Relativistic Classical and Quantum Mechanics

2

To develop the foundations of a manifestly covariant mechanics, we must first examine the Einstein notion of time and its physical meaning. We will then be in a position to introduce the relativistic quantum theory developed by Stueckelberg (1941) and Horwitz and Piron (1973). We describe in this chapter a simple and conceptual understanding of the Newton-Wigner problem (Newton 1949) presented above, a rigorous basis for the energy time uncertainty relation, as well as a simple explanation of the Landau-Peierls (Landau 1931) uncertainty relation between momentum and time. These applications provide a good basis for understanding the basic ideas of the relativistic quantum theory. Schieve and Trump (1999) have discussed at some length the associated manifestly covariant classical theory, but some basic aspects will be discussed here as well.

2.1 The Einstein Notion of Time

In this section, we shall carefully study the Einstein notion of time, the variable t which occurs in the Minkowski space and the Lorentz transformation.

We begin our study by returning to the basic thought experiment of Einstein (1922) Born (1962). Imagine a frame F with a set of synchronized clocks embedded, and a second frame F' with clocks embedded in it. Let us suppose that signals are sent, according to the clocks in F, at times τ_1 and τ_2 from F to F'. These signals are received by detectors in F' at times τ_1' and τ_2' according to the clocks embedded in F'. Then, we know, according to the phenomenology of the Michelson-Morley experiment and the formulation of the Lorentz transformation by Einstein, that

$$\tau_2' - \tau_1' = \frac{\tau_2 - \tau_1}{\sqrt{1 - \frac{v}{c}^2}} \tag{2.1}$$

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The interval $\tau_1 - \tau_2$ is called the proper time interval for the transmitter of the signals according to the clock interval in the frame F. The interval recorded in the relatively moving frame F' is the *Einstein time* $\Delta t = \tau_2' - \tau_1'$, corresponding to the time interval *observed* in the frame F' for the two events in F; the values assigned to the time of arrival of these events are read on clocks in the frame F'. It is therefore essential in this construction that the clocks embedded in F' be identical to the clocks in F, running at the same rate, or there would be no basis for comparison; the numbers τ_1' and τ_2', read off the clocks embedded in F' could otherwise be arbitrary.

We remark that if the clocks in F and F' that we consider have a varying self-energy caused by springs under tension or batteries with stored chemical energy, the rate of recording time of these clocks may be affected by the corresponding local concentration of energy density (as one may see from (2.12)). The standard universal clocks that we visualize as imbedded in each inertial frame must therefore be *ideal* clocks, in the sense that they contain no self-energy induced frequency shifts.

It is instructive, in this respect, to consider the gravitational redshift observed on a clock located at some point in the neighborhood of a very heavy planet, such as Jupiter. An interval of the time read on the face of such a clock Δt_J, its proper time, is determined, in general relativity, by the Einstein metric relation (we shall use units for which $\hbar = c = 1$ in the following)

$$\Delta s^2 = -g_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

where, at rest, the spatial interval $\Delta \mathbf{x}$ is understood to be zero, and Δs is the corresponding free fall proper time. Then,

$$\frac{\Delta t_J}{\Delta s} = \sqrt{-\frac{1}{g_{00}^J}};$$

the ratio of such a reading on Jupiter to that taken of a similar system (say, an ammonia molecule) on Earth is then, assuming that the corresponding interval Δs is the same at both locations (Weinberg 1972),

$$\frac{\Delta t_J}{\Delta t_E} = \sqrt{\frac{g_{00}^E}{g_{00}^J}},$$

in good agreement with experiment.

This calculation is remarkable in two respects; first, in that the interval of proper time between pulses of these clocks on Jupiter and the Earth must be the same for the cancellation of Δs when the two equations are divided one by the other, and second, in that somehow these clock mechanisms are responsive to a proper time that could be physically effective only if they were freely falling. Neither of the two systems are freely falling in this example, but are fixed in their respective gravitational fields.

The conceptual difficulties raised by this description of the phenomenon of the redshift may be resolved by considering the clocks in the two environments, on Jupiter and on the Earth, as machines evolving according to a universal time τ. The different gravitational field in the two cases causes the clocks to emit signals at different frequencies, according to the Einstein metric, as a result of the effect of the gravitational force on the equations of motion. Freely falling clocks may also be considered to be machines running according to this universal time. The absence of any gravitational (or other) force admits solutions which are a direct reflection of the universal time; we may therefore identify Δs, in this case, in the metric relation, with $\Delta\tau$, the universal time interval referred to in the thought experiment discussed above

We learn two essential points from these simple experiments. The first is that the Einstein time is defined as the result of measurement, and the second is that there must be an underlying time which is common to both frames in the first example in order to assign numerical values to the observed times that can be compared to the times associated with the emitted signals, and in the second example, to govern the dynamics of the clocks.

There appear, therefore, to be two types of time, an absolute time of clocks embedded in any system, independent of the state of the motion, and the second, the time that is the *outcome of a measurement*, as recorded in the detector (i.e., by the "observer") (Horwitz 1988). The notion of the Einstein time as an observable is completely analogous to the property $\mathbf{x}$ of location, corresponding to the position of a particle. When the particle is detected, the value $\mathbf{x}$ assigned to its position is given by the corresponding location on a standard ruler. For the Lorentz transformation relating intervals in space, the measure of length must be universally embedded in each frame, and the difference $\Delta\mathbf{x}'$, detected in a relatively moving frame, corresponding to an interval $\Delta\mathbf{x}$ in the original frame, is the outcome of measurement, induced by the dynamics of the relative motion. The spacetime coordinates of general relativity correspond to quantities that are observed by detectors; the general tensor properties under local diffeomorphisms, reflecting the covariance assumptions underlying general relativity, correspond to different physical situations, as for example, the Schwarzschild and Friedman-Robertson-Walker solutions of the Einstein equations (Schwarzschild 1916; Friedman 1924), where the coordinates are considered to be actual outcomes of measurement.[1]

These are the essential ingredients from which a manifestly covariant classical and quantum mechanics can be constructed (we shall confine ourselves here, for the most part, to the covariance characteristic of special relativity, although in a later chapter our considerations will be extended to applications in general relativity).

In classical nonrelativistic mechanics, the fact that the value assigned to the position of a particle $\mathbf{x}$ and the value of the momentum $\mathbf{p}$ are the outcomes of measurement

[1] Note that both time intervals, as well as space intervals, must be thought of as measured by geodesic projection (e.g. Weinberg 1972) since clocks and rulers brought to the location of the events would suffer distortion due to the gravitational field as well.

gives rise to the notion of a point in phase space describing the *state* of the particle. The state evolves, according to the theory of Hamilton and Lagrange by means of an evolution determined by the Hamilton equations, an elegant formulation of Newton's laws of motion (we denote the gradient formally by a partial derivative with respect to a vector),

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \\ \frac{d\mathbf{p}}{dt} &= -\frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}, \end{aligned} \tag{2.2}$$

where $H(\mathbf{x}, \mathbf{p})$ is the Hamiltonian of the system. Here the variables $\mathbf{x}$ and $\mathbf{p}$ are functions of the time t. These equations can be directly generalized to N particles, writing $\mathbf{x_i}$ in place of $\mathbf{x}$ and $\mathbf{p_i}$ in place of $\mathbf{p}$ for $i = 1, 2, 3, \ldots, N$, and the Hamiltonian is generally a function of all $6N$ variables. This structure, sometimes called *symplectic* because the formulas (2.2) have the symmetry of the symplectic group, is made possible due to the correlation between the variables established by the existence of the universal Newtonian time t.

Similarly, in the quantum theory, where a pure state of the system (in the simplist case) is determined by a wave function $\psi_t(\mathbf{x})$ (or $\psi_t(\mathbf{p})$), the Schrödinger equation governs the evolution of the system according to

$$i\frac{\partial \psi_t}{\partial t} = H(\mathbf{x}, \mathbf{p})\psi_t, \tag{2.3}$$

where the Hamiltonian is a function of the observables, i.e., the Hermitian operators, $\mathbf{x}$ and $\mathbf{p}$. This equation can be written in a representation (called the $\mathbf{x}$-representation) in which $\mathbf{x}$ is diagonal, i.e. numerical valued, and $\mathbf{p}$ is represented by $-i$ times the partial derivative with respect to $\mathbf{x}$, or conversely, in a representation called the $\mathbf{p}$-representation) in which in which $\mathbf{p}$ is diagonal, i.e. numerical valued, and $\mathbf{x}$ is represented by i times the partial derivative with respect to $\mathbf{p}$. This structure may be generalized to an N body system in the same way, for which the wave function in the $\mathbf{x}$ representation is a function of all the positions $\mathbf{x_i}$, $i = 1, 2, 3, \ldots, N$ at a given value of the universal Newtonian time t.

This description of the dynamics of systems of particles rests on the identification of the observables. In nonrelativistic dynamics, t is a *parameter* providing a framework for the correlation of different parts of a system as well as for its dynamical development.

If, as we have argued above, the time t is understood as an observable in relativistic dynamics, the set of observables assigned to each particle (often called an *event*) is comprised of all four Minkowski coordinates $x^\mu \equiv (t, x_1, x_2, x_3)$ as well as $p^\mu \equiv (E, p_1, p_2, p_3)$, along with others, such as the relativistic generalization of angular momentum (the Casimir operators of the Lorentz group, as we shall discuss further in Chap. 5). The construction of a dynamics to describe the motion of these fundamental objects, and some selected important applications of this dynamics, will be the subject of this book. I review in the following the arguments of Stueckelberg (1941) for the

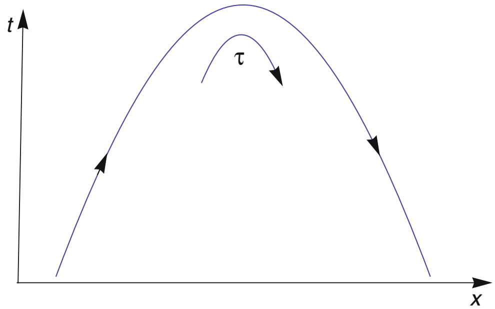

Fig. 2.1 Stueckelberg classical pair annihilation

construction of this theory and comment on an alternative, complementary, view (Horwitz 1973) leading to the same conclusions.

Stueckelberg (1941) first considered the classical spacetime diagram of the orbit, called a "worldline", of a free particle, expected to be simply a straight line. He then supposed that there is some force acting on the particle that makes the worldline bend during the interaction. He further supposed that the interaction may be strong enough to make the world line turn back and run in a direction opposite to that of the t axis, as shown in Fig. 2.1.

It is clear that Stueckelberg was thinking of this process as reflecting the effect of some dynamical laws on the evolution of the sequence of events constituting the worldline rather than a global manifestation of the worldline (the later work of Currie et al. (1963) showed that such a global dynamics of worldlines would, with some assumptions, suffer from a no-go theorem). In contrast to the view of Weyl (1952), who suggested that the particles we see are the intersection of the observer's plane of time with pre-existing world lines, comprising a static universe (see also discussion in Horwitz (1988)), with apparent motion generated by the effect of this plane cutting the worldlines at a succession of points in t, the worldline is envisaged here as generated by the motion of a single event moving according to dynamical laws, in a similar way to the formation of the orbit of a particle in nonrelativistic mechanics, generated as a function of the Newtonian time. Stueckelberg observed that in the extreme case of a reversal in the sense of time of this motion, the physical process of pair annihilation could be represented in the framework of *classical* mechanics if the path running backward in time were considered as an antiparticle. He, moreover, noted that the use of t as a parameter would be inadequate to describe this curve, but that an invariant parameter, which he called τ, along the curve, had to be introduced to construct a consistent description. Feynman (1950) followed a structure of this type in the construction of his spacetime diagrammatic approach to perturbation expansions in quantum electrodynamics, elegantly explained in a paper by Nambu (1950).

Horwitz and Piron (1973) further assumed, in order to treat many body systems, that this parameter is universal, as for the Newtonian time; it, in fact, plays the role of the universal time postulated by Newton in his Principia (Newton 1687).

The concept of a world time controlling the dynamical evolution in contrast to evolution in t is illustrated in Fig. 2.1. Along the curve, the parameter τ increases monotonically. The t axis of the diagram, however, consistently with the definition of t as the *measured* time in the laboratory, records the time on the clock in the laboratory at the moment when the signal is detected, which runs (in the absence of any other forces and on its mass shell) with τ; thus, the t measured in the laboratory records its evolution in τ. The sequence of τ values parametrizing the motion of the event along its world line in Fig. 2.1 is the same sequence along the t axis, reflected by values of t in the laboratory that coincide with τ. Close to the initial condition, the corresponding points (i.e. equal τ points) run along essentially the same t values, but as the system develops, the t values recorded in the laboratory as observed on the laboratory clock, and the t values detected as signals from the system under observation (with values read on the laboratory clock as well) diverge significantly. Thus, the character of the observable t becomes manifest as a consequence of the dynamics that affect its measured value.[2]

There is, however, another phenomenon illustrated in this diagram. During the period that the world line is deflected and curving, it passes through the light cone, becomes spacelike, and then becomes straight again in the final force-free region, but nevertheless, moving *backwards* in time. This inversion in the sequence in the final state cannot be attributed directly to forces acting on the system, but rather must be thought of as the positive monotonic evolution of the *antiparticle* in τ, forward in t. The figure therefore illustrates a profound physical transition. In the asymptotic region after the interaction, it represents the motion of an antiparticle in the positive direction of time, as maintained by Stueckelberg (1941) in agreement with the view adopted by Feynman (1950) and associated with CPT conjugation. Since CPT conjugation, as we shall discuss later in more detail, reverses the sign of momentum and energy (as well as the charge), the positive monotonic evolution of the antiparticle in τ is forward in t along the "outgoing" line. In this CPT conjugate picture the entire world line (taking into account the properties of CPT conjugation, such as a change in sign of the charge) is reversed in τ ordering, and the previously "incoming" line now runs backward in t; its CPT conjugate then runs forward in t, corresponding to the original incoming particle. The particle-antiparticle interpretation is not easily accessible in the interaction region, where the world line may be spacelike; the dynamics of the motion, however, is smoothly and unambiguously represented as a motion on spacetime according to τ (such a process can occur repeatedly as in neutrino oscillations and the evolution of the K and B meson systems; we shall study these processes in Chap. 4).

To pose an apparent paradox, one may think of cutting the world line at some point on the incoming line, absorbing the particle entirely, as suggested by Havas (1956); he remarked that this would destroy its continuation. However, that continuation is in

[2] This discussion is fundamental in understanding the essential distinction between the *measured* time of Einstein, which plays the role of a *coordinate* of a physical event, and the underlying absolute time τ governing the dynamical processes of evolution.

the *past* of t, leading to an apparent contradiction. He resolved this paradox by noting that the instrument that absorbed the particle is located at this point in spacetime for all τ, and therefore constitutes a change in initial conditions for the generation of this history. The antiparticle would therefore never have been produced. If the experiment records a particle and antiparticle, that antiparticle would have had to be generated elsewhere (e.g. at $t \to +\infty$) and would not be associated with this annihilation diagram.

One can approach the theory as we have presented it above from a somewhat different point of view (Horwitz 1973). We observe that in nature the mass of a particle generally depends on its state. One understands the decay of a neutron into proton, electron and neutrino (β decay) as associated with the fact that the neutron is heavier that the proton in free space. In a nucleus, however, the neutron generally does not decay. Moreover, the proton in a nucleus may decay into a neutron, positron and neutrino (inverse β decay), indicating that the proton is more massive that the neutron in that environment. As another example, calculations in quantum electrodynamics show that the difference between the mass of an electron in free space and in a Coulomb potential is not zero, making a contribution to the the Lamb shift (Lamb 1947) (see also the work of Davidson 2014, examining mass shifts in nuclei).

We therefore conclude that the observable mass of a particle, from the point of view of a *particle* theory (rather than investing the mass change in the surrounding fields) should be treated as a dynamical variable. Thus, in the momentum four vector, $\mathbf{p}$ and E should be considered as independent dynamical variables. The Fourier complement of this picture corresponds to the time t and position $\mathbf{x}$ necessarily being dynamical variables also (in accordance with our discussion of the physical meaning of these variables in special relativity above). Equations describing the distribution of these dynamical variables would then be static, with no parameter for the evolution of a state, and one must therefore introduce the notion of an invariant (universal in order to be able to treat the many body problem) variable τ with which to generate dynamical change. The resulting theory is then identical to that of Stueckelberg, with the additional postulate that τ is universal.

As a model for the structure of the dynamical laws that might be considered, Stueckelberg proposed a Lorentz invariant Hamiltonian for free motion of the form

$$K = \frac{p^\mu p_\mu}{2M}, \tag{2.4}$$

where M is considered a parameter, with dimension mass, associated with the particle being described, but is not necessarily its measured mass. In fact, the numerator (with metric $- + ++$),

$$p^\mu p_\mu = -m^2, \tag{2.5}$$

corresponds to the actual observed mass (according to the Einstein relation $E^2 = \mathbf{p}^2 + m^2$), where, in this context, m^2 is a dynamical variable.

The Hamilton equations, generalized covariantly to four dimensions, are then

$$\begin{aligned}\dot{x}^\mu &\equiv \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}\\ \dot{p}_\mu &\equiv \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu}.\end{aligned} \tag{2.6}$$

These equations are postulated to hold for any Hamiltonian model, including many types of interaction such as additive potentials or gauge fields (to be discussed in later chapters), and therefore a Poisson bracket may be defined in the same way as for the nonrelativistic theory. The construction is as follows. Consider the τ derivative of a function $F(x, p)$, i.e.,

$$\begin{aligned}\frac{dF}{d\tau} &= \frac{\partial F}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial F}{\partial p^\mu}\frac{dp^\mu}{d\tau}\\ &= \frac{\partial F}{\partial x^\mu}\frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p^\mu}\frac{\partial K}{\partial x_\mu}\\ &= \{F, K\},\end{aligned} \tag{2.7}$$

thus defining a Poisson bracket $\{F, G\}$ quite generally. The arguments of the non-relativistic theory then apply, i.e., that functions which obey the Poisson algebra isomorphic to their group algebras will have vanishing Poisson bracket with the Hamiltonian which has the symmetry of that group,and are thus conserved quantities, and the Hamiltonian itself is then (identically) a conserved quantity.

It follows from the Hamilton equations that for the free particle case

$$\dot{x}^\mu = \frac{p^\mu}{M} \tag{2.8}$$

and therefore, dividing the space components by the time components, cancelling the $d\tau$'s ($p^0 = E$ and $x^0 = t$),

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{E}, \tag{2.9}$$

the Einstein relation for the observed velocity. Furthermore, we see that

$$\dot{x}^\mu \dot{x}_\mu = \frac{p^\mu p_\mu}{M^2}; \tag{2.10}$$

with the definition of the invariant

$$ds^2 = -dx^\mu dx_\mu, \tag{2.11}$$

corresponding to proper time squared (for a timelike interval), this becomes

$$\frac{ds^2}{d\tau^2} = \frac{m^2}{M^2}. \tag{2.12}$$

Therefore, the proper time interval Δs of a particle along a trajectory parametrized by τ is equal to the corresponding interval $\Delta\tau$ only if $m^2 = M^2$, a condition we shall call "on mass shell".[3] The theory is, however, generally intrinsically "off-shell". We shall see that this property is essential for the resolution of the Newton-Wigner problem, and therefore for the possibility that the wave function has a local probability interpretation, and can be a candidate for a consistent relativistic quantum theory. There is, however, no obvious constraint, even for simple interacting models, such as in the potential model

$$K = \frac{p^\mu p_\mu}{2M} + V(x), \tag{2.13}$$

that would insure that the particle maintains a physical mass in the small neighborhood of some given value.[4] One might suppose that an electron, after interaction that could perturb the value of m^2, would result in a particle with a different mass; it has therefore been an explicit assumption in many successful applications (for example, in the two body bound state that we shall treat in Chap. 5) that there is a mechanism for returning the particle to the neighborhood of some equilibrium value of mass, such as a relaxation of free energy of the system in interaction with other particles or fields (e.g., a suggestion of Jordan 1980). It was found by Burakovsky and Horwitz (1996) that there may be a high temperature Bose-Einstein condensation, in the framework of statistical mechanics (to be discussed in Chap. 8), that causes a particle to stabilize its mass at some value determined by a chemical potential. More recently, Aharonovich and Horwitz (2011) have found that the electromagnetic self interaction of a charged particle can dynamically drive the particle to its mass shell. We shall assume in the following that there exists such a mechanism for every object that is recognized as a "particle" (even for reasonably sharp resonances) which stabilizes its mass, and discuss this question in more detail in later chapters. However, for the theory to be effective, this mass shell property can only be approximate, i.e., an absolutely sharp mass value would not be compatible with the structure of the theory, as will become clear below.

[3] In Galilean mechanics, due to the existence of a cohomology in the Lie algebra of the Galilean group, a definite value must be assigned to the value of the mass to achieve an irreducible representation (Sudarshan 1974). The Poincaré group does not have such a cohomology, and thus admits the full generality of the Stueckelberg theory. We discuss the Galilean limit in more detail in Chap. 10.

[4] An alternative covariant structure for a relativistic quantum theory, the so-called *constraint mechanics*, discussed in Appendix A of this chapter, based on the constraint theory developed by Dirac (1966) to deal with the quantization of gravity and gauge fields, extensively studied by Sudarshan et al. (1981a), Rohrlich (1981) and others (Llosa 1982), does have a mechanism for enforcing the asymptotic return of a particle to a given mass shell. This theory, however, necessarily makes use of a system of constraints of the *first class* (Itzykson 1980), a condition that makes the construction of a useful quantum theory very difficult (Horwitz 1982).

2.2 Classical Mechanics

To illustrate some of the properties of the covariant classical mechanics, consider the two body problem with invariant relative potential $V(x_1 - x_2)$, a Poincaré invariant potential (invariant under both the Lorentz group and translations); such a potential must be a function of $x^\mu x_\mu = \mathbf{x}^2 - t^2$, where we have called

$$x^\mu = x_1^\mu - x_2^\mu, \tag{2.14}$$

the relative spacetime coordinate, which we shall call x. The Stueckelberg Hamiltonian corresponding to this problem is (Horwitz 1973) (the assumption of the universality of τ made in this work, not explicitly made by Stueckelberg, is essential to the formulation of this problem)

$$K = \frac{p_1{}^\mu p_{1\mu}}{2M_1} + \frac{p_2{}^\mu p_{2\mu}}{2M_2} + V(x). \tag{2.15}$$

Since K does not depend on the total (spacetime) "center of mass"

$$X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2}, \tag{2.16}$$

the two body Hamiltonian can be separated into the sum of two Hamiltonians, one for the "center of mass" motion and the second for the relative motion, by defining the total momentum, which is absolutely conserved,

$$P^\mu = p_1^\mu + p_2^\mu \tag{2.17}$$

and the relative motion momentum

$$p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2} \tag{2.18}$$

Then, it is an identity that (as in the nonrelativistic two body problem)

$$\begin{aligned} K &= \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2M} + V(x), \\ &\equiv K_{CM} + K_{rel}, \end{aligned} \tag{2.19}$$

where $M = M_1 + M_2$ and $x = x_1 - x_2$; both K_{CM} and K_{rel} are constants of the motion.

We see in this construction the significance of defining τ as a universal parameter (Horwitz 1973). The potential function $V(x_1 - x_2)$ implicitly carries in it the information that the points x_1 and x_2 are at equal τ; this correlation makes it possible to consider pairs of points along the two world lines of the two particles as having well-defined interaction. A similar assumption is made in ordinary nonrelativistic dynamics; the implicit assumption in writing a potential function as $V(\mathbf{x}_1 - \mathbf{x}_2)$ is that the points $\mathbf{x}_1$ and $\mathbf{x}_2$ are taken at equal t along the orbits. This assumption is usually not made explicit, since the nonrelativistic Galilean world is always assumed to be at equal universal Newtonian time. Thus we see that the parameter τ, with the assumption of universality (Horwitz 1973), corresponds to the Newtonian time.

Although the t and x of Einstein undergo Lorentz transformations as they are perceived and measured in relatively moving inertial frames, the dynamical correlation provided by the invariant universal parameter τ is maintained independently of the state of motion.

Models parallel to those of the nonrelativistic theory can be constructed, for example, by replacing the $V(r)$ of nonrelativistic spherically symmetric models by $V(\rho)$. where $\rho = \sqrt{x^\mu x_\mu}$ for the relative coordinate x^μ spacelike, in accordance with our experience of the nonrelativistic two body problem. Moreover, for two timelike momenta, corresponding to particles with positive m^2, the relative momentum defined in (2.18) is generally spacelike since for not too large space components of the momenta, and for particles not too far from mass shell, the fourth components are then large and approximately equal to M_1 and M_2 respectively; the fourth component of the relative momentum carries a near cancellation, and the resulting vector is generally spacelike. The relativistic two-body problem therefore differs fundamentally from the nonrelativistic two body problem; in the latter case, separation of variables results in a center of mass motion accompanying what appears to be one particle in an external potential. In the relativistic case, the relative motion system is essentially *tachyonic*, i.e., it apppears to describe a "particle" with spacelike momentum (for which $\mathbf{p}/E > 1$, and thus light speed would be exceeded). The situation is not unphysical; we must realize that this is a relative motion of a two body system, and that the two particles being described can be properly timelike. If the theory were designed to rule out such tachyonic systems, we would not be able to study the two body case in the way we have described above.

For such a class of models, one may choose, for example,

$$V(\rho) = \frac{k}{\rho}, \tag{2.20}$$

corresponding to a Coulomb potential for $k = \pm e^2$, or a gravitational Kepler problem for $k = -GM_1M_2$. Since, according to (2.6), the Hamilton equations (written for each particle),

$$\frac{dt_i}{d\tau} = \frac{E_i}{M_i}, \tag{2.21}$$

if the $\{M_i\}$ are identified as the Galilean target masses of the particles (the Galilean group, as will be discussed further in Chap. 5, admits only a sharp mass, whereas the Poincaré group admits a continuum of possibilities (Sudarshan 1974), as occurs in the Stueckelberg theory), then the t values of all the particles may become identical in this limit, and the relative coordinate ρ goes over into the coordinate $r = |\mathbf{x}|$. Thus the Coulomb and Kepler models go over, as $c \to \infty$, precisely to the corresponding problems in the nonrelativistic theory.

We shall show in Chap. 5 that the corresponding relativistic quantum two body Coulomb problem can be solved exactly, and yields the nonrelativistic Schrödinger spectrum up to relativistic corrections ($\mathrm{O}(1/c^2)$).

For the classical case, a Lorentz invariant potential implies that the function

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\nu, \tag{2.22}$$

for which the Poisson bracket algebra is that of the Lorentz group, is conserved (its Poisson bracket with K vanishes). Therefore, the four linear cyclic combinations of $\{x_\lambda M_{\mu\nu}\}$ which vanish identically provide constraints on the orbits. Two of these relations are degenerate, and the remaining two restrict the Kepler motion to a plane. One finds, in contrast to Sommerfeld's (1921) conclusion, the resulting ellipse does not precess. The precession which Sommerfeld found in his search for the origin of the precession of the orbit of Mercury was due to his use of the noncovariant form $1/r$ for the potential. This problem is discussed in detail in Horwitz (1973) and in Trump (1999).

Another model of interest is that of the covariant harmonic oscillator, for which (for k some positive constant) (Feynman 1971; Kim 1977; Leutwyler 1977)

$$V = k\rho^2 = kx^\mu x_\mu. \tag{2.23}$$

The equations of motion separate into four independent second order equations, each of which correspond to a one dimensional oscillator, each following some elliptical path on spacetime, constituting an orbit which is bounded in the t direction; one may think of this as a continuing sequence of pair annihilation and creation processes (in relative motion) from the point of view of Stueckelberg's classical pair annihilation picture. In the corresponding quantum theory, this separation of variables leads to "ghost" states which must be suppressed by constraints. We shall see in Chap. 5 that this problem can be solved with no "ghost" states, obtaining the nonrelativistic oscillator spectrum (up to relativistic corrections).

From the point of view developed here, one sees the classical wave equations, such as the Klein-Gordon equation and the Dirac equation, as well as Maxwell's equations, as being essentially geometrical constraints rather than dynamical in this context. We shall be concerned here with developing the dynamics of systems evolving in a covariant way in spacetime.

2.3 The Quantum Theory

In this section we shall study the form of the quantum theory associated with Stueckelberg's dynamics in spacetime (Stueckelberg 1941; Horwitz 1973, to be called SHP in the following). We have argued that in a relativistically covariant theory, the space and time variables are observable, and therefore correspond to Hermitian operators in the quantum theory. The operator commutation relations are taken to be

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{2.24}$$

consistently with the Poisson bracket for the classical case, and the Lorentz covariant generalization of the nonrelativistic commutation relations $[x_i, p_j] = i\delta_{ij}$. With these commutation relations, the operator form of the definition (2.22) satisfies the commutation relations of the Lorentz group, just as the Poisson bracket relation for the classical case. To achieve this simple form for the generators of the Lorentz group in the quantum case, it is necessary that the Hilbert space be defined as $L^2(R^4, d^4x)$, as we define formally below; only in this way can the operator

$$E \rightarrow i\frac{\partial}{\partial t}$$

be considered as essentially self-adjoint. We shall discuss this operator form of the Lorentz group further in detail in Chap. 5.

The spectral decompositions of the self-adjoint operators x^μ or p^μ then provide representations of the quantum state, as explained, for example, in Dirac's book (Dirac 1930). The wave function is then a square integrable function on spacetime x (or p); its square modulus corresponds to the probability of finding an event per unit spacetime volume d^4x (or energy momentum space d^4p)at the point x (or p). In the x representation, for which x_μ is numerical valued, p^μ is represented by $-i\partial/\partial x_\mu$, and in the p representation, for which p^μ is numerical valued, x_μ is represented by $i\partial/\partial p^\mu$.

Stueckelberg assumed that the dynamical development of the wave function is governed by a Schrödinger-like equation, which we shall call the Stueckelberg-Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi_\tau(x) = K\psi_\tau(x) \tag{2.25}$$

where in the notation of Dirac (1930),

$$\psi_\tau(x) =< x|\psi_\tau > \tag{2.26}$$

and K is an operator function of x, p, which may correspond to the classical models discussed above. The wave function is assumed to be scalar; the representation of a particle with spin will be discussed in Chap. 3. Gauge field interactions, such as electromagnetism, can be accounted for by imposing gauge invariance, as we shall discuss in later chapters.

The Eq. (2.25) corresponds to unitary evolution, as for the nonrelativistic Schrödinger equation, where the evolution is generated by the operator (for K not explicitly dependent on τ)

$$U(\tau) = e^{-iK\tau}, \tag{2.27}$$

for which $\psi_\tau(x) = U(\tau)\psi(x)$.

The derivative of an expectation value of the observable F is then, as in the nonrelativistic quantum theory, consistent with the Poisson bracket formulation, i.e.

$$\frac{d}{d\tau}(\psi_\tau, F\psi_\tau) = -i(\psi_\tau, [F, K]\psi_\tau), \tag{2.28}$$

where $[F, K]$ is the commutator, with the correspondence defined by Dirac (1930)[5]

$$\{F, K\}_{PB} \rightarrow -i[F, K]. \tag{2.29}$$

[5]As Van Hove (1951) has pointed out, this correspondence is not applicable for higher order polynomials; both the Poisson bracket and the commutators are distributive in the Leibniz sense, but in the quantum case the algebra is not commutative, and it is not always possible to regroup factors as in the classical, commutative, case. The problem of consistent quantization has been studied under the name "geometric quantization" (Kostant 1970).

Since the "standard" bras and kets correspond to representations of the self-adjoint operators **x** and t, they are complete, and the scalar product (as for the expectation value in (2.28)) is given by

$$\begin{aligned} < \chi|\psi > &= \int d^4x < \chi|x >< x|\psi > \\ &= \int d^4x \chi(x)^* \psi(x). \end{aligned} \tag{2.30}$$

This is clearly a positive scalar product, defining the norm

$$\|\psi\|^2 = \int d^4x \psi(x)^* \psi(x) = \int d^4x |\psi(x)|^2, \tag{2.31}$$

as previously discussed in Chap. 1. This property, together with linear superposition over the complex numbers (which follows from the linearity of the scalar product) and boundedness of the norm, consistent with the Born probability interpretation, results in the proper structure of a Hilbert space and a consistent quantum theory.

The momentum representation, as in the nonrelativistic theory, is constructed from the Fourier transform

$$\psi(p) = \frac{1}{(2\pi)^2} \int d^4x e^{ip^\mu x_\mu} \psi(x), \tag{2.32}$$

with inverse

$$\psi(x) = \frac{1}{(2\pi)^2} \int d^4p e^{-ip^\mu x_\mu} \psi(p). \tag{2.33}$$

As we have noted in Sect. 1.2, the interpretation of the solutions of the Klein-Gordon equation as wave functions in a quantum theory encounters serious problems with localizability. In the theory of Stueckelberg, we have interpreted the wave function as the amplitude for the local probability density. It is therefore important to discuss the Newton-Wigner problem in the context of the Stueckelberg theory, and we turn to this question in the next section.

2.4 The Newton-Wigner Problem

Having defined the manifestly covariant quantum theory, we are now in a position to re-examine the Newton-Wigner problem (Newton 1949). From the viewpoint of this theory, we shall be able to understand the way the problem arises in the framework of theories which use equations of the type of those of Klein-Gordon and Dirac that impose a strict mass shell requirement.

We will show that the **x** operator in the Stueckelberg theory, corrected to extrapolate the occurrence of an event at some point in spacetime back to $t = 0$, as sought by Wigner and Newton, is exactly the Newton-Wigner position operator on each mass value (in the sense of a direct sum) under the integral defining the expectation value (Horwitz 1973).

Consider the expectation value of $\mathbf{x}$:

$$< x >= \int d^4 p \psi^*(\mathbf{p}, E) i \frac{\partial}{\partial \mathbf{p}} \psi(\mathbf{p}, E) \tag{2.34}$$

We now change variables, considering only $E \geq 0$, using the relation

$$E = \sqrt{\mathbf{p}^2 + m^2} \tag{2.35}$$

for m a new variable. Then,

$$dE = \frac{dm^2}{2E}, \tag{2.36}$$

where now E stands for the relation (2.35). Furthermore, if we want to think of the derivative in (2.34) as a straightforward derivative (it only acted on the first three arguments in ψ before the change of variables), we have to correct for its action on the fourth argument E, i.e., we must now write

$$\begin{aligned} i \frac{\partial}{\partial \mathbf{p}} &\rightarrow i \frac{\partial}{\partial \mathbf{p}} - i \frac{\partial E}{\partial \mathbf{p}} \frac{\partial}{\partial E} \\ &= i \frac{\partial}{\partial \mathbf{p}} - i \frac{\mathbf{p}}{E} \frac{\partial}{\partial E} \end{aligned} \tag{2.37}$$

when acting on $\psi(\mathbf{p}, E = \sqrt{p^2 + m^2})$.

We recognize that this extra term looks like velocity times time, the operator $i\partial/\partial E$. This corresponds to the displacement to get back to where a (virtual) world line would be at $t = 0$, if one imagines the semiclassical picture of a world line running through the point $(\mathbf{x}, t)$. This semiclassical interpretation of these operators, where the real information is encoded in the wave function, appears to be consistent. This extra term, however, in the quantum theory, should be symmetrized, so let us define the relativistic operator form of the Newton-Wigner operator in the context of the Stueckelberg theory as

$$x_{NW} = i \frac{\partial}{\partial \mathbf{p}} - \frac{1}{2}\{\mathbf{v}, t\}, \tag{2.38}$$

where $\mathbf{v} = \mathbf{p}/E$ and $t = i\partial/\partial E$. One must use the fact that when $\partial/\partial E$ acts on $\mathbf{p}/E$, it differentiates both this factor and the wave function that implicitly follows it. The last term in (2.38) is then

$$\frac{1}{2}\{\mathbf{v}, t\} = i \frac{\mathbf{p}}{E} \frac{\partial}{\partial E} - i \frac{\mathbf{p}}{2E^2}.$$

This is just the extra piece that came from the change of variables, plus a new term, which we saw is part of the Newton-Wigner operator displayed in Eq. (1.9). Thus, our operator (2.38), put into expectation value, can be seen as the expectation value of

$$\mathbf{x} \rightarrow \mathbf{x} - i \frac{\mathbf{p}}{2E^2},$$

as required by Newton and Wigner, but under the integral over all mass shells.

Therefore, the operator (2.38) may be represented as the Newton-Wigner operator under the integration over masses of an expectation value at each value of m.

The semiclassically expected value of the position of a particle as it passes $t = 0$ corresponds in this way to the Newton-Wigner operator.

We can understand from the point of view of the relativistic theory that position and mass, as the operator $\mathbf{x}$ and $m = \sqrt{E^2 - \mathbf{p}^2}$, are not compatible. The Klein Gordon theory does not consider the mass to be an operator; it is just a given number, corresponding to a point on the continuous spectrum of m. The Stueckelberg theory is completely local, consistent with our construction (2.38), and the interference phenomena we describe with the associated wave functions should predict the actual outcome of experiments. Such interference effects, predicted by Horwitz and Rabin (1976), have indeed been observed, as we shall discuss in Chap. 3 (the experiment of Lindner et al. 2005).

2.5 The Landau-Peierls Problem

In 1931, Landau and Peierls (1931) deduced a relation between dispersion in momentum and time of the form (we restore $\hbar$ and c in several formulas of this section to make the units clear)

$$\Delta p \Delta t \geq \hbar / c \tag{2.39}$$

concerning the time interval Δt during which the momentum of a particle is measured and the momentum dispersion of the state. According to Landau and Peierls, for any given dispersion of momentum in the state, there is a minimum interval of time necessary for measuring the outcomes predicted by knowledge of the state consistent with the relativistic bound on the velocities.

Landau and Peierls begin with the estimates of first order perturbation theory for the "almost conservation of energy", i.e.

$$|E - E'| \sim \hbar / \Delta t; \tag{2.40}$$

where, in perturbation theory, one argues that in sufficient time Δt, the initial energy E and the final energy E' after the transition are close. This relation corresponds to the well known estimate for the nonrelativistic energy time uncertainty relation.

Landau and Peierls, however, use this result, not a rigorous property of the wave functions of a particular state, to argue that if there is a dispersion in energy in the incoming state, and a dispersion in the outgoing state, the two sets of values must be restricted by this relation, for which the central values essentially cancel. Thus, one obtains

$$|\Delta E - \Delta E'| \sim \hbar / \Delta t. \tag{2.41}$$

They then use the relation (valid for both nonrelativistic and relativistic kinematics)

$$\Delta E = \frac{dE}{dP} \Delta P = v \Delta P; \tag{2.42}$$

using absolute conservation of momentum to assert that

$$\Delta P = \Delta P',$$

they then obtain

$$|(v - v')|\Delta P \sim \hbar/\Delta t. \tag{2.43}$$

This result implies a change in velocity from incoming to outgoing states. For a given ΔP, the smaller the time interval of measurement, the larger this velocity change must be. It is however, bounded by the velocity of light c, and one therefore obtains the relation (2.39).

Aharonov and Albert (1981) have understood this result in terms of causality. They argue that if a measurement is made in a short time Δt which restricts the particle to a range of momenta ΔP, the wave function must extend to $\Delta x \sim (\hbar/2\Delta P)$. The Landau-Peierls result then assures that $\Delta x \leq (c/2)\Delta t$. From the point of view of Aharonov and Albert, involving causality, as well as the use of a relativistic bound by Landau and Peierls, it is clear that the relation (2.39) should be associated with relativity.

Following the method used by Landau and Peierls for the relativistic Stueckelberg-Schrödinger equation (2.25), it would follow in the same way from first order perturbation theory that

$$|K - K'| \sim \hbar/\Delta\tau \tag{2.44}$$

Since $p^\mu p_\mu = -(\frac{E}{c})^2 - \mathbf{p}^2 = m^2c^2$, where m is the mass of the particle measured in the laboratory. The initial and final free Hamiltonians have the form

$$K = \frac{p^\mu p_\mu}{2Mc^2} = -\frac{m^2c^2}{2Mc^2} = -\frac{m^2}{2M}$$

and therefore the relation (2.44) becomes, for small Δm,

$$\begin{aligned}|\frac{m^2 - m'^2}{2M}| &\sim \frac{\hbar}{\Delta\tau} \\ &= \frac{|(m - m')|(m + m')}{2M} \cong |\Delta m|,\end{aligned}$$

for m close to its "mass shell" value M. We therefore find the relation (Burakovsky 1996)

$$\Delta m \Delta\tau \cong \hbar, \tag{2.45}$$

a mass-τ uncertainty relation. This result provides a justification for the for the generally assumed relation that the width of the mass dispersions of elementary particles as seen in decay modes is associated with the lifetime of the particle in its proper frame. If the particle is off shell due to additional interactions during the decay process, there would clearly be corrections.

As we have noted, such estimates are not rigorous, but carry the same semi-quantitative arguments used by Landau and Peierls, based on first order perturbation theory.

The $\Delta E \Delta t$ uncertainty relation in the SHP relativistic theory, on the other hand, follows rigorously from the commutation relation

$$[E, t] = i\hbar. \tag{2.46}$$

It is a general theorem in quantum mechanics that the dispersions of two self adjoint operators A and B in a given quantum state, defined by

$$\Delta A = \sqrt{< (A- < A >)^2 >}$$

and

$$\Delta B = \sqrt{< (B- < B >)^2 >}$$

are related by

$$\Delta A \Delta B \geq \frac{\hbar}{2} \mid < [A, B] > \mid .$$

It therefore follows from (2.46) that, as a rigorous property of the wave function representing the state of the system,

$$\Delta E \Delta t \geq \hbar/2. \tag{2.47}$$

In a similar way, it is possible to show that there is a simple and rigorous derivation of (2.39) in the framework of the manifestly covariant quantum theory we are working with here.

We have seen that the results of Newton and Wigner can be obtained in a straightforward way by defining an effective Newton-Wigner operator as in (2.38), with the semiclassical meaning of an extrapolation of the event position back to the value it would have at $t = 0$, interpreting the virtual velocity field contained in the wave function as associated (in expectation value) with an actual distribution that could be thought of as a collection of possible world lines. In the same way, we can construct an *effective time operator* by extrapolating the time of observation of an event back to the $x = 0$ axis, which one might think of as the location of a Geiger counter triggered by the passage of a world line through its position at $x = 0$. We therefore define a Landau-Peierls time operator as (Arshansky 1985)

$$t_{LP} = t - \frac{1}{2}\{\mathbf{x}; \frac{\mathbf{p}E}{p^2}\} \tag{2.48}$$

where $\frac{\mathbf{p}E}{p^2}$ is an *inverse velocity operator*, providing a shift in time for a virtual worldline (the semicolon implies both dot product as well as anticommutator). It then follows that

$$[t_{LP}, p] = -[\mathbf{x}, p] \cdot \frac{\mathbf{p}E}{p^2}.$$

But ($p \equiv \sqrt{\mathbf{p}^2}$)

$$[x_i, p] = i\hbar \frac{p_i}{p},$$

so that

$$[t_{LP}, p] = -i\hbar \frac{E}{p}. \tag{2.49}$$

It therefore follows from (2.49) that

$$\Delta t_{LP} \Delta p \geq \frac{1}{2} \hbar < E/p > . \tag{2.50}$$

The quantity E/p is the magnitude of the inverse velocity operator; if the virtual velocity p/E is bounded within the wave packet by the velocity of light c, we obtain the Landau-Peierls bound (2.39) as a rigorous property of the wave function describing the state of the system. There is, in principle, however, no bound on the occurrence of components of the wave function with values of p/E greater than one. On the other hand, application of the Ehrenfest theorem (Ehrenfest 1927), when it is valid, would rule out this possibility for the same causal reasons given by Landau and Peierls. The Ehrenfest theorem for the relativistic theory has the same structure as in the nonrelativistic theory, resulting in the classical Hamilton equations for the motion of the peak of the wave packet in spacetime. We review the argument in the following.

Consider a wave packet of the form (for free evolution)

$$\psi_\tau(x) = \frac{1}{(2\pi)^2} \int e^{i p^\mu x_\mu - i \frac{p^\mu p_\mu}{2M} \tau} \chi(p), \tag{2.51}$$

where $\chi(p)$, the momentum representation of the state, is a fairly sharp distribution in p^μ. The function $\chi(p)$ is modulus square normalized to one over integration on all four momenta if $\psi(x)$ is modulus square normalized to one over spacetime. For large τ, if one may assume that the values of x^μ also become large, the stationary phase values

$$x^\mu \sim \frac{p^\mu}{M} \tau \tag{2.52}$$

make the primary contribution, as in the nonrelativistic argument. The value of p^μ under the integral that contributes corresponds to the sharp peak value of the momentum space wave function, and the corresponding peak in the x^μ wave function describes the motion of a classical event, as described above in Eq. (2.8). In this case, a strong presence of spacelike momenta in the wavepacket could result in the evolution of the wordline in a spacelike direction, i.e., with $\frac{p}{E}$ exceeding light velocity. We could therefore, on the same causal grounds as Landau and Peierls, arguing that $< E/p >$ must be greater than $1/c$, rule out such a configuration, and arrive at the Landau-Peierls relation from (2.50).

However, as Zaslavsky (1985) has pointed out in the context of the nonrelativistic theory, the conditions for the validity of the Ehrenfest theorem degrade (in this case as a function of τ) due to the spreading of the wave packet as well as the effect of interactions on the structure of $\chi(p)$. Zaslavsky (1985) called the time for validity of the Ehrenfest theorem the "Ehrenfest time", and argued that for quantum systems for which the classical Hamiltonian induces chaotic behavior the Ehrenfest time is less. Therefore, dynamical effects may occur in the relativistic theory which could result in deviations from the Landau-Peierls bound. We shall discuss this subject further in Chap. 4.

In the classical construction of Stueckelberg (1941) in Fig. 2.1, the worldline of the particle passes through a region which is spacelike. In this region, the corresponding Landau-Peierls bound would be violated, with the contrary inequality

$$\Delta p \Delta t < \hbar/c, \tag{2.53}$$

implying that the wave function could be arbitrarily narrow in the t-direction for a given p distribution. Thus, this diagram could be described by a quantum wave packet which has normal Ehrenfest form for the incoming and outgoing lines, but may have a vertex which is very sharp in t over a small but finite distance. The spacetime diagrams discussed by Feynman (1949) may be thought of as an idealization of this limit. The example of neutrino oscillations and similar phenomena in the K and B meson systems, also providing an illustration of this effect, are discussed in Chap. 4.

The relation (2.48) was constructed from a semiclassical interpretation of the quantum observables, a procedure that was justified in our study of the Newton-Wigner problem. In that case, we began with the straightforward computation of the expectation value of the **x** operator, which has the same representation as in the nonrelativistic quantum mechanics. However, there is no corresponding analog in nonrelativistic quantum mechanics for a *time operator*; in the nonrelativistic quantum theory, t is a parameter of evolution, and its expectation value is a trivial identity (Ludwig 1982; Dirac 1930). We can, however, construct an argument analogous to that used for the Newton-Wigner problem within the framework of the relativistic theory, and show in the same way that the Landau-Peierls time operator (2.48) emerges from the mass-shell restriction of the expectation value of the relativistic time operator. To see this, consider the expectation value

$$< t >= \int d^4 p \psi^*(\mathbf{p}, E)(-i\frac{\partial}{\partial E})\psi(\mathbf{p}, E), \tag{2.54}$$

where we shall consider, for each value of m the magnitude of the momentum to be a function of E. Let us change the variables p^μ to the form (Ω, p, E), where Ω corresponds to the angular coordinate variables of $\mathbf{p}$, and define

$$p = \sqrt{E^2 - m^2} \tag{2.55}$$

Then,

$$d^4 p = p^2 d\Omega dp dE = -\frac{1}{2} p d\Omega dE dm^2. \tag{2.56}$$

We may then write

$$\begin{aligned} < t > = -\frac{1}{2} \int & p d\Omega dE dm^2 \psi^*(\sqrt{E^2 - m^2}, \Omega, E) \\ & \left[-i\frac{\partial}{\partial E}\psi(\sqrt{E^2 - m^2}, \Omega, E) + i\frac{E}{p}\frac{\partial}{\partial p}\psi(p, \Omega, E)|_{p=\sqrt{E^2-m^2}} \right], \end{aligned} \tag{2.57}$$

where the last term (containing the factor $(\partial p/\partial E = E/p)$ compensates for the fact that after the change of variables, $i\partial/\partial E$ acts on p as well as the last argument.

We now note that the Landau-Peierls operator (2.48) can be written as

$$t_{LP} = t - \frac{1}{2}\left[i\frac{\partial}{\partial \mathbf{p}} \cdot \frac{\mathbf{p}E}{p^2} + i\frac{\mathbf{p}E}{p^2}\frac{\partial}{\partial \mathbf{p}} \right] = -i\frac{\partial}{\partial E} - \frac{i}{2}\frac{E}{p^2} - i\frac{\mathbf{p}E}{p^2} \cdot \frac{\partial}{\partial \mathbf{p}}, \tag{2.58}$$

where we have used the fact that (most simply, carrying this out component by component)

$$\frac{\partial}{\partial \mathbf{p}} \cdot \frac{\mathbf{p}E}{p^2} = \frac{E}{p^2}.$$

If we take the expectation value of t_{LP} in place of t as in (2.53), one sees that the last term in (2.56) cancels with the last term in (2.58), resulting in

$$< t_{LP} >= -\frac{1}{2}\int p d\Omega dE dm^2 \psi^*\left(\sqrt{E^2 - m^2}, \Omega, E\right)\left[-i\frac{\partial}{\partial E} - \frac{i}{2}\frac{E}{p^2}\right]\psi\left(\sqrt{E^2 - m^2}, \Omega, E\right) \tag{2.59}$$

We now follow an argument similar to that used above for the Newton-Wigner problem to find the wave function of an event which occurs at a definite sharp *time*.

If $\psi_{t=0}(p)$ corresponds to a state for which an event is strictly localized to a point in time $t = 0$, the wave function $\psi_{t=t_0}$ must be orthogonal to it for $t_0 \neq 0$. Therefore,

$$\int d^4 p \psi^*_{t=t_0}(p)\psi_{t=0}(p) = 0 \tag{2.60}$$

for $t_0 \neq 0$. However, using the Poincaré group property $\psi_{t=t_0}(p) = e^{iEt_0}\psi_{t=0}$, we have

$$\int d^4 p e^{-iEt_0}|\psi_{t=0}(p)|^2 = 0, \tag{2.61}$$

implying that

$$\int d^3 p |\psi_{t=0}(p)|^2 = const \times (E),$$

or,

$$\int d\Omega p^2 dp |\psi_{t=0}(p)|^2 = const \times E \tag{2.62}$$

But, as pointed out above, $p^2 dp = -(1/2) p dm^2$, so that (2.62) becomes

$$-\frac{1}{2}\int p dm^2 d\Omega |\psi_{t=0}(p)|^2 = const \times E. \tag{2.63}$$

If the mass of the particle is concentrated at some value of m we conclude that

$$\int d\Omega |\psi_{t=0}(p)|^2 = \frac{1}{p} \times const, \tag{2.64}$$

or, for a spherically symmetric wave function,

$$\psi_{t=0}(p) \propto \frac{1}{\sqrt{p}}.$$

Shifting by translation in t, we see that

$$\psi_t(p) \propto (E^2 - m^2)^{-\frac{1}{4}} e^{iEt}. \tag{2.65}$$

This result corresponds to the necessary form of a wave function at some given value of m and concentrated at some value of t, the analog of the Newton-Wigner wave function for a particle concentrated at a given point $\mathbf{x}$. A simple computation shows that

$$-i\left(\frac{\partial}{\partial E} - \frac{iE}{2p^2}\right)\psi_t(p) = t\psi_t(p). \tag{2.66}$$

Thus, the operator that appears in the expectation value in (2.59) at each value of m in the foliation induced by the change of variables (2.56) corresponds to the analog of the Newton-Wigner position operator (1.9) for time, restricted to a given mass value.

Clearly, the Fourier transform of the function $\psi_{t_0}(p)$ of (2.65) (picking the localization point to be $t = t_0$) into the time domain by the kernel $\exp -iEt$ would not be localized in t, as for the Newton-Wigner problem in $\mathbf{x}$, and would therefore not form a viable quantum theory if, as we have assumed, the mass is concentrated at a fixed point. One could not use such wave functions to compute interference phenomena in time, as we shall discuss in Chap. 6.

We remark that, as for $\mathbf{x}_{NW}$, the Landau-Peierls operator t_{LP} is a constant of the free motion (as can be easily verified by computing their commutator with the free Hamiltonian). The (mean) intercepts of the virtual motions contained in the wave function, respectively to $t = 0$ and to $x = 0$ do not change under the free motion.

In the next chapter, we describe the basis for the construction of quantum states of particles with spin.

Appendix A

We describe here the basic ideas of the so-called constraint theory formulation of a many particle (many *event*) relativistic mechanics. In this theory, describing the positions $\{x_i^\mu\}$, and momenta $\{p_i^\mu\}$ for $i = 1, 2, 3, \ldots, N$ of the particles, a constraint is defined for each of the particles of the form (we use the metric $(-, +, +, +)$)

$$K_i = p^\mu{}_i p_{\mu_i} + m_i^2 + \phi_i(x, p), \tag{2.67}$$

where, on the constraint hypersurface $K_i \approx 0$, the $\phi_i(x, p)$ are functions of all the x's and p's, and the $\{m_i\}$ are the given masses of the particles. This set of N constraints restricts the motion to an N dimensional hypersurface in the $8N$ dimensional phase space.

The "first class" constraints K_i may act as generators of motion under Poisson bracket action (e.g. Itzyson 1980), thus defining the infinitesimal variations with respect to the corresponding parameters τ_i of the infinitesimal transformations of the coordinates and momenta by

$$\begin{aligned} \frac{dx_i}{d\tau_i} &= i\{K_i, x_i\}_{PB} \\ \frac{dp_i}{d\tau_i} &= i\{K_i, p_i\}_{PB}, \end{aligned} \tag{2.68}$$

providing a set of first order equations describing the motion on this hypersurface. This manifestly covariant formalism has the advantage that one may assume the interaction terms ϕ_i vanish asymptotically when the particles are far apart; the constraint conditions then enforce the particles to lie on mass shell ($p^{\mu}{}_i p_{\mu_i} + m_i^2 = 0$).

In order to construct a world line for the system on the range of these motions, one generally introduces another set of $N-1$ constraints, called *second class constraints*, forming surfaces with intersection along a line on the N dimensional hypersurface, and an Nth constraint which cuts this line and is a function of a single parameter τ, thus describing motion along this world line (Sudarshan 1981a). It is possible, however, to define these constraints in another way, by constructing a Hamiltonian of the form (Rohrlich 1981)

$$K = \Sigma_i \omega_i(x, p) K_i. \tag{2.69}$$

The Poisson bracket of this Hamiltonian with any observable $\mathcal{O}(x, p)$ then forms a linear combination

$$\frac{d\mathcal{O}}{d\tau} = \Sigma_i \omega_i \frac{d\mathcal{O}}{d\tau_i}, \tag{2.70}$$

where we have taken into account that the K_i vanish on the constraint hypersurface; the ω_i are then identified with $d\tau_i/d\tau$, with the τ_i considered as functions of the overall evolution parameter τ.

Although this approach is very elegant on a classical level, there are some difficulties in passing to the quantum theory. The condition $K_i = 0$ poses a difficult problem since, in general, the K_i have continuous spectrum, and the eigenstates would lie outside the Hilbert space. This problem can be treated by defining N Schrödinger type equations of the form (as for the treatment of cases with states in the continuous spectrum in the nonrelativistic theory)

$$i\frac{\partial \psi_{\tau_1,\tau_2,\ldots}}{\partial \tau_i} = K_i \psi_{\tau_1,\tau_2,\ldots} \tag{2.71}$$

but the combination $\Sigma_i \omega_i(x, p) K_i$ would, in general, not be Hermitian. The symmetric product with the ω_i's would not be useful, since the functions ω_i have no well-defined action on $\psi_{\tau_1,\tau_2,\ldots}$. Nevertheless, Rohrlich and the author succeeded in formulating a viable scattering theory in this framework (see references under Llosa 1982).

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