Chapter 2
Particle Orbit Theory

Abstract In which particle motion in assigned electric and magnetic fields is discussed and studied in detail in a few representative cases. Fundamental concepts such as guiding center motion, particle drifts, magnetic moment, and adiabatic invariants are introduced, and a brief discussion of the mirror force and magnetic bottles is presented.

This chapter is devoted to a description of the motion of individual charged particles in assigned magnetic and electric fields, which we assume to be given as a function of time and position. Such orbits can become quite complex even in this case, so gaining a general understanding of the main types of motions that may occur is an important pre-requisite to studying the dynamics of plasmas in their self-consistent electromagnetic fields. Let's begin with the simplest case, the motion of a charged particle in a constant, uniform, magnetic field.

2.1 Motion in a Uniform, Static Magnetic Field

Consider a magnetic field which is constant in space and time, so that with respect to a Cartesian system of coordinates the field reads $\boldsymbol{B} = (0, 0, B)$. The classical equation of motion for a particle of mass m and charge e_0 in such a field may be written as

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\ddot{\boldsymbol{r}} = \frac{e_0}{c}\boldsymbol{v} \times \boldsymbol{B}. \tag{2.1}$$

Because the force on the particle is perpendicular to its velocity,

$$\ddot{z} = 0$$

and motion along the field occurs at constant speed, $\dot{z} = v_\parallel$ is constant. Another immediate consequence is found taking the scalar product of the equation of motion with the velocity $\dot{\boldsymbol{r}}$:

$$m\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = 0,$$

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implying that the total kinetic energy energy

$$\tfrac{1}{2}\, m\dot{r}^2 = W = W_\parallel + W_\perp$$

is constant. However, since $W_\parallel = \tfrac{1}{2}\, m v_\parallel^2$ is separately conserved, we find that the kinetic energy of motion perpendicular to the field $W_\perp$ is also constant.

The trajectory of the particle may be found by integrating the equation of motion, Eq. (2.1), which rewritten in its three cartesian components reads:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \Omega\, v_y \tag{2.2a}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\Omega\, v_x \tag{2.2b}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0 \tag{2.2c}$$

where

$$\Omega = \frac{e_0 B}{m\, c}.$$

The solutions to Eq. (2.2a)–(2.2c) are given by:

$$v_x = v_\perp \cos(\Omega t + \alpha) \tag{2.3a}$$

$$v_y = -v_\perp \sin(\Omega t + \alpha) \tag{2.3b}$$

$$v_z = v_\parallel. \tag{2.3c}$$

Integrating Eq. (2.3a)–(2.3c) once more we obtain the trajectory as a function of time:

$$x = x_0 + \left(\frac{v_\perp}{\Omega}\right) \sin(\Omega t + \alpha) \tag{2.4a}$$

$$y = y_0 + \left(\frac{v_\perp}{\Omega}\right) \cos(\Omega t + \alpha) \tag{2.4b}$$

$$z = z_0 + v_\parallel t. \tag{2.4c}$$

The trajectory is a helix which positively charged particles sweep in the clockwise sense and negative particles in a counter-clockwise sense as seen projected on the (x, y) plane. The projected orbits are circular with radius $R_L = v_\perp/|\Omega|$, known as the *Larmor radius*. The gyration frequency $|\Omega|$ is known as the *cyclotron* frequency, also called the *gyrofrequency*.

The frequencies ω_{ce} and ω_{ci} defined in Chap. 1 are the values of $|\Omega|$ calculated for an electron and proton respectively. Let's proceed now to motion in combined electric and magnetic fields, starting from the case in which the fields are perpendicular (Figs. 2.1 and 2.2).

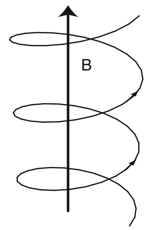

Fig. 2.1 Trajectory of a negatively charged particle in a uniform magnetic field

2.2 Motion in Orthogonal Electric and Magnetic Fields

Starting from the same configuration as above, we add a constant, homogeneous electric field $\boldsymbol{E}$ orthogonal to $\boldsymbol{B}$:

$$\boldsymbol{E} = (0, E, 0),$$
$$\boldsymbol{B} = (0, 0, B),$$

with E and B independent of position and time.

To solve Eq. (2.1) now we start from the y component of the equation of motion, Eq. (2.3c) which becomes:

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\Omega\, v_x + \frac{e_0\, E}{m} = -\Omega\,(v_x - c\frac{E}{B}). \tag{2.5}$$

It is natural to consider therefore a frame of reference S', moving at the speed $c\,E/B$ along the x direction, which is possible as long as $E \ll B$: in this case the Lorentz transformation reduces to the Galilean transformation

$$x' = x - c\,(E/B)t,$$
$$y' = y,$$
$$z' = z,$$
$$t' = t,$$

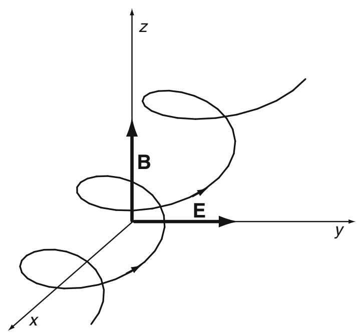

Fig. 2.2 Motion in orthogonal and constant electric and magnetic fields

and in the frame S' the equation of motion along y' becomes

$$\frac{dv'_y}{dt} = -\Omega\, v'_x,$$

identical to Eq. (2.2c). The trajectory written in the new, primed coordinates therefore coincides with that given by Eq. (2.4a) where y is replaced by y'. Returning to the original frame of reference we can see that the trajectory is just a superposition of the original helical trajectory and a uniform translation at speed cE/B along the x-axis as shown in (Fig. 2.2). To interpret this result consider that the effect of the electric field in the y-direction is to accelerate (or decelerate, depending on the charge) a particle along that axis. In the presence of a magnetic field, the increase (decrease) in speed along that direction leads to an increasing (decreasing) centripetal force deflecting the motion orthogonally to both $\boldsymbol{E}$ and $\boldsymbol{B}$. This force gradually changes the direction of motion eventually leading to a change of sign of v_y, so that motion in the y-direction is confined to a finite displacement. On the other hand the different radius of curvature of the gyrating motion as the particle moves along and against the direction of the electric field leads to the average uniform displacement orthogonal to the fields. The velocity associated with this motion is known as the drift velocity $\boldsymbol{v}_E$ and may be written in a coordinate independent way as

$$\boldsymbol{v}_E = c\,\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}. \tag{2.6}$$

When $E/B = O(1)$ relativistic corrections become important, and a different approach is required if $E > B$, since v_x in this case would become:

$$v_x = c\frac{E}{B} + v_{0x}\cos(\Omega t),$$

which exceeds the speed of light. With $E > B$ one must therefore use the relativistic equations of motion,

$$\frac{d\boldsymbol{p}}{dt} = e_0\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\Big)$$
$$\boldsymbol{p} = \gamma m\boldsymbol{v}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Consider the effects of a Lorentz transformation along the positive x-axis with speed v on electric and magnetic fields:

$$E'_x = E_x \qquad B'_x = B_x \tag{2.7a}$$
$$E'_y = \gamma(E_y - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z) \tag{2.7b}$$

$$E'_z = \gamma(E_z + \beta B_y) \qquad\qquad B'_z = \gamma(B_z - \beta E_y), \tag{2.7c}$$

where $\beta = v/c$.

For the configuration considered we have $E'_x = 0$, $E'_z = 0$, $E'_y = \gamma(E - \beta B)$. Therefore, for $E < B$, the choice $\beta = \frac{E}{B}$ makes the y component of the electric field in the S' frame vanish, and only a magnetic field along $z' = z$, $B'_z = \gamma(B - \beta E) = \gamma(B - E^2/B) = B/\gamma$ will remain, yielding the helical motion seen before. In the original frame S the motion will therefore be a superposition of the helical motion and the translational one. More generally, which kind of Lorentz transformation may be used to simplify the equations of motion may be understood by recalling the two Lorentz invariants for the electromagnetic field, $\boldsymbol{E} \cdot \boldsymbol{B}$ and $E^2 - B^2$. If the first of these vanishes, then it will always be possible to find a frame in which either E or B vanishes, depending on the sign of the second invariant $E^2 - B^2$. So if the electric and magnetic fields are perpendicular, but $E > B$, a new frame can be found with a vanishing magnetic field, corresponding to a Lorentz boost along the x direction with $\beta = B/E$. Once the equation of motion is solved in the new frame, one can transform the solution back to the original frame applying the inverse Lorentz transformation with $\beta = -B/E$ along the x' axis (see Problem 2.4).

A very important property of the drift speed $\boldsymbol{v}_E$ is that it does not depend on the sign of the charge of the particle in motion. Both electrons and protons move with the same speed under this drift, and therefore no current is generated by the so called "E cross B" drift within the plasma. The above discussion may be generalized to the case of any constant force $\boldsymbol{F}$ orthogonal to the magnetic field $\boldsymbol{B}$: it will give rise to a drift which is orthogonal both to the force and the magnetic field, whose magnitude is obtained by substituting $\boldsymbol{E}$ with $\boldsymbol{F}/e_0$ in Eq. (2.6):

$$\boldsymbol{v}_F = c\,\frac{\boldsymbol{F} \times \boldsymbol{B}}{e_0 B^2} \tag{2.8}$$

As a consequence, the drift from a non-electric force will give rise to an average current in the plasma.

So far we have only considered cases in which $E \neq B$. However, the case $E = B$ has particular importance since it is related to the motion of a particle in the field of an electromagnetic wave. The method of a Lorentz boost, used to treat the cases with $E \neq B$, is no longer applicable and we must solve directly the equation of motion of special relativity in the laboratory frame. The more familiar non relativistic case can be easily obtained by taking the appropriate limit.

We shall consider a particle of mass m and charge e_0, initially at rest at the origin of the coordinates, acted upon by a linearly polarized electromagnetic wave of frequency ω. We may then choose a frame of reference in which the electric field lies on the x-axis and the magnetic field on the y-axis, so that the waves propagates along the z-direction. Thus,

$$\boldsymbol{E} = (E_0 \cos\phi)\,\boldsymbol{e}_x, \quad \boldsymbol{B} = (E_0 \cos\phi)\,\boldsymbol{e}_y,$$

with $\phi = \omega(t - z/c)$.

The equations of relativistic dynamics are

$$\frac{\mathrm{d}\,(m\gamma\boldsymbol{v})}{\mathrm{d}t} = e_0(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}), \tag{2.9}$$
$$\frac{\mathrm{d}\,(m\gamma c)}{\mathrm{d}t} = e_0\boldsymbol{E} \cdot \boldsymbol{\beta} \tag{2.10}$$

where $\boldsymbol{\beta} = (\boldsymbol{v}/c)$. Introducing the quantity $\omega_E = (e_0 E_0/mc)$, Eqs. (2.9) and (2.10) give

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\beta_x) = \omega_E(1 - \beta_z)\cos\phi, \tag{2.11}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\beta_y 0) = 0, \tag{2.12}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\beta_z) = \omega_E\,\beta_x\cos\phi, \tag{2.13}$$
$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \omega_E\,\beta_x\cos\phi. \tag{2.14}$$

Equating Eqs. (2.13) and (2.14) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\gamma(1 - \beta_z)] = 0$$

and therefore $\gamma(1 - \beta_z) = K$ is a constant of motion.

To determine the trajectory of the particle, it is convenient to use the variable ϕ in place of t. Then

$$\boldsymbol{\beta} = \frac{1}{c}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\omega}{c}(1 - \beta_z)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\phi} = \frac{\omega}{c}(1 - \beta_z)\boldsymbol{r}',$$

where the prime indicates the derivative with respect to ϕ. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\boldsymbol{\beta}) = \frac{\mathrm{d}}{\mathrm{d}t}[\gamma\frac{\omega}{c}(1 - \beta_z)\boldsymbol{r}'] = \frac{\omega^2 K}{c}(1 - \beta_z)\boldsymbol{r}''.$$

Comparing the last expression with Eq. (2.9) we finally get the following set of differential equations:

$$x'' = \frac{c}{\omega K}\frac{\omega_E}{\omega}\cos\phi$$
$$y'' = 0$$
$$z'' = \frac{1}{K}\frac{\omega_E}{\omega}x'\cos\phi.$$

Since $\boldsymbol{r}(0) = 0$ and $\dot{\boldsymbol{r}}(0) = 0$ (the dot indicates the time-derivative), we have

$$\phi(0) = 0; \quad \dot{\phi}(0) = \omega[1 - \beta_z(0)] = \omega; \quad K = K(0) = 1,$$

and the initial conditions for the system above are:

$$x(0) = 0, \quad x'(0) = \dot{x}(0)/\dot{\phi}(0) = 0 \quad ; \quad y(0) = y'(0) = 0 \quad ; \quad z(0) = z'(0) = 0.$$

The second equation shows that $y = 0$: no motion occurs in the y-direction. A straightforward integration of the other two equations gives

$$x = -\Big(\frac{\omega_E}{\omega}\Big)\Big(\frac{c}{\omega}\Big)(1 - \cos\phi), \tag{2.15}$$

$$z = \Big(\frac{\omega_E}{2\omega}\Big)^2\Big(\frac{c}{\omega}\Big)(\phi - \tfrac{1}{2}\sin 2\phi) = \xi(\phi - \tfrac{1}{2}\sin 2\phi),$$

with

$$\xi = \Big(\frac{\omega_E}{2\omega}\Big)^2\Big(\frac{c}{\omega}\Big).$$

Inserting the definition of ϕ into the the expression for z we have

$$z = \omega\xi((t - z/c) - (\xi/2)\sin 2\phi,$$

or

$$z = \Big(\frac{\omega\xi}{1 + \omega\xi/c}\Big)t - \frac{\xi}{2(1 + \omega\xi/c)}\sin 2\phi. \tag{2.16}$$

The velocity along z is therefore a superposition of a constant speed plus a periodic term. The constant term corresponds to a value of β_z

$$\beta_z = \frac{\omega\xi/c}{1 + \omega\xi/c} = \frac{(\omega_E/2\omega)^2}{1 + (\omega_E/2\omega)^2},$$

which tends to unity when (ω_E/ω) is very large, namely when E_0 is very large or ω is very small. The first circumstance arises in the field of an intense laser beam or in linear acclerators. The second one has been considered in connection with the problem of the acceleration of cosmic rays in the vicinity of a pulsar. In fact, the pulsar emits a low frequency electromagnetic wave, which in principle could accelerate particles to very large energies. This model, however, presents a number of drawbacks and has been abandoned.

The periodic term in Eq. (2.16), coupled with the motion along x, Eq. (2.15), produces an eight-shaped trajectory in the (x, z) plane of a frame moving along z with a speed $c\beta_z$.

2.3 Motion in Slowly Variable Magnetic Fields

Because electric and magnetic fields in space are never uniform or constant in time, there are in general no exact integrals of motion to which one can resort to simplify the understanding of charged particle dynamics. However, when particle motion occurs in slowly variable magnetic fields, either in time or in space, the equation of particle motion, Eq. (2.1), may be solved approximately. In fact, the systematic expansion in terms of drift velocities of higher order is allowed if the the ratio of the Larmor radius R_L to the gradient scale L $R_L/L \ll 1$, or alternatively, if the characteristic frequency of time variation of the field $1/T$ is much smaller than the gyrofrequency $|\Omega|$, i.e. $|\Omega|T \gg 1$. When these conditions are satisfied, exact integrals of motion are replaced by approximate integrals or adiabatic invariants, quantities whose relative changes are bounded by inequalities more stringent than the ones above.

2.3.1 Charged Particle Orbits in the Presence of a Magnetic Fields with a Weak Gradient

Equation (2.8) may be used to understand particle motion by use of a local analysis in a magnetic field with gradients. Consider for example a magnetic field line having a certain curvature. The particle will locally gyrate around the magnetic field, which in view of the inequalities expressed above may be considered constant on the scale of the Larmor radius. However, the particle generally also has a motion parallel to the field $\boldsymbol{B}$, and, because of the curvature of the field line, the particle will feel in its own frame of reference an outward centrifugal force due to this motion $\boldsymbol{F} = (m\, v_\parallel^2\, \boldsymbol{R}_c)/(R_c^2)$, where R_c is the local radius of curvature. Equation (2.8) then shows that as a consequence of curvature a drift motion will arise with velocity

$$\boldsymbol{v}_C = \frac{mc\, v_\parallel^2 (\boldsymbol{R}_c \times \boldsymbol{B})}{e_0\, R_c^2\, B^2} = \frac{2\, c\, W_\parallel}{e_0\, R_c\, B} (\hat{\boldsymbol{R}}_c \times \hat{\boldsymbol{B}}), \tag{2.17}$$

where $\hat{\boldsymbol{B}}$ is the unit vector tangent to the local magnetic field, while $\hat{\boldsymbol{R}}_c$ is unit vector along the radius of curvature. The factor in parentheses is therefore a purely geometrical term which describes the local geometry of magnetic field lines. This motion of charged particles is called the *curvature drift*.

Consider now instead a magnetic field directed along one direction, whose magnitude depends on a coordinate along an axis orthogonal to the magnetic field $\boldsymbol{B}$ itself. An example is given by a field $\boldsymbol{B} = [0, 0, B(y)]$, where $B(y)$ satisfies the weak gradient condition

$$\frac{\mathrm{d}B/\mathrm{d}y}{B}\, R_L \ll 1.$$

The Lorentz force around a given point y_0 is given by:

$$\boldsymbol{F}(y) = \boldsymbol{F}(y_0 + \delta y) = \frac{e_0}{c}\left[B\left(-v_x \boldsymbol{e}_y + v_y \boldsymbol{e}_x\right)\right]_{y_0+\delta y},$$

where we are considering motions only in a plane perpendicular to the magnetic field $\boldsymbol{B}$.

Denoting y-derivatives with a prime, the field in the neighborhood of this point may be written $B(y_0 + \delta y) = B(y_0) + \delta y\, B'(y_0) = B_0 + \delta y\, B_0'$ so that

$$\begin{aligned}\boldsymbol{F}(y) = {} & \frac{e_0 B_0}{c}\left[-v_x \boldsymbol{e}_y + v_y \boldsymbol{e}_x\right] \\ & + \frac{e_0 B_0}{c}\left[-v_x \boldsymbol{e}_y + v_y \boldsymbol{e}_x\right] \delta y\, (B_0'/B_0). \end{aligned} \tag{2.18}$$

For δy of the same order as R_L one has by assumption that $\delta y\,(B_0'/B_0) \ll 1$. Therefore the first term in the equation describes the unperturbed circular motion while the second term provides the first order correction. We can thus assume that $v_x, v_y, \delta y = y - y_0$ are the same as those obtained from the motion in a constant magnetic field, which are periodic functions of the phase $\Phi = \Omega t + \alpha$. Averaging the force $\boldsymbol{F}$ over a gyration period, $P = 2\pi/|\Omega|$, that is calculating

$$\langle \boldsymbol{F} \rangle = \frac{1}{P}\int_0^P \boldsymbol{F}\, \mathrm{d}t,$$

we see that the only non zero contribution comes from the term $v_x \delta y B_0'$ on the right hand side of Eq. (2.18). As a result

$$\langle \boldsymbol{F} \rangle = -\frac{e_0}{c} v_\perp R_L \frac{1}{P}\int_0^P \cos^2(\Phi)\, \mathrm{d}t\, B_0'\, \boldsymbol{e}_y = -\frac{e_0 v_\perp}{2c}\, R_L B_0'\, \boldsymbol{e}_y = -\tfrac{1}{2}\, m v_\perp^2\, \frac{B_0'}{B_0}\, \boldsymbol{e}_y.$$

In other words, on averaging over a gyroperiod, we are left with a constant force perpendicular to $\boldsymbol{B}_0$ which, according to Eq. (2.8), must give rise to a drift orthogonal to both $\boldsymbol{B}_0$ and $\langle \boldsymbol{F} \rangle$. Substituting the expression for $\langle \boldsymbol{F} \rangle$ in Eq. (2.8) we find:

$$\boldsymbol{v}_G = \frac{c\, W_\perp\, B_0'}{e_0 B_0^2}\, \boldsymbol{e}_x,$$

whose general vector expression may be written as:

$$\boldsymbol{v}_G = \frac{c\, W_\perp (\boldsymbol{B}_0 \times \nabla)|\boldsymbol{B}_0|}{e_0 |\boldsymbol{B}_0|^3}. \tag{2.19}$$

This is known as the *gradient drift*.

When, as in the preceding discussion, the gradients are small with respect to the cyclotron radius and the fastest time-scale is given by the gyro-motion one calls the averaged position of the particle over one gyroperiod the guiding center of the particle. In terms of our previous discussion of drifts, one could say the particle executes a gyromotion around its guiding center while the guiding center itself moves following the drifts in question. In order for the so-called guiding center approximation to be valid, an additional condition must be established concerning the parallel motion: the distance a particle moves along the field during one gyroperiod must also be small respect to the scale of gradients, or $v_{\|} P \ll L$.

In these circumstances we may always write for the equations of motion along and across the magnetic field the equations:

$$\begin{aligned} m\frac{\mathrm{d}\boldsymbol{v}_{\|}}{\mathrm{d}t} &= \boldsymbol{F}_{\|} \\ m\frac{\mathrm{d}\boldsymbol{v}_{\perp}}{\mathrm{d}t} &= \boldsymbol{F}_{\perp} + \frac{e_0}{c}\big(\boldsymbol{v}_{\perp} \times \boldsymbol{B}\big). \end{aligned} \tag{2.20}$$

We may then write $\boldsymbol{v}_{\perp} = \boldsymbol{v}_{\Omega} + \boldsymbol{v}_d$ where $\boldsymbol{v}_{\Omega}$ is the velocity of the cyclotron motion around the guiding center and orthogonal to the local magnetic field while $\boldsymbol{v}_d$ is the drift velocity. We now expand

$$\boldsymbol{v}_d = \boldsymbol{v}_d^0 + \boldsymbol{v}_d^1 + \boldsymbol{v}_d^2 + \cdots$$

and expand order by order in the small gradient parameter the equations for the drifts:

$$m\big(\frac{\mathrm{d}\boldsymbol{v}_d^0}{\mathrm{d}t} + \cdots\big) = \frac{e_0}{c}\big(\boldsymbol{v}_d^1 + \cdots\big) \times \boldsymbol{B}, \tag{2.21}$$

so that, for example, considering that the time-dependence of the zero-order drift is a small quantity,

$$\boldsymbol{v}_d^1 = -\frac{mc}{e_0 B}\big(\frac{d\boldsymbol{v}_d^0}{dt} \times \frac{\boldsymbol{B}}{B}\big)$$

and so on and so forth. It is important to remark that quite generally the first order drift may differ in direction from the zero order drift, and must therefore be included to understand charged particle trajectories.

2.3.2 Magnetic Moment Conservation

Consider a magnetic field configuration with cylindrical symmetry in which the magnetic field has a strong axial component and again satisfies the conditions for weak gradients defined above.

The field may be written in cylindrical coordinates r, θ, z as

$$\boldsymbol{B} = [B_r(r,z), 0, B_z(r,z)]; \quad B_z \gg B_r \quad \rightarrow \quad B = (B_r^2 + B_z^2)^{1/2} \simeq B_z.$$

On the other hand the divergence free condition for the magnetic field $\nabla \cdot \boldsymbol{B} = 0$ may be written as:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, B_r) + \frac{\partial B_z}{\partial z} = 0,$$

which may be integrated to yield

$$\int_0^r \frac{\partial}{\partial r}(r\, B_r)\mathrm{d}r = -\int_0^r r\frac{\partial B_z}{\partial z}\mathrm{d}r \simeq -\langle\frac{\partial B}{\partial z}\rangle\frac{r^2}{2}.$$

Here we have taken into account the fact that r is of the same order as the Larmor radius R_L and that the magnetic field and the magnetic field gradients vary on a much larger scale. We therefore obtain:

$$B_r \simeq -\frac{r}{2}\frac{\partial B}{\partial z},$$

where we have now omitted the brackets $\langle\rangle$. The equation of motion of a particle in the direction parallel to the magnetic field becomes:

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\frac{e_0}{c}(\boldsymbol{v} \times \boldsymbol{B})_z = -\frac{e_0}{c}v_\theta\, B_r.$$

Remembering that the sense of particle gyration is opposite in sign to the charge we may write

$$v_\theta = -\frac{e_0}{|e_0|}\, v_\perp,$$

and therefore

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\frac{e_0}{c}\Big(-\frac{e_0}{|e_0|}\, v_\perp\Big)\Big(-\frac{r}{2}\frac{\partial B}{\partial z}\Big) \simeq -\Big(\frac{|e_0|}{c}R_L\, |\Omega|\, \frac{R_L}{2}\Big)\frac{\partial B}{\partial z} = -\Big(\frac{|e_0|}{c}\frac{\pi R_L^2}{P}\Big)\frac{\partial B}{\partial z},$$

where as before the gyroperiod $P = 2\pi/|\Omega|$.

The periodic gyromotion of a charged particle e_0 in the plane orthogonal to the magnetic field is electrically equivalent to a current carrying loop with $I = |e_0|/P$. According to Ampère's law, such a loop corresponds to a magnetic dipole with *magnetic moment* μ, given by:

$$\mu = \frac{I\, S}{c} = \frac{|e_0|}{c}\frac{\pi R_L^2}{P} = \frac{1}{2B}mv_\perp^2 = \frac{W_\perp}{B}. \tag{2.22}$$

We therefore get, along $\boldsymbol{B}$, the equation:

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\frac{\partial B}{\partial z},$$

which, multiplying by $v_\parallel$ leads to

$$\frac{\mathrm{d}W_\parallel}{\mathrm{d}t} = m v_\parallel \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu v_\parallel \frac{\partial B}{\partial z} = -\mu\frac{\mathrm{d}B}{\mathrm{d}t},$$

where we have used

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla.$$

On the other hand, since the magnetic field does no work on the particles and there is no explicit time-dependence of the field itself, the total energy of the particle is conserved:

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t} = -\frac{\mathrm{d}W_\parallel}{\mathrm{d}t} = \mu\frac{\mathrm{d}B}{\mathrm{d}t},$$

an expression which may also be found by writing the equation for perpendicular motion. We may now calculate the time- derivative of μ as follows:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(W_\perp/B) = \frac{1}{B}\frac{\mathrm{d}W_\perp}{\mathrm{d}t} - \frac{1}{B^2}W_\perp\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\mu}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \frac{1}{B^2}(\mu\, B)\frac{\mathrm{d}B}{\mathrm{d}t} = 0.$$

The magnetic moment is therefore a conserved quantity, provided gradient length-scales of the average field are small compared to the particle gyro-radii: this is what is meant by an *adiabatic invariant*. In the particle's frame of reference, the magnetic field changes, albeight slowly, with time, because of the motion of the particle. We can therefore expect that conservation of magnetic moment also holds for slow time-variations of the magnetic field.

Consider then such a slowly varying field, directed along the axis in a cylindrical coordinate system: $\boldsymbol{B} = [0, 0, B(t)]$, whose typical time variation T is much longer than the gyroperiod $2\pi/|\Omega|$. The time-variation induces an azimuthal electric field $\boldsymbol{E}$ which appears in the perpendicular components of the equation of motion. Taking the scalar product of the equation of motion with $\boldsymbol{v}_\perp$ we find:

$$m\,\frac{\mathrm{d}\boldsymbol{v}_\perp}{\mathrm{d}t}\cdot\boldsymbol{v}_\perp = e_0\boldsymbol{E}\cdot\boldsymbol{v}_\perp,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}m\,v_\perp^2) = e_0\boldsymbol{E}\cdot\boldsymbol{v}_\perp.$$

Integrating this over the gyroperiod, from time 0 to $P = 2\pi/|\Omega|$, the left hand side gives the variation of the kinetic energy of perpendicular motion over one gyroperiod $\Delta W_\perp$:

$$\Delta W_\perp = e_0 \int_0^P \boldsymbol{E} \cdot \boldsymbol{v}_\perp \, \mathrm{d}t = e_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}_\perp = e_0 \int_S (\nabla \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S},$$

where $\mathrm{d}\boldsymbol{r}_\perp = \boldsymbol{v}_\perp \mathrm{d}t$ while $\mathrm{d}\boldsymbol{S}$ is an oriented element of the surface upon which the cyclotron motion orbit rests.

From Maxwell's equations, we may rewrite:

$$\Delta W_\perp = -e_0 \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} \simeq \pi R_L^2 |e_0| \dot{B},$$

where we have been able to bring $< \partial \boldsymbol{B}/\partial t >= \dot{B}$ out of the integral because of the hypothesis of slow variations of the magnetic field and we have taken into account the sign coming from the orientation of particle orbits of different sign.
Substituting $R_L = |\boldsymbol{v}_\perp|/|\Omega|$ we obtain:

$$\Delta W_\perp = W_\perp \frac{2\pi}{|\Omega|} \frac{\dot{B}}{B} \ .$$

On the other hand, $(2\pi/|\Omega|)\dot{B}$ is the change of B over a gyroperiod, i.e. ΔB, so we can write:

$$\Delta W_\perp = W_\perp \frac{\Delta B}{B} \ ,$$

or, recalling Eq. (2.22),

$$\Delta(W_\perp/B) = \Delta(\mu) = 0, \tag{2.23}$$

Therefore, we have shown that the magnetic moment is an adiabatic invariant even in the case of slowly time-varying magnetic fields.

2.3.3 Magnetic Mirrors and Magnetic Bottles

The concept of the conservation of magnetic moment finds application in the study of magnetic mirrors and magnetic bottles as plasma confinement devices. Let's consider again a magnetic field, for example with axial symmetry, which increases in intensity along the positive direction of the axial direction z as shown in Fig. 2.3.

Since the magnetic field magnitude B increases with z, the constancy of μ also implies that the kinetic energy in perpendicular motion $W_\perp$ must increase. Conservation of energy, $W = W_\parallel + W_\perp = constant$, then means that any increase

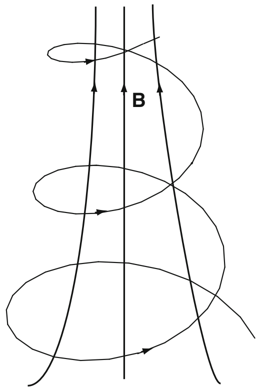

Fig. 2.3 Trajectory in a spatially varying magnetic field

in $W_\perp$ must be accompanied by a decrease in $W_\parallel$. Thus, it may happen that for some sufficiently large value of $B = B_R$, $W_\parallel$ vanishes. In such case, the charged particle, which cannot continue its trajectory along positive z, is reflected. This kind of configuration is called a *magnetic mirror*.

Introducing the pitch angle ϑ, defined as the angle formed by the velocity vector $\boldsymbol{v}$ with the axial direction z, that is with the direction of the dominant magnetic field component B, we find $v_\perp = v \sin\vartheta$ so that the magnetic moment invariance may be written in the form:

$$\tfrac{1}{2}mv^2 \frac{\sin^2\vartheta}{B} = \text{constant} \quad \text{i.e.} \quad \frac{\sin^2\vartheta}{B} = \text{constant}$$

because of the conservation of total energy $\frac{1}{2}mv^2$. By evaluating the constants at the reflection point we obtain:

$$\sin^2\vartheta = \frac{B}{B_R}. \tag{2.24}$$

Now if the field has a maximum somewhere, $B_{\max}$, obviously $B_R < B_{\max}$ must hold: indeed, if the particle is able to reach the position where $B = B_{\max}$ with $v_\parallel \neq 0$ it will continue in the same direction, as B decreases beyond the point $B = B_{\max}$ and therefore $W_\parallel$ increases at the expense of $W_\perp$. From Eq. (2.24) it follows that we can write the condition for reflection as

$$\sin^2\vartheta = \frac{B}{B_R} \geqslant \frac{B}{B_{\max}}.$$

Particles that start off with a pitch angle $\sin^2\vartheta < B/B_{\max}$ will not be reflected by the magnetic mirror.

If we now consider the illustrated magnetic field structure, comprised of two separate magnetic mirrors encompassing the same axial field, we see that it is possible

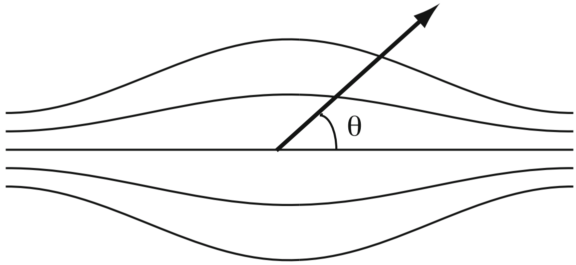

Fig. 2.4 A magnetic bottle configuration

to confine particles with sufficiently large pitch angles in the region between the two mirrors (Fig. 2.4).

Denoting with $B = B_0$ the lowest value of the magnetic field, the particles that will be confined will be those with

$$\sin^2 \vartheta_0 \geqslant \frac{B_0}{B_{\max}} = 1/R,$$

the quantity $R = B_{\max}/B_0$ being called the *mirror ratio*. The other particles, those with angles falling witihn a cone defined by ϑ_0, known as the *loss cone*, will not be confined. The probability $\mathcal{P}$ of losing a particle depends on the ratio between the solid angles swept by the loss cone and 2π, i.e.

$$\mathcal{P} = \frac{1}{2\pi} \int\limits_{\Omega_0} \sin\vartheta \, d\vartheta \, \mathrm{d}\varphi = \int\limits_0^{\vartheta_0} \sin\vartheta \, \mathrm{d}\vartheta = 1 - \sqrt{1 - 1/R} \simeq 1/2R \quad \text{for} \quad R \gg 1.$$

In a real situation, the loss process will never stop, because collisions between charged particles will continually feed particles into the loss cone. For this reason, the *magnetic bottle* is not an efficient confinement device with regards to possible future nuclear fusion machines.

A magnetic bottle configuration in which the intensity of the magnetic field varies slowly with time possesses another adiabatic invariant, the so-called *longitudinal adiabatic invariant J*, defined as

$$J = \int\limits_{s_1}^{s_2} v_\parallel \, \mathrm{d}s,$$

where s_1 and s_2 are the reflection points for the parallel motion whose position is assumed to change on time-scales much slower than the transit time inside the bottle. One may show (see, e.g. Boyd and Sanderson: *The Physics of Plasmas*, p. 28) that J is also an adiabatic invariant.

An interesting and important application of the existence of this invariant is connected to understanding the acceleration of cosmic rays. A simple model may be developed as follows: an ensemble of charged particles finds itself in an environment where magnetic clouds are present and whose behaviour may be schematized as being equivalent to moving magnetic mirrors. In the regions between clouds, the magnetic field has values much smaller than in the clouds, so that particles which start out with a sufficiently large pitch angle are confined in the region between clouds. If s_0 is a measure of the distance between clouds at a certain time t and s_0' that at a successive moment in time t', the invariance of J implies that

$$v_\parallel\, s_0 \simeq v_\parallel'\, s_0' \quad \text{or} \quad v_\parallel' \simeq v_\parallel \frac{s_0}{s_0'},$$

where $v_\parallel$ and $v_\parallel'$ are the average velocities respectively at times t an t'. It follows that the energy $W_\parallel'$ is equal to $W_\parallel' = W_\parallel\,(s_0/s_0')^2$ and increases if clouds approach each other while it decreases if they increase their separation. Since $W_\perp = \mu\, B = constant$
(μ is a constant because it is also an adiabatic invariant and B does not change much as the clouds and particles move around), we see that the total energy $W = W_\parallel + W_\perp$ also increases as clouds approach and vice-versa. Since the relative motion of clouds must be considered a random variable, one might think that on average energy gains and losses might balance out. This is wrong however, since over a given interval of time the number of head on collisions between clouds is greater than the number of overtaking ones. Their relative frequencies are, in fact, proportional to $(v_\parallel + v_C)/(v_\parallel - v_C)$, where v_C is the speed of the cloud. The net effect is therefore a gain of energy. This acceleration mechanism for cosmic rays was proposed by Enrico Fermi in 1949 and is known as the second order Fermi mechanism. It is not very efficient because of the particles that are scattered into the loss cone. Today it is thought that a much more efficient mechanism is given by the so-called first order Fermi mechanism where particles are accelerated as they repeatedly cross magnetic shock waves.

Problems

2.1. Consider a magnetic field configuration with a slight curvature: $\boldsymbol{B} \equiv (0, B_y(z), B_{0z})$, and B_{0z} constant while B_y and $\mathrm{d}B_y/\mathrm{d}z$ are small quantities. Show that there is a drift along the x direction with speed

$$v_c = -\frac{m v_\parallel^2 c}{e_0 B_{0z}^2}(\mathrm{d}B_y/\mathrm{d}z).$$

and show that this coincides with what one finds from Eq. (2.17).

2.2. Consider the motion of a proton and an electron at the surface of the Earth in the presence of a horizontal uniform magnetic field with an intensity of $B_0 \simeq 0.3$ G. Calculate the drift speed and directions knowing that the magnetic field is directed from South to North. Assuming that the electron density n_e is equal to that of protons n_p calculate thecurrent $\boldsymbol{J}$ resuting from this drift. Finally, show that

$$\frac{\boldsymbol{J}}{c} \times \boldsymbol{B} + (n_e m_e + n_p m_p)\boldsymbol{g} = 0,$$

where $\boldsymbol{g}$ is the gravitational acceleration vector at the surface of the Earth.

This example shows how, in the presence of gravity, a horizontal magnetic field induces a current in the plasma whose effect is to suspend the plasma in the magnetic field, cancelling out gravity. In reality, for a plasma volume of finite extent, there is a separation of charge induced at the lateral boundaries by the current $\boldsymbol{J}$. Show that the generated polarization electric field provides a downward E cross B drift motion, whose time derivative leads the plasma to fall through the magnetic field with acceleration $\boldsymbol{g}$. This points to the fundamental role of *boundary conditions* in establishing the correct behavior of a plasma.

2.3. The Earth's magnetic field may be considered with good approximation to be a dipole field, whose intensity in the magnetic equatorial plane is given by $B = B_0(R_E/R)^3$ where $B_0 = 0.3$ G, R_E is the radius of the Earth and R the Earth's radius. Show that because of the gradient drift in the dipole field, Eq. (2.19), a particle with pitch angle $\theta = 90°$carries out a circular orbit around the Earth and find an expression for this orbit in terms of the Energy of the particle E and its charge q. Calculate the period of the orbit for a proton and an electron with an energy of 1 keV at a distance of $R = 2R_E$. Compare this drift speed with the one due to the gravitational field directly and with the gravitational orbital speed at the same height.

2.4. Consider the motion of a charged particle in the presence of orthogonal uniform electric (along the y-direction) and magnetic (along the z-direction) fields where the electric field is greater than the magnetic field $E > B$. Solve the problem by moving to a frame moving along the x direction with $\beta = B/E$. Show that in this frame the magnetic field vanishes and solve the equation of motion explicitely. Move back to the original frame of reference and describe the particle orbit in this frame.

Solutions

2.1. Calling $\Omega = e_0 B_{0z}/mc$ e $\Omega_y = e_0 B_y/mc$, the equations of motion become:

$$\ddot{x} = \Omega \dot{y} - \Omega_y \dot{z},$$
$$\ddot{y} = -\Omega \dot{x},$$
$$\ddot{z} = \Omega_y \dot{x}.$$

Taking the time derivative and neglecting terms that are quadratic in small quantities we find:

$$\dddot{x} + \Omega^2 \dot{x} = -\Omega_y' v_\parallel^2,$$

$$\dddot{y} + \Omega^2 \dot{y} = \Omega \Omega_y v_\parallel.$$

A particular (non oscillating solution) of this equation is given by

$$\dot{x} = -\frac{\Omega_y'}{\Omega^2} = -\frac{m v_\parallel^2 c}{e_0 B_{0z}^2}(\mathrm{d}B_y/\mathrm{d}z),$$

providing the requested drift speed.

From the definition of the radius of curvature

$$\frac{\partial \boldsymbol{B}}{\partial s} = -\frac{\boldsymbol{R}_C}{R_C^2},$$

and taking into acocunt that $\partial \boldsymbol{B}/\partial s \simeq (\mathrm{d}B_y/\mathrm{d}z)\boldsymbol{e}_y$, while $\boldsymbol{B} \simeq B_z \boldsymbol{e}_z$, one finds that the above result is equivalent to that found using Eq. (2.17).

2.2. The drift speed is given by Eq. (2.8) where the gravitational acceleration is orthogonal to the magnetic field, so that

$$|v_{p,e}| = \frac{c m_{p,e} g}{eB} = 1.86 \times 10^{-4}\,\mathrm{cm/s},\ \text{(electron)} = 3.42 \times 10^{-1}\mathrm{cm/s},\ \text{(proton)}.$$

The velocity is towards the East (West) for a proton (electron). The current from this drift is also directed Eastwards and has a magnitude $|\boldsymbol{J}| = c n_p (m_p + m_e) g/B$.

2.3. The gradient drift for this case reduces to

$$|v_G| = \frac{cE}{q}\frac{1}{B^2}|dB/dr| = \frac{cE}{qB_0}\frac{3R^2}{R_E^3}$$

directed towards the West IEast) for protons (electrons). For a particle with the elctron or proton charge and an energy of 1 keV at a height of $R = 2R_E$ we find $|v_G| = 6.27 \times 10^6$ cm/s.

2.4. In the reference frame moving with $v_x = cB/E$ one finds from Eq. (2.7) that $B_z' = \Gamma(B_z - \beta E_y) = 0$ while $E_y' = \Gamma(E_y - \beta B_z) = E/\Gamma$ where here $\beta = B/E$ and $\Gamma = 1/\sqrt{1 - B^2/E^2}$. In the new, primed frame, the relativistic equation of motions are given by [see Eqs. 2.9) and (2.10)]

$$\frac{\mathrm{d}(\gamma' \beta_x')}{\mathrm{d}t'} = \frac{\mathrm{d}(\gamma' \beta_z')}{\mathrm{d}t'} = 0; \quad \frac{\mathrm{d}(\gamma' \beta_y')}{\mathrm{d}t'} = \omega_E'; \quad \frac{\mathrm{d}\gamma'}{\mathrm{d}t'} = 0,$$

with $\omega'_E = (e_0 E'/mc) = \Omega_E/\Gamma$. If the particle starts from rest at the origine $\beta'_x = \beta'_z = 0$, $\beta' = \beta'_y$ and $\gamma' = (1-\beta'^2)^{-1/2}$. From the remaining two equations we get:

$$\beta' \frac{\mathrm{d}\gamma'}{\mathrm{d}t'} + \gamma' \frac{\mathrm{d}\beta'}{\mathrm{d}t'} = \omega'_E = \omega'_E \beta'^2 + \gamma' \frac{\mathrm{d}\beta'}{\mathrm{d}t'},$$

or

$$\omega'_E (1-\beta'^2) = \gamma' \frac{\mathrm{d}\beta'}{\mathrm{d}t'} \quad \Longrightarrow \quad \frac{\mathrm{d}\beta'}{\mathrm{d}t'} = \frac{\omega'_E}{\gamma'^3}.$$

Integrating the last equation we obtain

$$\beta' = \frac{\tau'}{\sqrt{1+\tau'^2}} \quad \Longrightarrow \gamma'^2 = 1 + \tau'^2,$$

where $\tau' = \omega'_E t'$. Writing

$$v'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \omega'_E \frac{\mathrm{d}y'}{\mathrm{d}\tau'} = c\beta' = \frac{c\tau'}{\sqrt{1+\tau'^2}},$$

we see that the particle's speed increases in the y'-direction, i.e. along $\boldsymbol{E}'$, approaching c when $t' \to \infty$.

Integrating the above equation we finally find

$$y' = \frac{c}{\omega'_E}\left(\sqrt{1+\tau'^2} - 1\right),$$

which, coupled with $x' = z' = 0$, gives the trajectory of the particle in the primed frame. Transforming back to the original frame by means of a Lorentz transformation vith $V = -B/E$, we easily find

$$x = Vt; \quad y = \left(\frac{c\Gamma}{\omega_E}\right)\left[\left(1 + \frac{{\omega_E}^2 t^2}{\Gamma^4}\right)^{1/2} - 1\right].$$

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