Preface

This book is addressed to mathematicians working in analysis, probability, and various applications. The aim is to provide an understandable introduction to the basic theory of nonstandard analysis in Part I, and then illuminate some of the most striking applications. Much of the book, in particular Part I, can be used in a graduate course; problems are posed in those chapters. After Part I, each chapter takes up a different field for the application of nonstandard analysis, beginning with a gentle introduction that even nonexperts can read with profit. The remainder of each chapter is then addressed to experts, showing how to use nonstandard analysis in the search for solutions of open problems and how to obtain rich new structures that produce deep insight into the field under consideration.

The applications discussed here are in functional analysis including operator theory, topology applied to compactifications, probability theory including stochastic processes, economics including game theory and financial mathematics, and combinatorial number theory. In all of these areas, the intuitive notion of an infinitely small or infinitely large quantity plays an essential and helpful role in the creative process. For example, Brownian motion is often thought of as a random walk with infinitesimal increments; the spectrum of a selfadjoint operator is viewed as the set of "almost eigenvalues"; an ideal economy consists of an infinite number of agents each having an infinitesimal influence on the economy. Already at the level of calculus, one often views the integral as an infinitely large sum of infinitesimal quantities.

Of course, the notion of an infinitesimal quantity has been used in mathematics for over 2200 years. Although it was one of the leading ideas employed during the period of mathematical development from Leibniz and Newton to Cauchy, it eluded rigorous treatment until the work of Abraham Robinson between 1957 and 1966, culminating in 1966 with the publication of his book "Non-standard Analysis". That work finally established a rigorous foundation for the use of infinitesimals in mathematics. More precisely, Robinson constructed an ordered field extension ${}^*\mathbb{R}$ of the reals, $\mathbb{R}$, so that all of the sequences, functions and, indeed, relations of real analysis had a unique extension to the equivalent structure built from ${}^*\mathbb{R}$, and all statements true for the real structure remained true for the extended objects in the extended structure. This essential property, known as the Transfer Principle, is the pivotal result of Robinson's nonstandard analysis. With subsequent contributions to this new discipline from many mathematicians in the late 1960s, including a new result in standard functional analysis obtained by Robinson together with A.R. Bernstein, it became clear that nonstandard analysis was much more than a foundation of infinitesimal calculus. It promised to become a powerful tool in all branches of analysis.

Then, an important breakthrough was initiated by W.A.J. Luxemburg's 1969 paper developing nonstandard hulls, and was extended with the work of C.W. Henson and L.C.R. Moore. The prior construction of ultrapowers of the reals begun by E. Hewitt and generalized to ultraproducts of Banach spaces by J.L. Krivine in the mid-1960s had become a powerful tool in functional analysis. Nonstandard functional analysis using nonstandard hulls, discussed here in Manfred Wolff's Chap. 4, is a far reaching generalization of these applications of ultraproducts. Chapter 4 deals with old and new applications of nonstandard analysis to the theory of Banach spaces and linear operators. In particular it considers the structure theory of Banach spaces, basic operator theory, strongly continuous semigroups of operators, approximation theory of operators and their spectra, and the Fixed Point Property.

The early 1970s produced another breakthrough with P.A. Loeb's construction of a measure space out of a "hyperfinite" discrete analogue of finite probability spaces. The simplest example was based on a coin toss of length H for an infinitely large natural number H. Immediately, Loeb's general construction was successfully used by R.M. Anderson to formulate Brownian motion exactly as a random walk with infinitesimal increments based on that coin toss. The general procedure to extract a measure space out of some given "internal" one is now called Loeb measure theory. This theory is briefly introduced at the end of Chap. 3, and it is fully explored starting from first principles in Horst Osswald's Chap. 6. Applications to stochastic processes, including the Itô integral as well as the Malliavin calculus, are further detailed in Chap. 7, while Yeneng Sun's Chap. 8 contains an application solving the measurability problem that arises when one considers an infinite number of equally weighted independent agents, and more generally, a continuum of independent random variables. Most of the results of Chap. 8 come from Sun's recent papers and are based on the richness of the Loeb construction applied to product spaces.

Successful applications of nonstandard analysis have occurred in many applied areas such as mathematical physics (an example is L. Arkeryd's research starting in the early 1980s on gas kinetics and the Boltzmann equation) and mathematical economics. Work in the latter area was initiated with the seminal 1975 paper of D.J. Brown and A. Robinson on nonstandard exchange economies. There is a need in economic theory for models of economies with a very large number of equally weighted agents, each of which has only a negligible influence on the economy. Standard models have taken as the set of agents the unit interval [0,1] supplied with Lebesgue measure. A more natural model is the "hyperfinite set" of agents used by Brown and Robinson. Yeneng Sun's Chap. 9 of this book describes many uses of this model when combined with Loeb measure theory; it also shows why Lebesgue spaces must fail for many economic applications.

Other applications of nonstandard analysis include the exploitation of compactness using the “S-topology” on the nonstandard extension of a topological space to obtain quite general and intuitive constructions of compactifications; see Chap. 5 by Insall, Loeb and Marciniak. Important and extensive applications of nonstandard analysis to combinatorial number theory are the subject of Jin and Di Nasso’s Chaps. 10 and 11 in this book.

For the reader just learning nonstandard analysis, we point again to our Part I that begins with a simple form of nonstandard analysis, suitable for the results of calculus and basic real analysis. The presentation is intended to give the reader a feeling for the fundamental arguments of nonstandard analysis with a minimum use of model theory. The reader who begins with no background in mathematical logic should easily pick up what is needed to continue.

Chapter 2 of Part I is devoted to general nonstandard analysis and presents the heart of Robinson’s theory. It is written so that the interested reader learns all that is needed for later applications without being forced to read detailed model theoretic constructions, some of which are postponed to the appendix of the chapter. Part I concludes in Chap. 3 with further applications.

The authors have been asked from time to time about the relation between Robinson’s nonstandard analysis and the subject of “internal set theory,” initiated by Edward Nelson in the 1977 Bulletin of the American Mathematical Society. A good recent development of that framework with applications can be found in Nader Vakil’s text cited here in the references to Chap. 2. What is the difference? The Robinson framework adds to the standard mathematical “world” a second “nonstandard” mathematical world. There are no infinitely large integers in the standard world, but they do exist in the nonstandard world. Robinson chose the name “nonstandard analysis” because the nonstandard world is used to analyze the standard one. Internal set theory, on the other hand, works with only the nonstandard world, but recognizes some elements of that world as being “standard”. Important developments in the Robinson framework have taken objects formed in the nonstandard world and adjoined them to the standard world. For example, equivalence classes of “remote points” become compactifying boundary points of standard topological spaces. Quotients of points in the nonstandard extension of Banach spaces become new Banach spaces in the standard world. Measure spaces formed from nonstandard point-sets become rich measure spaces in the standard world. These constructions do not make sense in internal set theory because there is no standard world.

In working through the foundations for nonstandard analysis presented in this book, the reader will gain many new and helpful insights into the enterprise of mathematics. Once these foundations are understood, research formulated in the framework of internal set theory can be easily understood with just some translation of terminology. The editors have found, however, that the reverse is not generally true. Therefore, the reader may best be served by starting here at least with Part I, or with a similar introduction to Robinson’s theory.

The editors would like to thank the contributors to this second edition for their outstanding contributions to this project. We also thank Erik Talvila, who made numerous helpful suggestions for improvements of the first edition. Finally, the editors dedicate this book to one of the founders of nonstandard analysis; he is our mentor, colleague, and friend, W.A.J. (Wim) Luxemburg.

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