_* - p_2(1-m_4)Y_* - c - d_2 - m_2) & -p_2(1-m_4)I_* \\ 0 & 0 & -d_3-d_4 \end{bmatrix} \times \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix}$$
$$+ (\tau_k+\mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1p_1(1-m_3)Y_* & q_2(1-m_4)p_2Y_* & q_1p_1(1-m_3)S_* + q_2(1-m_4)p_2I_* \end{bmatrix} \times \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix},$$

we have considered, $\tau = (\tau_k + \mu)$, $\mu = 0$ gives the hopf bifurcation value for the mathematical model with delay as promised in the previous section. Normalizing delay $\tau$ by timescaling $t \to \frac{t}{\tau}$ the model is written in the Banach Space $\mathbb{C} \equiv \mathbb{C}([-1,0], \mathbb{R}^3)$. By the Riesz representation theorem, we found that there exists a matrix function whose components are bounded variation function $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$L_\mu\phi = \int_\Omega d\eta(\theta,\mu)\phi(\theta)$,$\phi \in \mathbb{C}$,$\Omega \in [-1, 0)$.

We can choose

$$\eta(\theta,\mu) = (\tau_k+\mu)\begin{bmatrix} -\frac{rS_*}{k} & -\left(\frac{r}{k}+\beta\right)S_* & (-p_1(1-m_3)S_*) \\ \beta I_* & (\beta S_* - p_2(1-m_4)Y_* - c - d_2 - m_2) & -p_2(1-m_4)I_* \\ 0 & 0 & -d_3-d_4) \end{bmatrix} \times \delta(\theta)$$
$$- (\tau_k+\mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1p_1(1-m_3)Y_* & q_2p_2(1-m_4)Y_* & q_1(1-m_3)p_1S_* + q_2(1-m_4)p_2I_* \end{bmatrix} \times \delta(\theta+1).$$

where $\delta(\theta)$ denotes the dirac delta function, viz.,

$\delta(\theta)=\begin{cases}0, & \theta\neq 0\\ 1, & \theta\doteq 0,\end{cases}$

for $\phi\in\mathbb{C}^1([-1,0],\mathbb{R}^3)$, define

$A(\mu)\phi(\theta)=\begin{cases}\frac{d\phi(\theta)}{d\theta} & \theta\in[-1,0)\\ \int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta) & \theta\doteq 0\end{cases}$

or $A(\mu)\phi(\theta)=\begin{cases}\frac{d\phi(\theta)}{d\theta}, & -1\leq\theta<0\\ \int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta), & \theta=0\end{cases}$

and

$\mathbb{R}(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0)\\ F(\mu,\phi), & \theta\doteq 0\end{cases}=\begin{cases}0, & -1\leq\theta<0\\ F(\mu,\phi), & \theta\doteq 0\end{cases}$

with these symbols, $\dot{u(t)}=L_\mu(\mu_t)+F(\mu,\mu_t)$ may be written as

$$\dot{u(t)}=A\mu(\mu_t)+\mathbb{R}(\mu)\mu_t \tag{9}$$

which is an abstract differential equation. Where $u_t(\theta)=u(t+\theta),-1\leq\theta<0$. Now we come to operator theory, for $\psi\in\mathbb{C}^1\left([0,1],(\mathbb{R}^3)^*\right)$ we define $A^*$, adjoint operator of $A$,

$A^*\psi(S)=\begin{cases}-\frac{d\psi(S)}{dS} & S\in(0,1]\\ \int_{-1}^{0}d\eta^T(S,\mu)\psi(-S) & S=0.\end{cases}$

And a bilinear product $<\psi(S),\phi(\theta)>=\overline{\psi}(0)\phi(0)-\int_1^0\int_{\xi=0}^{\theta}\overline{\psi}^T(\xi-\theta)d\eta(\theta)$ $\phi(\xi)d\xi$ where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and $A_*$ are adjoint operators. Now $\pm i\omega_0\tau_k$ are eigen values of $A(0)$. Hence they are eigenvalues of $A^*$ also. To determine the poincare normal form of the operator $A$, we first need to evaluate the eigenvectors of $A(0)$ and $A^*$ corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$ respectively. Suppose that $q(\theta)=(1,\alpha_1,\alpha_2)^T\exp(i\omega_0\tau_k\theta)$ is the eigen vector of $A(0)$ corresponding to $i\omega_0\tau_k$, then we have $A(0)q(\theta)=i\omega_0 q(\theta)$ from the definition of $A(0)$,we have

$$\left[\begin{bmatrix}-\frac{rS_*}{k} & -\left(\frac{r}{k}+\beta\right)S_* & (-p_1(1-m_3)S_*)\\ \beta I_* & (\beta S_*-p_2(1-m_4)Y_*-c-d_2-m_2) & -p_2(1-m_4)I_*\\ 0 & 0 & -d_3-d_4\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ q_1p_1(1-m_3)Y_* & q_2p_2(1-m_4)Y_* & q_1p_1(1-m_3)S_*+q_2(1-m_4)p_2I_*\end{bmatrix}\times\exp(i\omega_0\tau_k)\right]\begin{pmatrix}1\\ \alpha_1\\ \alpha_2\end{pmatrix}$$
$$=i\omega_0\begin{pmatrix}1\\ \alpha_1\\ \alpha_2\end{pmatrix},$$

$$\left[\begin{bmatrix}-\frac{rS_*}{k} & -\left(\frac{r}{k}+\beta\right)S_* & (-p_1(1-m_3)S_*)\\ \beta I_* & (\beta S_*-p_2(1-m_4)Y_*-c-d_2-m_2) & -p_2(1-m_4)I_*\\ 0 & 0 & -d_3-d_4\end{bmatrix}\times\begin{pmatrix}1\\ \alpha_1\\ \alpha_2\end{pmatrix}+\right.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_1 p_1(1-m_3)Y_* & q_2 p_2(1-m_4)Y_* & q_1 p_1(1-m_3)S_* + q_2 p_2(1-m_4)I_* \end{bmatrix} \times \exp(i\omega_0\tau_k) \Bigg] \begin{pmatrix} \exp(-i\omega_0\tau_k) \\ \alpha_1 \exp(-i\omega_0\tau_k) \\ \alpha_2 \exp(-i\omega_0\tau_k) \end{pmatrix}$$

$$= i\omega_0 \begin{pmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We obtain, $\alpha_1 = \frac{-p_2(1-m_4)I_*(i\omega_0+\frac{rS_*}{k})-p_1(1-m_3)\beta S_* I_*}{p_2(1-m_4)(\frac{r}{k}+\beta)S_* I_* - p_1(1-m_3)S_*(i\omega_0-\beta S_*+c+d_2+m_2+p_2(1-m_4)Y_*)}$,

$\alpha_2 = \frac{q_1 p_1(1-m_3)Y_* \exp(-i\omega_0\tau_k + q_2 p_2(1-m_4)Y_* \exp(-i\omega_0\tau_k)}{i\omega_0+d_3+d_4-q_1 p_1(1-m_3)S_*+q_2 p_2(1-m_4)I_*}$.

Next, suppose that $q_*(s) = B(1, \alpha_1^*, \alpha_2^*)\exp(i\omega_0\tau_k s)$ is the eigen vector of $A^*$ corresponding to $-i\omega_0\tau k$ similarly, we have,

$\alpha_1^* = \frac{-p_1(1-m_3)(\frac{r}{k}+\beta)S_* - p_2(1-m_4)(i\omega_0-\frac{rS_*}{k})}{p_2(1-m_4)\beta I_* - p_1(1-m_3)(i\omega_0+\beta S_*-c-d_2-m_2-p_2(1-m_4)Y_*)}$,

$\alpha_2^* = \frac{-p_1(1-m_3)S_* - p_2(1-m_4)I_*\alpha_1^*}{-i\omega_0+d_3+d_4-(q_1 p_1(1-m_3)S_*+q_2 p_2(1-m_4)I_*)\exp(-i\omega_0\tau_k)}$,

where $B$ has to be calculated. We have the conditions

$\left.\begin{array}{l} < q^*, q(\theta) >= 1 \\ < q^*, \overline{q(\theta)} >= 0 \end{array}\right\}$ which may be verified.

$< q^*, q(\theta) >= \overline{q^*}(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\infty} \overline{q^*}^T(\xi-\theta)d\eta(\theta)q(\xi)d\xi$

$= \overline{B}(1, \overline{\alpha_1}^*, \overline{\alpha_2}^*)(1, \alpha_1, \alpha_2)^T - \int_{-1}^{0}\int_{\xi=0}^{\infty} \overline{B}(1, \overline{\alpha_1}^*, \overline{\alpha_2}^*)\exp(-i\omega_0\tau_k(\xi-\theta))$ $d\eta(\theta) \times (1, \alpha_1, \alpha_2)^T \exp(i\omega_0\tau_k\xi)d\xi = \overline{B}\{1 + \alpha_1\overline{\alpha_1}^* + \alpha_2\overline{\alpha_2}^* - \int_{-1}^{0}(1, \overline{\alpha_1}^*, \overline{\alpha_2}^*)$ $\exp(i\omega_0\tau_k)d\eta(\theta)(1, \alpha_1, \alpha_2)^T\} = \overline{B}\{1+\alpha_1\overline{\alpha_1}^* + \alpha_2\overline{\alpha_2}^* + \tau_k[q_2 p_2(1-m_4)\overline{\alpha_2}^* Y_* +$ $q_2 p_2(1-m_4)\alpha_1\overline{\alpha_2}^* Y_* + (q_1 p_1(1-m_3)S_* + q_2 p_2(1-m_4)I_*)\alpha_2\overline{\alpha_2}^*]\exp(-i\omega_0\tau_k)\}$

which gives:

$$\overline{B} = \frac{1}{\{1 + \alpha_1\overline{\alpha_1}^* + \alpha_2\overline{\alpha_2}^* + \tau_k[q_2 p_2(1-m_4)\overline{\alpha_2}^* Y_* + q_2 p_2(1-m_4)\alpha_1\overline{\alpha_2}^* Y_* + (q_1 p_1(1-m_3)S_* + q_2 p_2(1-m_4)I_*)\alpha_2\overline{\alpha_2}^*]\exp(-i\omega_0\tau_k)\}}.$$

## *5.1 Stability of Bifurcated Periodic Solutions*

We first compute the coordinates to describe the Center Manifold $\mathbb{C}_0$ at $\mu = 0$. Let $u_t$ be the solution of $\dot{u}(t) = L_\mu(u_t) + F(\mu, \mu_t)$ and define, $z(t) =< q^*, u_t >$, $q^*$ being the eigenvalue of $A^*$. And $W(t, \theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}$ on the Center Manifold $\mathbb{C}_0$, we have, $W(t, \theta) = W(z(t), \overline{z(t)}, \theta)$, where,

$W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{11}(\theta)z\overline{z} + W_{30}\frac{z^3}{\lfloor 3} + \cdots$

In fact, $z$ and $\overline{z}$ are local coordinates for the Center Manifold $\mathbb{C}_0$ in the direction of $q^*$ and $\overline{q}^*$ respectively. The existence of $\mathbb{C}_0$ will provide an opportunity to reduce the system $\dot{u}(t) = L_\mu(u_t) + F(\mu, \mu_t)$ into an Ordinary Differential Equation ODE( in a single complex variable z) on $\mathbb{C}_0$ which is very interesting. $u_t$ is the solution of system under consideration. $u_t \in \mathbb{C}_0$, we have

$\dot{z}(t) =< q^*, \dot{u_t} >$

$=< q^*, A(u_t) + R(u_t) >$

$= < q^*, A(u_t) > + < q^*, R(u_t) >$
$= < A^* q^*, (u_t) > + < q^*, R(u_t) >$
$= i\omega_0 \overline{\tau} z + \overline{q}^* \cdot F(0, W(t,0) + 2Re[z(t)q(\theta)])$

Rewrite it as $\dot{z}(t) = i\omega_0 \overline{\tau} z + g(z, \overline{z})$, where $g(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{02}(\theta)\frac{\overline{z}^2}{2} + g_{11}(\theta) z\overline{z} + g_{21}\frac{\overline{z}z^2}{\lfloor 3} + \cdots$

The above two equations give us $g(z,\overline{z}) = (\overline{q}^*)^T F(z, \overline{z})$

$$= \tau_k \overline{B}(1, \alpha_1^*, \alpha_2^*) \begin{pmatrix} -\frac{r}{k}u_1^2(t) - (\frac{r}{k} + \beta)u_1(t)u_2(t) - p_1(1-m_3)u_1(t)u_3(t) \\ \beta u_1(t)u_2(t) - p_2(1-m_4)u_2(t)u_3(t) \\ p_1 q_1(1-m_3)u_1(t-1)(t)u_3(t-1) + p_2 q_2(1-m_4)u_1(t-1)u_2(t-1) \end{pmatrix}$$

Further,
$u(t+\theta) = W(t,\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta)$,
$u_1(t) = z + \overline{z} + W^{(1)}(t,0)$,
$u_2(t) = \alpha_1 z + \overline{\alpha_1 z} + W^{(2)}(t,0)$,
$u_3(t) = \alpha_2 z + \overline{\alpha_2 z} + W^{(3)}(t,0)$,
$u_1(t-1) = z\exp(-i\omega_0\tau_k) + \overline{z}\exp(i\omega_0\tau_k) + W^{(1)}(t,-1)$,
$u_2(t-1) = \alpha_1 z\exp(-i\omega_0\tau_k) + \overline{\alpha_1 z}\exp(i\omega_0\tau_k) + W^{(2)}(t,-1)$,
$u_3(t-1) = \alpha_2 z\exp(-i\omega_0\tau_k) + \overline{\alpha_2 z}\exp(i\omega_0\tau_k) + W^{(3)}(t,-1)$. Hence, $g(z,\overline{z}) = \tau_k\overline{B}[-\frac{r}{k}u_1^2(t) - (\frac{r}{k}+\beta)u_1(t)u_2(t) - p_1(1-m_3)u_1(t)u_3(t) + \overline{\alpha_1}^*\{\beta u_1(t)u_2(t) - p_2(1-m_4)u_2(t)u_3(t)\} + \overline{\alpha_2}^*\{p_1(1-m_3)q_1u_1(t-1)(t)u_3(t-1) + p_2(1-m_4)q_2u_1(t-1)u_2(t-1)\}]$.

Putting the values of $u_1, u_2, u_3, u_1(t-1), u_2(t-1), u_3(t-1)$ etc. in $g(z,\overline{z})$, we get

$g(z,\overline{z}) = \tau_k\overline{B}\Big(-\frac{r}{k}[z+\overline{z}+W^{(1)}(t,0)]^2 - (\frac{r}{k}+\beta)[z+\overline{z}+W^{(1)}(t,0)][\alpha_1 z + \overline{\alpha_1 z} + W^{(2)}(t,0)] - p_1[z+\overline{z}+W^{(1)}(t,0)][\alpha_2 z + \overline{\alpha_2 z} + W^{(3)}(t,0)] + \overline{\alpha_1}^*(\beta[z+\overline{z}+W^{(1)}(t,0)][\alpha_1 z+\overline{\alpha_1 z}+W^{(2)}(t,0)] - p_2(1-m_4)[\alpha_1 z+\overline{\alpha_1 z}+W^{(2)}(t,0)][\alpha_2 z+\overline{\alpha_2 z}+W^{(3)}(t,0)]) + \overline{\alpha_2}^*(p_1(1-m_3)q_1[z\exp(-i\omega_0\tau_k)+\overline{z}\exp(i\omega_0\tau_k)+W^{(1)}(t,-1)][\alpha_2 z\exp(-i\omega_0\tau_k) + \overline{\alpha_2 z}\exp(i\omega_0\tau_k) + W^{(3)}(t,-1)] + p_2(1-m_4)q_2[z\exp(-i\omega_0\tau_k) + \overline{z}\exp(i\omega_0\tau_k) + W^{(1)}(t,-1)][\alpha_1 z\exp(-i\omega_0\tau_k) + \overline{\alpha_1 z}\exp(i\omega_0\tau_k) + W^{(2)}(t,-1)])\Big)$.

From this equation we can find the values of the coefficients $g_{20}(\theta)$, $g_{02}(\theta)$,$g_{11}(\theta)$, $g_{21}(\theta)$, etc., by comparing the same powers of $z$, we have

$g_{20} = 2\tau_k\overline{B}\{-\frac{r}{k} - (\frac{r}{k}+\beta)\alpha_1 - p_1(1-m_3)\alpha_2 + \beta\overline{\alpha_1}^*\alpha_1 - \overline{\alpha_1}^*\alpha_1\alpha_2 p_2 (1-m_4) + \overline{\alpha_2}^*(p_1(1-m_3)q_1\alpha_2 + p_2(1-m_4)q_2\alpha_1)\exp(-2i\omega_0\tau_k)\}$,

$g_{11} = \tau_k\overline{B}(-\frac{2r}{k} + (\overline{\alpha_1}^*)\beta + \overline{\alpha_2}^* p_2(1-m_4)q_2 - \frac{r}{k} + \beta)(\overline{\alpha_1} + \alpha_1) + (\overline{\alpha_2}^* p_1 (1-m_3)q_1 - p_1(1-m_3))(\overline{\alpha_2} + \alpha_2) - \overline{\alpha_1}^* p_2(\overline{\alpha_2}\alpha_1 + \overline{\alpha_1}\alpha_2))$,

$g_{02} = 2\tau_k\overline{B}\{-\frac{r}{k} - (\frac{r}{k}+\beta)\overline{\alpha_1} - p_1(1-m_3)\overline{\alpha_2} + \beta\overline{\alpha_1}^*\overline{\alpha_1} - \overline{\alpha_1}^*\overline{\alpha_1\alpha_2}p_2(1-m_4) + \overline{\alpha_2}^*(p_1(1-m_3)q_1\overline{\alpha_2} + p_2q_2(1-m_4)\overline{\alpha_1})\exp(2i\omega_0\tau_k)\}$,

$g_{21} = 2\tau_k\overline{B}(-\frac{r}{k}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) - (\frac{r}{k}+\beta)[W_{11}^{(2)}(0) + \alpha_1 W_{11}^{(1)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(0)] - p_1(1-m_3)[W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0) + \alpha_2 W_{11}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(0)] + \overline{\alpha_1}^*\beta[W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \alpha_1 W_{11}^{(1)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(1)}(0)]$

These coefficients are used in calculating $\mathbb{C}_0$, etc. Now we need to calculate $W_{20}(\theta)$ and $W_{11}(\theta)$. Now $\dot{u}_t = A(\mu)u_t + R(\mu)u_t$ and $z(t) = < q^*, u_t >, W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}$ gives us

$$\dot{W} = \dot{u_t} - zq - \dot{\overline{zq}}$$

$$= \begin{cases} AW - 2Re\overline{q}^*(0)F_0 q(\theta), & \text{for} -1 \leq \theta < 0 \\ AW - 2Re\overline{q}^*(0)F_0 q(\theta) + F_0, & \text{for} \theta = 0. \end{cases}$$

Rewrite the above equation as, $\dot{W} = AW + H(z, \overline{z}, \theta)$, where,

$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots$ Near to the origin on $\mathbb{C}_0$, $\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}}$ $(A - 2i\omega_0\tau_k)W_{20}(\theta) = -H_{20}(\theta)$ and $AW_{11}(\theta) = -H_{11}(\theta)$ hence for $-1 \leq \theta < 0$ we have, $H(z,\overline{z},\theta) = -2Re(\overline{q}^*(0)F_0 q(\theta)) = -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta)$, by comparing the coefficients of $z$, we have $H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta)$ and $H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta)$,

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta),$$
$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta).$$

Integrating, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}q(0)\exp(i\omega_0\tau_k\theta) + \frac{i\overline{g}_{20}\overline{q}(0)}{3\omega_0\tau_k}\exp(-i\omega_0\tau_k\theta) + E_1\exp(2i\omega_0\tau_k\theta),$$
$$W_{11}(\theta) = \frac{g_{21}}{i\omega_0\tau_k}q(0)\exp(i\omega_0\tau_k\theta) + \frac{i\overline{g}_{11}\overline{q}(0)}{\omega_0\tau_k}\exp(-i\omega_0\tau_k\theta) + E_2.$$

where $E_1$ and $E_2$ are to be determined. From definitions of $A$ and $(A - 2i\omega_0\tau_k)$ $W_{20}(\theta) = -H_{20}(\theta)$

$(A - 2i\omega_0\tau_k)W_{20}(\theta) = -H_{20}(\theta)$ gives us $\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0)$ which gives us $H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0)$

$$+2\tau_k\begin{pmatrix} -\frac{r}{k} - (\frac{r}{k}+\beta)\alpha_1 - p_1(1-m_3)\alpha_2 \\ \beta\alpha_1 - p_2(1-m_4)\alpha_1\alpha_2 \\ (q_1p_1(1-m_3)\alpha_2 + q_2(1-m_4)p_2\alpha_1)\exp(-2i\omega_0\tau_k) \end{pmatrix}.$$

Now, $(i\omega_0\tau_k I - \int_{-1}^{0}\exp(i\omega_0\tau_k\theta)d\eta(\theta))q(0) = 0$

$$\left(-i\omega_0\tau_k I - \int_{-1}^{0}\exp(-i\omega_0\tau_k\theta)d\eta(\theta)\right)\overline{q(0)} = 0$$

And we have $\left(2i\omega_0\tau_k I - \int_{-1}^{0}\exp(i\omega_0\tau_k\theta)d\eta(\theta)\right)$

$E_1 = 2\tau_k\begin{pmatrix} -\frac{r}{k} - (\frac{r}{k}+\beta)\alpha_1 - p_1(1-m_3)\alpha_2 \\ \beta\alpha_1 - p_2(1-m_4)\alpha_1\alpha_2 \\ (q_1p_1(1-m_3)\alpha_2 + q_2p_2(1-m_4)\alpha_1)\exp(-2i\omega_0\tau_k) \end{pmatrix}$, which leads to

$$\begin{pmatrix} 2i\omega_0+\frac{rS_*}{k} & S_*(\frac{r}{k}+\beta) & p_1(1-m_3)S_* \\ -\beta I_* & 2i\omega_0-\beta S_*+c+d_2+m_2+p_2(1-m_4)Y_* & p_2(1-m_4)I_* \\ -q_1(1-m_3)p_1Y_*\exp(-2i\omega_0\tau_k) & -q_2p_2(1-m_4)Y_*\exp(-2i\omega_0\tau_k) & \begin{array}{c}2i\omega_0+d_3+d_4-(q_1p_1(1-m_3)S_* \\ +q_2p_2(1-m_4)I_*)\exp(-2i\omega_0\tau_k)\end{array} \end{pmatrix}\times E_1$$
$$= 2\begin{pmatrix} -\frac{r}{k} - (\frac{r}{k}+\beta)\alpha_1 - p_1(1-m_3)\alpha_2 \\ \beta\alpha_1 - p_2(1-m_4)\alpha_1\alpha_2 \\ (q_1p_1(1-m_3)\alpha_2 + q_2p_2(1-m_4)\alpha_1)\exp(-2i\omega_0\tau_k) \end{pmatrix}.$$

$E_1$ can be calculated from this equation. Now, $\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0)$

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau_k \begin{pmatrix} -\frac{r}{k} - (\frac{r}{k} + \beta)Re(\alpha_1) - p_1(1-m_3)Re(\alpha_2) \\ \beta Re(\alpha_1) - p_2(1-m_4)Re(\alpha_1\alpha_2) \\ (q_1p_1(1-m_3)Re(\alpha_2) + q_2p_2(1-m_3)Re(\alpha_1) \end{pmatrix},$$

$$\begin{pmatrix} \frac{rS_*}{k} & S_*(\frac{r}{k} + \beta) & p_1(1-m_3)S_* \\ -\beta I_* & -\beta S_* + c + d_2 + m_2 + p_2(1-m_4)Y_* & p_2(1-m_4)I_* \\ -q_1p_1(1-m_3)Y_* & -q_2p_2(1-m_4)Y_* & d_3 + d_4 - (q_1p_1(1-m_3)S_* + q_2p_2(1-m_4)I_*) \end{pmatrix}$$

$$\times E_2 = 2\begin{pmatrix} -\frac{r}{k} - (\frac{r}{k} + \beta)Re(\alpha_1) - p_1(1-m_3)Re(\alpha_2) \\ \beta Re(\alpha_1) - p_2(1-m_4)Re(\alpha_1\alpha_2) \\ (q_1p_1(1-m_3)Re(\alpha_2) + q_2p_2(1-m_4)Re(\alpha_1) \end{pmatrix}.$$

$E_2$ can be obtained from this equation. By putting values of $E_1$ and $E_2$ we can obtain $W_{20}(\theta)$ and $W_{11}(\theta)$ and hence $g_{20}$,$g_{11}$, $g_{02}$, $g_{21}$ etc. Hence as stated in [5, 10], we can obtain the following values;

$$\begin{cases} c_1(0) = & \frac{i}{2\omega_0\tau_k}(g_{11}g_{20} - 2 \mid g_{11} \mid^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ \mu_2 = & -\frac{Re(c_1(0))}{Re(\lambda'(\tau_k))}, \\ \beta_2 = & 2Re(c_1(0)), \\ T_2 = & -\frac{1}{\omega_0\tau_k}[Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_k))], \end{cases} \tag{10}$$

which determine the direction and stability of the model with delay at the critical value $\tau_k$. Now, we state the following theorem due to [5, 10, 21], which is the main result of this section:

**Theorem 3** *(i) The sign of $\mu_2$ determined the direction of Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical).*

*(ii)The stability of bifurcated periodic solutions is determined by $\beta_2$: the periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$.*

*(iii)The period of bifurcated periodic solutions is determined by $T_2$: the period increases if $T_2 > 0$ and decreases if $T_2 < 0$.*

From part (i) of this theorem, it is clear that Hopf bifurcation is supercritical if either $Re(c_1(0)) < 0$ or $Re(\lambda'(\tau_k)) < 0$. Similarly, Hopf bifurcation is subcritical if $Re(\lambda'(\tau_k)) > 0$ and $Re(c_1(0)) > 0$.

## 6 Numerical Simulation

In this section, we consider a hypothetical set of parameters $P_1 = \{r = 0.8, k = 1, \beta = 1, p_1 = 0.12, p_2 = 6, m_1 = 0.02, m_2 = 0.06, m_3 = 0.5, m_4 = 0.2, d_2 = 0.05, d_3 = 0.6, d_4 = 0.5, c = 0.025, q_1 = 0.75, q_2 = 0.75\}$. We will focus on positive equilibrium. Calculation shows that $\widetilde{S} = .2339, \widetilde{I} = .2749, \widetilde{Y} = .0487$, thus model has the positive equilibrium $E_5(.2339, .2749, 0.0487)$. Also, $\Gamma = -0.5550$, $\Delta = 1.8$, $\Theta = -4.860$, $\Lambda = -0.380$, $A_1 = .3698$, $A_2 = .1647$, $A_3 = .0217$, therefore $\Gamma\widetilde{S} + \Delta\widetilde{I} + \Theta\widetilde{Y} + \Lambda = -0.0335 < 0$ and $A_1A_2 + A_3 = .0826 > 0$,

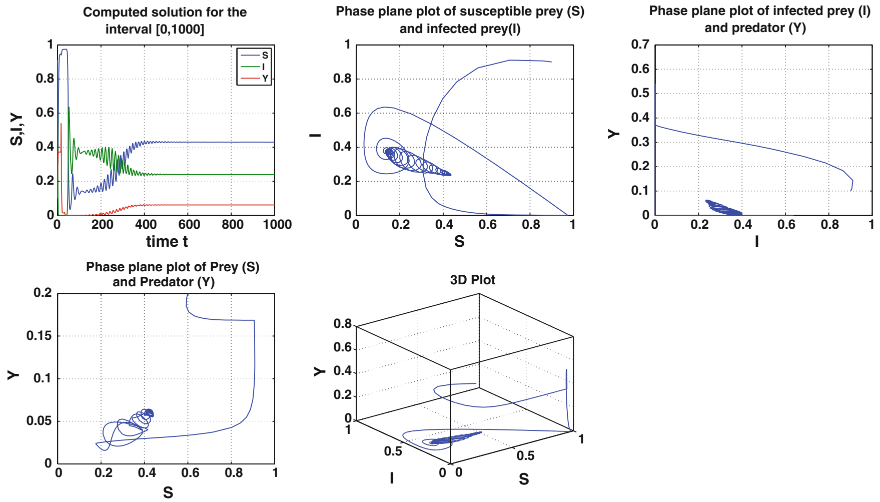


**Fig. 1** Solution of system (4) for initial function $S(0) = 0.6, I(0) = 0.2, Y(0) = 0.2$ with parameter set $P_1$, $\tau = 15.14 < \tau_0$, the positive equilibrium point is stable

hence $E_5(.2339, .2749, 0.0487)$ is stable. Indeed, we also have the jacobian matrix at $E_5$;

$$\begin{pmatrix} -0.1349 & -0.4210 & -0.0140 \\ 0.2749 & -0.1349 & -1.3195 \\ 0.0022 & 0.0029 & -0.1 \end{pmatrix},$$

this has the characteristics equation $\lambda^3 + 0.3698\lambda^2 + 0.1647\lambda + 0.0217$. It has three roots, viz.,

$$\begin{cases} -0.1020 + 0.3471i, \\ -0.1020 - 0.3471i, \\ -0.1658, \end{cases}$$

hence $E_5(0.2339, 0.2749, 0.0487)$ is stable. It is also calculated that $n_0 = -0.1941$, $n_2 = -1.1$, $n_1 = 0.6895$, $m_0 = 0.2158$, $m_1 = -0.5248$, $m_2 = 1.4698$. Therefore, $p_0 = 3.1633$, $q_0 = -0.4073$, $r_0 = 0.0089$, $h(z) = z^3 + 3.1633z^2 - 0.4073z + 0.0089$, $p_0^2 - 3q_0 = 11.2284 > 0$ and $z_1^* = 0.063$. From this $h(z_1^*) = 11.20998528$, hence $E_*$ is stable. Further $\omega_0 = 0.6382$ and $\tau_0 = 33.14$. Thus, Hopf bifurcation occurs as the $\tau$ passes through $\tau_0$ which is depicted by numerical simulation in Figs. 1 and 2.

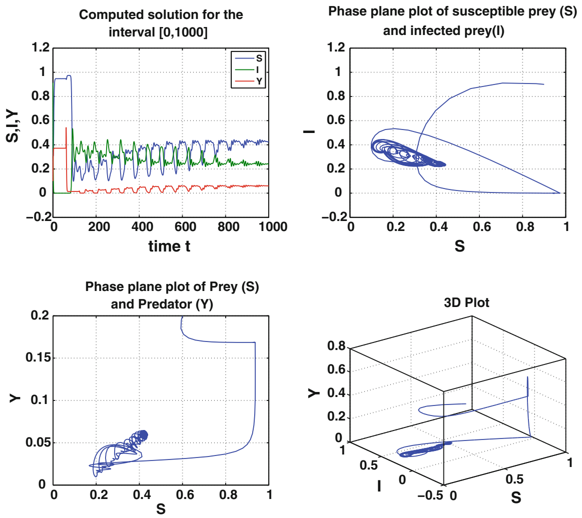


**Fig. 2** Solution of system (4) for initial function $S(0) = 0.6, I(0) = 0.2, Y(0) = 0.2$ with parameter set $P_1$, $\tau = 60.30 > \tau_0$, the positive equilibrium point is unstable

## 7 Discussion

In this paper, we have considered a delayed prey–predator system with infection. Migration has been allowed among prey population only. It is also considered that prey population has self-defence in the form of prey refuge. This decreases the availability of prey population for predation to predators. For instance, only $(1-m_3)S$ of sound prey are available for predation. Similarly, $(1 - m_4)I$ of infected prey are available for predation. Stability results have been investigated.

Similar to the study of [10], in this paper the time delay $\tau$ is the gestation period of predator. In our analysis this is found to be the bifurcation parameter. It is proved that beyond some specific value of $\tau$, Hopf-bifurcation occurs. The direction of Hopf-bifurcation and stability of bifurcated periodic solutions have been derived using the central manifold reduction technique and normal form theory.

In this paper, bifurcation of predator into two parts, viz., healthy predator and infected predator has been ignored. The same may be done in the future. Further, for simplification, parameters are taken as time independent. In real-life the parameters are time dependent, this may also considered in the future.

The main issue in applied mathematical modeling is to identify the real parameters. The present study is not a case study, hence real parameters are not available. Hence, the main scope of this study is to study a real eco-system and to identify the real/experimental parameters.

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