Chapter 2
An Introduction to General Nonstandard Analysis

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2.1 Superstructures

In this chapter, we develop the general framework of nonstandard analysis and the necessary logic for the transfer principle. We will begin each section with a brief summary for readers who want to postpone the technical details until a later reading. The summary will note any important definitions and results of the section that the reader should know before going on. For example, Definition 2.1.1 describing a superstructure and Remark 2.1.3 are important in this section. A reader who wants quickly to get to later applications may skip Sects. 2.5, 2.7, and 2.9.

The reader who has read the first chapter of this book will appreciate that Skolem functions will no longer be needed to replace the existential quantifier. The results obtained in the last chapter using our simple transfer principle will still be valid, since the transfer principle used here extends that simple one. The outline of this chapter is similar to that of Chap. 2 of the author's book with Albert E. Hurd, [4].

To work with general mathematical analysis, we need to consider sets, sets of sets, etc. All of these are constructed starting with a set of individuals. We think of an individual as an object different from a set. In particular, an individual contains no elements. We build our universe from the set X of individuals using the power set operation $\mathcal{P}$. The set X will always contain the natural numbers $\mathbb{N}$; usually it will contain $\mathbb{R}$.

Definition 2.1.1 Fix a set X containing $\mathbb{N}$. Let $V_0(X) = X$, and for each $n \in \mathbb{N}$, let $V_n(X) = V_{n-1}(X) \cup \mathcal{P}(V_{n-1}(X))$. The **superstructure** over X is the set $V(X) = \cup_{n=0}^{\infty} V_n(X)$. Entities in X are said to be of **rank** 0, and for $n \geq 1$, entities in $V_n(X) \setminus V_{n-1}(X)$ are said to be of **rank** n.

Example 2.1.2 Individuals are of rank 0. The number 7 and the set $\{7\}$ are in $V_1(X)$. The number 7, the set $\{7\}$, and the set of all finite subsets of $\mathbb{N}$ are in $V_2(X)$. Note that $V_1(X) \in V_2(X)$ and $V_1(X) \subset V_2(X)$.

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Remark 2.1.3 In this chapter's appendix, written by Horst Osswald, it will be shown that members of the set X of individuals can be coded so that they contain no elements of $V(X)$. That is if $b \in X$, there is no a with $a \in b$. For example, the equivalence class of Cauchy sequences of rational numbers with limit 7 is in $V(X)$. It is not, however, the same as the object in X we will call 7. Given a superstructure $V(X)$ and an entity b in $V(X)$ we will assume that only entities a in $V(X)$ can satisfy the relation $a \in b$. If we speak of an element a, with $a \in b$, a will automatically be in $V(X)$.

Definition 2.1.4 An ordered pair $\langle a, b\rangle$ is the set $\{\{a\}, \{a, b\}\}$.

Example 2.1.5 The ordered pair $\langle 3, \pi\rangle$ is the set $\{\{3\}, \{3, \pi\}\}$. The ordered pair $\langle 3, 3\rangle$ is the set $\{\{3\}\}$.

Definition 2.1.6 For $n \geq 1$, an ordered n-tuple $\langle x_1, \dots, x_n\rangle$ of entities $x_1, \dots, x_n$ is the set of ordered pairs $\{\langle 1, x_1\rangle, \dots, \langle n, x_n\rangle\}$. If $c_1, \dots, c_n$ are sets, we have $c_1 \times \cdots \times c_n = \{\langle x_1, \dots, x_n\rangle : x_i \in c_i \text{ for } 1 \leq i \leq n\}$. Moreover, $c^n = c \times \cdots \times c$ (n factors). For $n \geq 2$, an n-ary relation P on $c_1 \times \cdots \times c_n$ is a subset of $c_1 \times \cdots \times c_n$. We identify a 1-tuple $\langle x_1\rangle$with the corresponding element x_1, so a 1-ary relation on c is a subset of c.

Given a relation P as above, there will be a k such that each c_i is in $V_k(X)$. The relation P itself will be in $V_{k+4}(X)$. Functions are relations with the usual restriction. Suppose f is a function of n-variables, i.e., the domain consists of n-tuples $\langle x_i, \dots, x_n\rangle$ from $c_1 \times \cdots \times c_n$, with each variable in the domain and the range taking values in $V_k(X)$. Then f is an element of $V_{k+7}(X)$ since a typical element of f is a two tuple of the form $\langle\langle x_1, \dots, x_n\rangle, x_{n+1}\rangle$.

Lemma 2.1.7 *The n-tuple $\langle x_1, \dots, x_n\rangle = \langle y_1, \dots, y_n\rangle$ as sets if and only if $x_i = y_i$ for $i = 1, \dots, n$.*

Proof The proof is left to the reader (The only problem is if $x_i = i$). □

2.2 Language for Superstructures

In this section we describe the construction of formal statements in a formal language $\mathcal{L}_X$ about a superstructure $V(X)$. Given X, the language $\mathcal{L}_X$ for the superstructure $V(X)$ over X has the following symbols:

Connectives: $\neg, \wedge, \vee, \rightarrow, \longleftrightarrow$
Quantifiers: $\forall, \exists$
Parentheses: [,], (,), $<$, $>$
Constant Symbols: At least one name for each entity in $V(X)$.
Variable Symbols: A countable number of them will do.
Equality Symbol: Denotes equality for elements of X and set equality otherwise.
Set membership: $\in$.

We will not have terms in our language.

Definition 2.2.1 A **formula** of $\mathcal{L}_X$ is built up inductively with the following rules:

(a) If $x_1, \ldots, x_n$, x, and y are either constants or variables, then the following are formulas called atomic formulas: $x \in y$, $x = y$; $\langle x_1, \ldots, x_n \rangle \in y$; $\langle x_1, \ldots, x_n \rangle = y$; $\langle \langle x_1, \ldots, x_n \rangle, x \rangle \in y$; $\langle \langle x_1, \ldots, x_n \rangle, x \rangle = y$.
(b) If Φ and Θ are formulas, so are $\neg\Phi$, $\Phi \wedge \Theta$, $\Phi \vee \Theta$, $\Phi \rightarrow \Theta$, and $\Phi \longleftrightarrow \Theta$.
(c) If x is a variable symbol and y is either a variable symbol or a constant symbol and Φ is a formula that does not already contain a formula of the form $(\forall x \in z)\Theta$ or $(\exists x \in z)\Theta$, or $(\forall y \in z)\Theta$ or $(\exists y \in z)\Theta$, then $(\forall x \in y)\Phi$ and $(\exists x \in y)\Phi$ are formulas.

Definition 2.2.2 A variable x is **bound** in a formula Φ if it occurs in Φ and every occurrence takes the form $(\forall x \in z)\Theta$ or $(\exists x \in z)\Theta$. Here, z may be a constant or a variable. A variable occurring in a formula Φ but not bound in Φ is called a **free** variable in Φ. A **sentence** in $\mathcal{L}_X$ is a formula in which all variables are bound.

2.3 Interpretation of the Language for Superstructures

In this section we give the rules for interpreting the formal language $\mathcal{L}_X$. The rules are as follows:

(a) The atomic sentences $a \in b$, $\langle a_1, \ldots, a_n \rangle \in b$, $\langle \langle a_1, \ldots, a_n \rangle, c \rangle \in b$ are true or hold in $V(X)$ if the entities corresponding to the names a, $\langle a_1, \ldots, a_n \rangle$, or, respectively, $\langle \langle a_1, \ldots, a_n \rangle, c \rangle$ belong to the object named by b. The atomic sentences $a = b$, $\langle a_1, \ldots, a_n \rangle = b$, $\langle \langle a_1, \ldots, a_n \rangle, c \rangle = b$ are true or hold in $V(X)$ if the entities corresponding to the names a, $\langle a_1, \ldots, a_n \rangle$, or, respectively, $\langle \langle a_1, \ldots, a_n \rangle, c \rangle$ are identical to the object named by b.
(b) If Φ and Θ are sentences, then

    (i) $\neg\Phi$ is true in $V(X)$ if Φ is not true (does not hold) in $V(X)$;
    (ii) $\Phi \wedge \Theta$ is true in $V(X)$ if both Φ and Θ are true in $V(X)$;
    (iii) $\Phi \vee \Theta$ is true in $V(X)$ if either Φ or Θ is true in $V(X)$;
    (iv) $\Phi \rightarrow \Theta$ is true if either Φ is not true or Θ is true in $V(X)$;
    (v) $\Phi \longleftrightarrow \Theta$ is true if Φ and Θ are either both true or both not true in $V(X)$.

(c) Let $\Phi = \Phi(x)$ be a formula in which x either does not occur and all variables are bound or x is the only free variable. Given a constant a, we will write $\Phi(a)$ for Φ with all occurrences of x replaced by a. Let b be a constant naming a set $\beta \in V(X)$.

    (i) $(\forall x \in b)\Phi$ is true in $V(X)$ if for all entities $\alpha \in \beta$, $\Phi(a)$ is true in $V(X)$, where a is any constant naming α.
    (ii) $(\exists x \in b)\Phi$ is true in $V(X)$ if there is an entity $\alpha \in \beta$ such that $\Phi(a)$ is true in $V(X)$, where a is any constant naming α.

Remark 2.3.1 Although we do not have terms in our formal language, we will use $x_{n+1} = f(x_1, \ldots, x_n)$ as shorthand for $\langle\langle x_1, \ldots, x_n\rangle, x_{n+1}\rangle \in f$ and $y = f(x)$ as shorthand for $\langle x, y\rangle \in f$.

Example 2.3.2 The sentence that says that every nonzero real number has a multiplicative inverse has the following form using R to denote the set of real numbers and P to denote the product function:

$$(\forall x \in R)[\neg(x = 0) \to (\exists y \in R)[\langle\langle x, y\rangle, 1\rangle \in P]].$$

With our shorthand, this sentence has the form

$$(\forall x \in R)[\neg(x = 0) \to (\exists y \in R)[P(x, y) = 1]].$$

Using S to denote the sum function, the distributive law has the form

$$\begin{aligned}(\forall x \in R)(\forall y \in R)(\forall z \in R)(\forall \alpha \in R)(\forall \beta \in R)(\forall \gamma \in R)(\forall \delta \in R)\\ [[[S(y, z) = \alpha] \wedge [P(x, \alpha) = \beta] \wedge [P(x, y) = \gamma] \wedge [P(x, z) = \delta]]\\ \to [S(\gamma, \delta) = \beta]].\end{aligned}$$

Remark 2.3.3 Note that we are missing the composition of terms. We will usually use ordinary mathematical sentences employing terms. One should keep in mind, however, that these are shorthand for more complicated parts of sentences of $\mathcal{L}_X$.

Example 2.3.4 To say that a function f defined on $\mathbb{R}$ is continuous at a, let R^+ be the symbol used to denote the strictly positive real numbers. Let ρ denote the distance function, i.e., $\rho(x, y) = |x - y|$, and let I denote strict inequality, i.e., $\langle x, y\rangle \in I$ iff $x < y$. With $f(a) = b$, the sentence

$$\begin{aligned}(\forall \varepsilon \in R^+)(\exists \delta \in R^+)(\forall x \in R)(\forall \alpha \in R)(\forall \beta \in R)(\forall \gamma \in R)\\ [[\rho(x, a) = \alpha \wedge \langle\alpha, \delta\rangle \in I \wedge f(x) = \beta \wedge f(a) = b \wedge \rho(\beta, b) = \gamma]\\ \to [\langle\gamma, \varepsilon\rangle \in I]].\end{aligned}$$

is abbreviated with the sentence

$$(\forall \varepsilon \in R^+)(\exists \delta \in R^+)(\forall x \in R)[|x - a| < \delta \to |f(x) - f(a)| < \varepsilon].$$

Problem: Write sentences in the language for the real numbers expressing the commutative and associative laws for addition.

Answer: Let S denote the sum function. The commutative law for addition is expressed by

$$\begin{aligned}&(\forall x \in R)(\forall y \in R)(\forall \alpha \in R)(\forall \beta \in R)[[[S(x, y) = \alpha] \wedge [S(y, x) = \beta]]\\ &\quad \to [\alpha = \beta]].\end{aligned}$$

The associative law for addition is expressed by

$$(\forall x \in R)(\forall y \in R)(\forall z \in R)(\forall \alpha \in R)(\forall \beta \in R)(\forall \gamma \in R)(\forall \delta \in R)$$
$$[[[S(y, z) = \alpha] \wedge [S(x, \alpha) = \beta] \wedge [S(x, y) = \gamma] \wedge [S(\gamma, z) = \delta]] \rightarrow [\beta = \delta]].$$

Problem: Write a sentence in the language for the real numbers saying that for a given function f, $\lim_{x \to a \in A} f(x) = L$.

Answer: We use R^+ to denote the strictly positive real numbers, ρ to denote the distance function, i.e., $\rho(x, y) = |x - y|$, and I to denote strict inequality, i.e., $\langle x, y \rangle \in I$ iff $x < y$. The sentence has the form

$$(\forall \varepsilon \in R^+)(\exists \delta \in R^+)(\forall x \in A)(\forall \alpha \in R)(\forall \beta \in R)(\forall \gamma \in R)$$
$$[[\neg[x = a] \wedge \rho(x, a) = \alpha \wedge \langle \alpha, \delta \rangle \in I \wedge f(x) = \beta \wedge \rho(\beta, L) = \gamma]$$
$$\rightarrow [\langle \gamma, \varepsilon \rangle \in I]].$$

2.4 Monomorphisms and the Transfer Principle

Just as in Chap. 1, where we worked with $\mathbb{R}$ and ${}^*\mathbb{R}$, we now will work back and forth between X and its nonstandard extension ${}^*X$. We will in fact work with the superstructures $V(X)$ and $V({}^*X)$ using a "$*$-mapping ", which is a one-to-one mapping from $V(X)$ into, but not onto, $V({}^*X)$. The abstraction of the basic properties of the mapping $*$ originates with the work of Robinson and Zakon in [10]. For the moment, we write Y for ${}^*X$.

We assume X and Y are two sets of individuals. We associate the superstructure $V(X)$ and language $\mathcal{L}_X$ with X and the superstructure $V(Y)$ and language $\mathcal{L}_Y$ with Y. Each constant symbol in $\mathcal{L}_X$ names something in $V(X)$, and there is at least one such symbol for each entity in $V(X)$. A similar statement is true for Y. We work with a one-to-one map $*$ from $V(X)$ into $V(Y)$. For each $a \in V(X)$, we write ${}^*a$ for $*(a)$. In this chapter, we will use the same symbol for ${}^*a$ and for its name. The main results and definitions in this section are Definitions 2.4.1 and 2.4.3 along with Remark 2.4.4, Theorem 2.4.5, and Example 2.4.8.

Definition 2.4.1 If Φ is a formula in $\mathcal{L}_X$, the **$*$-transform** of Φ, ${}^*\Phi$, is the formula in $\mathcal{L}_Y$ obtained by replacing each constant c in Φ with ${}^*c$.

Example 2.4.2 The $*$-transform of the sentence that says there is a multiplicative inverse in the real numbers is

$$(\forall x \in {}^*R)[\neg(x = {}^*0) \rightarrow (\exists y \in {}^*R)[\langle \langle x, y \rangle , {}^*1 \rangle \in {}^*P]].$$

Definition 2.4.3 (*Robinson–Zakon* [10]) The injection $*$ from $V(X)$ into $V(Y)$ is called a **monomorphism** if the following conditions hold.

(i) $^*(\emptyset) = \emptyset$, *where* $\emptyset$ *denotes the empty set.*
(ii) *If* $a \in X$, *then* $^*a \in Y$.
(iii) *If* a *has rank* n, $^*a$ *has rank* n.
(iv) *If* $a \in {}^*V_n(X)$ *for* $n \geq 1$ *and* $b \in a$, *then* $b \in {}^*V_{n-1}(X)$.
(v) *(Transfer Principle) For each sentence* Φ *in* L_X, *if* Φ *holds in* $V(X)$ *then* $^*\Phi$ *holds in* $V(Y)$.

Remark 2.4.4 It can be shown (the reader is invited to construct the proof) that Properties (i)–(iv) follow from the Transfer Principle, i.e., Property (v). We have listed them here as separate properties to help in understanding the notion of a monomorphism. We use them in the next section to help with the particular monomorphism constructed there. Property (iv) will be interpreted to say, among other things, that elements of "internal sets" are internal. Given Property (ii), we will assume from now on that $^*a = a$ for each individual $a \in X$; in particular, for $n \in \mathbb{N}$, $^*n = n$. That is, we will assume that $X \subseteq Y$. We will write a for both $a \in X$ and $^*a \in Y$. The Transfer Principle has the following consequence.

Theorem 2.4.5 (Downward Transfer Principle) *For each sentence* Φ *in* $\mathcal{L}_X$, *if* $^*\Phi$ *holds in* $V(Y)$ *then* Φ *holds in* $V(X)$.

Proof If $\neg\Phi$ holds in $V(X)$, then $^*(\neg\Phi) = \neg(^*\Phi)$ holds in $V(Y)$. □

We sometimes state the transfer principle as follows: Φ holds in $V(X)$ if and only if $^*\Phi$ holds in $V(Y)$. Recall that the domain of a binary relation P is the set of all x for which there is a y with $\langle x, y\rangle \in P$. The range of P is the set of all y for which there is an x with $\langle x, y\rangle \in P$.

Proposition 2.4.6 *We have the following properties of the* $*$*-mapping.*

(a) Let $a, b, a_1, \ldots, a_n$ *be fixed entities in* $V(X)$. *Then*

(i) $^*\{a_1, \ldots, a_n\} = \{^*a_1, \ldots, {}^*a_n\}$;
(ii) $^*\langle a_1, \ldots, a_n\rangle = \langle ^*a_1, \ldots, {}^*a_n\rangle$;
(iii) $a \in b$ *iff* $^*a \in {}^*b$;
(iv) $a = b$ *iff* $^*a = {}^*b$;
(v) $a \subseteq b$ *iff* $^*a \subseteq {}^*b$;
(vi) For $n \in \mathbb{N}$, $^*(\cup_{i=1}^n a_i) = \cup_{i=1}^n {}^*a_i$, $^*(\cap_{i=1}^n a_i) = \cap_{i=1}^n {}^*a_i$;
(vii) $^*(a_1 \times a_2 \times \cdots \times a_n) = {}^*a_1 \times {}^*a_2 \times \cdots \times {}^*a_n$.

(b) If P *is a relation on* $a_1 \times a_2 \times \cdots \times a_n$, *then* $^*P$ *is a relation on* $^*a_1 \times {}^*a_2 \times \cdots \times {}^*a_n$. *Moreover, if* $n = 2$, *and* a *and* b *are the domain and range of* P, *then* $^*a$ *and* $^*b$ *are the domain and range of* $^*P$.
(c) If f *is a function from* a *into* b, *then* $^*f$ *is a function from* $^*a$ *into* $^*b$, *such that* $\forall c \in a$, $^*[f(c)] = {}^*f(^*c)$. *If* f *maps* a *onto* b, $^*f$ *maps* $^*a$ *onto* $^*b$. *If* f *is one-to-one (i.e., injective), so is* $^*f$.

Proof of Part a:
(i) Let $s = \{a_1, \ldots, a_n\}$, and transform the sentence $(\forall x \in s)[x = a_1 \vee \cdots \vee x = a_n]$. Also transform sentences $a_1 \in s, \ldots, a_n \in s$.
(ii)

$$\begin{aligned} {}^*\langle a_1, \ldots, a_n\rangle &= {}^*\{\{\{1\}, \{1, a_1\}\}, \ldots, \{\{n\}, \{n, a_n\}\} \\ &= \{{}^*\{\{1\}, \{1, a_1\}\}, \ldots, {}^*\{\{n\}, \{n, a_n\}\} \\ &= \{\{{}^*\{1\}, {}^*\{1, a_1\}\}, \ldots, \{{}^*\{n\}, {}^*\{n, a_n\}\} \\ &= \{\{\{1\}, \{1, {}^*a_1\}\}, \ldots, \{\{n\}, \{n, {}^*a_n\}\}. \end{aligned}$$

(iii) and (iv) are clear.
(v) Transform $(\forall x \in a)[x \in b]$.
(vi) Left to the reader.
(vii) We show ${}^*(a \times b) = {}^*a \times {}^*b$. Transform

$$(\forall z \in (a \times b))(\exists x \in a)(\exists y \in b)[\langle x, y\rangle = z]$$

to show that ${}^*(a \times b) \subseteq {}^*a \times {}^*b$. Transform

$$(\forall x \in a)(\forall y \in b)(\exists z \in (a \times b))[\langle x, y\rangle = z]$$

to show ${}^*a \times {}^*b \subseteq {}^*(a \times b)$.

Proof of Part b: To show ${}^*P$ is a relation on ${}^*a_1 \times {}^*a_2 \times \cdots \times {}^*a_n$, transform the sentence

$$(\forall x \in P)(\exists x_1 \in a_1) \ldots (\exists x_n \in a_n)[\langle x_1, \ldots, x_n\rangle = x].$$

Thus we know for $n = 2$ and a and b the domain and range of P, that ${}^*a$ contains the domain of ${}^*P$ and ${}^*b$ contains the range of ${}^*P$. To show that ${}^*a$ and ${}^*b$ are the domain and range of ${}^*P$, transform

$$(\forall x \in a)(\exists y \in b)[\langle x, y\rangle \in P] \quad \text{and} \quad (\forall y \in b)(\exists x \in a)[\langle x, y\rangle \in P].$$

Proof of Part c: Let f be a function from a into b. By the transform of

$$(\forall x \in a)(\forall y \in b)(\forall z \in b)[[\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f] \rightarrow [y = z]]$$

and Part b, ${}^*f$ is a function from ${}^*a$ into ${}^*b$. If $\langle c, d\rangle \in f$, then $\langle {}^*c, {}^*d\rangle \in {}^*f$, whence $\forall c \in a$, ${}^*[f(c)] = {}^*f({}^*c)$. The rest is left to the reader. □

When $a \in X$, we will, as already noted, associate a with ${}^*a$. In general however, as the next example shows, we will have to be more careful. For this and the next example, we assume $*$ is a monomorphism between $V(\mathbb{R})$ and $V({}^*\mathbb{R})$.

Example 2.4.7 Let $\mathcal{I}$ denote the set of closed and bounded intervals in $\mathbb{R}$. Then $\mathcal{I} \in V_2(\mathbb{R})$. Moreover, the following sentences are true for $V(\mathbb{R})$:

$$(\forall x \in \mathcal{I})(\exists a, b \in \mathbb{R})(\forall y \in \mathbb{R})[a \le y \le b \longleftrightarrow y \in x]$$

$$(\forall a, b \in \mathbb{R})[a \le b \to [(\exists x \in \mathcal{I})(\forall y \in \mathbb{R})[a \le y \le b \longleftrightarrow y \in x]]$$

Therefore, ${}^*\mathcal{I}$ contains the extensions of standard intervals as well as new ones. For example, if $\varepsilon \simeq 0$ is positive, then $[\varepsilon, 2\varepsilon]$ is in ${}^*\mathcal{I}$ but it is not the extension of any standard interval. Even for a standard interval $[a, b]$, with $a \neq b$, there are points in ${}^*[a, b]$ not in the original interval $[a, b]$. Thus we can not directly associate $[a, b]$ and ${}^*[a, b]$.

Example 2.4.8 The Downward Transfer Principle can often be used to avoid arguments by contradiction. For example, to show that $s_n \to L$ if $\forall \eta \in {}^*\mathbb{N}_\infty$, ${}^*s_\eta \simeq L$, we note that for a given $\varepsilon > 0$, the $*$-transform of the sentence

$$\Phi = (\exists k \in \mathbb{N})(\forall n \in \mathbb{N})[n \ge k \to |s_n - L| < \varepsilon]$$

is true in $V({}^*\mathbb{R})$; just let $k \in {}^*\mathbb{N}_\infty$. It follows that Φ holds in $V(\mathbb{R})$.

Problem: Assume $*$ is a monomorphism between $V(\mathbb{R})$ and $V({}^*\mathbb{R})$, and use downward transfer to show that f is uniformly continuous on A if $\forall x \in {}^*A$, $\forall y \in {}^*A$ with $x \simeq y$, we have ${}^*f(x) \simeq {}^*f(y)$.

Answer: Fix $\varepsilon > 0$. We want to show the truth of the sentence

$$\Phi = (\exists \delta \in \mathbb{R}^+)(\forall x \in A)(\forall y \in A)[|x - y| < \delta \to |f(x) - f(y)| < \varepsilon].$$

Now ${}^*\Phi$ is true for $V({}^*\mathbb{R})$; just take $\delta \simeq 0$. Therefore, Φ is true for $V(\mathbb{R})$.

2.5 Ultrapower Construction of Superstructures and Monomorphisms

We now fix $V(X)$, an index set I, and an ultrafilter $\mathcal{U}$ in I. From these we will construct a new superstructure $V({}^*X)$ from $V(X)$ and a corresponding monomorphism. We may have $I = \mathbb{N}$, we may even have $\mathcal{U}$ fixed at some $i_0 \in I$; i.e., $U \in \mathcal{U}$ if and only if $i_0 \in U$. If $\mathcal{U}$ is fixed in this way, we will see that we get nothing new. In the next section we will see that by assuming additional properties for I and $\mathcal{U}$ we obtain a monomorphism with the desired properties for a nonstandard extension.

Because an ultrafilter corresponds to a finitely additive measure on the power set of I, with that measure taking only the values 0 and 1, we say a property holds almost everywhere or a.e. if it holds on some $U \in \mathcal{U}$.

Given $S \in V(X)$, we write ΠS for the set of all maps from I into S. For each such map a, we write a_i for $a(i)$. Two maps a and b are equivalent (with respect to $\mathcal{U}$) and we write $a =_{\mathcal{U}} b$ if $a_i = b_i$ a.e. (that is, $a_i = b_i$ for all i in some $U \in \mathcal{U}$.) The relation $=_{\mathcal{U}}$ is an equivalence relation. We write $\Pi_{\mathcal{U}} S$ for the set of equivalence classes, and $[a]$ for the equivalence class containing the mapping a. If $b \in V(X)$, we write $\overline{b}$ for the constant function $b_i = b$. Let $V_{-1}(X)$ denote the empty set. The reader should at least note the following two definitions and related remark.

Definition 2.5.1 The **bounded ultrapower** $\Pi^0_{\mathcal{U}} V(X)$ of $V(X)$ is defined by setting

$$\Pi^0_{\mathcal{U}} V(X) := \cup_{n=0}^{\infty} \Pi_{\mathcal{U}}[V_n(X) \setminus V_{n-1}(X)].$$

We denote by e the mapping from $V(X)$ into $\Pi^0_{\mathcal{U}} V(X)$ defined at each $b \in V(X)$ by $e(b) = [\,\overline{b}\,]$. When $[a], [b] \in \Pi^0_{\mathcal{U}} V(X)$, we write $[a] \in_{\mathcal{U}} [b]$ if $a_i \in b_i$ a.e.

Remark 2.5.2 Recall that $\overline{5}$ is the constant function 5 on I, and $[\,\overline{5}\,]$ is the corresponding equivalence class. While it is true that $[\,\overline{5}\,] \in_{\mathcal{U}} e(\mathbb{N})$, the relation $\in_{\mathcal{U}}$ is not the set membership relation we want. We need another map "M" so that we can replace $e(\mathbb{N})$ in $\Pi^0_{\mathcal{U}} V(X)$ with the set of numbers

$$M(e(\mathbb{N})) = \{[r] \in \Pi_{\mathcal{U}} X : r_i \in \mathbb{N} \quad \text{a.e.}\} = \{[r] \in \Pi_{\mathcal{U}} X : [r] \in_{\mathcal{U}} [\,\overline{\mathbb{N}}\,]\}.$$

Similarly, $e(\mathcal{P}(\mathbb{N})) = [\,\overline{\mathcal{P}(\mathbb{N})}\,]$ is the equivalence class formed from the sequence $\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{N}), \ldots, \mathcal{P}(\mathbb{N}), \ldots$. This, however, is not a collection of sets of numbers. While it is true that $[\,\overline{\mathbb{N}}\,] \in_{\mathcal{U}} [\,\overline{\mathcal{P}(\mathbb{N})}\,]$, this is not the set membership we want. Once we have used the map M at the level 1 to get true sets of numbers, we will then use M to replace $[\,\overline{\mathcal{P}(\mathbb{N})}\,]$ with a true collection of sets of numbers, $M(e(\mathcal{P}(\mathbb{N})))$. In particular, the set ${}^*\mathbb{N} = M(e(\mathbb{N}))$, which we will think of as the set of nonstandard natural numbers, will be a member of the collection of sets $M(e(\mathcal{P}(\mathbb{N})))$ in the usual sense.

Definition 2.5.3 Let ${}^*X = \Pi_{\mathcal{U}} X = \Pi_{\mathcal{U}} V_0(X)$, and let $V({}^*X)$ be the associated superstructure built on the set of individuals ${}^*X$. The **Mostowski Collapsing Function** M is a mapping of $\Pi^0_{\mathcal{U}} V(X)$ into $V({}^*X)$ defined by induction on the level n as follows:

(i) For each element $[a] \in \Pi_{\mathcal{U}} V_0(X) = {}^*X,\ M([a]) = [a]$.
(ii) Given $[b] \in \Pi_{\mathcal{U}}[V_n(X) \setminus V_{n-1}(X)]$, $n \geq 1$, we set

$$M([b]) = \{M([a]) : [a] \in \cup_{k=0}^{n-1} \Pi_{\mathcal{U}}[V_k(X) \setminus V_{k-1}(X)] \text{ and } [a] \in_{\mathcal{U}} [b]\}.$$

We finish this section by showing that the map $* = M \circ e : V(X) \to V({}^*X)$ is a monomorphism. On first reading, the reader may wish to skip the following technical proofs and go on to the next section.

Proposition 2.5.4 *We have the following properties for $*$, e, and M:*

(i) e and M are 1:1 maps, so $* = M \circ e$ is a 1:1 map of $V(X)$ into $V({}^*X)$.
(ii) e maps X into ${}^*X$, and the restriction $M|{}^*X$ is the identity map (by definition).
(iii) $e(X) = [\,\overline{X}\,]$ (by definition), and $M([\,\overline{X}\,]) = {}^*X$, so $*(X) = {}^*X$.

(iv) *If* $\emptyset$ *is the empty set, then for no* $[a]$ *is it true that* $[a] \in_{\mathcal{U}} e(\emptyset) = [\,\overline{\emptyset}\,]$, *so* $*(\emptyset) = M(e(\emptyset)) = \emptyset$.
(v) $a \in b$ *in* $V(X)$ *iff* $e(a) \in_{\mathcal{U}} e(b)$; $[a] \in_{\mathcal{U}} [b]$ *iff* $M([a]) \in M([b])$.
(vi) *For* $n \geq 1$, e *maps* $V_n(X)\backslash V_{n-1}(X)$ *into* $\Pi_{\mathcal{U}}[V_n(X)\backslash V_{n-1}(X)]$ *and* M *maps* $\Pi_{\mathcal{U}}[V_n(X)\backslash V_{n-1}(X)]$ *into* $V_n({}^*X)\backslash V_{n-1}({}^*X)$. *It follows that if* a *has rank* $n \geq 0$, $*(a)$ *has rank* n.
(vii) *If* $M([b]) \in *(V_n(X))$ *and* $M([a]) \in M([b])$, *then* $M([a]) \in *(V_{n-1}(X))$.

Proof (i) If $a \neq b$, then $\forall i \in I$, $\overline{a}_i \neq \overline{b}_i$, so $e(a) = [\,\overline{a}\,] \neq [\,\overline{b}\,] = e(b)$. Therefore, e is 1:1. To show M is 1:1, assume $[a] \neq [b]$. Since M is the identity map on ${}^*X$, if either $[a]$ or $[b]$ is in ${}^*X$, $M([a]) \neq M([b])$. Otherwise, $\exists U \in \mathcal{U}$ such that either $a_i\backslash b_i \neq \emptyset\ \forall i \in U$ or $b_i\backslash a_i \neq \emptyset\ \forall i \in U$; assume the latter. Then $\exists\ [c]$ such that $c_i \in b_i\backslash a_i$ a.e. Therefore, $M([c]) \in M([b])$, but $M([c]) \notin M([a])$, so $M([a]) \neq M([b])$. Thus M is 1:1.

(ii), (iii), (iv), and (v) are clear.

(vi) By definition, if $[a] \in \Pi_{\mathcal{U}} V_0(X) = {}^*X$, then $M([a]) = [a] \in V_0({}^*X) = {}^*X$. Fix $n \geq 1$, and assume that for each $k < n$, if $[b] \in \Pi_{\mathcal{U}}[V_k(X)\backslash V_{k-1}(X)]$, then $M([b]) \in V_k({}^*X)\backslash V_{k-1}({}^*X)$. Given $[b] \in \Pi_{\mathcal{U}}[V_n(X)\backslash V_{n-1}(X)]$ and $M([a]) \in M([b])$, we have $[a] \in \cup_{k=0}^{n-1}\Pi_{\mathcal{U}}[V_k(X)\backslash V_{k-1}(X)]$, so by assumption, $M([a]) \in V_{n-1}({}^*X)$. Moreover, $\exists[c]$ such that $\forall i \in I$, $c_i \in V_{n-1}(X)\backslash V_{n-2}(X)$ and $c_i \in b_i$. Therefore, $M([c]) \in V_{n-1}({}^*X)\backslash V_{n-2}({}^*X)$, and thus, $M([b]) \in V_n({}^*X)\backslash V_{n-1}({}^*X)$.

(vii) Assume $M([b]) \in *(V_n(X))$ and $M([a]) \in M([b])$, then $[b] \in_{\mathcal{U}} e(V_n(X))$, and $[a] \in_{\mathcal{U}} [b]$. Therefore, there is a set $U \in \mathcal{U}$ such that $\forall i \in U$, $a_i \in b_i \in V_n(X)$. For these same i, $a_i \in V_{n-1}(X)$, whence $[a] \in_{\mathcal{U}} e(V_{n-1}(X))$, so $M([a]) \in *(V_{n-1}(X))$. □

Now, except for the transfer principle, we have shown that $* = M \circ e$ is a monomorphism from $V(X)$ into $V({}^*X)$. We will write ${}^*a$ for $*(a)$. The next proposition is used to establish the transfer principle.

Proposition 2.5.5 *Let* $[a]$, $[a^1], \ldots, [a^n]$, $[b]$, *and* $[c]$ *be elements of the bounded ultrapower* $\Pi^0_{\mathcal{U}} V(X)$; *here, any or all of these may be of the form* $e(d) = [\,\bar{d}\,]$.

(i) $M([a])$ $(=$ *or* $\in)$ $M([c])$ *iff* a_i $(=$ *or* $\in)$ c_i *a.e.*
(ii) $\{M([a^1]), \ldots, M([a^n])\}$ $(=$ *or* $\in)$ $M([c])$ *iff* $\{a_i^1, \ldots, a_i^n\}$ $(=$ *or* $\in)$ c_i *a.e.*
(iii) $\langle M([a^1]), \ldots, M([a^n])\rangle$ $(=$ *or* $\in)$ $M([c])$ *iff* $\langle a_i^1, \ldots, a_i^n\rangle$ $(=$ *or* $\in)$ c_i *a.e.*
(iv) $\langle\langle M([a^1]), \ldots, M([a^n])\rangle, M([b])\rangle$ $(=$ *or* $\in)$ $M([c])$ *iff* $\langle\langle a_i^1, \ldots, a_i^n\rangle, b_i\rangle$ $(=$ *or* $\in)$ c_i *a.e.*

Proof We will assume that $n = 2$ for the proof of (ii) and (iii).

(i) **(for $=$)** Since M is 1:1, $M([a]) = M([c])$ iff $[a] = [c]$ iff $a_i = c_i$ a.e.
(i) **(for $\in$)** $M([a]) \in M([c])$ iff $[a] \in_{\mathcal{U}} [c]$ iff $a_i \in c_i$ a.e.
(ii) **(for $=$)** Assume $\{a_i, b_i\} = c_i$ a.e. Then

$$\begin{aligned} M([c]) &= \{M([y]) : y_i \in \{a_i, b_i\} \text{ a.e.}\} \\ &= \{M([y]) : y_i = a_i \text{ a.e.}\} \cup \{M([y]) : y_i = b_i \text{ a.e.}\} \\ &= \{M([a]), M([b])\}. \end{aligned}$$

Conversely, if $M([c]) = \{M([a]), M([b])\}$, then $a_i \in c_i$ a.e. and $b_i \in c_i$ a.e. If $d_i \in c_i$ a.e. then either $d_i = a_i$ a.e. or $d_i = b_i$ a.e.; that is, c_i has only two points a.e. Therefore, $c_i = \{a_i, b_i\}$ a.e.

(ii) **(for $\in$)** Choose representatives for $[a]$ and $[b]$, and let $d_i = \{a_i, b_i\}\ \forall i \in I$. Then $M([d]) = \{M([a]), M([b])\}$. Therefore, $\{a_i, b_i\} \in c_i$ a.e. iff $d_i \in c_i$ a.e. iff $M([d]) \in M([c])$ iff $\{M([a]), M([b])\} \in M([c])$.

(iii) **(for $=$)** Choose representatives for $[a]$ and $[b]$, and $\forall i \in I$, let $d_i = \{\{1\}, \{1, a_i\}\}$ and $e_i = \{\{2\}, \{2, b_i\}\}$. By Part ii, $M([d]) = \{\{M([\,\overline{1}\,]\}, \{M([\,\overline{1}\,]), M([a])\}\} = \{\{1\}, \{1, M([a])\}\}$ since $M([\,\overline{1}\,]) = 1$. A similar result holds for $M([e])$. Again by Part (ii), $\langle a_i, b_i\rangle = c_i$ a.e. iff $\{d_i, e_i\} = c_i$ a.e. iff $\{M([d]), M([e])\} = M([c])$ iff $\langle M([a]), M([b])\rangle = M([c])$.

(iii) **(for $\in$)** This is the same as ((ii) $\in$) with $\{\,\}$ replaced with $\langle\,\rangle$.

(iv) Apply (iii) to $\langle d_i, b_i\rangle$ and $\langle M([d]), M([b])\rangle$, where $d_i = \left\langle a_i^1, \dots, a_i^n\right\rangle$ a.e., so $M([d]) = \left\langle M([a^1]), \dots, M([a^n])\right\rangle$. □

Notation. If $\Phi(x_1, \dots, x_n)$ is a formula with variables $x_1, \dots, x_n$, either free in Φ or not appearing in Φ and $c^1, \dots, c^n$ are constants, then $\Phi(c^1, \dots, c^n)$ is Φ with each x_i appearing in Φ replaced by c^i.

To establish the transfer principle for $*$, we need a theorem of Łŏs (pronounced "Wash").

Theorem 2.5.6 (Łŏs) *Let $\Phi(x_1, \dots, x_n)$ be a formula in the language $\mathcal{L}_X$ with $x_1, \dots, x_n$ a set of variables containing all of the free variables in Φ. Fix $[a^1], \dots,$ $[a^n]$ in $\Pi^0_{\mathcal{U}} V(X)$. Then ${}^*\Phi(M([a^1]), \dots, M([a^n]))$ holds in $V({}^*X)$ iff $\Phi(a_i^1, \dots, a_i^n)$ holds in $V(X)$ for almost all $i \in I$.*

Remark 2.5.7 Note that the previous proposition has established the result for atomic formulas. For example, if $\Phi(x, y)$ is the formula $x \in y$, then $M([a]) \in M([c])$ iff $a_i \in c_i$ a.e. If $\Phi(x)$ is the formula $x \in N$, then $M([a]) \in {}^*N = M([\,\overline{N}\,])$ iff $a_i \in N$ a.e. If $\Phi(x_1, \dots, x_n, y)$ is the formula $\{x_1, \dots, x_n, 5, A\} \in y$, where A is a standard set, then we have $\{M([a^1]), \dots, M([a^n]), M([\,\overline{5}\,]), M([\,\overline{A}\,])\} \in M([c])$ iff $\{a_i^1, \dots, a_i^n, 5, A\} \in c_i$ a.e.

Proof of Łŏs' Theorem. The proof is by an induction argument similar to the induction used in the construction of formulas. Remember, we are establishing an equivalence; i.e., one sentence holds for $V({}^*X)$ iff related sentences indexed by $i \in I$ hold a.e. for $V(X)$.

(1) The previous proposition has established the equivalence for atomic formulas.
(2) Assume the equivalence has been established for the formulas $\Phi(x_1, \dots, x_n)$ and $\Theta(x_1, \dots, x_n)$. Given this assumption, we establish the equivalence for $\neg\Phi$, and

$\Phi \wedge \Theta$. This will also establish the equivalence for $\Phi \vee \Theta = \neg(\neg\Phi \wedge \neg\Theta)$, $\Phi \to \Theta = \neg\Phi \vee \Theta$, and $\Phi \longleftrightarrow \Theta = [\Phi \to \Theta] \wedge [\Theta \to \Phi]$. For $\neg\Phi$, we have

$$^*(\neg\Phi)(M([a^1]), \ldots, M([a^n])) = \neg(^*\Phi)(M([a^1]), \ldots, M([a^n])).$$

The latter is true in $V(^*X)$ iff $\{i \in I : \Phi(a_i^1, \ldots, a_i^n) \text{ holds}\} \notin \mathcal{U}$ iff $\{i \in I : \neg\Phi(a_i^1, \ldots, a_i^n) \text{ holds}\} \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. (Note we need an equivalence here, not just Φ a.e. $\to {}^*\Phi$.) For $\Phi \wedge \Theta$, note that the following are equivalent:

$$^*(\Phi \wedge \Theta)(M([a^1]), \ldots, M([a^n]))$$

$$^*\Phi(M([a^1]), \ldots, M([a^n])) \wedge {}^*\Theta(M([a^1]), \ldots, M([a^n]))$$

$$\{i : \Phi(a_i^1, \ldots, a_i^n) \text{ holds}\} \in \mathcal{U} \text{ and } \{i : \Theta(a_i^1, \ldots, a_i^n) \text{ holds}\} \in \mathcal{U}$$

$$\{i : \Phi(a_i^1, \ldots, a_i^n) \text{ holds}\} \cap \{i : \Theta(a_i^1, \ldots, a_i^n) \text{ holds}\} \in \mathcal{U}$$

$$\{i : (\Phi \wedge \Theta)(a_i^1, \ldots, a_i^n) \text{ holds}\} \in \mathcal{U}.$$

We have used the fact that a superset of a set in $\mathcal{U}$ is in $\mathcal{U}$.

(3) Assume we have the equivalence for $\Phi(x_1, \ldots, x_n, z, y)$, and d is a constant and z a variable that either does not appear in Φ or is free in Φ. We will establish the result for $(\exists y \in d)\Phi$ and $(\exists y \in z)\Phi$. This will give the result since $(\forall y \in d)\Phi = \neg(\exists y \in d)\neg\Phi$ and $(\forall y \in z)\Phi = \neg(\exists y \in z)\neg\Phi$.
For $(\exists y \in z)\Phi$, we must fix $[a^1], \ldots, [a^n], [c] \in \Pi^0_{\mathcal{U}} V(X)$ and then show that $(\exists y \in M([c]))^*\Phi(M([a^1]), \ldots, M([a^n]), M([c]), y)$ holds in $V(^*X)$ if and only if $(\exists y \in c_i)\Phi(a_i^1, \ldots, a_i^n, c_i, y)$ holds in $V(X)$ for almost all $i \in I$. (The proof of $(\exists y \in d)\Phi$ is a special case of this where we replace $M([c])$ with $M([\,\overline{d}\,])$ and c_i with d; i.e., the constant sequence $\overline{d}$ replaces c.) Assume that

$$(\exists y \in M([c]))^*\Phi(M([a^1]), \ldots, M([a^n]), M([c]), y)$$

holds in $V(^*X)$. Then there is an $M([a])$ such that

$$[(M([a]) \in M([c])] \wedge [^*\Phi(M([a^1]), \ldots, M([a^n]), M([c]), M([a]))]$$

holds in $V(^*X)$, so

$$\{i \in I : (a_i \in c_i) \wedge \Phi(a_i^1, \ldots, a_i^n, c_i, a_i) \text{ holds}\} \in \mathcal{U}.$$

Therefore, the larger set

$$\{i \in I : (\exists y \in c_i)\Phi(a_i^1, \ldots, a_i^n, c_i, y) \text{ holds}\} \in \mathcal{U}.$$

To show the converse, assume there is a set $U_0 \in \mathcal{U}$ such that

$$U_0 = \{i \in I : (\exists y \in c_i)\Phi(a_i^1, \ldots, a_i^n, c_i, y)\}.$$

For each $i \in I$, pick an element $a_i \in c_i$, but make the choice so that for all $i \in U_0$, $\Phi(a_i^1, \ldots, a_i^n, c_i, a_i)$ holds. There is a $U_1 \in \mathcal{U}$ such that for *some* $n \in \mathbb{N}$ and all $i \in U_1$, $a_i \in V_n(X) - V_{n-1}(X)$. Choose any $\alpha \in V_n(X)\backslash V_{n-1}(X)$ and replace a_i with α for $i \notin U_1$. Then for this sequence $a = \langle a_i : i \in I\rangle$,

$$\{i \in I : (a_i \in c_i) \wedge \Phi(a_i^1, \ldots, a_i^n, c_i, a_i) \text{ holds}\} \supseteq U_0 \cap U_1 \in \mathcal{U}.$$

Therefore,

$$[(M([a]) \in M([c])] \wedge [^*\Phi(M([a^1]), \ldots, M([a^n]), M([c]), M([a]))]$$

holds in $V(^*X)$. If follows that

$$(\exists y \in M([c]))^*\Phi(M([a^1]), \ldots, M([a^n]), M([c]), y)$$

holds in $V(^*X)$.

(4) The theorem now follows by induction. □

Theorem 2.5.8 *The map* $* : V(X) \to V(^*X)$ *defined by* $* = M \circ e$ *is a monomorphism.*

Proof It only remains to show that if Φ is a sentence in $\mathcal{L}_X$ that is true for $V(X)$, then $^*\Phi$ is true for $V(^*X)$. Since Φ is a sentence, Φ has no free variables, only constants and bound variables. Since Φ is true for all $i \in I$, $^*\Phi$ is true for $V(^*X)$. □

Problem: Recall that given $[a], [b] \in \Pi^0_{\mathcal{U}} V(X)$, we write $[a] \in_{\mathcal{U}} [b]$ if $a_i \in b_i$ a.e. Show that this relation is well-defined, that is, that it is independent of the choice of representative from $[a]$ and $[b]$.

Answer: Assume a_i and $\widetilde{a}_i$ represent $[a]$ and b_i and $\widetilde{b}_i$ represent $[b]$. Also assume there is a set $U \in \mathcal{U}$ such that $\forall i \in U\ a_i \in b_i$. We know that there are sets V and W in $\mathcal{U}$ such that for all $i \in V$ we have $a_i = \widetilde{a}_i$, while for all $i \in W$ we have $b_i = \widetilde{b}_i$. Now $U \cap V \cap W$ is in $\mathcal{U}$, and for all i in this set, $\widetilde{a}_i \in \widetilde{b}_i$.

Problem: Given a mapping a from I into $V_n(X)$, show that for some $k \le n$, $a_i \in V_k(X)\backslash V_{k-1}(X)$ a.e.

Answer: For each $k \le n$, let $I_k = \{i \in I : a_i \in V_k(X)\backslash V_{k-1}(X)\}$. Then I is the disjoint finite union of the sets I_k, $1 \le k \le n$, so one and only one of the sets I_k is in $\mathcal{U}$.

2.6 Special Index Sets Yielding Enlargements

We have shown how to construct a monomorphism $*$ using a superstructure $V(X)$, an index set I, and an ultrafilter $\mathcal{U}$. To get more, however, we need additional assumptions. The construction of an enlargement starting with the material after Example 2.6.5 and ending with Theorem 2.6.7 can be omitted on first reading.

First we note that if $\mathcal{U}$ is not free, then $\mathcal{U}$ is fixed at some $i_0 \in I$; i.e., the singleton set $\{i_0\} \in \mathcal{U}$. In this case, every sequence a is equivalent to the constant sequence $\overline{a}_{i_0}$ since $a_i = a_{i_0}$ a.e. Therefore, ${}^*X = X$ and we get nothing new. We assume from now on that the ultrafilter $\mathcal{U}$ is free; i.e., for every $i \in I$, the set $\{i\} \notin \mathcal{U}$.

Definition 2.6.1 All entities in $V(X)$ and entities in $V({}^*X)$ of the form ${}^*b$ for $b \in V(X)$ are called **standard**.

Example 2.6.2 The sets $[0, 1]$ and ${}^*[0, 1]$ are standard entities even though ${}^*[0, 1]$ contains nonstandard numbers.

Note that in interpreting a sentence ${}^*\Phi$, the only entities α that arise are related by a finite $\in$ chain to a standard entity ${}^*b$. Such an entity α is of the form $M([a])$. There are, however, entities in $V({}^*X)$ that are not of this form. An example is the set $\mathbb{N}$. We will show that if $\langle A_i \rangle$ is a sequence of sets such that for each $j \in \mathbb{N}$, the equivalence class containing the constant sequence j is in A_i for all i in some element of the ultrafilter, then the equivalence class $[\langle A_i \rangle]$ contains unlimited natural numbers. We now consider "hyperfinite" sets; these are extremely important for the applications of nonstandard analysis.

Definition 2.6.3 For each $A \in V(X)\backslash X$, let $\mathcal{P}_F(A)$ denote the finite subsets of A. The collection of **hyperfinite** or $*$-finite sets in $V({}^*X)$ is $\cup_{n=0}^{\infty} {}^*\mathcal{P}_F(V_n(X)) = \cup\{{}^*\mathcal{P}_F(A) : A \in V(X)\backslash X\}$.

Definition 2.6.4 Given a superstructure $V(X)$ and a monomorphism $*$, we say that $V({}^*X)$ is an **enlargement** of $V(X)$ if for each set $A \in V(X)$ there is a set $B \in {}^*\mathcal{P}_F(A)$ such that for every $a \in A$, ${}^*a \in B$, i.e., B contains all of the standard entities in ${}^*A$. It follows from the transfer principle that if A is not a finite set, then there are elements of ${}^*A$ that are not in B.

Example 2.6.5 For $\eta \in {}^*\mathbb{N}_\infty$, the "initial segment" $\{1, 2, \ldots, \eta\}$ is a hyperfinite set; it contains $\mathbb{N}$.

Fix $V(X)$. We now construct an index set J and an ultrafilter $\mathcal{U}$ on J such that the corresponding superstructure $V({}^*X)$ is an enlargement of $V(X)$. Since X is an infinite set (containing $\mathbb{N}$), it will follow that ${}^*X \neq X$. Let J be the collection of all nonempty finite sets belonging to $V(X)$. Note that each element of J is in $V_n(X)$ for some n. For each $a \in J$, let $J_a = \{b \in J : a \subseteq b\}$. Let $\mathcal{F}$ be the collection of all subsets of J such that for each $A \in \mathcal{F}$ there is an $a \in J$ with $J_a \subseteq A$.

Proposition 2.6.6 *The collection $\mathcal{F}$ is a free filter on J.*

Proof For each $A \in \mathcal{F}$ there is an $a \in J$ with $a \in J_a \subseteq A$, so $A \neq \emptyset$. Given A and B in $\mathcal{F}$ and $a, b \in J$, with $J_a \subseteq A$ and $J_b \subseteq B$,

$$J_a \cap J_b = J_{a \cup b} \subseteq A \cap B,$$

and any subset of J containing a set in $\mathcal{F}$ is in $\mathcal{F}$. Therefore, $\mathcal{F}$ is a filter. To show $\mathcal{F}$ is free, fix $a \in J$ and find some $b \in J$ with $a \cap b = \emptyset$. Then J_b itself is in $\mathcal{F}$, and the set a is not in J_b. □

Now use Zorn's Lemma to obtain a free ultrafilter $\mathcal{U}$ on J such that $\mathcal{U} \supset \mathcal{F}$. Note that for each $a \in J$, $J_a \in \mathcal{U}$.

Theorem 2.6.7 *If $V(^*X)$ is constructed from $V(X)$ using J and $\mathcal{U}$, then it is an enlargement of $V(X)$.*

Proof Let A be a nonempty set belonging to $V(X)$. Choose an $a_0 \in A$. Define a map $\Gamma : J \to \mathcal{P}_F(A)$ by setting $\Gamma_a = (a \cap A) \cup \{a_0\}$ for each $a \in J$. Since $A \in V_m(X)$ for some $m \in \mathbb{N}$, there is an $n \in \mathbb{N}$ and a set $U_0 \in \mathcal{U}$ such that Γ_a has rank n for all $a \in U_0$. Choose any $a_1 \in U_0$, and replace Γ_a with Γ_{a_1} for $a \notin U_0$. Let $B = M([\Gamma])$. Then $B \in {}^*\mathcal{P}_F(A)$ since $\forall a \in J$, $\Gamma_a \in \mathcal{P}_F(A)$. Fix $c \in A$. We must show that ${}^*c \in B$. Since the singleton $\{c\}$ is a finite subset of A, $J_{\{c\}} \in \mathcal{U}$, so $c \in \Gamma_a$ for all $a \in J_{\{c\}} \cap U_0$, that is, a.e. Therefore, ${}^*c \in B$. □

Definition 2.6.8 A binary relation P is **concurrent** or finitely satisfiable on a set A contained in its domain if for each $n \in \mathbb{N}$ and each finite set $\{x_1, \ldots, x_n\} \subseteq A$ there is a y in the range of P such that $\langle x_i, y\rangle \in P$ for $1 \leq i \leq n$. A relation P is concurrent if it is concurrent on all of its domain.

Example 2.6.9 The relations $\leq$ in $\mathbb{N}$ and $\subseteq$ in $\mathcal{P}_F(\mathbb{N})$ are concurrent relations.

Concurrent relations are of interest because of the following property.

Theorem 2.6.10 *Given X and a monomorphism $*$, the following are equivalent:*

(i) $V(^*X)$ is an enlargement of $V(X)$.
(ii) Given any concurrent relation $P \in V(X)$, there is an element c in the range of ${}^*P$ such that $\langle {}^*a, c\rangle \in {}^*P$ for every a in the domain of P.

Proof (i→ii) Let A be the domain of P, and let $B \subseteq {}^*A$ be a hyperfinite set containing ${}^*a$ for each $a \in A$. By transfer of the concurrency condition for P, there is a c in the range of ${}^*P$ such that $\langle b, c\rangle \in {}^*P$ for each $b \in B$, in particular, $\langle {}^*a, c\rangle \in P$ for each $a \in A$.

(ii→i) Fix a set $A \in V(X)$. Since $\subseteq$ in $\mathcal{P}_F(A)$ is a concurrent relation, there is a hyperfinite set B that contains the extension of every standard finite subset of A. In particular, $\forall a \in A$, ${}^*\{a\} = \{{}^*a\} \subset B$, so ${}^*a \in B$. □

Corollary 2.6.11 *If A is an infinite set in $V(X)$ and $V(^*X)$ is an enlargement, then there is a nonstandard $b \in {}^*A$.*

Proof The relation $\neq$ is concurrent in A. □

Remark 2.6.12 The results of the last chapter are valid for an enlargement of a superstructure $V(X)$ when X contains $\mathbb{R}$.

Problem: Assume $\{O_\alpha : \alpha \in A\}$ is an open covering of $S \subseteq \mathbb{R}$ with no finite subcovering. Show that there is a point $b \in {}^*S$ such that b is not in the monad of any $x \in S$.

Answer: The relation P such that $\langle O_\alpha, y\rangle \in P$ if $y \in S \backslash O_\alpha$ is concurrent, so $\exists b \in {}^*S$ such that $b \in {}^*S \backslash {}^*O_\alpha$ $\forall \alpha \in A$. If $x \in S$, then $x \in O_\alpha$ for some α, and so $m(x) \subseteq {}^*O_\alpha$. It follows that $b \notin m(x)$ for any $x \in S$.

Problem: Let A be a collection of sets with $A \in V(X)$. Assume A has the finite intersection property. That is, the intersection of any finite number of elements of A is not empty. Suppose that $V({}^*X)$ is an enlargement. Show that the **intersection monad** $\mu(A) := \bigcap_{a \in A} {}^*a$ is not empty.

Answer: Take a hyperfinite $B \subseteq {}^*A$ with ${}^*a \in B$ for each $a \in A$. Then $\exists y \in \bigcap_{b \in B} b \subseteq \bigcap_{a \in A} {}^*a$.

2.7 A Result in Infinite Graph Theory

As an application of the existence of an enlargement, we give an easy proof of a result of de Bruijn and Erdős [3].

Definition 2.7.1 A graph (A, E) consists of a set A of "vertices" and a binary, symmetric relation E on $A \times A$. If $\langle x, y\rangle \in E$, we say that x and y are connected by an edge. The graph (A, E) is k-colorable if there is a map $f : A \to \{1, \ldots, k\}$ (the set of k colors) such that if $\langle a, b\rangle \in E$, then $f(a) \neq f(b)$. If $B \subseteq A$ and $E|B$ is the restriction of the relation E to $B \times B$, then $(B, E|B)$ is called a subgraph of A. The cardinality of a graph (A, E) is that of the set A.

Theorem 2.7.2 (de Bruijn and Erdős) *If each finite subgraph of an infinite graph is k-colorable, then the graph itself is k-colorable.*

Proof Let (A, E) be the infinite graph, and let $\mathcal{F}$ denote the set of all finite subsets of A. The following sentence, given here in shorthand, holds for the original superstructure

$$(\forall F \in \mathcal{F})(\exists g : F \to \{1, \ldots, k\})(\forall x, y \in F)[\langle x, y\rangle \in E \to g(x) \neq g(y)].$$

Let B be a hyperfinite subset of ${}^*A$ such that $\forall a \in A$, ${}^*a \in B$. Then $B \in {}^*\mathcal{F}$, so $\exists g : B \to \{1, \ldots, k\}$ such that $\forall x, y \in B$, if $\langle x, y\rangle \in {}^*E$, then $g(x) \neq g(y)$. In particular, $\forall\, a, b \in A$, if $\langle a, b\rangle \in E$, then $\langle {}^*a, {}^*b\rangle \in {}^*E$, so $g({}^*a) \neq g({}^*b)$. Let f denote the restriction of g to A; that is, $f(a) = g({}^*a)$. Then f is a k-coloring of A. □

2.8 Internal and External Sets

In this section we make the important distinction between internal and external objects in $V({}^*X)$. We also establish some additional properties that will be used in applications of nonstandard analysis. The reader may wish to skip over the proofs of Theorems 2.8.4 and 2.8.11 on first reading.

Definition 2.8.1 An entity a in $V({}^*X)$ is called **internal** if for some standard set $b \in V(X)$, $a \in {}^*b$. All other entities in $V({}^*X)$ are called **external**.

This means that the internal entities are the elements of standard entities. Of course, if $b \in V(X)$, then $b \in V_{n+1}(X)$ for some n, so if $a \in {}^*b$, then $a \in {}^*V_n(X)$. Thus internal entities are elements of ${}^*V_n(X)$ for some n. On the other hand, suppose that b is internal but not a standard entity, and that $a \in b$. Then $b \in {}^*V_{n+1}(X)$ for some n, so by transfer of the sentence

$$(\forall y \in V_{n+1}(X))(\forall x \in y)[x \in V_n(X)],$$

it follows that $a \in {}^*V_n(X)$, whence a is internal.

In interpreting the transfer ${}^*\Phi$ of a sentence Φ, only constants naming standard objects are used, so only internal entities come up in the interpretation.

If $a \in {}^*b$, then we can obtain information about a by transferring sentences of the form $(\forall x \in b)[\cdots]$. If a is external, however, then the transfer principle does not yield information in this way about a.

Example 2.8.2 To show that the set $\mathbb{N}$ is external in $V({}^*X)$, let us suppose the contrary, which means that $\mathbb{N} \in {}^*\mathcal{P}(\mathbb{N})$. Then the set ${}^*\mathbb{N}\backslash\mathbb{N} = {}^*\mathbb{N}_\infty$ is also internal by transfer of the sentence

$$(\forall A \in \mathcal{P}(\mathbb{N}))[\mathbb{N}\backslash A \in \mathcal{P}(\mathbb{N})].$$

It follows by transfer of the sentence

$$(\forall A \in \mathcal{P}(\mathbb{N}))[A \neq \emptyset \to (\exists m \in A)(\forall k \in A)[m \leq k]],$$

that there is a first element m in ${}^*\mathbb{N}_\infty$. In this case, $m-1$ is the last element of $\mathbb{N}$, which is impossible. This contradiction is a proof that $\mathbb{N}$ is external.

For applications of the Transfer Principle, it is clearly important to know when an entity in $V({}^*X)$ is internal. We have already given the proof for the following criterion.

Theorem 2.8.3 *The set of internal elements of $V({}^*X)$ is the set $\cup_{n=0}^{\infty}{}^*V_n(X)$.*

This result is not much help in determining when a set is internal. The next result is considerably more useful. Recall that $\mathcal{L}_{{}^*X}$ is the language for $V({}^*X)$. A formula in $\mathcal{L}_{{}^*X}$ is called standard if all of the constants are names of standard entities; it is called internal if all of the constants are names of internal entities.

Theorem 2.8.4 (Keisler's Internal Definition Principle) *Let $\Phi(x)$ be an internal formula in $\mathcal{L}_{*X}$ for which x is the only free variable. Let A be an internal set. Then the set $\{a \in A : \Phi(a)$ holds in $V(^*X)\}$ is internal.*

Proof Let $c_1, \ldots, c_n$ be the constants in $\Phi(x)$; denote Φ by $\Phi(c_1, \ldots, c_n, x)$. Fix $k \in \mathbb{N}$ so that $A, c_1, \ldots, c_n$ are all in $^*V_k(X)$. The sentence in $\mathcal{L}_X$

$$(\forall x_1, \ldots, x_n, y \in V_k(X))(\exists z \in V_{k+1}(X))(\forall x \in V_k(X))$$

$$[x \in z \longleftrightarrow [x \in y \wedge \Phi(x_1, \ldots, x_n, x)]]$$

holds in $V(X)$. Its transfer says that $\{a \in A : \Phi(a) \text{ holds}\} \in {}^*V_{k+1}(X)$. □

Theorem 2.8.5 *If A and B are internal, so are $A \cup B$, $A \cap B$, $A \backslash B$, $A \times B$.*

Proof For $\cup$, assume that $A, B \in {}^*V_{n+1}(X)$ and transfer

$$(\forall W, Y \in V_{n+1}(X))(\exists Z \in V_{n+1}(X))(\forall x \in V_n(X))$$

$$[x \in Z \longleftrightarrow x \in W \vee x \in Y].$$

For $\cap$, replace $\vee$ with $\wedge$. For $\backslash$ replace $x \in W \vee x \in Y$ with $x \in W \wedge x \notin Y$. The proof for $\times$ is left to the reader. □

We have already shown that $\mathbb{N}$ and $^*\mathbb{N}_\infty$ are external. As a consequence, we have the following result, where $\mathbb{Z}$ denotes the integers.

Theorem 2.8.6 *In an enlargement of a structure $V(X)$ with $\mathbb{R} \subseteq X$, the sets $\mathbb{N}$, $^*\mathbb{N}_\infty$, $\mathbb{R}$, $\mathbb{Z}$, $^*\mathbb{Z}_\infty$, $^*\mathbb{R}_\infty$, $m(0)$ are all external.*

Proof It follows from the fact that $\mathbb{N}$ and $^*\mathbb{N}_\infty$ are external that $\mathbb{Z}$ and $^*\mathbb{Z}_\infty$ are also external. Since $\mathbb{Z} = \mathbb{R} \cap {}^*\mathbb{Z}$, $\mathbb{R}$ is external. Since $^*\mathbb{N}_\infty = {}^*\mathbb{R}_\infty \cap {}^*\mathbb{N}$, $^*\mathbb{R}_\infty$ is external. Since for $x \neq 0$, $x \in m(0)$ iff $1/x \in {}^*\mathbb{R}_\infty$, $m(0)$ is external. □

Remark 2.8.7 One should not think that external entities are "bad" in any sense. They are just not the subject of the transfer principle. A review of the last chapter shows the utility of such external objects as $m(0)$, $^*\mathbb{N}_\infty$, and the standard part map.

Problem: Recall that $\mathcal{P}_F$ denotes the finite power set operation. Show that in general, $\mathcal{P}_F(^*A) \subseteq {}^*\mathcal{P}_F(A)$, but $^*\mathcal{P}(A) \subseteq \mathcal{P}(^*A)$.

Answer: A truly finite subset of $^*A$ is internal, so it is in $^*\mathcal{P}_F(A)$. If, for example, the set has three elements, then the following sentence is true by transfer

$$(\forall x \in {}^*A)(\forall y \in {}^*A)(\forall z \in {}^*A)(\exists w \in {}^*\mathcal{P}_F(A))[\{x, y, z\} = w].$$

Thus $\mathcal{P}_F(^*A) \subseteq {}^*\mathcal{P}_F(A)$. Not every hyperfinite set is finite, so the inclusion does not go the other way. If $S \in {}^*\mathcal{P}(A)$, then by the transfer principle, $\forall s \in S$, $s \in {}^*A$, so $S \in \mathcal{P}(^*A)$. Since external subsets of $^*A$ are not in $^*\mathcal{P}(A)$, we can only say that $^*\mathcal{P}(A) \subseteq \mathcal{P}(^*A)$.

Problem: It was shown in the last chapter that $(^*\mathbb{R}, ^*+, ^*\cdot, ^*<)$ forms an ordered field extension of the real number system. Use the transfer principle to show that it is complete with respect to internal sets. That is, if an internal set has an upper bound in $^*\mathbb{R}$, it has a least upper bound in $^*\mathbb{R}$.

Problem: Use the transfer principle to show the following about the ordered field $(^*\mathbb{R}, ^*+, ^*\cdot, ^*<)$:

(1) Each internal non empty set $B \subseteq {^*\mathbb{N}}$ has a smallest element with respect to $^*<$.
(2) (Internal Transfinite Induction) For each internal $B \subseteq {^*\mathbb{N}}$, the following implication holds:

$$1 \in B \text{ and } \forall x(x \in B \to x + 1 \in B) \to B = {^*\mathbb{N}}.$$

Here is an additional property of monomorphisms that allows one to use the transfer principle for certain internal sets.

Definition 2.8.8 A monomorphism $*$ is **comprehensive** if for any sets C, D in $V(X)$ and any map $h : C \longmapsto {^*D}$, there is an internal map $H : {^*C} \longmapsto {^*D}$ such that $H(^*a) = h(a)$ for each $a \in C$. A monomorphism $*$ is called **denumerably comprehensive** if it is comprehensive with the restriction that C must be a countable set in $V(X)$.

Proposition 2.8.9 *A monomorphism $*$ is denumerably comprehensive if and only if any ordinary sequence $\{A_n : n \in \mathbb{N}\}$ from an internal, but not necessarily standard, set S can be extended to an internal sequence $\{A_n : n \in {^*\mathbb{N}}\}$ in S.*

Proof Suppose $*$ is a denumerably comprehensive monomorphism. Let $C = \mathbb{N}$, and let $\{A_n : n \in \mathbb{N}\}$ be an ordinary sequence of elements taken from an internal set S. Recall that each A_n must then be internal. For some k, we have $S \in {^*V_k(X)}$, so there is an internal sequence $\{A_n : n \in {^*\mathbb{N}}\}$ that extends the original sequence and takes values in $^*V_k(X)$. If we consider the internal set $\{n \in {^*\mathbb{N}}\colon A_n \notin S\}$, there must be a first element n_0, which must be unlimited. By setting $A_n = A_{n_0-1}$ for all $n \geq n_0$, we may assume that $A_n \in S$ for all $n \in {^*\mathbb{N}}$. For the converse, we note that if C is countably infinite, then there is an enumeration $\{c_n : n \in \mathbb{N}\}$ of C. That enumeration has a nonstandard extension $\{c_n : n \in {^*\mathbb{N}}\}$. The composition $c_n \to n \to A_n, n \in {^*\mathbb{N}}$, gives the desired extension of the mapping $c_n \to n \to A_n$, $n \in \mathbb{N}$. □

Remark 2.8.10 Starting with [5] and then [1], the property of denumerable comprehensiveness has played a central role in the application of nonstandard analysis to measure theory. The next result applies in particular to the enlargement defined in Theorem 2.6.7.

Theorem 2.8.11 *Monomorphisms constructed using ultrapowers are comprehensive.*

Proof Let $\mathcal{U}$ be an ultrapower with index set I. Fix sets C, D in $V(X)$ and a map $h : C \to {}^*D$. For each element $M([b]) \in {}^*D$, let $S(M([b]))$ be a representative element of the equivalence class $[b]$. We may assume that $S(M([b]))(i) \in D\ \forall i \in I$. Now for each $i \in I$, let k_i be the mapping from C into D given by

$$k_i = \{\langle a, S(h(a))(i)\rangle : a \in C\}.$$

There is an $n \in \mathbb{N}$ and a set U in the ultrafilter such that $\forall i \in U$, k_i has rank n. Pick some $i_0 \in U$, and let $k_i = k_{i_0}$ for $i \notin U$. Let $[k]$ denote the equivalence class containing k. For each $i \in I$, k_i is a function from C into D, so $H = M([k])$ is an internal function from ${}^*C$ into ${}^*D$. We need only show that $H({}^*\alpha) = h(\alpha)$ for all $\alpha \in C$. Fix $\alpha \in C$ and recall that ${}^*\alpha = M([\,\overline{\alpha}\,])$ and $h(\alpha)$ is an element $M([b]) \in {}^*D$. By definition, $H({}^*\alpha) = M([c])$, where

$$c_i = k_i\,(\overline{\alpha}_i) = S(h(\overline{\alpha}_i))(i) = S(h(\alpha))(i) = b_i \quad \text{a.e.},$$

so $H({}^*\alpha) = h(\alpha)$. □

Here is an important principle for the applications of nonstandard analysis. It uses the externality of $\mathbb{R}$ and $\mathbb{N}$ in $V({}^*X)$ to obtain information about internal subsets of ${}^*\mathbb{R}$.

Theorem 2.8.12 (Spillover Principle) *Let A be an internal subset of ${}^*\mathbb{R}$.*

(i) If A contains all standard natural numbers, then A contains all elements of ${}^*\mathbb{N}$ less than some unlimited natural number.
(ii) If A contains all unlimited natural numbers, then A contains all elements of ${}^*\mathbb{N}$ greater than some standard natural number.
(iii) If A contains the positive infinitesimals, then A contains all elements of ${}^*\mathbb{R}^+$ smaller than some standard positive real number.
(iv) Assume that for each unlimited natural number H there exists an unlimited natural number $K < H$ such that $K \in A$. Then A contains a standard natural number.

Proof (i) Since A is internal, we cannot have $A = \mathbb{N}$. If ${}^*\mathbb{N}\backslash A$ is not empty, it has a first element, which must be unlimited.
(ii) Here, ${}^*\mathbb{N}\backslash A$ is bounded above by an unlimited integer, so it has a least upper bound M, which must be limited. Let n be the first integer strictly bigger than M. It follows that A contains all elements of ${}^*\mathbb{N}$ greater than n.
(iii) Let B be the set $\{n \in {}^*\mathbb{N} : \forall x \text{ with } 0 < x < 1/n\,,\, x \in A\}$. Then B is internal and contains ${}^*\mathbb{N}_\infty$, so it contains some finite m.
(iv) Since the set $A \cap {}^*\mathbb{N}$ is internal and non-empty, it contains a smallest element k, which must, by assumption, be limited. □

Here is a very useful corollary of the Spillover Principle. Abraham Robinson said he wanted this principle written on his tombstone, but it isn't.

Theorem 2.8.13 (Robinson's Sequential Lemma) *Let $\langle s_n : n \in {}^*\mathbb{N}\rangle$ be an internal sequence in ${}^*\mathbb{R}$ such that $s_n \simeq 0$ for all $n \in \mathbb{N}$. Then for some $\eta \in {}^*\mathbb{N}_\infty$, $s_n \simeq 0$ for all $n \leq \eta$.*

Proof Let $A = \{n \in {}^*\mathbb{N} : |s_n| < 1/n\}$. Then A is internal and contains $\mathbb{N}$, so it contains all elements of ${}^*\mathbb{N}$ smaller than some unlimited η. Therefore, $s_n \simeq 0$ for all unlimited $n \leq \eta$, and we already know that $s_n \simeq 0$ for all limited n. □

Remark 2.8.14 This fact does not go the other way: $s_n = 1/n$ is infinitesimal for all unlimited n, but it is not infinitesimal for any limited n.

Robinson used his lemma in [8] to verify his construction of Banach limits. That construction, given below, has been extended to vector valued sequences by the author and Horst Osswald in [6].

Definition 2.8.15 Let ℓ_∞ denote the set of standard bounded real-valued sequences. A linear map $L : \ell_\infty \to \mathbb{R}$ is called a Banach limit if for each $\sigma \in \ell_\infty$, $L(\sigma)$ is a value between the $\liminf \sigma$ and the $\limsup \sigma$ and $L(\sigma) = L(T(\sigma))$ where $T(\sigma)(n) = \sigma(n+1)$.

Robinson's nonstandard construction of a Banach limit picks $\eta \in {}^*\mathbb{N}_\infty$ and sets

$$L_\eta(\sigma) = \mathrm{st}\left(\frac{1}{\eta}\sum_{n=1}^{\eta} {}^*\sigma(n)\right).$$

Here, $\Sigma_{n=1}^{\eta}$ is the nonstandard extension, evaluated at η, of the usual summation operator that sums a sequence from 1 to k.

Theorem 2.8.16 *For any $\eta \in {}^*N_\infty$, L_η is a Banach Limit.*

Proof The map L_η is clearly linear. The sum of any finite set of limited numbers divided an unlimited integer is infinitesimal. Moreover, given any $\varepsilon > 0$, a bounded sequence, after a finite number of terms, takes values between its lim inf $-\varepsilon$ and its lim sup $+\varepsilon$. Therefore, for all limited $m \in \mathbb{N}$,

$$\left|\frac{1}{\eta}\sum_{n=1}^{\eta} {}^*\sigma(n) - \frac{1}{\eta - m}\sum_{n=m+1}^{\eta} {}^*\sigma(n)\right|$$

$$\leq \left|\frac{1}{\eta}\sum_{n=1}^{\eta} {}^*\sigma(n) - \frac{1}{\eta - m}\sum_{n=1}^{\eta} {}^*\sigma(n)\right| + \left|\frac{1}{\eta - m}\sum_{n=1}^{m} {}^*\sigma(n)\right|$$

$$\simeq \frac{m}{\eta - m}\left|\frac{1}{\eta}\sum_{n=1}^{\eta} {}^*\sigma(n)\right| \simeq 0.$$

It follows that for any $\sigma \in \ell_\infty$ and any $\varepsilon > 0$, $\liminf(\sigma) - \varepsilon \le L_\eta(\sigma) \le \limsup(\sigma) + \varepsilon$, whence the inequality is true with $\varepsilon = 0$. It is clear that shifting a bounded sequence σ to the left by 1 changes $\frac{1}{\eta}\sum_{n=1}^{\eta} {}^*\sigma(n)$ by only an infinitesimal amount, and thus does not change the value of $L_\eta(\sigma)$. $\square$

2.9 Saturation

We will need one more general property of monomorphisms when dealing with mathematical objects such as Banach spaces and topological spaces. It is a useful "compactness" property. Fix an uncountable cardinal number κ. Let us say that a set A is κ**-small** if its cardinal number satisfies the inequality $\mathrm{card}(A) < \kappa$. Recall that a binary relation P is concurrent or finitely satisfiable on a set A contained in its domain if for each $n \in \mathbb{N}$ and each finite set $\{x_1, \ldots, x_n\} \subseteq A$ there is a y in the range of P such that $\langle x_i, y\rangle \in P$ for $1 \le i \le n$.

Definition 2.9.1 A nonstandard superstructure $V({}^*X)$ is κ **-saturated** if for each internal binary relation $P \in V({}^*X)$ and each κ-small subset A (internal or external) of the domain of P such that P is concurrent on A there is an element b in the range of P such that $\langle a, b\rangle \in P$ for each $a \in A$.

For κ-saturation, the cardinal number κ should be determined by $V(X)$; it must not depend on objects in $V({}^*X)$, because these can change depending on the monomorphism. It is sufficient, therefore, to fix κ as the first cardinal number larger than the cardinality of the original superstructure $V(X)$. One then says that $V({}^*X)$ is **polysaturated**. Horst Osswald presents a construction of polysaturated models in the appendix, i.e., the next section, of this chapter. A construction of saturated models can also be found in Stroyan and Luxemburg's book [12].

Theorem 2.9.2 *Suppose the nonstandard superstructure $V({}^*X)$ is* κ***-saturated*** *and A is a set in $V(X)$ such that* $\mathrm{card}(A) < \kappa$*. Then there is a hyperfinite set B such that for each $a \in A$, ${}^*a \in B$. If D is a set in $V(X)$ and $h : A \longmapsto {}^*D$, then there is an internal map $H : {}^*A \longmapsto {}^*D$ such that $H({}^*a) = h(a)$ for each $a \in A$. In particular, if $V({}^*X)$ is* ***polysaturated****, then it is concurrent and an enlargement.*

Proof The binary relation P such that for each $c \in {}^*A$ and each hyperfinite subset B of ${}^*A$, $\langle c, B\rangle \in P$ if and only if $c \in B$ is concurrent on the set $\{{*a} : a \in A\}$. The binary relation Q on the set of internal mappings with domain contained in ${}^*A$ and range contained in ${}^*D$ such that $\langle g, \widehat{g}\rangle \in Q$ if and only if $\widehat{g}$ is an extension of g is concurrent on the following set of mappings with singleton domains: $\{{*a} \longmapsto h(a) : a \in A\}$. An internal map G extending all of these maps can be further extended by mapping any $c \in {}^*A$ outside the domain of G to a fixed element of ${}^*D$. $\square$

Remark 2.9.3 In applications, one often starts with a topological space $(X, \mathcal{T})$ and works with just a κ-saturated enlargement, where $\kappa > \mathrm{card}(\mathcal{T})$.

Theorem 2.9.4 *A nonstandard superstructure $V({}^*X)$ is κ-saturated if and only if for each internal set C and every (internal or external) κ-small collection $\mathcal{B}$ consisting of internal subsets of C such that $\mathcal{B}$ has the finite intersection property, there is an $a \in \cap\{B : B \in \mathcal{B}\}$.*

Proof Assume first that $V({}^*X)$ is κ-saturated, that $\mathcal{B} \subseteq {}^*\mathcal{P}(C)$ is κ-small, and that $\mathcal{B}$ has the finite intersection property. Then the relation $P := \{\langle B, a\rangle \mid a \in B \in {}^*\mathcal{P}(C)\}$ is internal and concurrent on $\mathcal{B} \subseteq \text{domain}(P)$. By the assumption, there exists an a such that $\langle B, a\rangle \in P$ for each $B \in \mathcal{B}$. Now assume that P is an internal relation, and that P is concurrent on a κ-small set $A \subseteq \text{domain}(P)$. For each $a \in A$ set

$$S_a := \{b \mid \langle a, b\rangle \in P\}\,.$$

Then $\mathcal{B} := \{S_a \mid a \in A\}$ is κ-small and has the finite intersection property. By the assumption, there is a $b \in \bigcap \mathcal{B}$. It follows that $\langle a, b\rangle \in P$ for each $a \in A$. □

Theorem 2.9.5 *For $V({}^*X)$, $\aleph_1$-saturation is equivalent to being denumerably comprehensive.*

Proof First, assume that $V({}^*X)$ is at least $\aleph_1$-saturated. Fix an internal set S, and let $\langle a_n : n \in \mathbb{N}\rangle$ be an ordinary sequence of elements from S. We must show that this sequence can be extended to an internal sequence $\langle a_n : n \in {}^*\mathbb{N}\rangle$ in S. For each $n \in \mathbb{N}$, Let B_n be the collection of internal maps F from ${}^*\mathbb{N}$ into S such that $\forall i \le n$, $F(i) = a_i$. Then B_n is internal. Moreover, $\mathcal{B} := \{B_n \mid n \in \mathbb{N}\}$ is κ-small and has the finite intersection property. Therefore, there exists an internal $F : {}^*\mathbb{N} \to S$ with $F \in \bigcap \mathcal{B}$, whence $F(n) = a_n$ for each $n \in \mathbb{N}$. The proof of the converse is left to the reader. □

In the following we assume $V({}^*X)$ be κ-saturated, where $\kappa \ge \aleph_1$.

Proposition 2.9.6 *If A is an infinite but κ-small set, then A is external.*

Proof Assume that A is internal. Then $\{\langle a, b\rangle \mid a, b \in A \text{ and } a \ne b\}$ is internal and concurrent on A. Therefore, there exists an element $b \in A$ such that $a \ne b$ for all $a \in A$. □

Proposition 2.9.7 *Let $\kappa >$ card$(V(X))$ and let A be a set in $V(X)$. Let ${}^*[A] := \{{}^*a : a \in A\}$. Then ${}^*[A] \subseteq {}^*A$ and ${}^*[A] \ne {}^*A$ if and only if A is infinite.*

Proof If $a \in A$, then by transfer, ${}^*a \in {}^*A$. Thus, ${}^*[A] \subseteq {}^*A$. Suppose A is infinite. Since ${}^*[A]$ is infinite and κ-small, by the previous result, ${}^*[A]$ is external. Since ${}^*A$ is internal, ${}^*[A] \ne {}^*A$. On the other hand, if A is finite, say $A = \{a_1, \ldots, a_k\}$, then,

$${}^*A = \left\{{}^*a_1, \ldots, {}^*a_k\right\} = {}^*[A]. \square$$

Part (i) of the next result shows that each κ-small internal cover of an internal set A contains a finite subcover of A; Part ii implies that the standard part of an internal finitely additive measure on an internal algebra is always σ-additive:

Proposition 2.9.8 *Fix an internal set C.*

(i) If $A \subseteq C$ is internal and $\mathcal{B}$ is a κ -small subset of ${}^*\mathcal{P}(C)$ such that $A \subseteq \bigcup \mathcal{B}$. Then there exist finitely many $B_1, \ldots, B_k \in \mathcal{B}$ with $A \subseteq B_1 \cup \cdots \cup B_k$.
(ii) If $(A_k)_{k \in \mathbb{N}}$ is a strictly increasing or strictly decreasing sequence of internal subsets of C, then $\bigcup_{k \in \mathbb{N}} A_k$ and $\bigcap_{k \in \mathbb{N}} A_k$ are external.

Proof (i) Assume that the assertion fails. Then the internal relation $R := \{\langle B, a \rangle \mid a \in A,\ a \notin B \in {}^*\mathcal{P}(C)\}$ is concurrent on $\mathcal{B}$. Since $V({}^*X)$ be κ-saturated, $A \subsetneq \bigcup \mathcal{B}$.
(ii) Assume that $B := \bigcup_{k \in \mathbb{N}} A_k$ is internal. Since $\mathbb{N}$ is κ-small, by (i), there exists an $m \in \mathbb{N}$ such that $B \subseteq A_1 \cup \cdots \cup A_m = A_m \subsetneq A_{m+1} \subseteq B$, which is a contradiction. The rest follows from DeMorgan's Law and the fact that the complement of an external set is external. □

Corollary 2.9.9 *The sets $\mathbb{N}$, ${}^*\mathbb{N}_\infty$, ${}^*\mathbb{R}_\infty$, $m(0)$ are external.*

Proof We have seen that $\mathbb{N}$ is external. This also follows from the fact that $\mathbb{N}$ is a κ-small set. It now follows that ${}^*\mathbb{N}_\infty = \bigcap_{k \in \mathbb{N}} \{m \in {}^*\mathbb{N} \mid k < m\}$ is external. With a similar proof, it follows that ${}^*\mathbb{R}_\infty$ and $m(0)$ are external. □

Problem: Assume $V({}^*X)$ is $\aleph_1$-saturated. Also assume that A is an internal set such that $\mathbb{N} \cap A$ is infinite. Show that for each unlimited natural number M there exists an unlimited natural number K with $K \leq M$ and $K \in A$.
Answer: Fix an unlimited $M \in {}^*\mathbb{N}$. For each $n \in \mathbb{N}$, let

$$b_n := \left\{k \in {}^*\mathbb{N} \mid n \leq k \leq M \text{ and } k \in A\right\}.$$

The b_n's have the finite intersection property, so there is a $K \in \bigcap b_n$. It follows that $K \in {}^*\mathbb{N}_\infty \cap A$ and $K \leq M$.

We conclude this section by listing the numbers of the major tools needed for applications. They are the following: Part (v) of 2.4.3, 2.4.5, 2.6.4, 2.8.4, 2.8.8, 2.8.9, 2.8.12, 2.9.1. For further background, there is the initiating work of [9] and then [7]. Also we note the text of Vakil [13], that employs a weaker form of nonstandard analysis discussed in this book's preface.

Appendix: Nonstandard Models

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As promised in Sect. 2.9, we shall now show that for each superstructure V of cardinality κ there exists a κ^+-saturated superstructure W and a monomorphism * from V into W. (κ^+ is the smallest cardinality greater than κ.) By the way, in model theory a monomorphism is often called an elementary embedding.

The proof is elementary. This means that the ambitious existence of countably incomplete κ^+ good ultrafilter is not used as for example in the proof of Theorem 6.1.8 in the book of Chang and Keisler [2].

The idea of the proof is similar to the proof in algebra that each field k has an algebraically closed extension K. In the algebraic proof one has to find roots of polynomials; here we have to find elements in the intersection of families of internal sets. Roughly speaking, in algebra one constructs a suitable transfinite increasing chain $(k_\alpha)_{\alpha<\gamma}$ of fields k_α with $k_0 = k$ such that, if α is a successor ordinal, then k_α contains exactly one root of each irreducible polynomial over $k_{\alpha-1}$. If α is a limit number, then k_α is the union of the preceding fields. The union of all the k_α provides an algebraically closed extension of k. Here we construct a transfinite elementary chain; in the successor step we apply the compactness theorem, which says that for each superstructure and each family of internal sets in the superstructure having the finite intersection property there exists a superstructure—with the same formal properties—in which the intersection of the whole family is non-empty. In the limit case we take the so called elementary limit of the preceding structures. In brief, we are going to prove the following

Theorem 2.9.10 *Fix a superstructure $V(X)$ over X of cardinality κ. There exists a superstructure $V(Y)$ over a set Y such that the following two principles hold:*

(T) (The Transfer Principle) There exists a monomorphism * from $V(X)$ into $V(Y)$ with $Y = {}^*X$.

(PS) (Polysaturation) $V({}^*X)$ is κ^+ -saturated.

The proof of this theorem, essentially adopted from the first edition of this book, slightly modifies Sacks [11] corresponding proof for first order logic.

Models

In order to work with long chains of superstructures and monomorphisms between them, it is convenient to consider only the internal part of a superstructure, which we will call a model.

Let $\mathbb{N}_0$ denote the nonnegative integers. A sequence $V := (V_n)_{n\in\mathbb{N}_0}$ is called a **weak model** if $V_0 \neq \emptyset$ and $V_n \subset \mathcal{P}(V_0 \cup \cdots \cup V_{n-1})$ for each $n \geq 1$. The entities in V_0 are called the **individuals** in V, the entities in V_n with $n \geq 1$ are called the **sets** in V.

Since individuals and sets must be treated differently in the proof of Theorem 2.9.10, we think of an individual as an object different from a set, in particular, it is different from the empty set. Moreover, since we are not interested in the elements of an individual, we assume that individuals do not contain any elements. Because of the extensionality axiom, "global" individuals don't exist in Zermelo-Fraenkel (ZF)-set theory. In many books on poly-saturated models the existence of such "global" individuals is presupposed. Here we suggest to use "local" individuals instead. Local individuals are individuals relative to the model according to equations (M 1) and (M 2) below. These local individuals are sufficient to lay the foundation for the theory of poly-saturated models and they exist in ZF-set theory.

A weak model $V = (V_n)_{n\in\mathbb{N}_0}$ is called a **model** if

(M 1) $V_0 \cap \bigcup_{1\le n} V_n = \emptyset$, which means that individuals in a model are different from sets within the model.

(M 2) $b \cap \bigcup_{n\in\mathbb{N}_0} V_n = \emptyset$ for each $b \in V_0$, which means that individuals in a model are empty relative to the model.

A model $V := (V_n)_{n\in\mathbb{N}_0}$ is called a **standard model** if $V_n = \mathcal{P}(V_0 \cup \cdots \cup V_{n-1})$ for each $n \ge 1$. Recall that, if $V := (V_n)_{n\in\mathbb{N}_0}$ is a standard model and $X := V_0$, then $V(X) = \bigcup_{n\in\mathbb{N}_0} V_n$ has been called a superstructure over X. If V is a standard model, by induction it is easy to see that $V_n(X) = X \cup V_n$ for each $n \in \mathbb{N}_0$, where $V_n(X)$ is defined in 2.2.1 letting

$$V_0(X) := X \text{ and } V_{n+1}(X) := V_n(X) \cup \mathcal{P}(V_n(X)).$$

This shows that $V_n(X)$ results from V_n by adding the set X of individuals – if V is a standard model.

Let $S := (S_n)_{n\in\mathbb{N}_0}$ be a sequence of sets. We use the following notation:

$$S_{<\infty} := \bigcup_{n\in\mathbb{N}_0} S_n \text{ and } S_{\le n} := S_0 \cup \cdots \cup S_n.$$

From Weak Models to Models

As was promised in Remark 2.1.3, we will now show that the individuals of a weak model V can be renamed in such a way that V becomes a model. Let us call a (weak) model $V := (V_n)_{n\in\mathbb{N}_0}$ a **(weak) model over** X if $V_0 = X$.

Proposition 2.9.11 *For each set $Y \neq \emptyset$ there exists a bijection c from Y onto X such that each weak model over X is already a model over X.*

Proof Set

$$\widetilde{Y} := \{\{b\} \mid b \in Y\}.$$

Then $\mathrm{card}(\widetilde{Y}) = \mathrm{card}(Y)$. Define by induction:

$$Y_0 := \widetilde{Y},\ Y_{n+1} := \mathcal{P}(Y_n),\ Y_\infty := \bigcup_{n\in\mathbb{N}_0} Y_n,$$

and

$$X := \{\{\{\{y\}, Y_\infty\}\} \mid y \in Y\}.$$

Now define $c(y) := \{\{\{y\}, Y_\infty\}\}$ and rename the elements y of Y by $c(y) \in X$. Notice that

(1) Y_∞ is an infinite set, $\operatorname{card}(a) = 1$ and $\operatorname{card}(\{a, Y_\infty\}) = 2$ for each $a \in \widetilde{Y} \cup X$. It follows that
(2) the function $i : a \mapsto \{\{a, Y_\infty\}\}$ defines a bijection from $\widetilde{Y}$ onto X. Moreover, $Y_0 = \widetilde{Y} \notin X$. Since $\emptyset \in Y_n$ for each $n \geq 1$, $Y_n \notin X$ for each $n \in \mathbb{N} \cup \{\infty\}$.
Let $V := (V_n)_{n\in\mathbb{N}_0}$ be a weak model over X. By induction over $n \in \mathbb{N}_0$, we see that $Y_n \notin V_k$ for each $n \in \mathbb{N}_0$ and each $k \in \{0, \dots, n\}$. It follows that
(3) $Y_\infty \notin V_k$ for each $k \in \mathbb{N}_0$.
Now we prove that V is a model over X :
(4) We first show that $X \cap \bigcup_{n\geq 1} V_n = \emptyset$:
Assume that $k \geq 1$ and $\{\{a, Y_\infty\}\} \in V_k$ for some $a \in \widetilde{Y}$. Then $\{a, Y_\infty\} \in V_i$ for some $i < k$. By (1), $\{a, Y_\infty\} \notin X = V_0$. Therefore, there exists a $j < i$ with $Y_\infty \in V_j$, contradicting (3).
(5) Finally, we prove that $b \cap \bigcup_{n\in\mathbb{N}_0} V_n = \emptyset$ for each $b \in X$:
Fix $b = \{\{a, Y_\infty\}\} \in X$. Assume that there exist $n \in \mathbb{N}_0$ and $x \in V_n$ with $x \in b$. Then $x = \{a, Y_\infty\}$. By (1), $x \notin X = V_0$, thus, $n \geq 1$. It follows that there exists a $k < n$ with $Y_\infty \in V_k$, contradicting (3). □

It depends on the theory we have in mind, which objects we take either as individuals or as sets. If we are not interested in the shape of elements of a mathematical entity, then this entity can be chosen as an individual. For example, in arbitrary Banach spaces the elements may be individuals, but in the Lebesgue L^p-spaces the elements are functions, thus sets. In general, the real numbers should be individuals, but if we want to study real numbers as equivalence classes of Cauchy-sequences of rationals, then real numbers become sets and now the rationals may be chosen to be individuals.

Languages for Models

Monomorphisms between superstructures are stronger than the usual injective homomorphisms or homeomorphic embeddings. A monomorphism not only preserves the algebraic or topological structure, it preserves every mathematical property. In order to make the phrase “mathematical property” precise, we need a mathematical language strong enough to formalize every mathematical statement.

The introduction of this language and its interpretation is similar to the approach at the beginning of this chapter. However, to keep this section self-contained, we repeat the necessary basic facts in a form suitable for the proof of our main result. Given a model $V := (V_n)_{n\in\mathbb{N}_0}$, the alphabet of the language $\mathcal{L}_V$ has the following symbols:

Logical symbols: $\vee$, $\neg$, $\exists$, $=$, $\in$.
Variables: A countable number of them will do.
Parameters: The elements of $V_{<\infty}$ are the parameters in $\mathcal{L}_V$.
Auxiliary symbols: Parentheses “(”,“)”, point “·” and comma “,”.

We assume that all these symbols are pairwise distinct.

A **sentence** in $\mathcal{L}_V$ is built up inductively from these rules:

(a) If $a, b \in V_{<\infty}$, then $(a \in b)$ and $(a = b)$ are sentences in $\mathcal{L}_V$.

Remark 2.9.12 It should be mentioned that in the sentence $a \in b$ the term a may be a set, in particular, a is an n-tuple of elements in $V_{<\infty}$ (see Part (4) of Proposition 2.9.14). See also the proof of Part (10) in that proposition. It would follow that b is an n -placed relation.

(b) If A and B are sentences in $\mathcal{L}_V$, then $(A \vee B)$ and $(\neg A)$ are sentences in $\mathcal{L}_V$.
(c) Let A be a sentence in $\mathcal{L}_V$ and let a, b be parameters in $\mathcal{L}_V$. If x is a variable, not occurring in A, then $(\exists x \in a A_b(x))$ is a sentence in $\mathcal{L}_V$. Here $A_b(x)$ is the string of signs of the alphabet of $\mathcal{L}_V$ that results from A by replacing each b, where b occurs in A, with x.
(d) The set of sentences of $\mathcal{L}_V$ is the smallest set having the properties (a), (b) and (c).

A **formula** in $\mathcal{L}_V$ results from a sentence in $\mathcal{L}_V$ by replacing some parameter with a variable x, not occurring in the sentence. We then say that x is **free** in the formula. We shall write $A(x)$, to indicate that x is free in the formula A.

As usual,we use the following abbreviations:

$$(A \to B) \text{ for } ((\neg A) \vee B),\ (A \wedge B) \text{ for } (\neg((\neg A) \vee (\neg B))),$$
$$(A \leftrightarrow B) \text{ for } ((A \to B) \wedge (B \to A)),$$
$$(\forall x \in a A_b(x)) \text{ for } (\neg(\exists x \in a(\neg A_b(x)))).$$

In order to save parentheses we agree that $\neg$, $\exists$, $\forall$ bind stronger than $\wedge$, $\wedge$ binds stronger than $\vee$, $\vee$ binds stronger than $\to$, and $\to$ binds stronger than $\leftrightarrow$. The relation "binds stronger" is transitive. Moreover, we will use the following shorthand

$$\exists x_1, \dots, x_k \in a\ A \text{ for } \exists x_1 \in a \dots \exists x_k \in a\ A,$$
$$\forall x_1, \dots, x_k \in a\ A \text{ for } \forall x_1 \in a \dots \forall x_k \in a\ A.$$

Interpretation of the Language

Fix a model $V := (V_n)_{n \in \mathbb{N}_0}$. Here is the truth predicate for sentences in $\mathcal{L}_V$:

(a) Fix $a, b \in V_{<\infty}$.

  (i) The sentence $a = b$ is true in V if $a = b$ in the sense of common set theory.
  (ii) The sentence $a \in b$ is true in V if $a \in b$ in the sense of common set theory.

By the definition of models, $a \in b$ can never become true in V if b is an individual of V.

(b) Let A, B be sentences of $\mathcal{L}_V$.

  (i) $\neg A$ is true in V if A is not true in V.
  (ii) $A \vee B$ is true in V if A is true in V or B is true in V.

(iii) $\exists x \in a \; A(x)$ is true in V if there exists a $c \in a$ such that $A(x)(c)$ is true in V. Here the sentence $A(x)(c)$ results from the formula $A(x)$ by replacing x with c.

Following a common practice in model theory, we will write $V \models A$ to denote that A is true in V.

We conclude this section with a remark concerning individuals.

Remark 2.9.13 Recall that if $\emptyset$ is a set in the model V, then a is an individual in V iff

$$a \neq \emptyset \quad \text{and } V \models \neg \exists x (x \in a).$$

(Here $\exists x (x \in a)$ is a shorthand for the cumbersome formula $\exists x \in a (x \in a)$.) In particular, when we assume that the positive integers are individuals in V, we mean they are coded so that they never appear as sets in the model.

Models Closed Under Definition

We will now study models having nice closure properties: A model $V := (V_n)_{n \in \mathbb{N}_0}$ is called **closed under definition** if for each formula $A(x)$ in $\mathcal{L}_V$ and each $n \in \mathbb{N}_0$

$$\left\{a \in V_{\leq n} \mid V \models A(x)(a)\right\} \in V_{n+1}.$$

Proposition 2.9.14 *Suppose that V is closed under definition and the positive integers are individuals in V, coded according to Proposition 2.9.11. Fix $n \geq 1$. Then*

(1) $V_1 \subseteq \cdots \subseteq V_n \subseteq \ldots$, *thus* $V_{\leq n} = V_0 \cup V_n$.
(2) If F is a finite subset of $V_{\leq n-1}$, then $F \in V_n$.
(3) If $a, b \in V_{\leq n-1}$, then $\langle a, b\rangle := \{\{a, b\}, \{b\}\} \in V_{n+1}$.
(4) Fix $a_1, \ldots, a_k \in V_{\leq n-1}$. Then $(a_1, \ldots, a_n) := \{\langle 1, a_1\rangle, \ldots, \langle k, a_k\rangle\} \in V_{n+2}$.
(5) Fix $a_1, \ldots, a_k \in V_n$. Then $a_1 \cup \cdots \cup a_k \in V_n$ and $a_1 \cap \cdots \cap a_k \in V_n$.
(6) If $a, b \in V_n$, then $a \backslash b \in V_n$.
(7) $\emptyset \in V_n$.
(8) $V_{\leq n-1} \in V_n$.
(9) $V_{n-1} \in V_n$.
(10) Fix $a_1, \ldots, a_k \in V_n$. Then

$$a_1 \times \cdots \times a_k = \{(\alpha_1, \ldots, \alpha_k) \mid \alpha_1 \in a_1 \wedge \cdots \wedge \alpha_k \in a_k\} \in V_{n+3}.$$

Proof (1) Let $c \in V_n$. Then

$$c = \left\{a \in V_{\leq n} \mid V \models a \in c\right\} = \left\{a \in V_{\leq n} \mid V \models (x \in c)\,(a)\right\} \in V_{n+1}.$$

(2) Let $F = \{a_1, \ldots, a_k\}$. Then

$$F = \left\{a \in V_{\leq n-1} \mid V \models a = a_1 \vee \cdots \vee\ a = a_k\right\} \in V_n.$$

(3) and (4) These follow from (2).
(5) Since $a_1, \ldots, a_k \subseteq V_{\leq n-1}$,

$$a_1 \cup \cdots \cup\ a_k = \left\{a \in V_{\leq n-1} \mid V \models a\ \in\ a_1 \vee \cdots \vee\ a\ \in\ a_k\right\} \in V_n.$$

The proof for "$\cap$" is similar.
(6) The proof of (6) is similar to the proof of (5).
(7) This is true, because

$$\emptyset = \left\{a \in V_{\leq n-1} \mid V \models \neg a = a\right\} \in V_n.$$

(8) $V_{\leq n-1} = \left\{a \in V_{\leq n-1} \mid V \models a = a\right\} \in V_n$.
(9) By (8), $V_0 = V_{\leq 0} \in V_1$. Let $n > 1$. Then, by (1), (6) and (8),

$$V_{n-1} = V_{\leq n-1} \backslash V_0 \in V_n.$$

(10) Using (4), we obtain:

$$a_1 \times \cdots \times\ a_k = \left\{a \in V_{\leq n+2} \mid V \models \exists x_1 \in a_1 \ldots \exists x_k \in a_k (a = (x_1, \ldots, x_k))\right\} \in V_{n+3},$$

where we have used the following abbreviations:
  $a = (x_1, \ldots, x_k)$ for $\forall x \in V_{\leq n+1}\,(x\ \in\ a \leftrightarrow x = \langle 1, x_1\rangle \vee \cdots \vee x = \langle k, x_k\rangle)$,
  $x = \langle i, x_i\rangle$ for $\forall z \in V_{\leq n}(z\ \in\ x \leftrightarrow z = \{i, x_i\} \vee z = \{x_i\})$,
  $z = \{i, x_i\}$ for $\forall y \in V_{\leq n-1}(y\ \in\ z \leftrightarrow y = i \vee y = x_i)$,
  $z = \{x_i\}$ for $\forall y \in V_{\leq n-1}(y\ \in\ z \leftrightarrow y = x_i)$. $\square$

Elementary Embeddings

Fix a model $V = (V_n)_{n\in\mathbb{N}_0}$ closed under definition and a second model $W = (W_n)_{n\in\mathbb{N}_0}$. Suppose $*$ is a mapping from $V_{<\infty}$ into $W_{<\infty}$, and recall that, if A is a sentence (formula) in $\mathcal{L}_V$, then the $*$-transform of A, ${}^*A$, is the sentence (formula) in $\mathcal{L}_W$ obtained by replacing each parameter a in A by $*(a)$. We will write ${}^*a$ instead of $*(a)$. A mapping $* : V_{<\infty} \to W_{<\infty}$ is called an **elementary embedding** from V into W if (E 1) and (E 2) are true:
  (E 1) For each $n \in \mathbb{N}_0$, ${}^*V_n = W_n$. It follows that $W_n \in W_{<\infty}$.
  (E 2) (Transfer Principle) For each sentence A in $\mathcal{L}_V$

$$V \models A \Leftrightarrow W \models {}^*A.$$

Proposition 2.9.15 *Let $*$ be an elementary embedding from V into W and assume that the positive integers are individuals in V, coded according to Proposition 2.9.11. We obtain for each $n \in \mathbb{N}_0$:*

(1) The restriction $* \restriction V_n$ *maps* V_n *into* W_n*, and* W_n *is a set in* W.
(2) The map $*$ *is injective and a* ***homomorphism****, that is, for* $a, b \in V_{<\infty}$,

$$a \in b \Leftrightarrow {}^*a \in {}^*b.$$

(3) $*(V_{\leq n}) = W_{\leq n} \in W_{n+1}$.
(4) W *is closed under definition.*
(5) Let $A(x)$ *be a formula in* $\mathcal{L}_V$*. Then*

$$^*\left\{a \in V_{\leq n} \mid V \models A(x)(a)\right\} = \left\{b \in W_{\leq n} \mid W \models \left({}^*(A(x))\right)(b)\right\}.$$

(6) If $\{a_1, \ldots, a_k\}$ *is a finite subset of* $V_{<\infty}$*, then* $^*\{a_1, \ldots, a_k\} = \{{}^*a_1, \ldots, {}^*a_k\}$*, in particular,* $^*\emptyset = \emptyset$*. It follows that* $^*(a_1, \ldots, a_k) = ({}^*a_1, \ldots, {}^*a_k)$ *for all* $a_1, \ldots, a_k \in V_{<\infty}$.
(7) Let $a_1, \ldots, a_k, a$ *be sets in* V*. Let* f *be a set in* V *and a mapping from* $a_1 \times \cdots \times a_k$ *into* a*. Then* $^*f$ *is a mapping from* $^*a_1 \times \cdots \times {}^*a_k$ *into* $^*a$ *such that for all* $(b_1, \ldots, b_k) \in a_1 \times \cdots \times a_k$,

$$^*(f(b_1, \ldots, b_k)) = {}^*f({}^*b_1, \ldots, {}^*b_k)).$$

If, in addition, f *is surjective onto* a *or injective or bijective, then* $^*f$ *is surjective onto* $^*a$*, injective or bijective, respectively.*

Proof (1) Since V is closed under definition, $V_n \in V_{n+1}$ and $V_n \neq \emptyset$ for all $n \in \mathbb{N}_0$. Let $a \in V_n$. Since $V \models a \in V_n$, $W \models {}^*a \in W_n$. Therefore, W_n is a set in W and $^*a \in W_n$.

(2) Let $a, b \in V_{<\infty}$. Then

$$a = (\in)\, b \Leftrightarrow V \models a = (\in)\, b \Leftrightarrow W \models {}^*a = (\in)\, {}^*b \Leftrightarrow {}^*a = (\in)\, {}^*b.$$

(3) If $n = 0$, the result follows from (E 1). Let $n \geq 1$. Set

$$A := \forall x \in V_{\leq n}(x \in V_0 \vee x \in V_n) \wedge \forall x \in V_0(x \in V_{\leq n}) \wedge \forall x \in V_n(x \in V_{\leq n}).$$

Since A is true in V, $^*A$ is true in W. It follows that $^*V_{\leq n} = W_0 \cup W_n$. Since $\forall x \in V_k(x \in V_{k+1})$ is true in V for each $k \geq 1$ and therefore the *-image of this sentence is true in W for all $k \geq 1$, by (E 1), $W_{\leq n} = W_0 \cup W_n$.

(4) Let $A(x)$ be a formula in $\mathcal{L}_W$. There is an $m \in \mathbb{N}_0$ such that the parameters $b_1, \ldots, b_k$ occurring in $A(x)$ belong to $W_{\leq m}$. Since V is closed under definition,

$$V \models \forall x_1, \ldots, x_k \in V_{\leq m} \exists z \in V_{n+1} \forall x \in V_{\leq n}(x \in z \leftrightarrow A(x, x_1, \ldots, x_k)))$$

where $A(x, x_1, \ldots, x_k)$ results from $A(x)$ by replacing b_i with x_i, $i = 1, \ldots, k$. We may assume that $x_1, \ldots, x_k$ do not occur in $A(x)$ and are pairwise different. Since $^*V_{\leq n} = W_{\leq n}$ and $^*V_n = W_n$, we obtain the fact that

$$W \models \forall x_1, \dots, x_k \in W_{\leq m} \exists z \in W_{n+1} \forall x \in W_{\leq n} (x \in z \leftrightarrow A(x, x_1, \dots, x_k)).$$

It follows that for $b_1, \dots, b_k$ there is a $B \in W_{n+1}$ such that for $a \in W_{\leq n}$, $a \in B$ iff $W \models A(x)(a)$. Therefore, $B = \left\{a \in W_{\leq n} \mid W \models A(x)(a)\right\}$.

(5) Define $B := \left\{a \in V_{\leq n} \mid V \models A(x)(a)\right\}$. Then $B \in V_{n+1}$ and

$$V \models \forall x \in V_{\leq n} (x \in B \leftrightarrow A(x)).$$

Since $W \models \forall x \in W_{\leq n} (x \in {}^*B \leftrightarrow {}^*A(x))$ and ${}^*B \in W_{n+1}$, we obtain

$${}^*B = \left\{b \in W_{\leq n} \mid W \models {}^*A(x)(b)\right\}.$$

(6) Let $\{a_1, \dots, a_k\} \subseteq V_{\leq n}$. Then, by (1), $\{{}^*a_1, \dots, {}^*a_k\} \subseteq W_{\leq n}$ and, by (5),

$${}^*\{a_1, \dots, a_k\} = {}^*\left\{a \in V_{\leq n} \mid V \models a = a_1 \vee \dots \vee\ a = a_k\right\} =$$

$$\left\{b \in W_{\leq n} \mid W \models b = {}^*a_1 \vee \dots \vee\ b = {}^*a_k\right\} = \left\{{}^*a_1, \dots, {}^*a_k\right\}.$$

Moreover, we obtain for each $n \geq 1$,

$${}^*\emptyset = {}^*\left\{a \in V_{\leq n-1} \mid V \models \neg a = a\right\} = \left\{b \in W_{\leq n-1} \mid W \models \neg b = b\right\} = \emptyset.$$

(7) We identify the k-placed functions f with the 2-placed relations, where the second argument is uniquely determined by the first argument, which is a k-tuple. Therefore, if f is a k-placed function, we can use $x_{k+1} = f(x_1, \dots, x_k)$ as a shorthand for $((x_1, \dots, x_k), x_{k+1}) \in f$.
We may assume that there exists an $n \geq 1$ such that $a_1, \dots, a_k, a, f \in V_n$. Then $V \models A_1 \wedge A_2 \wedge A_3$, where

$$A_1 := \forall x_1 \in a_1 \dots \forall x_k \in a_k \exists y \in a(((x_1, \dots, x_k), y) \in f).$$

$$A_2 := \forall x_1, \dots, x_k, y \in V_{\leq n}(((x_1, \dots, x_k), y) \in f \to x_1 \in a_1 \wedge \dots \wedge x_k \in a_k \wedge y \in a).$$

$$A_3 := \forall x_1, \dots, x_k, y, y' \in V_{\leq n}(((x_1, \dots, x_k), y) \in f \wedge \big((x_1, \dots, x_k), y'\big) \in f \to y = y').$$

A_1 means that domain$(f) \supseteq a_1 \times \dots \times a_k$; A_2 means that range$(f) \subseteq a$ and domain$(f) \subseteq a_1 \times \dots \times\ a_k$; A_3 means that f is a function. Since $V \models A_1 \wedge A_2 \wedge A_3$ and therefore $W \models {}^*A_1 \wedge {}^*A_2 \wedge {}^*A_3$, we see that ${}^*f$ is a mapping from ${}^*a_1 \times \dots \times {}^*a_k$ into ${}^*a$. Assume that $f(b_1, \dots, b_k) = b$. Then $((b_1, \dots, b_k), b) \in f$ and $(({}^*b_1, \dots, {}^*b_k), {}^*b) \in {}^*f$. Therefore, ${}^*f({}^*b_1, \dots, {}^*b_k) = {}^*b = {}^*(f(b_1, \dots, b_k))$. The other parts of assertion (7) can be proved in a similar way. □

Since * is injective, we may identify each individual a in V with the individual ${}^*a$ in W.

Saturated Models

Let us call a non-empty family $\mathcal{B}$ of sets **deep** if $\mathcal{B}$ has the **finite intersection property**, which means that the intersection of all finitely many sets in $\mathcal{B}$ is nonempty. Let γ be an uncountable cardinal. A model $V = (V_n)_{n\in\mathbb{N}_0}$, which is closed under definition, is called **γ-saturated** if for each $n \geq 1$

$$\bigcap \mathcal{B} \neq \emptyset \text{ for each deep } \mathcal{B} \subseteq V_n \text{ with } \operatorname{card}(\mathcal{B}) < \gamma.$$

It follows that the internal sets in a γ-saturated model can be treated as though they were γ-compact (see Sect. 2.9).

From Pre-models to Models

In order to deal with ultrapowers and limits of elementary chains under the same roof, we introduce the notion "pre-model": A triple $((P_n)_{n\in\mathbb{N}_0}, \sim, E)$ is called a **pre-model** if the following conditions are true:

(PM 1) $P_0 \neq \emptyset$ and $P_0 \cap \bigcup_{n\geq 1} P_n = \emptyset$.

(PM 2) $\sim$ is an equivalence relation on $P_{<\infty}$ such that for all $a, b \in P_{<\infty}$ with $a \sim b$ and all $n \in \mathbb{N}_0$:

$$a \in P_n \Rightarrow b \in P_n.$$

(PM 3) The relation E is a subset of $P_{<\infty} \times P_{1\leq}$ such that for $a, a', b, b' \in P_{<\infty}$ with $a' \sim a$ and $b' \sim b$:

$$a\, E\, b \Rightarrow a'\, E\, b'.$$

(PM 4) Transitivity. If $n \geq 1$ and $a \in P_n$, then

$$b\, E\, a \Rightarrow b \in P_{\leq n-1}.$$

(PM 5) Extensionality. Fix $a, b \in P_{1\leq}$. Then $a \sim b$ if $(cEa \Leftrightarrow cEb)$ for all $c \in P_{<\infty}$.

Later on we shall be concerned with two important examples of pre-models.

Fix a pre-model $P = ((P_n)_{n\in\mathbb{N}_0}, \sim, E)$. Using the Mostowski Collapsing Function (see Definition 2.5.3), we define inductively on $n \in \mathbb{N}_0$ sets V_n and their elements $\overline{a}^n$ such that (V_n) becomes a model generated by P.

For each $a \in P_0$ set $\widetilde{a}^0 := \{b \in P_0 | \, b \sim a\}$. According to Proposition 2.9.11, we can rename each $\widetilde{a}^0$ to $\overline{a}^0$ such that the function $\widetilde{a}^0 \mapsto \overline{a}^0$ defines a bijection between $\left\{\widetilde{a}^0 \mid a \in P_0\right\}$ and $\left\{\overline{a}^0 \mid a \in P_0\right\}$, and such that each weak model over

$$V_0 := \{\overline{a}^0 \mid a \in P_0\}$$

is a model over V_0.

Now let $n \geq 1$ and assume that $\overline{a}^k$ is already defined for each $k < n$ and each $a \in P_k$. Moreover, assume that $V_k := \{\overline{a}^k \mid a \in P_k\}$ for each $k < n$. If $a \in P_n$, set

$$\overline{a}^n := \{\overline{c}^k \mid k < n, c \in P_k \text{ and } c \; E \; a\}.$$

and

$$V_n := \{\overline{a}^n \mid a \in P_n\}.$$

Since $P_0 \neq \emptyset$, we have $V_0 \neq \emptyset$. Since for $n \geq 1$, $\overline{a}^n \subset V_{\leq n-1}$, we see that $V_n \subset \mathcal{P}(V_{\leq n-1})$. It follows that $V := (V_n)_{n \in \mathbb{N}_0}$ is a model over V_0. We say that the model V is **generated** by $((P_n)_{n \in \mathbb{N}_0}, \sim, E)$.

Proposition 2.9.16 *Fix $n, m \in \mathbb{N}_0$ and $a \in P_n$ and $b \in P_m$. Then*

$$\overline{a}^n = \overline{b}^m \Leftrightarrow a \sim b.$$

Proof We prove this result by induction on $\mu_{m,n} := \max\{n, m\}$.
Let $\mu_{m,n} = 0$. Then, by the definition of $\overline{a}^0$, $\overline{a}^0 = \overline{b}^0 \Leftrightarrow \widetilde{a}^0 = \widetilde{b}^0 \Leftrightarrow a \sim b$. For the induction step let $\mu_{m,n} > 0$.

Assume that $\overline{a}^n = \overline{b}^m$. Since $V_0 \cap V_{1\leq} = \emptyset$, we have $1 \leq n, m$. Suppose that $c\, E\, a$ holds. By (PM 4), there exists an $i < n$ with $c \in P_i$. It follows that $\overline{c}^i \in \overline{a}^n = \overline{b}^m$. Therefore, there exists $j < m$ and an element $d \in P_j$ with $d\, E\, b$ and $\overline{d}^j = \overline{c}^i$. Since $\mu_{i,j} < \mu_{m,n}$, by the induction hypothesis, $d \sim c$. From (PM 3) it follows that $c\, E\, b$ holds. Therefore, $c\, E\, a$ implies $c\, E\, b$ and vice versa. By (PM 5), $a \sim b$.

Now assume that $a \sim b$. Since $m > 0$ or $n > 0$, by (PM 1) and (PM 2), $1 \leq m, n$. In order to show that $\overline{a}^n = \overline{b}^m$, let $\overline{c}^i \in \overline{a}^n$ with $i < n$, $c \in P_i$ and $c\, E\, a$. By (PM 3), $c\, E\, b$. By (PM 4), there exists a $j < m$ with $c \in P_j$. It follows that $\overline{c}^j \in \overline{b}^m$. Since $\mu_{i,j} < \mu_{m,n}$, by the induction hypothesis, $\overline{c}^i = \overline{c}^j \in \overline{b}^m$. This proves that $\overline{a}^n \subseteq \overline{b}^m$. The proof of $\overline{a}^n \supseteq \overline{b}^m$ is similar. □

In view of the previous proposition, we may define for each $a \in P_{<\infty}$,

$$\overline{a} := \overline{a}^n \text{ if } a \in P_n.$$

We obtain the following

Corollary 2.9.17 *Fix $a, b \in P_{<\infty}$. Then*

(1) $\overline{a} = \overline{b} \Leftrightarrow a \sim b$.
(2) $\overline{a} \in \overline{b} \Leftrightarrow a\, E\, b$.

Proof We only need to prove (2). Assume that $\overline{a} \in \overline{b}$. Since V is a model, $\overline{b}$ is a set in V. Therefore, there exist $i \in \mathbb{N}_0$ and $a' \in P_i$ with $a'\, E\, b$ and $\overline{a'} = \overline{a}$. By (1), $a' \sim a$. By (PM 3), $a\, E\, b$. Now assume that $a\, E\, b$. Since $b \in P_n$ for some $n \geq 1$, by (PM 4), there exists an $i < n$ with $a \in P_i$. By the definition of $\overline{b}$, $\overline{a} \in \overline{b}$. □

Ultrapowers

The aim of this section is the proof of the compactness theorem. We assume that the reader is familiar with filters and ultrafilters (see Sect. 1.2). First we will construct a pre-model generating the ultrapower of a model V:

Fix a non-empty set I, an ultrafilter D on I and a model $V = (V_n)_{n\in\mathbb{N}_0}$, which is closed under definition. We shall use the following abbreviations.

If $(a_i)_{i\in I}$ is an I-sequence, then we will write (a_i) instead of $(a_i)_{i\in I}$. If $\mathcal{F}(i)$ is an assertion about elements i of I, then we shall write $\mathcal{F}(i)$ a.e. instead of $\{i \in I \mid \mathcal{F}(i)\} \in D$. For each $n \in \mathbb{N}_0$ set

$$F_n := \{(a_i) \mid a_i \in V_n \text{ a.e.}\}.$$

On $F_{<\infty} = \bigcup_{n\in\mathbb{N}_0} F_n$ we define a relation $\sim$ by setting

$$(a_i) \sim (b_i) :\Leftrightarrow a_i = b_i \text{ a.e.}$$

and we define a relation $E \subseteq F_{<\infty} \times F_{1\leq}$ by setting,

$$(a_i)\ E\ (b_i)\ :\Leftrightarrow a_i \in b_i \text{ a.e.}$$

Using the ultrafilter properties one can easily prove the following

Lemma 2.9.18 *The triple* $\big((F_n)_{n\in\mathbb{N}_0}, \sim, E\big)$ *is a pre-model.*

The model generated by $((F_n)_{n\in\mathbb{N}_0}, \sim, E)$ is called the D-**ultrapower of** V and is denoted by $\Pi_D(V)$.

Now we shall show that the D-ultrapower of V has the same formal properties as V. Let A be a (sentence) formula in $\mathcal{L}_{\Pi_D(V)}$. By $(A^i)_{i\in I}$ we denote one of the I-sequences of sentences (formulas) in $\mathcal{L}_V$ that result from A by replacing each parameter $\overline{(a_i)}$ in A by a_i. Notice that, if $(\widetilde{A}^i)_{i\in I}$ is another result, then $\widetilde{A}^i = A^i$ a.e.

Theorem 2.9.19 (Theorem of Łŏs) *Fix a sentence A in $\mathcal{L}_{\Pi_D(V)}$. Then,*

$$\Pi_D(V) \models A \Leftrightarrow V \models A^i \textit{ a.e.}$$

Proof By induction over the definition of the sentences in $\mathcal{L}_{\Pi_D(V)}$.

(1) (a) Let A be the sentence $(\overline{(a_i)} \in \overline{(b_i)})$. Then

$$\Pi_D(V) \models \overline{(a_i)} \in \overline{(b_i)} \Leftrightarrow \overline{(a_i)} \in \overline{(b_i)} \Leftrightarrow a_i \in b_i \text{ a.e.} \Leftrightarrow V \models A^i \text{ a.e.}$$

(b) If $A = (\overline{(a_i)} = \overline{(b_i)})$, then the proof is similar to the proof under (a).

(2) (a) Let $A = B \vee C$ be a formula of $\mathcal{L}_{\Pi_D(V)}$. Assume that the assertion is true for B and C.

Suppose that $\Pi_D(V) \models A$. Then $\Pi_D(V) \models B$ or $\Pi_D(V) \models C$. By the induction hypothesis, $V \models B^i$ a.e. or $V \models C^i$ a.e. Therefore, $V \models B^i \vee C^i$ a.e. The result follows, since A^i equals $B^i \vee C^i$ a.e.

Suppose that $V \models B^i \vee C^i$ a.e. Since D is an ultrafilter, $V \models B^i$ a.e. or $V \models C^i$ a.e. By the induction hypothesis, $\Pi_D(V) \models B$ or $\Pi_D(V) \models C$, thus, $\Pi_D(V) \models B \vee C$.

(b) Let $A = \neg B$.

Assume that $\Pi_D(V) \models A$, thus, $\Pi_D(V) \not\models B$. By the induction hypothesis, $\{V \models B^i\} \notin D$. Since D is an ultrafilter $\{V \not\models B^i\} \in D$, thus, $V \models \neg B^i$ a.e.

Assume that $V \models \neg B^i$ a.e. Since $\emptyset \notin D$, $\{V \vDash B^i\} \notin D$. The induction hypothesis implies $\Pi_D(V) \models \neg B$.

(c) Let $A = \exists x \in \overline{(a_i)} B(x)$.

If $\Pi_D(V) \models A$, then there is a $\overline{(b_i)} \in \overline{(a_i)}$ with $\Pi_D(V) \models B(x)(\overline{(b_i)})$. By the induction hypothesis, $V \models B(x)^i(b_i)$ a.e. Since $b_i \in a_i$ a.e., $V \models A^i$ a.e.

Suppose that $V \models A^i$ a.e., that is, $Y := \{i \in I \mid V \models \exists x \in a_i(B(x)^i)\} \in D$. We choose an I-sequence (b_i) with $V \models b_i \in a_i \wedge B(x)^i(b_i)$ for each $i \in Y$ and set $b_i = \emptyset$ if $i \notin Y$. Since $V \models B(x)^i(b_i)$ a.e., by the induction hypothesis, $\Pi_D(V) \models B(x)(\overline{(b_i)})$. Since $b_i \in a_i$ a.e., $\overline{(b_i)} \in \overline{(a_i)}$. It follows that $\Pi_D(V) \models A$. □

An important consequence of Łŏs' theorem is the existence of an elementary embedding from V into the ultrapower $\Pi_D(V)$ of V.

For each $b \in V_{<\infty}$ let (b) be the constant I-sequence (b_i) with $b_i = b$ for each $i \in I$. The mapping $^* : V_{<\infty} \to \Pi_D(V)_{<\infty}$ is defined by setting

$$^*b := \overline{(b)}.$$

Corollary 2.9.20 *The function $*$ is an elementary embedding from V into $\Pi_D(V)$.*

Proof We have to check the properties (E 1) and (E 2).

(E 1) Since $V_n \in V_{n+1}$, $(V_n) \in F_{n+1}$, thus $^*V_n = \overline{(V_n)} \in \Pi_D(V)_{<\infty}$. By Corollary 2.9.17 (2), and since V_n is a set in V, we obtain

$$\overline{(a_i)} \in \overline{(V_n)} \Leftrightarrow (a_i)\ E\ (V_n) \Leftrightarrow \overline{(a_i)} \in \Pi_D(V)_n.$$

This proves that $^*V_n = \Pi_D(V)_n$.

(E 2) Let A be a sentence in $\mathcal{L}_V$. Then $^*A$ is a sentence in $\mathcal{L}_{\Pi_D(V)}$ and $(^*A)^i = A$ a.e. By the theorem of Łŏs, we obtain

$$V \models A \Leftrightarrow V \models (^*A)^i \text{ a.e.} \Leftrightarrow \Pi_D(V) \models {}^*A. \square$$

Recall from Proposition 2.9.15 (4) that the model $\Pi_D(V)$ is closed under definition. Using the ultrapower construction given above, we can prove the Compactness Theorem. The proof is the same as in first order model theory (see [2]).

Proposition 2.9.21 (The Compactness Theorem) *Let $V = (V_n)_{n\in\mathbb{N}_0}$ be a model which is closed under definition, let $n \geq 1$ and let $\mathcal{B}$ be a deep subset of V_n. Then there exists a model $W = (W_n)_{n\in\mathbb{N}_0}$ and an elementary embedding * from V into W, such that $\bigcap {}^*[\mathcal{B}] \neq \emptyset$. (Notice that $\bigcap {}^*[\mathcal{B}] = \bigcap \{{}^*A \mid A \in \mathcal{B}\}$.)*

Proof We may assume that $\mathcal{B}$ is closed under finite intersections. Let I be the set of finite non-empty subsets of $\mathcal{B}$. Since $\mathcal{B}$ is deep, for each $i \in I$ there exists an $a_i \in V_{\leq n-1}$ such that $a_i \in \bigcap i \in \mathcal{B}$. For each set $A \in \mathcal{B}$ define

$$\widetilde{A} := \{i \in I \mid a_i \in A\} \subset I,$$

and

$$G := \{\widetilde{A} \mid A \in \mathcal{B}\} \subseteq \mathcal{P}(I).$$

Then $\emptyset \notin G$, because $\{A\} \in \widetilde{A}$ for each $A \in \mathcal{B}$. Since $\mathcal{B} \neq \emptyset$, $G \neq \emptyset$. Now let $A, B \in \mathcal{B}$. Then $\widetilde{A \cap B} \in G$ and it is easy to see that $\widetilde{A \cap B} \subset \widetilde{A} \cap \widetilde{B}$. It follows that $F(G) := \{B \subseteq I \mid \exists C \in G(C \subseteq B)\}$ is a filter. Fix an ultrafilter $D \supseteq F(G)$ on I. Let $\Pi_D(V)$ be the D-ultrapower of V and let $* : V_{<\infty} \to \Pi_D(V)_{<\infty}$ be the elementary embedding introduced previously. Since $a_i \in V_{\leq n-1}$ and D is an ultrafilter, $(a_i) \in F_{\leq n-1}$, thus $\overline{(a_i)} \in \Pi_D(V)_{\leq n-1}$. In order to show that $\overline{(a_i)} \in \bigcap {}^* [\mathcal{B}]$, fix $A \in \mathcal{B}$. Then

$$\{i \in I \mid V \models a_i \in A\} = \{i \in I \mid a_i \in A\} = \widetilde{A} \in G \subset F(G) \subset D.$$

By the theorem of Łŏs, $\Pi_D(V) \models \overline{(a_i)} \in \overline{(A)} = {}^*A$. □

Elementary Chains and Their Elementary Limits

Here we extend the notions “elementary chain” and “elementary limit” to our notion of a model. Let λ be an ordinal number different from 0 and let $(V^\alpha)_{\alpha<\lambda}$ be a λ-sequence of models $V^\alpha = (V_n^\alpha)_{n\in\mathbb{N}_0}$, which are closed under definition. Then the pair $\big((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda}\big)$ is called an **elementary chain** if for each $\alpha \leq \beta \leq \gamma < \lambda$

(EC 1) $*_{\alpha\beta}$ is an elementary embedding from V^α into V^β.

(EC 2) $*_{\alpha\alpha}(a) = a$ for each $a \in V^\alpha_{<\infty}$.

(EC 3) $*_{\beta\gamma} \circ *_{\alpha\beta} = *_{\alpha\gamma}$.

Starting from an elementary chain C we define a pre-model such that the generated model becomes the elementary limit of C. To this end fix an elementary chain $\big((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda}\big)$. We define for each $n \in \mathbb{N}_0$,

$$P_n := \{(a, \alpha) \mid \alpha < \lambda \text{ and } a \in V_n^\alpha\}.$$

Two elements $(a, \alpha), (b, \beta) \in P_{<\infty} = \bigcup_{n\in\mathbb{N}_0} P_n$ are called **equivalent** if there exists a $\gamma < \lambda$ with $\alpha, \beta \leq \gamma$ such that $*_{\alpha\gamma}(a) = *_{\beta\gamma}(b)$, in which case we shall write $(a, \alpha) \sim (b, \beta)$.

For $(a, \alpha) \in P_{<\infty}$ and $(b, \beta) \in P_{1\leq}$ we set $(a, \alpha)\ E\ (b, \beta)$ if there exists a $\gamma < \lambda$ with $\alpha, \beta \leq \gamma$ such that $*_{\alpha\gamma}(a) \in *_{\beta\gamma}(b)$. The simple proof of the next result is left to the reader:

Lemma 2.9.22 *$((P_n)_{n\in\mathbb{N}_0}, \sim, E)$ is a pre-model.*

The model, generated by the pre-model $((P_n)_{n\in\mathbb{N}_0}, \sim, E)$, is called the **elementary limit** of the elementary chain $((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$ and is denoted by $V^\lambda = (V_n^\lambda)_{n\in\mathbb{N}_0}$.

We shall now collect some immediate consequences of the construction of the elementary limit (see Corollary 2.9.17 (1) and (2)).

Proposition 2.9.23 *Fix an elementary chain $((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$ and let V^λ be its elementary limit. Fix $n \in \mathbb{N}_0$. Then*

(1) $V_n^\lambda = \{\overline{(a, \alpha)} \mid (a, \alpha) \in P_n\} = \{\overline{(a, \alpha)} \mid \alpha < \lambda$ *and* $a \in V_n^\alpha\}$.
Fix $\alpha, \beta < \lambda, a \in V_{<\infty}^\alpha, b \in V_{<\infty}^\beta$.

(2) $\overline{(a, \alpha)} = \overline{(b, \beta)}$ *iff there is a γ with $\alpha, \beta \leq \gamma < \lambda$ and $*_{\alpha\gamma}(a) = *_{\beta\gamma}(b)$. It follows that $\overline{(a, \alpha)} = \overline{(*_{\alpha\beta}(a), \beta)}$ if $\alpha \leq \beta < \lambda$.*

(3) $\overline{(a, \alpha)} \in \overline{(b, \beta)}$ *iff there is a $\gamma < \lambda$ with $\alpha, \beta \leq \gamma$ and $*_{\alpha\gamma}(a) \in *_{\beta\gamma}(b)$.*

(4) $\overline{(V_n^\alpha, \alpha)} = \overline{(V_n^\beta, \beta)}$*, because for each γ with $\alpha, \beta \leq \gamma < \lambda$*

$$*_{\alpha\gamma}(V_n^\alpha) = V_n^\gamma = *_{\beta\gamma}(V_n^\beta).$$

Definition 2.9.24 Fix an elementary chain $((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$ and its elementary limit V^λ. For each $\alpha \leq \lambda$ define a mapping $*_{\alpha\lambda} : V_{<\infty}^\alpha \to V_{<\infty}^\lambda$ by setting

$$*_{\alpha\lambda}(a) := \overline{(a, \alpha)} \text{ for } \alpha < \lambda \text{ and } *_{\lambda\lambda}\left(\overline{(a, \alpha)}\right) := \overline{(a, \alpha)}.$$

Proposition 2.9.25 *Fix $\alpha < \lambda$. Then $*_{\alpha\lambda}$ is an elementary embedding from V^α into V^λ. It follows that, $((V^\alpha)_{\alpha<\lambda+1}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda+1})$ is an elementary chain.*

Proof To prove (E 1), fix $n \in \mathbb{N}_0$. We have to show that $V_n^\lambda = *_{\alpha\lambda}(V_n^\alpha)$:

"$\subseteq$" Fix $x \in V_n^\lambda$. Then there exist $\beta < \lambda$ and $b \in V_n^\beta$ with $x = \overline{(b, \beta)}$. By (3) and (4),

$$x = \overline{(b, \beta)} \in \overline{(V_n^\beta, \beta)} = \overline{(V_n^\alpha, \alpha)} = *_{\alpha\lambda}V_n^\alpha.$$

"$\supseteq$" Fix $x \in *_{\alpha\lambda}(V_n^\alpha) = \overline{(V_n^\alpha, \alpha)}$. Since V_n^α is a set in V^α, $\overline{\left(V_n^\alpha, \alpha\right)}$ is a set in V^λ. Therefore, there exist $\beta < \lambda$ and $b \in V_{<\infty}^\beta$ with $x = \overline{(b, \beta)}$. By Corollary 2.9.17 (2), $(b, \beta)\ E\ \left(V_n^\alpha, \alpha\right)$, thus there exists a $\gamma \geq \alpha, \beta$ with $*_{\beta\gamma}(b) \in *_{\alpha\gamma}(V_n^\alpha) = V_n^\gamma$. We obtain

$$x = \overline{(b, \beta)} = \overline{(*_{\beta\gamma}(b), \gamma)} \in V_n^\lambda.$$

This proves (E 1). To prove (E 2), we will prove by induction on the definition of the sentences A in $\bigcup_{\alpha<\lambda} \mathcal{L}_{V^\alpha}$ that if A is in $\mathcal{L}_{V^\alpha}$ then

$$V^\alpha \models A \Leftrightarrow V^\lambda \models {}^{*_{\alpha\lambda}}A.$$

(1) (a) Let $A = (a \in b)$. Then

$$V^\alpha \models A \Leftrightarrow a \in b \Leftrightarrow (a, \alpha)\, E\, (b, \alpha) \Leftrightarrow \overline{(a, \alpha)} \in \overline{(b, \alpha)}$$

$$\Leftrightarrow *_{\alpha\lambda}(a) \in *_{\alpha\lambda}(b) \Leftrightarrow V^\lambda \models {}^{*_{\alpha\lambda}}A.$$

The second "$\Leftarrow$ " of the previous computation can be seen as follows: If $(a, \alpha)\, E\, (b, \alpha)$, then $*_{\alpha\gamma}(a) \in *_{\alpha\gamma}(b)$ for some $\gamma \geq \alpha$, thus $a \in b$.

(b) If $A = (a = b)$, then the proof is similar to the proof of (1) (a).

(2) If $A = (B \vee C)$ or $A = \neg B$, then the assertion follows immediately from the induction hypothesis.

Let $A = \exists x \in a B(x)$.
Assume that $V^\alpha \models A$. Then there is a $b \in a$ with $V^\alpha \models B(x)(b)$. By the induction hypothesis, $V^\lambda \models {}^{*_{\alpha\lambda}}B(x)\left(\overline{(b, \alpha)}\right)$. Since $\overline{(b, \alpha)} \in \overline{(a, \alpha)}$, $V^\lambda \models {}^{*_{\alpha\lambda}}A$.
Assume that $V^\lambda \models {}^{*_{\alpha\lambda}}A$. Then $V^\lambda \models {}^{*_{\alpha\lambda}}B(x)\left(\overline{(b, \delta)}\right)$ for some $\overline{(b, \delta)} \in \overline{(a, \alpha)}$. Set $\gamma := \max\{\alpha, \delta\}$. Since $\overline{(*_{\delta\gamma}(b), \gamma)} = \overline{(b, \delta)}$, $V^\lambda \models {}^{*_{\gamma\lambda}}({}^{*_{\alpha\gamma}}B(x)\left(*_{\delta\gamma}(b)\right))$. By the induction hypothesis, $V^\gamma \models {}^{*_{\alpha\gamma}}B(x)\left(*_{\delta\gamma}(b)\right)$. Since $*_{\delta\gamma}(b) \in *_{\alpha\gamma}(a)$, we obtain the fact that $V^\gamma \models \exists x \in *_{\alpha\gamma}(a)\, {}^{*_{\alpha\gamma}}B(x)$, that is, $V^\gamma \models {}^{*_{\alpha\gamma}}\exists x \in a B(x)$. Since the parameters of $\exists x \in a B(x)$ belong to $V^\alpha_{<\infty}$, we obtain $V^\alpha \models \exists x \in a B(x)$. $\square$

Existence of Polysaturated Nonstandard Models

In order to finish the proof of Theorem 2.9.10, we need one more lemma.

Lemma 2.9.26 *Fix a model $W := (W_n)_{n\in\mathbb{N}_0}$ that is closed under definition. Then there exist a model U and an elementary embedding $*$ from W into U such that for each $n \geq 1$ and each deep $\mathcal{B} \subseteq W_n$, $\bigcap {}^*[\mathcal{B}] \neq \emptyset$.*

Proof There exists a cardinal number θ and a listing $(\mathcal{B}_\alpha)_{\alpha<\theta}$ of all deep subsets $\mathcal{B}_\alpha \subseteq W_n$ for some $n \geq 1$. By transfinite recursion, we define an elementary chain $((W^\alpha)_{\alpha<\theta}, (*_{\alpha\beta})_{\alpha\leq\beta<\theta})$ in the following way: Set

$$W^0 := W,\ *_{00} := \text{identity} \restriction W_{<\infty}.$$

Assume that $\lambda < \theta$ and that W^α, $*_{\alpha\beta}$ are already defined for each $\alpha \leq \beta < \lambda$ such that the following conditions $(1, \lambda)$ and $(2, \lambda)$ hold:

$$(1, \lambda)\ ((W^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda}) \text{ is an elementary chain,}$$

$$(2, \lambda)\ \bigcap{}^{*_{0\alpha}}\left[\mathcal{B}_\beta\right] \neq \emptyset \text{ for each } \beta < \alpha < \lambda.$$

In order to construct W^λ and $*_{\alpha\lambda}$ for each $\alpha < \lambda$, we have to consider two cases:

Case 1: $\lambda = \gamma + 1$ is a successor ordinal.

Since $\mathcal{B}_\gamma$ is deep and $*_{0\gamma}$ is an elementary embedding from W^0 into W^γ, ${}^{*_{0\gamma}}\left[\mathcal{B}_\gamma\right]$ is deep. By the compactness theorem, there exist a model W^λ and an elementary embedding $*_{\gamma\lambda}$ from W^γ into W^λ, such that $\bigcap{}^{*_{\gamma\lambda}\circ *_{0\gamma}}\left[\mathcal{B}_\gamma\right] \neq \emptyset$. We now define for each $\alpha \leq \gamma$:

$$*_{\alpha\lambda} := *_{\gamma\lambda} \circ *_{\alpha\gamma} \text{ and } *_{\lambda\lambda} := \text{identity} \upharpoonright W^\lambda_{<\infty}.$$

Notice that $(1, \lambda + 1)$ and $(2, \lambda + 1)$ are true.

Case 2: λ is a limit ordinal.

Let W^λ be the elementary limit of $((W^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$. For each $\alpha < \lambda$, the elementary embedding $*_{\alpha\lambda}$ from W^α into W^λ is defined in Definition 2.9.24. Set $*_{\lambda\lambda} :=$ identity $\upharpoonright W^\lambda_{<\infty}$. Notice that $(1, \lambda + 1)$ and $(2, \lambda + 1)$ are true.

We thus obtain an elementary chain $((W^\lambda)_{\lambda<\theta}, (*_{\alpha\lambda})_{\alpha\leq\lambda<\theta})$. Let $U := W^\theta$ be its elementary limit. Set ${}^* := *_{0\theta}$. By Proposition 2.9.25, $*$ is an elementary embedding from W into U. It is easy to check that $\bigcap{}^*[\mathcal{B}] \neq \emptyset$ if $\mathcal{B}$ is a deep subset of W_n with $n \geq 1$. □

Now we are able to finish the proof of Theorem 2.9.10: fix a superstructure $V(X)$ of cardinality κ. Let $V = (V_n)_{n\in\mathbb{N}_0}$ be the standard model over X. Then $V(X) = V_{<\infty}$. Let κ^+ be the smallest cardinal number greater than κ. Then κ^+ is a **regular** cardinal, that is, for each $\rho < \kappa^+$ and each ρ-sequence $(\alpha_\beta)_{\beta<\rho}$ in κ^+, $\sup_{\beta<\rho}\alpha_\beta < \kappa^+$. By transfinite recursion, we construct again an elementary chain $((V^\alpha)_{\alpha<\kappa^+}, (*_{\alpha\beta})_{\alpha\leq\beta<\kappa^+})$: Set

$$V^0 := V \text{ and } *_{00} := \text{identity} \upharpoonright V_{<\infty}.$$

Fix an ordinal λ strictly between 0 and κ^+ and assume that there already exists an elementary chain $((V^\alpha)_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$ such that

$$\text{for } \alpha, \beta < \lambda, \alpha < \beta,\ n \geq 1 \text{ and each deep } \mathcal{B} \subseteq V^\alpha_n,\ \bigcap{}^{*_{\alpha\beta}}[\mathcal{B}] \neq \emptyset \qquad (\diamond\lambda)$$

We now define V^λ and $*_{\alpha\lambda}$ for each $\alpha \leq \lambda$:

First assume that $\lambda = \gamma + 1$ is a successor ordinal.

Then, by Lemma 2.9.26, there exists a model V^{λ} and an elementary embedding $*_{\gamma\lambda}$ from V^{γ} into V^{λ}, such that for each $n \geq 1$ and each deep $\mathcal{B} \subseteq V_n^{\gamma}$, $\bigcap^{*_{\gamma\lambda}}[\mathcal{B}] \neq \emptyset$. For each $\alpha < \lambda$ set

$$*_{\alpha\lambda} := *_{\gamma\lambda} \circ *_{\alpha\gamma} \text{ and } *_{\lambda\lambda} := \text{identity} \restriction V^{\lambda}_{<\infty}.$$

Note that $((V^{\alpha})_{\alpha<\lambda+1}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda+1})$ is an elementary chain such that $\diamondsuit(\lambda+1)$ is true.

Now assume that λ is a limit number.

Let V^{λ} be the elementary limit of $((V^{\alpha})_{\alpha<\lambda}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda})$. For $\alpha \leq \lambda$ let $*_{\alpha\lambda}$ be the elementary embedding from V^{α} into V^{λ}, defined in Definition 2.9.24. Notice that $((V^{\alpha})_{\alpha<\lambda+1}, (*_{\alpha\beta})_{\alpha\leq\beta<\lambda+1})$ is an elementary chain and $\diamondsuit(\lambda+1)$ is true.

Let $W := V^{\kappa^+}$ be the elementary limit of $((V^{\alpha})_{\alpha<\kappa^+}, (*_{\alpha\beta})_{\alpha\leq\beta<\kappa^+})$. Set $^* := *_{0\kappa^+}$ Then * is an elementary embedding from V into W. To prove that W is κ^+ -saturated, fix $n \geq 1$ and a deep set $\mathcal{B} \subseteq W_n$ with $\text{card}(\mathcal{B}) < \kappa^+$. Since κ^+ is a regular cardinal, there exists a $\delta < \kappa^+$ such that

$$\mathcal{B} \subseteq \left\{\overline{(b,\beta)} \mid \beta < \delta \text{ and } b \in V_n^{\beta}\right\}.$$

Set $\mathcal{B}' := \left\{^{*_{\beta\delta}}(b) \mid \beta < \delta,\ b \in V_n^{\beta} \text{ and } \overline{(b,\beta)} \in \mathcal{B}\right\}$. Notice that $^{*_{\delta\kappa^+}}[\mathcal{B}'] = \mathcal{B}$ and that $\mathcal{B}'$ is deep. From the construction of $((V^{\alpha})_{\alpha<\kappa^+}, (*_{\alpha\beta})_{\alpha\leq\beta<\kappa^+})$ it follows that $\bigcap^{*_{\delta(\delta+1)}}[\mathcal{B}'] \neq \emptyset$. Since $*_{(\delta+1)\kappa^+}$ is an elementary embedding from $V^{\delta+1}$ into $V^{\kappa^+} = W$, by (EC 3),

$$\bigcap \mathcal{B} = \bigcap^{*_{\delta\kappa^+}}[\mathcal{B}'] = \bigcap^{*_{(\delta+1)\kappa^+} \circ *_{\delta(\delta+1)}}[\mathcal{B}'] \neq \emptyset.$$

Now set $Y := W_0$. Then $Y = {}^*V_0 = {}^*X$, and $V(Y)$ is a superstructure over Y with $W_{<\infty} \subseteq V(Y)$.

Finally, we will prove that the internal entities in $V(Y)$ coincide with the elements in $W_{<\infty}$: Let $B \in V(Y)$ be internal. Then there exists an $A \in V(X)\backslash X = V_{1\leq}$ with $B \in {}^*A$. Since ${}^*A \in W_{1\leq}$, we obtain the fact that $B \in W_{<\infty}$. Now assume that $B \in W_{<\infty}$. Then $B \in W_n = {}^*V_n$ for some $n \in \mathbb{N}_0$. This proves that B is internal. □

We end this appendix with an important application of Theorem 2.9.10.

Theorem 2.9.27 *Fix an internal set A in $V({}^*X)$, a set I in $V(X)$ and a mapping $f : I \to A$. Then there exists an internal mapping $F : {}^*I \to A$ with $f(i) = F({}^*i)$ for all $i \in I$. Since the mapping $*$ is injective, we may identify $i \in I$ with ${}^*i \in {}^*I$, thus F is an internal extension of f.*

Proof If I is standard finite, then f is internal, thus f is its own internal extension (see Proposition 2.9.14 Part (2)). Therefore, we may assume that I is infinite, thus $A \neq \emptyset$. Let $\mathcal{E}(I)$ denote the set of all finite subsets of I. For all $E \in \mathcal{E}(I)$ define,

$$\mathcal{B}_E := \left\{F : {}^*I \to A \mid F \text{ is internal and } \forall i \in E \left(f(i) = F({}^*i)\right)\right\}.$$

Note that $\mathcal{B}_E$ is internal for each $E \in \mathcal{E}(I)$ and that there exists an internal $F : {}^*I \to A$ such that $f(i) = F({}^*i)$ for all $i \in E$. Therefore, $\{\mathcal{B}_E \mid E \in \mathcal{E}(I)\}$ has the finite intersection property. Since its cardinality is smaller than κ^+, there exists an $F \in \mathcal{B}_E$ for all $E \in \mathcal{E}(I)$. This F is an internal extension of f. $\square$

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