Chapter 2
Finite Noncommutative Spaces

In this chapter (and the next) we consider only finite discrete topological spaces. However, we will stretch their usual definition, which is perhaps geometrically not so interesting, to include the more intriguing finite *noncommutative* spaces. Intuitively, this means that each point has some internal structure, described by a particular noncommutative algebra. With such a notion of finite noncommutative spaces, we search for the appropriate notion of maps between, and (geo)metric structure on such spaces, and arrive at a diagrammatic classification of such finite noncommutative geometric spaces. Our exposition of the finite case already gives a good first impression of what noncommutative geometry has in store, whilst having the advantage that it avoids technical complications that might obscure such a first tour through noncommutative geometry. The general case is subsequently treated in Chap. 4.

2.1 Finite Spaces and Matrix Algebras

Consider a finite topological space X consisting of N points (equipped with the discrete topology):

$$1\bullet \qquad 2\bullet \qquad \cdots\cdots \qquad N\bullet$$

The first step towards a noncommutative geometrical description is to trade spaces for their corresponding function algebras.

Definition 2.1 A (*complex*, *unital*) *algebra* is a vector space A (over $\mathbb{C}$) with a bilinear associative product $A \times A \to A$ denoted by $(a, b) \mapsto ab$ (and a unit 1 satisfying $1a = a1 = a$ for all $a \in A$).

A $*$-*algebra* (or, *involutive algebra*) is an algebra A together with a conjugate-linear map (the *involution*) $* : A \to A$ such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in A$.

W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*,
Mathematical Physics Studies, DOI: 10.1007/978-94-017-9162-5_2,

In this book, we restrict to unital algebras, and simply refer to them as algebras.

In the present case, we consider the $*$-algebra $C(X)$ of $\mathbb{C}$-valued functions on the above finite space X. It is equipped with a pointwise linear structure,

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda(f(x)),$$

for any $f, g \in C(X)$, $\lambda \in \mathbb{C}$ and for any point $x \in X$, and with pointwise multiplication

$$fg(x) = f(x)g(x).$$

There is an involution given by complex conjugation at each point:

$$f^*(x) = \overline{f(x)}.$$

The C in $C(X)$ stands for continuous and, indeed, any $\mathbb{C}$-valued function on a finite space X with the discrete topology is automatically continuous.

The $*$-algebra $C(X)$ has a rather simple structure: it is isomorphic to the $*$-algebra $\mathbb{C}^N$ with each complex entry labeling the value the function takes at the corresponding point, with the involution given by complex conjugation of each entry. A convenient way to encode the algebra $C(X) \simeq \mathbb{C}^N$ is in terms of diagonal $N \times N$ matrices, representing a function $f : X \to \mathbb{C}$ as

$$f \rightsquigarrow \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f(N) \end{pmatrix}.$$

Hence, pointwise multiplication then simply becomes matrix multiplication, and the involution is given by hermitian conjugation.

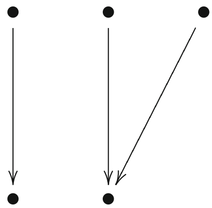
If $\phi : X_1 \to X_2$ is a map of finite discrete spaces, then there is a corresponding map from $C(X_2) \to C(X_1)$ given by pullback:

$$\phi^* f = f \circ \phi \in C(X_1); \quad (f \in C(X_2)).$$

Note that the pullback ϕ^* is a $*$-homomorphism (or, $*$-algebra map) under the pointwise product, in that

$$\phi^*(fg) = \phi^*(f)\phi^*(g), \quad \phi^*(\bar{f}) = \overline{\phi^*(f)}, \quad \phi^*(\lambda f + g) = \lambda\phi^*(f) + \phi^*(g).$$

For example, let X_1 be the space consisting of three points, and X_2 the space consisting of two points. If a map $\phi : X_1 \to X_2$ is defined according to the following diagram,



then

$$\phi^* : \mathbb{C}^2 \simeq C(X_2) \to \mathbb{C}^3 \simeq C(X_1)$$

is given by

$$(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, \lambda_2).$$

Exercise 2.1 *Show that $\phi : X_1 \to X_2$ is an injective (surjective) map of finite spaces if and only if $\phi^* : C(X_2) \to C(X_1)$ is surjective (injective).*

Definition 2.2 A (*complex*) *matrix algebra* A is a direct sum

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some positive integers n_i and N. The involution on A is given by hermitian conjugation, and we simply refer to the $*$-algebra A with this involution as a matrix algebra.

Hence, we have associated a matrix algebra $C(X)$ to the finite space X, which behaves naturally with respect to maps between topological spaces and $*$-algebras. A natural question is whether this procedure can be inverted. In other words, given a matrix algebra A, can we obtain a finite discrete space X such that $A \simeq C(X)$? Since $C(X)$ is always commutative but matrix algebras need not be, we quickly arrive at the conclusion that the answer is negative. This can be resolved in two ways:

(1) Restrict to *commutative* matrix algebras.
(2) Allow for more morphisms (and consequently, more isomorphisms) between matrix algebras, e.g. by generalizing $*$-homomorphisms.

Before explaining each of these options, let us introduce some useful definitions concerning representations of finite-dimensional $*$-algebras (which are not necessarily commutative) which moreover extend in a straightforward manner to the infinite-dimensional case (cf. Definitions 4.26 and 4.27). We first need the prototypical example of a $*$-algebra.

Example 2.3 Let H be an (finite-dimensional) inner product space, with inner product $(\cdot, \cdot) \to \mathbb{C}$. We denote by $L(H)$ the $*$-algebra of operators on H with product given by composition and the involution is given by mapping an operator T to its adjoint T^*.

Note that $L(H)$ is a normed vector space: for $T \in L(H)$ we set

$$\|T\|^2 = \sup_{h \in H} \{(Th, Th) : (h, h) \leq 1\}.$$

Equivalently, $\|T\|$ is given by the square root of the largest eigenvalue of T^*T .

Definition 2.4 A *representation* of a finite-dimensional $*$-algebra A is a pair (H, π) where H is a (finite-dimensional, complex) inner product space and π is a $*$-algebra map

$$\pi : A \to L(H).$$

A representation (H, π) is called *irreducible* if $H \neq 0$ and the only subspaces in H that are left invariant under the action of A are $\{0\}$ or H.

We will also refer to a finite-dimensional inner product space as a *finite-dimensional Hilbert space*.

Example 2.5 Consider $A = M_n(\mathbb{C})$. The defining representation is given by $H = \mathbb{C}^n$ on which A acts by left matrix multiplication; hence it is irreducible. An example of a reducible representation is $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in M_n(\mathbb{C})$ acting in block-form:

$$a \in M_n(\mathbb{C}) \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in L(\mathbb{C}^n \oplus \mathbb{C}^n) \simeq M_{2n}(\mathbb{C})$$

which therefore decomposes as the direct sum of two copies of the defining representation. See also Lemma 2.15 below.

Exercise 2.2 *Given a representation (H, π) of a $*$-algebra A, the* **commutant** $\pi(A)'$ *of $\pi(A)$ is defined as*

$$\pi(A)' = \left\{ T \in L(H) : \pi(a)T = T\pi(a) \textit{ for all } a \in A \right\}.$$

(1) *Show that $\pi(A)'$ is also a $*$-algebra.*
(2) *Show that a representation (H, π) of A is irreducible if and only if the commutant $\pi(A)'$ of $\pi(A)$ consists of multiples of the identity.*

Definition 2.6 Two representations (H_1, π_1) and (H_2, π_2) of a $*$-algebra A are *unitarily equivalent* if there exists a unitary map $U : H_1 \to H_2$ such that

$$\pi_1(a) = U^* \pi_2(a) U.$$

Definition 2.7 The *structure space* $\widehat{A}$ of A is the set of all unitary equivalence classes of irreducible representations of A.

We end this section with an illustrative exercise on passing from representations of a $*$-algebra to matrices over that $*$-algebra.

Exercise 2.3 (1) *If A is a unital $*$-algebra, show that the $n \times n$-matrices $M_n(A)$ with entries in A form a unital $*$-algebra.*

(2) *Let $\pi : A \to L(H)$ be a representation of a $*$-algebra A and set $H^n = H \oplus \cdots \oplus H$ (n copies). Show that the following defines a representation $\tilde{\pi} : M_n(A) \to L(H^n)$ of $M_n(A)$:*

$$\tilde{\pi}\left((a_{ij})\right) = \left(\pi(a_{ij})\right); \quad ((a_{ij}) \in M_n(A)).$$

(3) *Let $\tilde{\pi} : M_n(A) \to L(H^n)$ be a representation of the $*$-algebra $M_n(A)$. Show that the following defines a representation $\pi : A \to L(H^n)$ of the $*$-algebra A:*

$$\pi(a) = \tilde{\pi}\left(a\mathbb{I}_n\right)$$

where $\mathbb{I}_n$ is the identity in $M_n(A)$.

2.1.1 Commutative Matrix Algebras

We now explain how option (1) on page 11 above resolves the question raised by constructing a space from a commutative matrix algebra A. A natural candidate for such a space is, of course, the structure space $\widehat{A}$, which we now determine. Note that any commutative matrix algebra is of the form $A \simeq \mathbb{C}^N$, for which by Exercise 2.2(2) any irreducible representation is given by a map of the form

$$\pi_i : (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C}$$

for some $i = 1, \ldots, N$. We conclude that $\widehat{A} \simeq \{1, \ldots, N\}$.

We conclude that there is a *duality* between finite spaces and commutative matrix algebras. This is nothing but a finite-dimensional version of *Gelfand duality* (see Theorem 4.28 below) between compact Hausdorff topological spaces and unital commutative C^*-algebras. In fact, we will see later (Proposition 4.25) that any finite-dimensional C^*-algebra is a matrix algebra, which reduces Gelfand duality to the present finite-dimensional duality.

2.1.2 Noncommutative Matrix Algebras

The above trade of finite discrete spaces for finite-dimensional commutative $*$-algebras does not really make them any more interesting, for the $*$-algebra is always of the form $\mathbb{C}^N$. A more interesting perspective is given by the noncommutative alternative, viz. option (2) on page 11. We thus aim for a duality between finite spaces and *equivalence classes* of matrix algebras. These equivalence classes are described by a generalized notion of isomorphisms between matrix algebras, also known as Morita equivalence.

Let us first recall the notion of an algebra (bi)module.

Definition 2.8 Let A, B be algebras (not necessarily matrix algebras). A *left* A*-module* is a vector space E that carries a left representation of A, i.e. there is a bilinear map $A \times E \ni (a, e) \mapsto a \cdot e \in E$ such that

$$(a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \quad (a_1, a_2 \in A, e \in E).$$

Similarly, a *right* B*-module* is a vector space F that carries a right representation of B, i.e. there is a bilinear map $F \times B \ni (f, b) \mapsto f \cdot b \in F$ such that

$$f \cdot (b_1 b_2) = (f \cdot b_1) \cdot b_2; \quad (b_1, b_2 \in B, f \in F).$$

Finally, an $A - B$*-bimodule* E is both a left A-module and a right B-module, with mutually commuting actions:

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \quad (a \in A, b \in B, e \in E).$$

When no confusion can arise, we will also write ae instead of $a \cdot e$ to denote the left module action.

There is a natural notion of (left) A**-module homomorphism** as a linear map $\phi : E \to F$ that respect the representation of A:

$$\phi(a \cdot e) = a \cdot \phi(e); \quad (a \in A, e \in E).$$

Similarly for right modules and bimodules.

We introduce the following notation:

- ${}_A E$ for a left A-module E;
- F_B for a right B-module F;
- ${}_A E_B$ for an $A - B$-bimodule E.

Exercise 2.4 *Check that a representation* $\pi : A \to L(H)$ *of a* $*$*-algebra* A *(cf. Definition 2.4) turns* H *into a left* A*-module* ${}_A H$.

Exercise 2.5 *Show that* A *is itself an* $A - A$*-bimodule* ${}_A A_A$*, with left and right actions given by the product in* A.

If E is a right A-module, and F is a left A-module, we can form the *balanced tensor product*:

$$E \otimes_A F := E \otimes F / \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A, e_i \in E, f_i \in F \right\}.$$

In other words, the quotient imposes A-linearity of the tensor product, i.e. in $E \otimes_A F$ we have

$$ea \otimes_A f = e \otimes_A af; \quad (a \in A, e \in E, f \in F).$$

Definition 2.9 Let A, B be matrix algebras. A *Hilbert bimodule* for the pair (A, B) is given by an $A - B$-bimodule E together with a B-valued inner product $\langle \cdot, \cdot \rangle_E : E \times E \to B$ satisfying

$$\langle e_1, a \cdot e_2 \rangle_E = \langle a^* \cdot e_1, e_2 \rangle_E; \quad (e_1, e_2 \in E, a \in A),$$

$$\langle e_1, e_2 \cdot b \rangle_E = \langle e_1, e_2 \rangle_E b; \quad \langle e_1, e_2 \rangle_E^* = \langle e_2, e_1 \rangle_E; \quad (e_1, e_2 \in E, b \in B),$$

$$\langle e, e \rangle_E \geq 0 \text{ with equality if and only if } e = 0; \quad (e \in E).$$

The set of Hilbert bimodules for (A, B) will be denoted by $\mathsf{KK}_f(A, B)$.

In the following, we will also write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_E$, unless confusion might arise.

Exercise 2.6 *Check that a representation $\pi : A \to L(H)$ (cf. Definiton 2.4 and Exercise 2.4) of a matrix algebra A turns H into a Hilbert bimodule for $(A, \mathbb{C})$.*

Exercise 2.7 *Show that the $A - A$-bimodule given by A itself (cf. Exercise 2.5) is an element in $\mathsf{KK}_f(A, A)$ by establishing that the following formula defines an A-valued inner product $\langle \cdot, \cdot \rangle_A : A \times A \to A$:*

$$\langle a, a' \rangle_A = a^* a'; \quad (a, a' \in A).$$

Example 2.10 More generally, let $\phi : A \to B$ be a $*$-algebra homomorphism between matrix algebras A and B. From it, we can construct a Hilbert bimodule E_ϕ in $\mathsf{KK}_f(A, B)$ as follows. Let E_ϕ be B as a vector space with the natural right B-module structure and inner product (cf. Exercise 2.7), but with A acting on the left via the homomorphism ϕ:

$$a \cdot b = \phi(a)b; \quad (a \in A, b \in E_\phi).$$

Definition 2.11 The *Kasparov product* $F \circ E$ between Hilbert bimodules $E \in \mathsf{KK}_f(A, B)$ and $F \in \mathsf{KK}_f(B, C)$ is given by the balanced tensor product

$$F \circ E := E \otimes_B F; \quad (E \in \mathsf{KK}_f(A, B), F \in \mathsf{KK}_f(B, C)),$$

so that $F \circ E \in \mathsf{KK}_f(A, C)$, with C-valued inner product given on elementary tensors by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F, \tag{2.1.1}$$

and extended linearly to all of $E \otimes F$.

Note that this product is associative up to isomorphism.

Exercise 2.8 *Show that the association $\phi \rightsquigarrow E_\phi$ from Example 2.10 is* natural *in the sense that*

(1) $E_{\mathrm{id}_A} \simeq A \in \mathsf{KK}_f(A, A)$,
(2) *for $*$-algebra homomorphisms $\phi : A \to B$ and $\psi : B \to C$ we have an isomorphism*

$$E_\psi \circ E_\phi \equiv E_\phi \otimes_B E_\psi \simeq E_{\psi\circ\phi} \in \mathsf{KK}_f(A, C),$$

that is, as $A - C$-bimodules.

Exercise 2.9 *In the above definition:*

(1) *Check that $E \otimes_B F$ is an $A - C$-bimodule.*
(2) *Check that $\langle \cdot, \cdot \rangle_{E\otimes_B F}$ defines a C-valued inner product.*
(3) *Check that $\langle a^*(e_1 \otimes f_1), e_2 \otimes f_2\rangle_{E\otimes_B F} = \langle e_1 \otimes f_1, a(e_2 \otimes f_2)\rangle_{E\otimes_B F}$.*

Conclude that $F \circ E$ is indeed an element of $\mathsf{KK}_f(A, C)$.

Let us consider the Kasparov product with the Hilbert bimodule for (A, A) given by A itself (cf. Exercise 2.7). Then, since for $E \in \mathsf{KK}_f(A, B)$ we have $E \circ A = A \otimes_A E \simeq E$, the bimodule ${}_A A_A$ is the identity element with respect to the Kasparov product (up to isomorphism). This motivates the following definition.

Definition 2.12 Two matrix algebras A and B are called *Morita equivalent* if there exist elements $E \in \mathsf{KK}_f(A, B)$ and $F \in \mathsf{KK}_f(B, A)$ such that

$$E \otimes_B F \simeq A, \quad F \otimes_A E \simeq B,$$

where $\simeq$ denotes isomorphism as Hilbert bimodules.

If A and B are Morita equivalent, then the representation theories of both matrix algebras are equivalent. More precisely, if A and B are Morita equivalent, then a right A-module is sent to a right B-module by tensoring with $\_ \otimes_A E$ for an invertible element E in $\mathsf{KK}_f(A, B)$.

Example 2.13 As seen in Exercises 2.4 and 2.6, the vector space $E = \mathbb{C}^n$ is an $M_n(\mathbb{C}) - \mathbb{C}$-bimodule; with the standard $\mathbb{C}$-valued inner product it becomes a Hilbert module for $(M_n(\mathbb{C}), \mathbb{C})$. Similarly, the vector space $F = \mathbb{C}^n$ is a $\mathbb{C} - M_n(\mathbb{C})$-bimodule by right matrix multiplication. An $M_n(\mathbb{C})$-valued inner product is given by

$$\langle v_1, v_2\rangle = \bar{v}_1 v_2^t \in M_n(\mathbb{C}).$$

We determine the Kasparov products of these Hilbert bimodules as

$$E \otimes_{\mathbb{C}} F \simeq M_n(\mathbb{C}); \quad F \otimes_{M_n(\mathbb{C})} E \simeq \mathbb{C}.$$

In other words, $E \in \mathsf{KK}_f(M_n(\mathbb{C}), \mathbb{C})$ and $F \in \mathsf{KK}_f(\mathbb{C}, M_n(\mathbb{C}))$ are each other's inverse with respect to the Kasparov product. We conclude that $M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent.

This observation leads us to our first little result.

Theorem 2.14 *Two matrix algebras are Morita equivalent if and only if their structure spaces are isomorphic as finite discrete spaces, i.e. have the same cardinality.*

Proof Let A and B be Morita equivalent. Thus there exists Hilbert bimodules ${}_AE_B$ and ${}_BF_A$ such that

$$E \otimes_B F \simeq A, \quad F \otimes_A E \simeq B.$$

If $[(\pi_B, H)] \in \widehat{B}$ then we can define a representation π_A by setting

$$\pi_A : A \to L(E \otimes_B H); \quad \pi_A(a)(e \otimes v) = ae \otimes v. \tag{2.1.2}$$

Vice versa, we construct $\pi_B : B \to L(F \otimes_A W)$ from $[(\pi_A, W)] \in \widehat{A}$ by setting $\pi_B(b)(f \otimes w) = bf \otimes w$ and these two maps are one another's inverse. Thus, $\widehat{A} \simeq \widehat{B}$ (see Exercise 2.10 below).

For the converse, we start with a basic result on irreducible representations of $M_n(\mathbb{C})$.

Lemma 2.15 *The matrix algebra $M_n(\mathbb{C})$ has a unique irreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.*

Proof It is clear from Exercise 2.2 that $\mathbb{C}^n$ is an irreducible representation of $A = M_n(\mathbb{C})$. Suppose H is irreducible and of dimension K, and define a linear map

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{K \text{ copies}} \to H^*; \quad \phi(a_1, \ldots, a_K) \to e^1 \circ a_1^t + \cdots + e^K \circ a_K^t$$

in terms of a basis $\{e^1, \ldots e^K\}$ of the dual vector space H^*. Here $v \circ a$ denotes pre-composition of $v \in H^*$ with $a \in A$, acting on H. This is a morphism of $M_n(\mathbb{C})$-modules, provided a matrix a acts on the dual vector space H^* by sending $v \mapsto v \circ a^t$. It is also surjective, so that the dual map $\phi^* : H \to (A^K)^*$ is injective. Upon identifying $(A^K)^*$ with A^K as A-modules, and noting that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as A-modules, it follows that H is a submodule of $A^K \simeq \oplus^{nK} \mathbb{C}^n$. By irreducibility $H \simeq \mathbb{C}^n$. $\square$

Now, if A, B are matrix algebras of the following form.

$$A = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}), \quad B = \bigoplus_{j=1}^M M_{m_j}(\mathbb{C}),$$

then $\widehat{A} \simeq \widehat{B}$ implies that $N = M$. Then, define

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i},$$

with A acting by block-diagonal matrices on the first tensor and B acting in a similar way by right matrix multiplication on the second leg of the tensor product. Also, set

$$F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i},$$

with B now acting on the left and A on the right. Then, as above,

$$\begin{aligned} E \otimes_B F &\simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left(\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i} \right) \otimes \mathbb{C}^{n_i} \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i} \simeq A, \end{aligned}$$

and similarly we obtain $F \otimes_A E \simeq B$, as required.

Exercise 2.10 *Fill in the gaps in the above proof:*

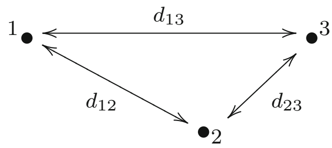
(a) *Show that the representation π_A defined by (2.1.2) is irreducible if and only if π_B is.*
(b) *Show that the association of the class $[\pi_A]$ to $[\pi_B]$ through (2.1.2) is independent of the choice of representatives π_A and π_B.*

We conclude that there is a duality between finite spaces and Morita equivalence classes of matrix algebras. By replacing $*$-homomorphisms $A \to B$ by Hilbert bimodules for (A, B), we introduce a much richer structure at the level of morphisms between matrix algebras. For example, any finite-dimensional inner product space defines an element in $\mathsf{KK}_f(\mathbb{C}, \mathbb{C})$, whereas there is only one map from the corresponding structure space consisting of one point to itself. When combined with Exercise 2.10 we conclude that Hilbert bimodules form a proper extension of the $*$-morphisms between matrix algebras.

2.2 Noncommutative Geometric Finite Spaces

Consider again a finite space X, described as the structure space of a matrix algebra A. We would like to introduce some geometry on X and, in particular, a notion of a metric on X.

Thus, the question we want to address is how we can (algebraically) describe distances between the points in X, say, as embedded in a metric space. Recall that a metric on a finite discrete space X is given by an array $\{d_{ij}\}_{i,j\in X}$ of real non-negative entries, indexed by a pair of elements in X and requiring that $d_{ij} = d_{ji}$, $d_{ij} \leq d_{ik} + d_{kj}$, and $d_{ij} = 0$ if and only if $i = j$:



Example 2.16 If X is embedded in a metric space (e.g. Euclidean space), it can be equipped with the induced metric.

Example 2.17 The **discrete metric** on the discrete space X is given by:

$$d_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

In the commutative case, we have the following remarkable result, which completely characterizes the metric on X in terms of linear algebraic data. It is the key result towards a *spectral* description of finite geometric spaces.

Theorem 2.18 *Let d_{ij} be a metric on the space X of N points, and set $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$, so that $\widehat{A} \simeq X$. Then there exists a representation π of A on a finite-dimensional inner product space H and a symmetric operator D on H such that*

$$d_{ij} = \sup_{a\in A} \{|a(i) - a(j)| : \|[D, \pi(a)]\| \leq 1\}. \tag{2.2.1}$$

Proof We claim that this would follow from the equality

$$\|[D, \pi(a)]\| = \max_{k\neq l} \left\{ \frac{1}{d_{kl}} |a(k) - a(l)| \right\}. \tag{$*$}$$

Indeed, if this holds, then

$$\sup_{a} \{|a(i) - a(j)| : \|[D, a]\| \leq 1\} \leq d_{ij}.$$

The reverse inequality follows by taking $a \in A$ for fixed i, j to be $a(k) = d_{ik}$. Then, we find $|a(i) - a(j)| = d_{ij}$, while $\|[D, \pi(a)]\| \leq 1$ for this a follows from the reverse triangle inequality for d_{ij}:

$$\frac{1}{d_{kl}} |a(k) - a(l)| = \frac{1}{d_{kl}} |d_{ik} - d_{il}| \leq 1.$$

We prove (∗) by induction on N. If $N = 2$, then on $H = \mathbb{C}^2$ we define a representation $\pi : A \to L(H)$ and a hermitian matrix D by

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix}, \quad D = \begin{pmatrix} 0 & (d_{12})^{-1} \\ (d_{12})^{-1} & 0 \end{pmatrix}.$$

It follows that $\|[D, a]\| = (d_{12})^{-1}|a(1) - a(2)|$.

Suppose then that (∗) holds for N, with representation π_N of $\mathbb{C}^N$ on an inner product space H_N and symmetric operator D_N; we will show that it also holds for $N + 1$. We define

$$H_{N+1} = H_N \oplus \bigoplus_{i=1}^{N} H_N^i$$

with $H_N^i := \mathbb{C}^2$. Imitating the above construction in the case $N = 2$, we define the representation π_{N+1} by

$$\pi_{N+1}(a(1), \ldots, a(N + 1)) = \pi_N(a(1), \ldots, a(N)) \\ \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a(N) & 0 \\ 0 & a(N+1) \end{pmatrix},$$

and define the operator D_{N+1} by

$$D_{N+1} = D_N \oplus \begin{pmatrix} 0 & (d_{1(N+1)})^{-1} \\ (d_{1(N+1)})^{-1} & 0 \end{pmatrix} \\ \oplus \cdots \oplus \begin{pmatrix} 0 & (d_{N(N+1)})^{-1} \\ (d_{N(N+1)})^{-1} & 0 \end{pmatrix}.$$

It follows by the induction hypothesis that (∗) holds for $N + 1$. □

Exercise 2.11 *Make the above proof explicit for the case $N = 3$. In other words, compute the metric of (2.2.1) on the space of three points from the set of data $A = \mathbb{C}^3$, $H = (\mathbb{C}^2)^{\oplus 3}$ with representation $\pi : A \to L(H)$ given by*

$$\pi(a(1), a(2), a(3)) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & \\ & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & \\ & a(3) \end{pmatrix},$$

and hermitian matrix

$$D = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix},$$

with $x_1, x_2, x_3 \in \mathbb{R}$.

Exercise 2.12 *Compute the metric on the space of three points given by formula* (2.2.1) *for the set of data $A = \mathbb{C}^3$ acting in the defining representation on $H = \mathbb{C}^3$, and*

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for some non-zero $d \in \mathbb{R}$.

Even though the above translation of the metric on X into algebraic data assumes commutativity of A, the distance formula itself can be extended to the case of a noncommutative matrix algebra A.

In fact, suppose we are given a $*$-algebra representation of A on an inner product space, together with a symmetric operator D on H. Then we can define a metric on the structure space $\widehat{A}$ by

$$d_{ij} = \sup_{a \in A} \left\{ |\mathrm{Tr} a(i) - \mathrm{Tr} a(j)| : \|[D, a]\| \leq 1 \right\}, \tag{2.2.2}$$

where i labels the matrix algebra $M_{n_i}(\mathbb{C})$ in the decomposition of A. This distance formula is a special case of Connes' distance formula (see Note 12 on page 72) on the structure space of A.

Exercise 2.13 *Show that the d_{ij} in* (2.2.2) *is a metric (actually, an* **extended metric**, *taking values in* $[0, \infty]$*) on $\widehat{A}$ by establishing that*

$$d_{ij} = 0 \iff i = j, \quad d_{ij} = d_{ji}, \quad d_{ij} \leq d_{ik} + d_{kj}.$$

This suggests that the above structure consisting of a matrix algebra A, a finite-dimensional representation space H, and a hermitian matrix D provides the data needed to capture a metric structure on the finite space $X = \widehat{A}$. In fact, in the case that A is commutative, the above argument combined with our finite-dimensional Gelfand duality of Sect. 2.1.1 is a reconstruction theorem. Indeed, we reconstruct a given metric space (X, d) from the data (A, H, D) associated to it.

We arrive at the following definition, adapted to our finite-dimensional setting.

Definition 2.19 A *finite spectral triple* is a triple (A, H, D) consisting of a unital $*$-algebra A represented faithfully on a finite-dimensional Hilbert space H, together with a symmetric operator $D : H \to H$.

We do not demand that A is a matrix algebra, since this turns out to be automatic:

Lemma 2.20 *If A is a unital $*$-algebra that acts faithfully on a finite-dimensional Hilbert space, then A is a matrix algebra of the form*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

Proof Since A acts faithfully on a Hilbert space it is a $*$-subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C})$; the only such subalgebras are themselves matrix algebras. □

Unless we want to distinguish different representations of A on H, the above representation will usually be implicitly assumed, thus considering elements $a \in A$ as operators on H.

Example 2.21 Let $A = M_n(\mathbb{C})$ act on $H = \mathbb{C}^n$ by matrix multiplication, with the standard inner product. A symmetric operator on H is represented by a hermitian $n \times n$ matrix.

We will loosely refer to D as a **finite Dirac operator**, as its infinite-dimensional analogue on Riemannian spin manifolds is the usual Dirac operator (see Chap. 4). In the present case, we can use it to introduce a 'differential geometric structure' on the finite space X that is related to the notion of **divided difference**. The latter is given, for each pair of points $i, j \in X$, by

$$\frac{a(i) - a(j)}{d_{ij}}.$$

Indeed, these divided differences appear precisely as the entries of the commutator $[D, a]$ for the operator D as in Theorem 2.18.

Exercise 2.14 *Use the explicit form of D in Theorem 2.18 to confirm that the commutator of D with $a \in C(X)$ is expressed in terms of the above divided differences.*

We will see later that in the continuum case, the commutator $[D, \cdot]$ corresponds to taking derivatives of functions on a manifold.

Definition 2.22 Let (A, H, D) be a finite spectral triple. The A-bimodule of Connes' differential one-forms is given by

$$\Omega^1_D(A) := \left\{ \sum_k a_k [D, b_k] : a_k, b_k \in A \right\}.$$

Consequently, there is a map $\mathrm{d} : A \to \Omega^1(A)$, given by $\mathrm{d}(\cdot) = [D, \cdot]$.

Exercise 2.15 *Verify that* d *is a derivation of a $*$-algebra, in that:*

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b); \quad \mathrm{d}(a^*) = -\mathrm{d}(a)^*.$$

Exercise 2.16 *Verify that $\Omega^1_D(A)$ is an A-bimodule by rewriting the operator $a(a_k[D, b_k])b$ ($a, b, a_k, b_k \in A$) as $\sum_k a'_k[D, b'_k]$ for some $a'_k, b'_k \in A$.*

As a first little result—though with an actual application to matrix models in physics—we compute Connes' differential one-forms for the above Example 2.21.

Lemma 2.23 *Let* $(A, H, D) = (M_n(\mathbb{C}), \mathbb{C}^n, D)$ *be the finite spectral triple of Example 2.21 with D a hermitian* $n \times n$ *matrix. If D is not a multiple of the identity, then* $\Omega^1_D(A) \simeq M_n(\mathbb{C})$.

Proof We may assume that D is a diagonal matrix: $D = \sum_i \lambda_i e_{ii}$ in terms of real numbers λ_i (not all equal) and the standard basis $\{e_{ij}\}$ of $M_n(\mathbb{C})$. For fixed i, j choose k such that $\lambda_k \neq \lambda_j$. Then

$$\left(\frac{1}{\lambda_k - \lambda_j} e_{ik}\right)[D, e_{kj}] = e_{ij}.$$

Hence, since $e_{ik}, e_{kj} \in M_n(\mathbb{C})$, any basis vector $e_{ij} \in \Omega^1_D(A)$. Since also $\Omega^1_D(A) \subset L(\mathbb{C}^n) \simeq M_n(\mathbb{C})$, the result follows. □

Exercise 2.17 *Consider the following finite spectral triple:*

$$\left(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}\right),$$

with $\lambda \neq 0$. *Show that the corresponding space of differential one-forms* $\Omega^1_D(A)$ *is isomorphic to the vector space of all off-diagonal* 2×2 *matrices.*

2.2.1 Morphisms Between Finite Spectral Triples

In a spectral triple (A, H, D) both the $*$-algebra A and a finite Dirac operator D act on the inner product space H. Hence, the most natural notion of equivalence between spectral triples is that of unitary equivalence.

Definition 2.24 Two finite spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2) are called *unitarily equivalent* if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$ such that

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a); \quad (a \in A_1), \\ UD_1U^* &= D_2. \end{aligned}$$

Exercise 2.18 *Show that unitary equivalence of spectral triples is an equivalence relation.*

Remark 2.25 A special type of unitary equivalence is given by the unitaries in the matrix algebra A itself. Indeed, for any such unitary element u the spectral triples (A, H, D) and (A, H, uDu^*) are unitarily equivalent. Another way of writing uDu^* is $D + u[D, u^*]$, so that this type of unitary equivalence effectively adds a differential one-form to D.

Following the spirit of our extended notion of morphisms between algebras, we might also deduce a notion of "equivalence" coming from Morita equivalence of the corresponding matrix algebras. Namely, given a Hilbert bimodule E in $\mathsf{KK}_f(B, A)$, we can try to construct a finite spectral triple on B starting from a finite spectral triple on A. This transfer of metric structure is accomplished as follows. Let (A, H, D) be a spectral triple; we construct a new spectral triple (B, H', D'). First, we define a vector space

$$H' = E \otimes_A H,$$

which inherits a left action of B from the B-module structure of E. Also, it is an inner product space, with $\mathbb{C}$-valued inner product given as in (2.1.1).

The naive choice of a symmetric operator D' given by $D'(e \otimes \xi) = e \otimes D\xi$ will not do, because it does not respect the ideal defining the balanced tensor product over A, being generated by elements of the form

$$ea \otimes \xi - e \otimes a\xi; \quad (e \in E, a \in A, \xi \in H).$$

A better definition is

$$D'(e \otimes \xi) = e \otimes D\xi + \nabla(e)\xi, \tag{2.2.3}$$

where $\nabla : E \to E \otimes_A \Omega^1_D(A)$ is some map that satisfies the **Leibniz rule**

$$\nabla(ea) = \nabla(e)a + e \otimes [D, a]; \quad (e \in E, a \in A). \tag{2.2.4}$$

Indeed, this is precisely the property that is needed to make D' a well-defined operator on the balanced tensor product $E \otimes_A H$:

$$D'(ea \otimes \xi - e \otimes a\xi) = ea \otimes D\xi + \nabla(ea)\xi - e \otimes D(a\xi) - \nabla(e)a\xi = 0.$$

A map $\nabla : E \to E \otimes_A \Omega^1_D(A)$ that satisfies Eq. (2.2.4) is called a **connection** on the right A-module E associated to the derivation $\mathrm{d} : a \mapsto [D, a]$ $(a \in A)$.

Theorem 2.26 *If (A, H, D) is a finite spectral triple and $E \in \mathsf{KK}_f(B, A)$, then (in the above notation) $(B, E \otimes_A H, D')$ is a finite spectral triple, provided that ∇ satisfies the compatibility condition*

$$\langle e_1, \nabla e_2\rangle_E - \langle \nabla e_1, e_2\rangle_E = \mathrm{d}\langle e_1, e_2\rangle_E; \quad (e_1, e_2 \in E). \tag{2.2.5}$$

Proof We only need to show that D' is a symmetric operator. Indeed, for $e_1, e_2 \in E$ and $\xi_1, \xi_2 \in H$ we compute

$$
\begin{aligned}
\langle e_1 \otimes \xi_1, D'(e_2 \otimes \xi_2)\rangle_{E\otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2\rangle_E \xi_2\rangle_H + \langle \xi_1, \langle e_1, e_2\rangle_E D\xi_2\rangle_H \\
&= \langle \xi_1, \langle \nabla e_1, e_2\rangle_E \xi_2\rangle_H + \langle \xi_1, \mathrm{d}\langle e_1, e_2\rangle_E \xi_2\rangle_H \\
&\quad + \langle D\xi_1, \langle e_1, e_2\rangle_E \xi_2\rangle_H - \langle \xi_1, [D, \langle e_1, e_2\rangle_E]\xi_2\rangle_H \\
&= \langle D'(e_1 \otimes \xi_1), e_2 \otimes \xi_2\rangle_{E\otimes_A H},
\end{aligned}
$$

using the stated compatibility condition and the fact that D is symmetric. □

Theorem 2.26 is our finite-dimensional analogue of Theorem 6.15, to be obtained below.

Exercise 2.19 *Let ∇ and ∇' be two connections on a right A-module E. Show that their difference $\nabla - \nabla'$ is a right A-linear map $E \to E \otimes_A \Omega^1_D(A)$.*

Exercise 2.20 *In this exercise, we consider the case that $B = A$ and also $E = A$. Let (A, H, D) be a spectral triple, we determine (A, H', D').*

(1) *Show that the derivation $\mathrm{d}(\cdot) = [D, \cdot] : A \to A \otimes_A \Omega^1_D(A) = \Omega^1_D(A)$ is a connection on A considered a right A-module.*
(2) *Upon identifying $A \otimes_A H \simeq H$, what is the operator D' of* Eq. (2.2.3) *when the connection ∇ on A is given by* d *as in (1)?*
(3) *Use* (1) *and* (2) *of this exercise to show that any connection $\nabla : A \to A \otimes_A \Omega^1_D(A)$ is given by*

$$\nabla = \mathrm{d} + \omega,$$

with $\omega \in \Omega^1_D(A)$.
(4) *Upon identifying $A \otimes_A H \simeq H$, what is the operator D' of* Eq. (2.2.3) *with the connection on A given as $\nabla = \mathrm{d} + \omega$.*

If we combine the above Exercise 2.20 with Lemma 2.23, we see that $\nabla = \mathrm{d} - D$ is an example of a connection on $M_N(\mathbb{C})$ (as a module over itself and with $\omega = -D$), since $\Omega^1_D(A) \simeq M_N(\mathbb{C})$. Hence, for this choice of connection the new finite spectral triple as constructed in Theorem 2.26 is given by $(M_N(\mathbb{C}), \mathbb{C}^N, D' = 0)$. So, Morita equivalence of algebras does not carry over to an equivalence relation on spectral triples. Indeed, we now have $\Omega^1_{D'}(M_N(\mathbb{C})) = 0$, so that no non-zero D can be generated from this spectral triple and the symmetry of this relation fails.

2.3 Classification of Finite Spectral Triples

Here we classify finite spectral triples on A modulo unitary equivalence, in terms of so-called **decorated graphs**.

Definition 2.27 A *graph* is an ordered pair $(\Gamma^{(0)}, \Gamma^{(1)})$ consisting of a set $\Gamma^{(0)}$ of *vertices* and a set $\Gamma^{(1)}$ of pairs of vertices (called *edges*).

Fig. 2.1 A node at $\mathbf{n}_i$ indicates the presence of the summand $\mathbb{C}^{n_i}$; the double node at $\mathbf{n}_j$ indicates the presence of the summand $\mathbb{C}^{n_j} \oplus \mathbb{C}^{n_j}$ in H

We allow edges of the form $e = (v, v)$ for any vertex v, that is, we allow loops at any vertex.

Consider then a finite spectral triple (A, H, D); let us determine the structure of all three ingredients and construct a graph from it.

The algebra: We have already seen in Lemma 2.20 that

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some $n_1, \dots, n_N$. The structure space of A is given by $\widehat{A} \simeq \{1, \dots, N\}$ with each integer $i \in \widehat{A}$ corresponding to the equivalence classes of the representation of A on $\mathbb{C}^{n_i}$. If we label the latter equivalence class by $\mathbf{n}_i$ we can also identify $\widehat{A} \simeq \{\mathbf{n}_1, \dots, \mathbf{n}_N\}$.

The Hilbert space: Any finite-dimensional faithful representation H of such a matrix algebra A is completely reducible (i.e. a direct sum of irreducible representations).

Exercise 2.21 *Prove this result for any $*$-algebra by establishing that the complement $W^{\perp}$ of an A-submodule $W \subset H$ is also an A-submodule of H.*

Combining this with the proof of Lemma 2.15, we conclude that the finite-dimensional Hilbert space representation H of A has a decomposition into irreducible representations, which we write as

$$H \simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes V_i,$$

with each V_i a vector space; we will refer to the dimension of V_i as the **multiplicity** of the representation labeled by $\mathbf{n}_i$ and to V_i itself as the **multiplicity space**. The above isomorphism is given by a unitary map.

To begin the construction of our decorated graph, we indicate the presence of a summand $\mathbf{n}_i$ in H by drawing a node at position $\mathbf{n}_i \in \widehat{A}$ in a diagram based on the structure space $\widehat{A}$ of the matrix algebra A (see Fig. 2.1 for an example). Multiple nodes at the same position represent multiplicities of the representations in H.

The finite Dirac operator: Corresponding to the above decomposition of H we can write D as a sum of matrices

$$D_{ij} : \mathbb{C}^{n_i} \otimes V_i \to \mathbb{C}^{n_j} \otimes V_j,$$

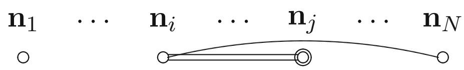


Fig. 2.2 The edges between the nodes $\mathbf{n}_i$ and $\mathbf{n}_j$, and $\mathbf{n}_i$ and $\mathbf{n}_N$ represent non-zero operators $D_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j} \otimes \mathbb{C}^2$ (multiplicity 2) and $D_{iN} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_N}$, respectively. Their adjoints give the operators D_{ji} and D_{Ni}

restricted to these subspaces. The condition that D is symmetric implies that $D_{ij} = D_{ji}^*$. In terms of the above diagrammatic representation of H, we express a non-zero D_{ij} and D_{ji} as a (multiple) edge between the nodes $\mathbf{n}_i$ and $\mathbf{n}_j$ (see Fig. 2.2 for an example).

Another way of putting this is as follows, in terms of decorated graphs.

Definition 2.28 A Λ-*decorated graph* is given by an ordered pair (Γ, Λ) of a finite graph Γ and a finite set Λ of positive integers, with a labeling:

- of the vertices $v \in \Gamma^{(0)}$ by elements $n(v) \in \Lambda$;
- of the edges $e = (v_1, v_2) \in \Gamma^{(1)}$ by operators $D_e : \mathbb{C}^{n(v_1)} \to \mathbb{C}^{n(v_2)}$ and its conjugate-transpose $D_e^* : \mathbb{C}^{n(v_2)} \to \mathbb{C}^{n(v_1)}$,

so that $n(\Gamma^{(0)}) = \Lambda$.

The operators D_e between vertices that are labeled by $\mathbf{n}_i$ and $\mathbf{n}_j$, respectively, add up to the above D_{ij}. Explicitly,

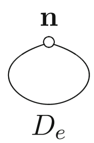
$$D_{ij} = \sum_{\substack{e=(v_1,v_2)\\ n(v_1)=\mathbf{n}_i\\ n(v_2)=\mathbf{n}_j}} D_e,$$

so that also $D_{ij}^* = D_{ji}$. Thus we have proved the following result.

Theorem 2.29 *There is a one-to-one correspondence between finite spectral triples modulo unitary equivalence and Λ-decorated graphs, given by associating a finite spectral triple (A, H, D) to a Λ-decorated graph (Γ, Λ) in the following way:*

$$A = \bigoplus_{n\in\Lambda} M_n(\mathbb{C}), \quad H = \bigoplus_{v\in\Gamma^{(0)}} \mathbb{C}^{n(v)}, \quad D = \sum_{e\in\Gamma^{(1)}} D_e + D_e^*.$$

Example 2.30 The following Λ-decorated graph



corresponds to the spectral triple $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$ of Example 2.21.

Exercise 2.22 *Draw the Λ-decorated graph corresponding to the spectral triple*

$$\left(A = \mathbb{C}^3, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \bar{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right); \quad (\lambda \neq 0).$$

Exercise 2.23 *Use Λ-decorated graphs to classify all finite spectral triples (modulo unitary equivalence) on the matrix algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$.*

Exercise 2.24 *Suppose that (A_1, H_1, D_1) and (A_2, H_2, D_2) are two finite spectral triples. We consider their direct sum and tensor product and give the corresponding Λ-decorated graphs.*

(1) *Show that $(A_1 \oplus A_2, H_1 \oplus H_2, (D_1, D_2))$ is a finite spectral triple.*
(2) *Describe the Λ-decorated graph of this* direct sum spectral triple *in terms of the Λ-decorated graphs of the original spectral triples.*
(3) *Show that $(A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes 1 + 1 \otimes D_2)$ is a finite spectral triple.*
(4) *Describe the Λ-decorated graph of this* tensor product spectral triple *in terms of the Λ-decorated graphs of the original spectral triples.*

Notes

Section 2.1 Finite Spaces and Matrix Algebras

1. The notation KK_f in Definition 2.9 is chosen to suggest a close connection to Kasparov's bivariant KK-theory [1], here restricted to the finite-dimensional case. In fact, in the case of matrix algebras the notion of a *Kasparov module* for a pair of C^*-algebras (A, B) (cf. [2, Sect. 17.1] for a definition) coincides (up to homotopy) with that of a Hilbert bimodule for (A, B) (cf. [3, Sect. IV.2.1] for a definition).
2. Definition 2.12 agrees with the notion of equivalence between arbitrary rings introduced by Morita [4]. Moreover, it is a special case of strong Morita equivalence between C^*-algebras as introduced by Rieffel [5].
3. Theorem 2.14 is a special case of a more general result on the structure spaces of Morita equivalent C^*-algebras (see e.g. [6, Sect. 3.3]).

Section 2.2 Noncommutative Geometric Finite Spaces

4. Theorem 2.18 can be found in [7].
5. The reconstruction theorem mentioned in the text before Definition 2.19 is a special case, to wit the finite-dimensional case, of a result by Connes [8] on a reconstruction of Riemannian (spin) manifolds from so-called spectral triples (cf. Definition 4.30 and Note 13 on page 72 below).
6. A complete proof of Lemma 2.20 can be found in [9, Theorem 3.5.4].

7. For a complete exposition on differential algebras, connections on modules, *et cetera*, we refer to [10, Chap. 8] and [11] and references therein.
8. The failure of Morita equivalence to induce an equivalence between spectral triples was noted in [12, Remark 1.143] (see also [13, Remark 5.1.2]). This suggests that it is better to consider Hilbert bimodules as *correspondences* rather than equivalences, as was already suggested by Connes and Skandalis in [14] and also appeared in the applications of noncommutative geometry to number theory (cf. [12, Chap. 4.3]) and quantization [15]. This forms the starting point for a categorical description of (finite) spectral triples themselves. As objects the category has finite spectral triples (A, H, D), and as morphisms it has pairs (E, ∇) as above. This category is the topic of [16, 17], working in the more general setting of spectral triples, hence requiring much more analysis as compared to our finite-dimensional case. The category of finite spectral triples plays a crucial role in the noncommutative generalization of spin networks in [18].

References

1. Kasparov, G.G.: The operator K-functor and extensions of C^*-algebras. Izv. Akad. Nauk SSSR Ser. Mat. **44**, 571–636, 719 (1980)
2. Blackadar, B.: K-Theory for Operator Algebras. Mathematical Sciences Research Institute Publications, Cambridge University Press, Cambridge (1998)
3. Landsman, N.P.: Mathematical Topics between Classical and Quantum Mechanics. Springer, New York (1998)
4. Morita, K.: Duality for modules and its applications to the theory of rings with minimum condition. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6**, 83–142 (1958)
5. Rieffel, M.A.: Morita equivalence for C^*-algebras and W^*-algebras. J. Pure Appl. Algebra **5**, 51–96 (1974)
6. Raeburn, I., Williams, D.P.: Morita Equivalence and Continuous-trace C^*-algebras, Volume 60 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI (1998)
7. Iochum, B., Krajewski, T., Martinetti, P.: Distances in finite spaces from noncommutative geometry. J. Geom. Phys. **37**, 100–125 (2001)
8. Connes, A.: On the spectral characterization of manifolds. J. Noncommut. Geom. **7**, 1–82 (2013)
9. Etingof, P., Golberg, O., Hensel, S., Liu, T., Schwendner, A., Vaintrob, D., Yudovina, E.: Introduction to Representation Theory, Volume 59 of Student Mathematical Library. American Mathematical Society, Providence, RI (2011) (With historical interludes by Slava Gerovitch)
10. Landi, G.: An Introduction to Noncommutative Spaces and their Geometry. Springer, Berlin (1997)
11. André, Y.: Différentielles non commutatives et théorie de Galois différentielle ou aux différences. Ann. Sci. École Norm. Sup. **34**(4), 685–739 (2001)
12. Connes, A., Marcolli, M.: Noncommutative Geometry. Quantum Fields and Motives. AMS, Providence (2008)
13. Venselaar, J.J.: Classification and equivalences of noncommutative tori and quantum lens spaces. PhD thesis, Utrecht University (2012)
14. Connes, A., Skandalis, G.: The longitudinal index theorem for foliations. Publ. Res. Inst. Math. Sci. **20**, 1139–1183 (1984)
15. Landsman, N.P.: Quantized Reduction as a Tensor Product. In: Quantization of Singular Symplectic Quotients, Volume 198 of Program Mathematics, pp. 137–180, Birkhäuser, Basel (2001)

16. Mesland, B.: Bivariant K-theory of groupoids and the noncommutative geometry of limit sets. PhD thesis, Universität Bonn (2009)
17. Mesland, B.: Unbounded bivariant K-theory and correspondences in noncommutative geometry. J. Reine Angew. Math. **691**, 101–172 (2014)
18. Marcolli, M., van Suijlekom, W.D.: Gauge networks in noncommutative geometry. J. Geom. Phys. **75**, 71–91 (2014)

http://www.springer.com/978-94-017-9161-8

Noncommutative Geometry and Particle Physics
van Suijlekom, W.
2015, XVI, 237 p. 28 illus., 2 illus. in color., Hardcover
ISBN: 978-94-017-9161-8