

Contents

Part I Thermodynamics, Statistical Mechanical Models and Phase Transitions

1	Thermodynamics	5
1.1	Formulae and Variables	5
1.2	The Field-Density Representation	8
1.3	The Thermodynamic Limit	9
1.4	Particular Response Functions	10
1.4.1	Magnetic Systems	11
1.4.2	Fluid Systems	11
2	Statistical Mechanics	13
2.1	Distributions	13
2.1.1	Quantum Systems	14
2.1.2	The Connection to Thermodynamics	15
2.2	Variations of the Probability Function	16
2.3	Coupling Representations	17
2.3.1	The Case $n_f = 2$	19
2.4	Lattice Systems	21
2.4.1	Site-Variable Models	22
2.4.2	Edge-Variable Models	22
2.5	Correlation Functions and Symmetry Properties	23
2.5.1	A General Hamiltonian	23
2.5.2	Correlation Functions	24
2.5.3	Symmetry Properties	26
3	A Survey of Models	29
3.1	Upper and Lower Critical Dimensions	29
3.2	The Quantum Heisenberg Model	30
3.2.1	One-Dimensional Chains	31

3.3	Classical Vector Models	33
3.4	The Gaussian and Spherical Models	34
3.5	Ising Models	36
3.5.1	The Spin- $\frac{1}{2}$ Ising Model	36
3.5.2	The Spin-1 Ising Model.	44
3.6	State-Difference Models.	45
3.6.1	The Classical XY Model	45
3.6.2	The Ashkin–Teller Model	53
3.6.3	Potts Models	54
3.6.4	The Standard Potts Model	55
3.7	Chirality	57
3.7.1	Chiral Potts Models	58
3.7.2	An Extended 3-State Potts Model on the Triangular Lattice	59
3.8	Vertex Models	60
3.8.1	The Eight-Vertex Model	61
3.8.2	The Six-Vertex Model.	72
3.9	Dimer Models	80
3.9.1	The Modified KDP Model Equivalence	83
3.9.2	The Ising Model Equivalence.	85
4	Phase Transitions and Scaling Theory	89
4.1	The Geometry of Phase Transitions.	89
4.1.1	A Two-Dimensional Phase Space	90
4.1.2	A Three-Dimensional Phase Space	93
4.2	Universality, Fluctuations and Scaling	94
4.2.1	Universality	95
4.2.2	Scaling for the Ising Model	97
4.3	General Scaling Formulation	100
4.3.1	The Kadanoff Scaling Hypothesis	100
4.3.2	First-Order Transitions.	104
4.3.3	Effective Exponents	105
4.3.4	The Nightingale–’T Hooft Scaling Hypothesis	109
4.3.5	Constraints on Scaling.	110
4.3.6	Scaling Operators and Dimensions	115
4.3.7	Correlation Functions	118
4.3.8	Variable Scaling Exponents	119
4.3.9	Densities and Response Functions.	121
4.4	Critical Point and Coexistence Curve	122
4.4.1	Critical Exponents.	124
4.4.2	Exponent Inequalities	125
4.5	Scaling for a Critical Point.	125
4.5.1	Scaling Fields for the Critical Point.	126
4.5.2	Approaches to the Critical Point	128

4.5.3	Experimental Variables	129
4.5.4	The Density and Response Functions	130
4.5.5	Asymptotic Forms.	131
4.5.6	Critical Exponents and Scaling Laws.	132
4.5.7	Correlation Scaling at a Critical Point	134
4.6	Tricritical Point	136
4.7	Scaling for a Tricritical Point	141
4.7.1	Scaling Fields for the Tricritical Point	141
4.7.2	Tricritical Exponents and Scaling Laws	143
4.8	Corrections to Scaling	146
4.9	Scaling and Universality	148
4.10	Finite-Size Scaling	152
4.10.1	The Finite-Size Scaling Field	153
4.10.2	The Shift and Rounding Exponents.	155
4.10.3	Universality and Finite-Size Scaling	157
4.11	Conformal Invariance	159
4.11.1	From Scaling to the Conformal Group.	159
4.11.2	Correlation Functions for $d \geq 2$	159
4.11.3	Universal Amplitudes for $d = 2$	161
4.11.4	Schramm–Loewner Evolution.	163

Part II Classical Approximation Methods

5	Phenomenological Theory and Landau Expansions	169
5.1	Classical Methods.	169
5.1.1	A First-Order Transition	171
5.1.2	Metastability	174
5.2	The Van der Waals Equation	175
5.3	Landau Expansions with One Order Parameter.	176
5.3.1	The Spin- $\frac{1}{2}$ Ising Model.	182
5.4	Landau Expansions with Two Order Parameters	183
5.4.1	The Spin-1 Ising Model.	183
5.4.2	The 3-State Potts Model	184
5.5	Landau Theory for a Tricritical Point	188
5.5.1	Tricritical Exponents	190
5.6	Ginzburg–Landau Theory	193
5.6.1	A Critical Point	193
5.6.2	The Gaussian Approximation	194
5.6.3	Gaussian Critical Exponents.	197
6	Mean-Field Theory	205
6.1	The Ising Ferromagnet.	205
6.1.1	Mean-Field Fluctuations	209

6.2	A Model for Metamagnetism	212
6.2.1	The Paramagnetic State	215
6.2.2	The Antiferromagnetic State	216
6.2.3	A Neighbourhood of the Critical Curve	217
6.2.4	The First-Neighbour Antiferromagnet: $\lambda = 0$	221
6.2.5	The First-Order Transition	223
6.2.6	A Neighbourhood of the Tricritical Point	226
7	Cluster-Variation Methods	229
7.1	Improving Mean-Field Theory	229
7.2	The KHDeB Hierarchy of Approximations	231
7.2.1	Distribution Numbers	231
7.2.2	Extensive Quantities	233
7.2.3	The Hamiltonian and Free Energy	235
7.2.4	The Entropy	235
7.2.5	Minimization	237
7.2.6	Labelling Configurations	238
7.3	The Bethe-Pair Approximation for the Ising Model	239
7.4	Reduction to the Mean-Field Approximation	241
7.5	3-State Potts Model on a Triangular Lattice	243
7.6	A Lattice Model for Fluid Water	246

Part III Exact Results

8	Algebraic Methods	259
8.1	The Thermodynamic Limit	259
8.2	The Infinite-System Approach	261
8.3	Lower Bounds for Phase Transitions: The Peierls Method	263
8.3.1	The Simple Lattice Fluid	269
8.4	Grand Partition Function Zeros and Phase Transitions	271
8.4.1	Ruelle's Theorem	273
8.4.2	The Yang-Lee Circle Theorem	276
8.4.3	Systems with Pair Interactions	279
9	Transformation Methods	283
9.1	Related Systems	283
9.2	The Wegner Transformation	284
9.2.1	Duality for the ν -State Potts Model	286
9.2.2	Duality for the Spin- $\frac{1}{2}$ Ising Model	290
9.2.3	The Weak-Graph Transformation	291
9.3	The Regular Square-Lattice Eight-Vertex Model	293
9.3.1	Symmetry Properties and Transformations	294
9.3.2	The Case of Region I	297
9.3.3	Regions and Variables	301

9.4	The Star-Triangle Transformation	304
9.4.1	The ν -State Potts Model	306
9.4.2	The Spin- $\frac{1}{2}$ Ising Model	307
10	Edge-Decorated Ising Models	311
10.1	Primary and Secondary Sites	311
10.2	Super-Exchange or Bond-Dilution.	312
10.2.1	Critical Properties and Exponent Renormalization.	314
10.3	A Ferrimagnet	319
10.3.1	The Zero-Field Axis	320
10.3.2	Non-Zero Field.	322
10.4	A Competing-Interaction Magnetic Model	324
10.5	Decoration with Orientable Molecules	326
10.6	A Decorated Lattice Fluid	332
10.6.1	Case I: A Single Vapour/Liquid Transition.	335
10.6.2	Case II: A Water-Like Model.	337
10.6.3	Case III: Maxithermal, Critical Double and Cuspoidal Points.	339
11	Transfer Matrices: Incipient Phase Transitions	345
11.1	The Transfer Matrix Formulation	345
11.1.1	The Eigen Problem	346
11.1.2	The Partition Function.	347
11.1.3	Correlation Functions and Lengths	348
11.2	Incipient Phase Transitions.	353
11.3	Using Symmetry Properties	354
11.3.1	Block-Diagonalization of the Transfer Matrix.	355
11.3.2	Applications.	359
11.4	Analysis in the Complex Plane: The Wood Method	367
11.4.1	Evolution of Partition Function Zeros	367
11.4.2	Connection Curves and Cross-Block Curves.	369
11.4.3	The Spin- $\frac{1}{2}$ Square-Lattice Ising Model	372
11.4.4	Critical Points and Exponents.	376
12	Transfer Matrices: Exactly Solved Models	381
12.1	A General Eight-Vertex Model.	381
12.1.1	A Generalized Star-Triangle Transformation.	382
12.1.2	The Solution to the GST Transformation and the Elliptic Variable Formulation	383
12.1.3	Z-Invariance.	387
12.1.4	Edge Variables and Matrix Formulation.	391
12.1.5	Square-Lattice Models.	393

12.2	Square-Lattice Ising Models	394
12.2.1	The Modified Checkerboard Ising Model	399
12.2.2	Properties of the Transfer Matrices	403
12.2.3	The Reduction to Regular Ising Models	407
12.2.4	Transfer Matrix Eigenvectors	409
12.2.5	Notational Changes	410
12.2.6	Transfer Matrix Eigenvalues	412
12.2.7	The Standard Model	421
12.3	The Square-Lattice Eight-Vertex Model	431
12.3.1	The Low-Temperature Zone $\mathcal{R}_L(\text{I})$	431
12.3.2	The Low-Temperature Zone $\mathcal{R}_L(\text{III})$	433
12.3.3	The Transfer Matrix	434
12.3.4	Analysis in Terms of Pauli Matrices	437
12.3.5	Analysis of the Transfer Matrix	442
12.3.6	The VQ Equation	447
12.3.7	The Free Energy and Magnetization	475
12.3.8	Critical Behaviour	478
12.3.9	The Coupling Form and the Ising Model Limit	482
12.3.10	The Six-Vertex Model	485
12.3.11	The Eight-Vertex Model and Universality	491
13	Dimer Models	495
13.1	The Dimer Partition Function	495
13.2	Superposition Polynomials and Pfaffians	496
13.2.1	The Square-Lattice Case	500
13.2.2	The Honeycomb-Lattice Case	504
13.3	Vertex and Ising Model Equivalences	510
13.3.1	The Five-Vertex Model	510
13.3.2	The Honeycomb-Lattice Anisotropic Ising Model	512
13.4	K-Type and O-Type Transitions	514

Part IV Series and Renormalization Group Methods

14	Series Expansions	521
14.1	The Task and the Methods	521
14.2	Moment Expansions	524
14.2.1	At Low Temperatures	525
14.2.2	At High Temperatures	530
14.2.3	Duality for Graphs	534
14.3	Cumulant Expansions	536
14.3.1	The Low-Temperature Case	539
14.3.2	The High-Temperature Case	539
14.4	The Finite-Cluster Method	540

14.5	The Finite-Lattice Method	543
14.5.1	Block-Formation and Accuracy.	544
14.5.2	Constructing Block Partition Functions	548
14.5.3	Calculating the Series	551
14.6	The Analysis of Series: Second-Order Transitions.	555
14.6.1	Late-Term Analysis.	556
14.6.2	The Ratio Method.	557
14.6.3	Padé Approximants	559
14.6.4	Differential and Algebraic Approximants	562
14.7	The Analysis of Series: First-Order Transitions.	565
15	Real-Space Renormalization Group Theory	567
15.1	The Basic Elements of the Renormalization Group	567
15.2	RG Transformations and Weight Functions	570
15.3	Fixed Points and the Linear Renormalization Group	574
15.4	Free Energy and Densities	577
15.5	Decimation for the Ising Model	579
15.5.1	In One Dimension	579
15.5.2	In Two Dimensions.	585
15.6	The Kosterlitz–Thouless Transition	588
15.7	Upper-Bound and Lower-Bound Approximations	594
15.7.1	An Upper-Bound Method	595
15.7.2	A Lower-Bound Method	599
15.8	Finite-Lattice Approximations.	603
15.9	Variational Approximations	607
15.10	Phenomenological Renormalization.	609
15.10.1	The Square-Lattice Ising Model	611
15.10.2	Other Models	613
15.10.3	More Than One Coupling	614
15.11	Other Renormalization Group Methods	615

Part V Mathematical Appendices

16	Graphs and Lattices	619
16.1	Graphs	619
16.1.1	Introduction	619
16.1.2	The Cyclomatic Number	621
16.1.3	Triangulation of Graphs.	622
16.1.4	Oriented Graphs	622
16.1.5	The Dual Graph	623
16.2	Lattices	624
16.2.1	Types of Regular Lattices	624
16.2.2	Lattice Transformations	628
16.3	Rapidity Graphs and Lattices	633

16.4	Lattice Graphs	637
16.4.1	Augmented Graphs and the Whitney Polynomial	638
16.4.2	Hopping Matrices and the Canonical Flux Distribution	638
16.4.3	Embeddings and Topologies.	639
16.4.4	Lattice Constants	640
16.4.5	Partially-Ordered Sequences of Graphs and the T Matrix	644
16.4.6	Generating the Partially-Ordered Sequence.	647
16.4.7	Incorporating Sublattices	651
16.4.8	The Guggenheim–McGlashan Approach	654
16.4.9	Further Results	656
17	Algebra	659
17.1	Catastrophe Theory	659
17.1.1	Equivalence and Determinancy	659
17.1.2	Critical Points, Codimension and Unfoldings	662
17.1.3	Symmetry Considerations.	668
17.2	Matrix Algebra	670
17.2.1	Diagonalizability.	671
17.2.2	Commutativity	672
17.2.3	Reducibility	673
17.2.4	Theorems of Perron and Frobenius	673
17.2.5	Direct Products and Traces.	675
17.2.6	Defective Matrices	675
17.2.7	Groups of Matrices	676
17.3	Groups and Representations	676
17.3.1	Representations.	678
17.3.2	Permutation Representations and Equivalence Classes	682
17.3.3	Block Diagonalization Within an Equivalence Class. . .	684
17.3.4	Symmetry Groups.	687
17.4	The Conformal Group	691
17.5	Some Transformations in the Complex Plane	693
17.6	Algebraic Functions	695
17.7	Determinants of Cyclic Matrices.	700
18	Analysis	703
18.1	Fourier Transforms in d Dimensions	703
18.1.1	Discrete Finite Lattices	703
18.1.2	A Continuous Finite Volume	705
18.1.3	A Continuous Infinite Volume	707
18.1.4	Integrals Involving Bessel Functions	708
18.1.5	Lattice Green's Functions	710

18.2	Doubly-Periodic and Quasi-Periodic Functions	711
18.3	Elliptic Integrals and Functions.	714
18.3.1	Elliptic Integrals	714
18.3.2	Jacobi Theta Functions	717
18.3.3	Jacobi Elliptic Functions	720
18.3.4	Transformations in the Elliptic Modulus	724
18.3.5	The Modified Amplitude Function	726
18.3.6	Nome Series.	727
18.3.7	Special Results and Functions for Chap. 12	728
18.3.8	Baxter's Modified Theta Functions	731
18.4	The Potts Delta Function	737
18.4.1	The $\mu = 0$ Case	740
18.4.2	The $\mu \neq 0$ Case	742
18.5	Padé, Differential and Algebraic Approximants.	743
18.5.1	Padé Approximants	743
18.5.2	Dlog Padé Approximants	748
18.5.3	Differential Approximants	750
18.5.4	Algebraic Approximants	754
	References and Author Index.	757
	Index	783

Equilibrium Statistical Mechanics of Lattice Models

Lavis, D.

2015, XVII, 793 p. 101 illus., Hardcover

ISBN: 978-94-017-9429-9