Chapter 2
Basic Properties, Estimation and Prediction Under Exponentiated Distributions

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2.1 Introduction

The class of distributions $\Im = \{H: H(x) = [G(x)]^{\alpha}\}$, where H is defined by (1.1.5), shall be called class of exponentiated distributions. The PDF and SF, corresponding to H, are given, respectively, by

$$h(x) = \alpha[G(x)]^{\alpha-1} g(x), \tag{2.1.1}$$

where $g(x)$ is the PDF corresponding to $G(x)$ and

$$R_H(x) = 1 - [G(x)]^{\alpha}. \tag{2.1.2}$$

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In terms of survival functions (SFs) $R_H(x) = 1 - H(x)$ and $R_G(x) = 1 - G(x)$, corresponding to $H(x)$ and $G(x)$, we could either have

$$R_H(x) = 1 - [1 - R_G(x)]^{\alpha}, \tag{2.1.3}$$

or simply write

$$R_H(x) = [R_G(x)]^{\alpha}. \tag{2.1.4}$$

Notice that SF (2.1.3) corresponds to CDF (1.1.5) whereas SF $R_H(x)$, in (2.1.4), is obtained by exponentiating SF $R_G(x)$ by α. If, in (2.1.3), $\alpha = N$, a positive integer, the SF of the minimum of N independently, identically distributed (iid) random variables from G is obtained.

Cramer and Kamps (1996) were concerned with obtaining and studying the properties of the model parameters in a sequential k-out-of-n structure based on the exponentiated SF (2.1.4) after indexing the parameter α by $i, i = 1, \ldots, n$ and writing (2.1.4) in terms of the CDF's. That is

$$H_i(x) = 1 - [1 - G(x)]^{\alpha_i}, i = 1, \ldots, n. \tag{2.1.5}$$

The specific choice of CDF's, as given by (2.1.5), in the definition of sequential order statistics with CDF G and positive real numbers $\alpha_1, \ldots, \alpha_n$ leads to the following important cases [see Cramer and Kamps (1996)]:

(i) $\alpha_1 = \cdots = \alpha_i \Rightarrow$ ordinary order statistics.
(ii) $\alpha_i = \gamma_i/(n - i + 1) \Rightarrow$ generalized order statistics.
(iii) $\alpha_i = k/(n - i + 1) \Rightarrow k$th record values.
(iv) $\alpha_i = (N + 1 - i - \sum_{\ell=1}^{i-1} R_\ell)/(n - i + 1) \Rightarrow$ progressive type II censoring, where $\alpha_i = \gamma_i/(n - i + 1)$ and $N = n + \sum_{\ell=1}^{n} R_\ell, (R_1, \ldots, R_n)$ is the censoring scheme at the beginning of the experiment.

Nagaraja and Hoffman (2001) used (1.1.5) as a record model and described the exact as well as the asymptotic distributions of the inter-arrival times of upper record values from the G^{α} record model when $\{X_n, n \geq 1\}$ is a sequence of independent random variables such that $X_n \sim G^{\alpha}$. Hoffman and Nagaraja (2000) studied the model in which $X_i \sim G^{\alpha_i}, i \geq 1$, are independent random variables assuming that the number of observations is random and independent of the observations and that the $\alpha_i's$ are positive constants. They called this model a *random G^{α} model*. Hoffman and Nagaraja (2002) introduced the *random power record model* where for every $n \geq 1$, the joint CDF of $X_1, \ldots, X_n$ of a sequence $\{X_n, n \geq 1\}$ of random variables, not necessarily independent nor identically distributed, is given by

$$H_n(x_1, \ldots, x_n) = E\{G^{\alpha_1}(x_1), \ldots, G^{\alpha_n}(x_n)\}, x_i \in R, i = 1, \ldots, n,$$

where the expectation is taken with respect to the $\alpha's$ which are assumed to be almost sure finite positive random variables. They established a hierarchical relationship between several previously studied record models and showed that this model incorporates all of them.

2.2 Properties of the Exponentiated Class of Distributions

Motivated by the fact that any (absolutely continuous) SF $R_G(x)$ can be written in the form

$$R_G(x) \equiv R_G(x;\beta) = \exp[-u(x;\beta)] \equiv \exp[-u(x)], 0 \le a < x < b \le \infty, \qquad (2.2.1)$$

where $u(x) = -\ln R_G(x)$, is such that $R_G(x)$ is a SF, so that $u(x)$ is a continuous, monotone increasing, differentiable function of x such that $u(x) \to 0$ as $x \to a^+$ and $u(x) \to \infty$ as $x \to b^-$, in which a and b are real numbers that may assume the values 0 and ∞, respectively. We shall use (2.2.1), sometimes, instead of the direct use of $G(x)$.

Remarks

1. The expression for $R_G(x)$, given by (2.2.1), holds true for any distribution defined over the whole real line if a is allowed to assume the value $-\infty$. However, we shall restrict the domain to the positive half of the real line, or subset of it, as given in (2.2.1), which is more appropriate for x to be used as time variable.
2. Class (2.2.1) includes all SF's with positive support (or subset of it). In particular, it includes among others, the Weibull (exponential and Rayleigh as special cases), compound Weibull (or Burr type XII), (compound exponential (or Lomax) and compound Rayleigh as special cases), beta, Pareto I, Gomperz and compound Gompertz SF's.
3. Although (2.1.3) and (2.1.4) are both exponentiated models, they are not quite the same. Substituting (2.2.1) in (2.1.3) and (2.1.4), we obtain, for $x > 0$,

$$R_H(x) = 1 - \{1 - \exp[-u(x)]\}^{\alpha} \qquad (2.2.2)$$

and

$$R_H(x) = \exp[-\alpha u(x)]. \qquad (2.2.3)$$

Notice that SF (2.2.3) is of the same form as that given by (2.1.4) with, say, $u^*(x) = \alpha u(x)$. So, we shall concentrate on the class of SFs (2.2.2).

4. It is easy to see that if $Z = -\ln G(X)$, where X has CDF $H(x) = [G(x)]^{\alpha}$ then Z has the exponential distribution with HRF α.

5. Suppose that $H(x) = [G(x)]^{\alpha}$. Gupta et al. (1998) showed that:
   if $\alpha > 1$ and G admits increasing HRF, then F admits increasing HRF and
   if $\alpha < 1$ and G admits decreasing HRF, then F admits decreasing HRF.

2.2.1 Moments

The CDF and PDF corresponding to SF (2.2.2) are given, for $x > 0$, by

$$H(x) = (1 - \exp[-u(x)])^{\alpha}, \tag{2.2.4}$$

and

$$h(x) = \alpha u'(x) \exp[-u(x)]\{1 - \exp[-u(x)]\}^{\alpha-1}. \tag{2.2.5}$$

It can be shown that the ℓth moment of a random variable X following CDF (2.2.4) is given by

$$E(X^{\ell}) = \ell \sum_{j=1}^{v} c_j I_j(\ell), \tag{2.2.6}$$

where

$$v = \begin{cases} \alpha = 1, 2, 3, \ldots \\ \infty, \quad \alpha \text{ is a positive non}-\text{integral value}, \end{cases} \tag{2.2.7}$$

$$\begin{aligned} c_j &= (-1)^{j-1}\alpha(\alpha-1)\ldots(\alpha-j+1)/j!, \\ I_j(\ell) &= \int_0^{\infty} x^{\ell-1} \exp[-ju(x)]dx. \end{aligned} \tag{2.2.8}$$

In the non-exponentiated case ($\alpha = N = 1$), the ℓth moment of $H(x) = 1 - \exp[-u(x)]$ is

$$E(X^{\ell}) = \ell I_1(\ell) = \ell \int_0^{\infty} x^{\ell-1} \exp[-u(x)]dx.$$

Result (2.2.6) can be shown by observing that integration by parts yields

$$E(X^{\ell}) = \ell \int_0^{\infty} x^{\ell-1} R_H(x)dx,$$

where, from (2.2.2), $R_H(x) = \sum_{j=1}^{N} c_j \exp[-ju(x)]$, c_j is given by (2.2.8) when $\alpha = N$ is an integer ≥1 and $R_H(x) = \sum_{j=1}^{\infty} c_j \exp[-ju(x)]$, c_j is given by (2.2.8) when α takes positive non-integral values. Substituting $R_H(x)$ in the integral of $E(X^\ell)$, we obtain (2.2.6).

Remark

1. If $j - 1 = i$, then (2.2.6) becomes $E(X^\ell) = \ell \sum_{i=0}^{v-1} c_i I_i(\ell)$,

   where

$$c_i = (-1)^i \alpha(\alpha - 1)\ldots(\alpha - i)/(i+1)! = \frac{\alpha}{i+1} c_i^*,$$
$$c_i^* = (-1)^i (\alpha - 1)\ldots(\alpha - i)/i!,$$
$$I_i(\ell) = \int_0^\infty x^{\ell-1} e^{-(i+1)u(x)} dx$$

   Therefore $E(X^\ell) = \ell\alpha \sum_{i=0}^{v-1} \frac{c_i^*}{i+1} I_i(\ell)$.

   For example, if $u(x) = \beta x$ (the exponential baseline distribution) and $v = \alpha$ is a positive integer, then

$$E(X^\ell) = \frac{\ell!\alpha}{\beta^\ell} \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha - 1}{i} \left(\frac{1}{i+1}\right)^{\ell+1}.$$

   This result agrees with that given in Gradshteln and Ryshlik (1980), p. 1077.

   Table (2.1) gives the ℓth moment $E(X^\ell)$ for some members of class $\Im$, where, for $j = 1, \ldots, v$, c_j is given by (2.2.8) and v by (2.2.7). The letter E preceding the name of the distribution stands for exponentiated. The letter C for compound, W for Weibull, Ray for Rayliegh, Par 1 for Pareto type1 and Gomp for Gompertz.

2.2.2 Quantiles

The quantile x_q of the absolutely continuous distribution (2.2.4) is given by

$$x_q = u^{-1}[-\ln(1 - q^{1/\alpha})], \tag{2.2.9}$$

where $u^{-1}(.)$ is the inverse function of $u(.)$. This is true since the quantile is the value of x_q satisfying $q = H(x_q) = \{1 - \exp[-u(x_q)]\}^\alpha$. Table (2.2) displays the medians of some members of the exponentiated class $\Im$. It may be observed that in the non-exponentiated case $(\alpha = 1)$ the median reduces to $median = u^{-1}(\ln 2)$.

Table 2.1 ℓth moments of some members of the exponentiated class $\mathfrak{F}$

Distribution	$u(x)$	ℓth moment
EW$(\alpha, \beta_1, \beta_2)$	$\beta_1 x^{\beta_2}$	$\frac{\Gamma(1+\ell/\beta_2)}{\beta_1^{\ell/\beta_2}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell/\beta_2}}\right)$
EE(α, β)	βx	$\frac{\Gamma(1+\ell)}{\beta^{\ell}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell}}\right)$
ERay(α, β)	βx^2	$\frac{\Gamma(1+\ell/2)}{\beta^{\ell/2}} \sum_{j=1}^{v} \left(\frac{c_j}{j^{\ell/2}}\right)$
ECW(Burr XII) $(\alpha, \beta_1, \beta_2, \delta)$	$\beta_2 \ln(1+\delta x^{\beta_1})$	$\frac{\Gamma(1+\ell/\beta_1)}{\delta^{\ell/\beta_1}} \sum_{j=1}^{v} \left(\frac{c_j \Gamma(j\beta_2 - \ell/\beta_1}{\Gamma(j\beta_2)}\right)$
ECE(ELomax) (α, β, δ)	$\beta \ln(1+\delta x)$	$\frac{\Gamma(1+\ell)}{\delta^{\ell}} \sum_{j=1}^{v} \left(\frac{c_j \Gamma(j\beta_2 - \ell)}{\Gamma(j\beta_2)}\right)$
ECRay(α, β, δ)	$\beta \ln(1+\delta x^2)$	$\frac{\Gamma(1+\ell/2)}{\delta^{\ell/2}} \sum_{j=1}^{v} \left(\frac{c_j \Gamma(j\beta_2 - \ell/2)}{\Gamma(j\beta_2)}\right)$
EPar I$(\alpha, \beta_1, \beta_2)$	$-\ln(x/\beta_1)^{\beta_2}, (x > \beta_1^{1/\beta_2})$	$\ell \beta_1^{\beta_2+\ell/\beta_2} \sum_{j=1}^{v} \left(\frac{c_j}{(j\beta_2-\ell)\beta_1^{j}}\right)$
EBeta(α, β)	$-\ln(1-x^{\beta}), 0<x<1$	$\Gamma(1+\ell/\beta) \sum_{j=1}^{v} \left(\frac{c_j \Gamma(1+j)}{\Gamma(1+j+\ell/\beta)}\right)$
EGomp$(\alpha, \beta_1, \beta_2)$	$\beta_1 \exp(\beta_2 x)$	$\frac{1}{\beta_2^{\ell}} \sum_{j=1}^{v} c_j I_j(\ell)$ [a]
ECGomp$(\alpha, \beta_1, \beta_2, \delta)$	$\delta \ln\left[1+\frac{e^{\beta_2 x}-1}{\beta_1 \beta_2}\right]$	$\ell \sum_{j=1}^{v} c_j I_j^*(\ell)$ [b]

[a] $I_j(\ell) = \int_0^{\infty} \left[\ln(1+\frac{z}{j\beta_1}\right]^{\ell} e^{-z} dz, \quad z = j\beta_1(e^{\beta_2 x} - 1).$

[b] $I_j^*(\ell) = \int_0^{\infty} \frac{x^{\ell-1}}{\left[(\beta_1\delta - 1) + e^{-\beta_2 x}\right]^{j\ell}} dx.$

2.2.3 Mode

The logarithm of PDF (2.2.5) is given by

$$\ln h(x) = \ln \alpha + \ln u'(x) - u(x) + (\alpha - 1) \ln [1 - e^{-u(x)}].$$

So that

$$\frac{h'(x)}{h(x)} = \frac{u''(x)}{u'(x)} - u'(x) + (\alpha - 1)\left[\frac{u'(x)e^{-u(x)}}{1 - e^{-u(x)}}\right] \Rightarrow$$
$$0 = u''(x)(1 - e^{-u(x)}) - [u'(x)]^2[1 - (\alpha - 1)e^{-u(x)}]$$

The value of x which satisfies this equation is a mode of the PDF, given by (2.2.5).

Table 2.2 Medians of some members of the exponentiated class $\Im$

Distribution	u(x)	$u^{-1}(y)$	Median
EW$(\alpha, \beta_1, \beta_2)$	$\beta_1 x^{\beta_2}$	$(y/\beta_1)^{1/\beta_2}$	$\left[\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta_1}\right]^{1/\beta_2}$
EE(α, β)	βx	y/β	$\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta}$
ERay(α, β)	βx^2	$(y/\beta)^{1/2}$	$\left[\ln\left(1-2^{-1/\alpha}\right)^{-1/\beta}\right]^{1/2}$
ECW(Burr XII)$(\alpha, \beta_1, \beta_2, \delta)$	$\beta_2 \ln(1+\delta x^{\beta_1})$	$[\frac{1}{\delta}(e^{y/\beta_2}-1)]^{1/\beta_1}$	$\left[\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta_2}-1\right\}/\delta\right]^{1/\beta_1}$
ECE(ELomax)(α, β, δ)	$\beta \ln(1+\delta x)$	$\frac{1}{\delta}(e^{y/\beta}-1)$	$\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta}-1\right\}/\delta$
ECRay(α, β, δ)	$\beta \ln(1+\delta x^2)$	$[(e^{y/\beta}-1)]^{1/2}$	$\left[\left\{\left(1-2^{-1/\alpha}\right)^{-1/\beta}-1\right\}/\delta\right]^{1/2}$
EPar I$(\alpha, \beta_1, \beta_2)$	$-\ln(x/\beta_1)^{\beta_2}, x>\beta_1^{1/\beta_2}$	$\beta_1^{1/\beta_2} e^{y/\beta_2}$	$\beta_1^{1/\beta_2}\left[1-2^{-1/\alpha}\right]^{-1/\beta_2}$
EBeta(α, β)	$-\ln(1-x^{\beta}), (0<x<1)$	$(1-e^{-y})^{1/\beta}$	$2^{-1/(\alpha\beta)}$
EGomp$(\alpha, \beta_1, \beta_2)$	$\beta_1 e^{\beta_2 x}$	$[\ln(1+y/\beta_1)]/\beta_2$	$\ln\left[1-\ln\left(1-2^{-1/\alpha}\right)^{1/\beta_1}\right]^{1/\beta_2}$
ECGomp$(\alpha, \beta_1, \beta_2, \delta)$	$\delta \ln\left[1+\frac{e^{\beta_2 x}-1}{\beta_1\beta_2}\right]$	$[\ln\{1+\beta_1\beta_2(e^{x/\beta_2}-1)\}]/\beta_2$	$\ln\left[1+\beta_1\beta_2\left\{\left(1-2^{-1/\alpha}\right)^{-1/\delta}-1\right\}\right]^{1/\beta_2}$

2.2.4 Hazard Rate Function

The hazard rate function (HRF) corresponding to the exponentiated CDF (1.1.5) is given, for $x > 0$, by

$$\lambda_H(x) = \frac{h(x)}{R_H(x)} = \frac{\alpha[G(x)]^{\alpha-1}g(x)}{1-[G(x)]^{\alpha}} = \alpha[1-\in_{\alpha}(x)]\lambda_G(x), \tag{2.2.10}$$

where $\lambda_G(x) = g(x)/R_G(x)$ and $\in_{\alpha}(x) = \frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}$.

Notice that, since $G(x)$ is a CDF on $[0,\infty)$, then

If $0<\alpha<1$, then $-\infty < \in_{\alpha}(x) \le 1 \Rightarrow 1-\in_{\alpha}(x) \ge \frac{1}{\alpha} \Rightarrow \lambda_H(x) \ge \lambda_G(x)$.

If $\alpha \ge 1$, then $\frac{\alpha-1}{\alpha} \le \in_{\alpha}(x) \le 1 \Rightarrow 0 \le \alpha[1-\in_{\alpha}(x)] \le 1 \Rightarrow 0 \le \lambda_H(x) \le \lambda_G(x)$. $\in_{\alpha}(0) = 1$ and $\in_{\alpha}(\infty) = \lim_{x\to\infty}\left[\frac{1-G^{\alpha-1}(x)}{1-G^{\alpha}(x)}\right] = \frac{\alpha-1}{\alpha}$. So that, $\frac{\alpha-1}{\alpha} \le \in_{\alpha}(x) \le 1$, for all $x \in [0,\infty)$. Hence, $0 \le \alpha[1-\in_{\alpha}(x)] \le 1$.

By differentiating $\lambda_H(x)$, given by (2.2.10) with respect to x and simplifying, it can be shown that, provided that $G(x)g'(x) < g^2(x)$,

H has an increasing hazard rate (IHR), if:

$$G^{\alpha}(x) > 1 - \frac{\alpha}{1-\{G(x)g'(x)/g^2(x)\}}. \tag{2.2.11}$$

H has a decreasing hazard rate (DHR), if:

$$G^{\alpha}(x) < 1 - \frac{\alpha}{1-\{G(x)g'(x)/g^2(x)\}}. \tag{2.2.12}$$

If equality holds, then critical points at which extrema for $H(x)$ may be obtained and so other shapes for the HRF of $H(x)$ are expected to take place.

2.2.5 Proportional Reversed Hazard Rate Function

The proportional reversed hazard rate function (PRHRF) of H, denoted by $\lambda_H^*(x)$ is defined by

$$\lambda_H^*(x) = \frac{d}{dx}[\ln H(x)] = \frac{h(x)}{H(x)}.$$

It may be noticed, from (2.2.10), that the HRF $\lambda_H(x)$ of H(x) is not proportional to the HRF $\lambda_G(x)$ of G(x). However, the PRHRF $\lambda_H^*(x)$ of H(x) can be seen to be proportional to the PRHRF $\lambda_G^*(x)$ of G(x). In fact,

$$\lambda_H^*(x) = \frac{h(x)}{H(x)} = \frac{\alpha[G(x)]^{\alpha-1}g(x)}{[G(x)]^{\alpha}} = \alpha\lambda_G^*(x). \tag{2.2.13}$$

This is why the exponentiated model is equivalently called PRHRM.

It may also be noted that $\lambda_H^*(x)$ dx provides the probability of failing in (x − dx, x), when a unit is found failed at time x. In general, the PRHRF has been found to be useful in estimating the SF for left censored data.

It can be seen that the CDF $H(x)$ can be written, in terms of the HRF $\lambda_H(x)$ and PRHRM $\lambda_H^*(x)$ of H as follows

$$H(x) = \frac{\lambda_H(x)}{\lambda_H(x) + \lambda_H^*(x)} \tag{2.2.14}$$

From which, the SF and PDF are given, respectively, by

$$R(x) = \frac{\lambda_H^*(x)}{\lambda_H(x) + \lambda_H^*(x)} \quad \text{and} \quad h(x) = \lambda_H(x)R(x) = \frac{\lambda_H^*(x)\lambda_H(x)}{\lambda_H(x) + \lambda_H^*(x)}.$$

2.2.6 Density Function of the rth m-Generalized Order Statistic

The PDF of the rth m-GOS based on an absolutely continuous CDF $H(x)$, whose SF is $R_H(x)$ and PDF is $h(x)$, and positive numbers $\gamma_1, \ldots, \gamma_r$ is given by (1.2.1).

The following theorem gives an expression for the PDF of the rth m-GOS based on an exponentiated distribution.

Theorem 2.1 *The PDF of the r*th *m-GOS based on an exponentiated distribution, whose CDF* $H(x) = [1 - e^{-u(x)}]^{\alpha}$ *and SF* $R_H(x) = 1 - [1 - e^{-u(x)}]^{\alpha}$, *is given from* (1.2.1) *in case* $(m \neq -1)$, *by*

$$f_{X_r^*}(x) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \left(1 - [1 - e^{-u(x)}]^{\alpha}\right)^{\gamma_r - 1} \alpha u'(x) e^{-u(x)} [1 - e^{-u(x)}]^{\alpha - 1} \left(1 - [1 - e^{-u(x)}]^{(m+1)\alpha}\right)^{r-1}. \tag{2.2.15}$$

where C_{r-1}, γ_r *are as given in Chap.* 1.

In the case $m = -1, \gamma_i = k, C_{r-1} = k^r$, *the PDF of the rth OURV is given by*

$$f_{X_r^*}(x) = \frac{k^r}{(r-1)!} [1 - (1 - e^{-u(x)})^{\alpha}]^{k-1} \alpha u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha - 1} [-\ln\{1 - (1 - e^{-u(x)})^{\alpha}\}]^{r-1}. \tag{2.2.16}$$

The PDF of the *r*th ordinary order statistic (OOS) is obtained by setting $k = 1$ *and* $m = 0$ in (2.2.15), or equivalently by the direct use of (1.2.2), to get

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{1}{B(r, n-r+1)} \left(1 - [1 - e^{-u(x)}]^{\alpha}\right)^{n-r} \alpha u'(x) e^{-u(x)} [1 - e^{-u(x)}]^{\alpha - 1} (1 - e^{-u(x)})^{\alpha(r-1)}. \\ &= \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} \frac{1}{B(r, n-r+1)} (1 - e^{-u(x)})^{\alpha(j+r)-1} \alpha u'(x) e^{-u(x)}, \\ &= \sum_{j=0}^{n-r} \omega_j h_j^*(x), \end{aligned} \tag{2.2.17}$$

where

$$\omega_j = \frac{(-1)^j n!}{(n-r-j)!(r-1)!j!(r+j)} \tag{2.2.18}$$

and

$$h_j^*(x) = \alpha(r+j) u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha(r+j)-1}. \tag{2.2.19}$$

Also, from (2.2.16), the PDF of OURV is obtained by setting $k = 1$ and $m = -1$ to get

$$f_{X_r^*}(x) = \frac{1}{(r-1)!} \alpha u'(x) e^{-u(x)} (1 - e^{-u(x)})^{\alpha - 1} [-\ln\{1 - (1 - e^{-u(x)})^{\alpha}\}]^{r-1}. \tag{2.2.20}$$

(i) An s-out-of-n structure functions if at least s of its components function, or equivalently, that the life of the s-out-of-n structure is the (n − s + 1) largest of the component lifetimes. So, if r is replaced by n − s + 1 in the PDF of the rth order statistic (2.2.18), we obtain the PDF of life of an s-out-of-n structure.

$$f_{n-s+1:n}(x) = \binom{n}{n-s+1}(n-s+1)\alpha u'(x)\mathrm{e}^{-u(x)} \left[1-\mathrm{e}^{-u(x)}\right]^{\alpha(n-s+1)-1}\left[1-\left(1-\mathrm{e}^{-u(x)}\right)^{\alpha}\right]^{s-1}. \tag{2.2.21}$$

(ii) The PDFs of a series (n-out-of-n) and parallel (1-out of n) structures are obtained, for $x > 0$, from (2.2.21), respectively, as follows:

$$f_{1:n}(x) = n\alpha u'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{\alpha-1}\left[1-(1-\mathrm{e}^{-u(x)})^{\alpha}\right]^{n-1}. \tag{2.2.22}$$

$$f_{n:n}(x) = n\alpha u'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{n\alpha-1}. \tag{2.2.23}$$

Notice that in the non-exponentiated case ($\alpha = 1$), $f_{1:n}(x) = nu'(x)\mathrm{e}^{-nu(x)}$ and $f_{n:n}(x) = nu'(x)\mathrm{e}^{-u(x)}(1-\mathrm{e}^{-u(x)})^{n-1}$, which agree with the PDFs of the minimum and maximum order statistics based on a population with CDF $1-\mathrm{e}^{-u(x)}$.

(iii) Expression (2.2.17) agrees with the expression obtained by Sarabia and Castillo (2005), for the PDF of the rth order statistic, with the appropriate parameters. This expression makes it easy to obtain the corresponding CDF, SF and moments.

(iv) Mudholkar and Hutson (1996) obtained asymptotic distributions of the extreme order statistics $X_{1:n}$ and $X_{n:n}$ and the extreme spacings $X_{2:n} - X_{1:n}$ and $X_{n::n} - X_{n-1:n}$.

2.3 Estimation of $\alpha, R(x_0), \lambda(x_0)$ (All Parameters of G are Known)

2.3.1 Maximum Likelihood Estimation of $\alpha, R(x_0), \lambda(x_0)$

In this section, the parameter α, SF $R(x_0)$ and HRF $\lambda(x_0)$, at x_0, are estimated using the maximum likelihood (ML) and Bayes methods.

Suppose that α is the only unknown parameter (that is all of the parameters of G are known). We are going to show that an unbiased estimator $\hat{\alpha}$ of α which is also consistent and asymptotically efficient, is given by

$$\hat{\alpha} = (n-1)/\sum_{i=1}^{n} Z_i. \tag{2.3.1}$$

It may be noticed that the transformation $Z = -\ln G(X)$, where X is distributed as $H(X) = [G(X)]^{\alpha}$, transforms X to an exponential random variable $Z$with HRF α, denoted by Exp (α). In fact

$$\begin{aligned} F_Z(z) &= P[Z \le z] = P[-\ln G(X) \le z] = P[X > G^{-1}(e^{-z})] = 1 - H_X[G^{-1}(e^{-z})] \\ &= 1 - e^{-\alpha z}, z > 0 \end{aligned}$$

Suppose that $X_1, < \cdots < X_r$ are the first r order statistics in a random sample of size n drawn from a population whose CDF is given by $H(X) = [G(X)]^{\alpha}$ (type II censoring). Let $Z_i = -\ln G(X_i)$, then $Z_{1:n} > \cdots > Z_{r:n}$ where $Z_{j:n}$ is the jth order statistic of a random sample $Z_1, \ldots, Z_n$ of size n from Exp (α). It then follows that the LF is given by

$$\begin{aligned} L(\alpha; \underline{z}) &\propto [\prod_{i=1}^{r} h(z_i)][H(z_r)]^{n-r} \\ &\propto [\prod_{i=1}^{r} \alpha e^{-\alpha z_i}][1 - e^{-\alpha z_r}]^{n-r} \\ &\propto \alpha^r e^{-\alpha T}[1 - e^{-\alpha z_r}]^{n-r}, \end{aligned} \tag{2.3.2}$$

where $\underline{z} = (z_1, \ldots, z_r)$ and

$$T = \sum_{i=1}^{r} Z_i = -\sum_{i=1}^{r} \ln G(X_i). \tag{2.3.3}$$

The log-likelihood function is given, from (2.3.2), by

$$\ell(\alpha, \underline{x}) \equiv \ln L(\alpha, \underline{x}) \propto r \ln \alpha - \alpha T + (n-r) \ln(1 - e^{-\alpha z_r}).$$

Differentiating both sides with respect to α and then equating to zero, we obtain

$$\frac{r}{\alpha} - T - \frac{(n-r) z_r e^{-\alpha z_r}}{1 - e^{-\alpha z_r}} = 0. \tag{2.3.4}$$

The solution of (2.3.4) is the MLE $\tilde{\alpha}_{ML}$ of α. Such solution could not be obtained analytically and numerical solution may be necessary.

In the complete sample case ($r = n$), it follows, from (2.3.4), that

$$\tilde{\alpha}_{ML} = n/T. \tag{2.3.5}$$

where $Z_i = -\ln[G(X_i)]$ are independently, identically distributed random variables from the exponential distribution with parameter α. It then follows that $T = \sum_{i=1}^{n} Z_i$ has a gamma (n, α) distribution. Therefore,

$$E(\tilde{\alpha}_{ML}) = E(n/T) = \int_0^\infty \frac{n}{t} \frac{\alpha^n}{\Gamma(n)} t^{n-1} \mathrm{e}^{-\alpha t} dt = \frac{n}{n-1}\alpha$$

So that, from (2.3.5),

$$\hat{\alpha}_{ML} = \frac{n-1}{n}\tilde{\alpha} = \frac{n-1}{T}, \tag{2.3.6}$$

is unbiased for α. Furthermore, it can be shown that the distribution (1.1.5) belongs to the exponential class, so that $\sum_{i=1}^{n} \ln[G(x_i)]$ is sufficient and complete for α. The efficiency of the estimator [see, for example, Hogg et al. (2005), p. 324] is given by $e = \frac{RCLB}{V(\hat{\alpha})} = 1 - \frac{2}{n} \to 1, as\ n \to \infty$. Notice that Rao-Cramer lower bound (*RCLB*) is the reciprocal of n times Fisher information $I(\alpha)$, given by the variance of the score function. That is, $RCLB = \frac{1}{nI(\alpha)} = \frac{1}{nV(\partial \ln h/\partial \alpha)} = \frac{\alpha^2}{n}$ and it can be shown that the variance of $\hat{\alpha}$ is given by $V(\hat{\alpha}) = \frac{\alpha^2}{n-2}$. The estimator $\hat{\alpha}$ is unbiased, consistent estimator for α. It then follows that $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{D} N(0, \alpha^2)$.

Remarks

1. If all of the parameters are unknown, the MLE's of the unknown parameters of G are obtained (by solving the likelihood equations involved) and then substituted in G to get $Z_i = -\ln[G(X_i)]$ and hence $\hat{\alpha}_{ML}$.
2. The invariance principle of MLEs can be used in estimating the SF and HRF by replacing the parameters by their estimates.
3. The above estimators of $\alpha, R(x_0), \lambda(x_0)$, may be of use when $G(x)$ is in 'standard form' or can be transformed to standard form, where all of its parameters are known and interest is in estimating α.
4. In the complete sample case, the MLE $\tilde{\alpha}_{ML} = -n/\sum_{i=1}^{n} \ln G(X_i)$ agrees with the result obtained by Gupta and Gupta (2007).

2.3.2 Bayes Estimation of $\alpha, R(x_0), \lambda(x_0)$

Assuming that the prior belief of the experimenter about α is gamma (b_1, b_2) with PDF

$$\pi(\alpha) \propto \alpha^{b_1-1}\mathrm{e}^{-b_2\alpha}, \alpha > 0, (b_1, b_2 > 0). \tag{2.3.7}$$

The posterior PDF is given, from (2.3.2) and (2.3.7) by

$$\begin{aligned} &\pi(\alpha|\underline{z}) \propto L(\alpha; \underline{z})\pi(\alpha) = A\alpha^{r+b_1-1}\mathrm{e}^{-(b_2+T)\alpha}[1-\mathrm{e}^{-\alpha z_r}]^{n-r} \\ &\Rightarrow \pi(\alpha|\underline{z}) = A\sum\nolimits_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-T_{0j_1}\alpha], \end{aligned} \tag{2.3.8}$$

where

$$T_{0j_1} = b_2 + j_1 z_r + \sum_{i=1}^{r} z_i \tag{2.3.9}$$

and A is a normalizing constant, which can be shown to be given by

$$A = \frac{1}{\Gamma(r+b_1)S_0}, \tag{2.3.10}$$

$$S_0 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{0j_1}^{r+b_1}}\right), C_{j_1} = (-1)^{j_1}\binom{n-r}{j_1} \tag{2.3.11}$$

and T_{0j_1} is given by (2.3.9).

Based on squared error loss function, the Bayes estimators of $\alpha, R(x_0), \lambda(x_0)$ were obtained in AL-Hussaini (2010a), using (1.3.1), as follows

$$\left.\begin{aligned} \hat{\alpha}_{SEL} &= E(\alpha|\underline{z}) = \frac{(r+b_1)S_1}{S_0}, \\ \hat{R}_{SEL}(z_0) &= E[R(z_0)|\underline{z}] = 1 - \frac{S_2}{S_0}, \\ \hat{\lambda}_{SEL}(z_0) &= E[\lambda(z_0)|\underline{z}] = \frac{(r+b_1)\lambda_G^*(z_0)S_3}{S_0} \end{aligned}\right\} \tag{2.3.12}$$

where S_0 and C_{j_1} are given by (2.3.11) and $\lambda_G^*(z_0) = \frac{g(z_0)}{G(z_0)}$,

$$S_1 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{0j_1}^{r+b_1+1}}\right), S_2 = \sum_{j_1=0}^{n-r}\left(\frac{C_{j_1}}{T_{1j_1}^{r+b_1}}\right), S_3 = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\left(\frac{C_{j_1}}{T_{j_1,j_2}^{r+b_1+1}}\right), \tag{2.3.13}$$

$$T_{1j_1} = T_{0j_1} - \ln G(z_0), \tag{2.3.14}$$

$$T_{j_1,j_2} = T_{0j_1} - (j_2+1)\ln G(z_0). \tag{2.3.15}$$

For proof, see AL-Hussaini (2010a). This development shall be called 'standard Bayes method' (SBM).

Remarks

1. In the complete sample case, the Bayes estimator $\hat{\alpha}_{SEL} = \frac{n+b_1}{b_2 - \sum_{i=1}^{n} \ln G(X_i)}$, based on the SEL function, agrees with the result obtained by AL-Hussaini (2010b).
2. In the complete sample case, $\hat{\alpha}_{ML}$ and $\hat{\alpha}_{SEL}$ coincide for non-informative prior of α (the case in which $b_1 = b_2 = 0$).
3. It may be observed that $\tilde{\alpha}_{SEL} \to \tilde{\alpha}_{ML}$ as $n \to \infty$, indicating that $\tilde{\alpha}_{SEL}$ has the same properties as $\tilde{\alpha}_{ML}$ for large values of n.

The following theorem gives the Bayes estimates under the LINEX loss function, using standard Bayes method (SBM).

Theorem 2.2 *Based on LINEX loss function, the Bayes estimators of* $\alpha, R(x_0), \lambda(x_0)$ *are given, using* (1.3.4), *by the following*:

$$\hat{\alpha}_{LNX} = -\frac{1}{\kappa} \ln \int_0^\infty \mathrm{e}^{-\alpha\kappa} \pi(\alpha|\underline{z}) d\alpha = -\frac{1}{\kappa} \ln \left(\frac{S_1^*}{S_0} \right), \tag{2.3.16}$$

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa} \ln \int_0^\infty \mathrm{e}^{-R(x_0)\kappa} \pi(\alpha|\underline{z}) d\alpha = -\frac{1}{\kappa} \ln \left(\frac{S_2^*}{S_0} \right), \tag{2.3.17}$$

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa} \ln \int_0^\infty \mathrm{e}^{-\lambda(x_0)\kappa} \pi(\alpha|\underline{z}) d\alpha = -\frac{1}{\kappa} \ln \left(\frac{S_3^*}{S_0} \right), \tag{2.3.18}$$

where S_0 *is given by* (2.3.11),

$$S_1^* = \sum_{j_1=0}^{n-r} \frac{C_{j_1}}{(\kappa + T_{0j_1})^{r+b_1}}, \tag{2.3.19}$$

$$S_2^* = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \mathrm{e}^{-\kappa} \frac{\kappa^{j_2}}{j_2!} \frac{C_{j_1}}{[T_{0j_1} - j_2 \ln G(z_0)]^{r+b_1}}, \tag{2.3.20}$$

$$S_3^* = \sum C \left(\frac{[\lambda_G^*(z_0)]^{j_2} \Gamma(r + b_1 + j_2)}{\Gamma(r + b_1)[T_{0j_1} - (j_2 + j_3) \ln G(z_0)]^{r+b_1+j_2}} \right), \tag{2.3.21}$$

where T_{0j_1} *is given by* (2.3.9),

$$\sum = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \quad \text{and} \quad C = C_{j_1} C_{j_2} C_{j_3} \tag{2.3.22}$$

C_{j_1} is given by (2.3.11),

$$C_{j_2} = \frac{(-1)^{j_2}\kappa^{j_2}}{j_2!}, \quad C_{j_3} = \binom{j_2 + j_3 - 1}{j_3}. \tag{2.3.23}$$

Proof Such estimators can be seen to be obtained, by using (1.3.4) and the posterior PDF (2.3.8), as follows:

$$\begin{aligned}
\hat{\alpha}_{LNX} &= -\frac{1}{\kappa}\ln \int_0^\infty \mathrm{e}^{-\alpha\kappa}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln A \sum_{j_1=0}^{n-r} C_{j_1} \int_0^\infty \alpha^{r+b_1-1}\mathrm{e}^{-\alpha[T_{0j_1}+\kappa]}d\alpha \\
&= -\frac{1}{\kappa}\ln\left(\frac{S_1^*}{S_0}\right),
\end{aligned}$$

where A is given by (2.3.10), S_0 by (2.3.11) and S_1^* by (2.3.19).

$$\begin{aligned}
\hat{R}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln \int_0^\infty \mathrm{e}^{-R_H(z_0)\kappa}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln \int_0^\infty \mathrm{e}^{-[1-\{G(z_0)\}^\alpha]\kappa}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln \sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \mathrm{e}^{j_2\alpha\ln G(z_0)}\pi(\alpha|\underline{z})d\alpha, \\
&= -\frac{1}{\kappa}\ln \sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \mathrm{e}^{j_2\alpha\ln G(z_0)}A\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\mathrm{e}^{-T_{0j_1}\alpha}d\alpha, \\
&= -\frac{1}{\kappa}\ln A\sum_{j_1=0}^{n-r} C_{j_1}\sum_{j_2=0}^\infty \mathrm{e}^{-\kappa}\frac{\kappa^{j_2}}{j_2!}\int_0^\infty \alpha^{r+b_1-1}\mathrm{e}^{-(T_{0j_1}-j_2\ln G(z_0)\alpha}d\alpha, \\
&= -\frac{1}{\kappa}\ln\left(\frac{S_2^*}{S_0}\right),
\end{aligned}$$

where S_0 is given by (2.3.11) and S_2^* by (2.3.20).

Finally, the LINEX estimator of $\lambda_H(x_0)$ is given by

$$\begin{aligned}
\hat{\lambda}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln\int_0^\infty \mathrm{e}^{-\lambda(z_0)\kappa}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\int_0^\infty \sum_{j_2=0}^{\infty}\frac{[-\lambda(z_0)\kappa]^{j_2}}{j_2!}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\int_0^\infty \sum_{j_2=0}^{\infty}\frac{(-\kappa)^{j_2}}{j_2!}\left[\frac{\alpha[G(z_0)]^{\alpha-1}g(z_0)}{1-[G(z_0)]^{\alpha}}\right]^{j_2}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^{\infty}[\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}\int_0^\infty \alpha^{j_2}[G(z_0)]^{j_2\alpha}\{1-[G(z_0)]^{\alpha}\}^{-j_2}\pi(\alpha|\underline{z})d\alpha \\
&= -\frac{1}{\kappa}\ln\sum_{j_2=0}^{\infty}[\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}\sum_{j_3=0}^{\infty}C_{j_3}\int_0^\infty \alpha^{j_2}[G(z_0)]^{(j_2+j_3)\alpha}\pi(\alpha|\underline{z})d\alpha
\end{aligned}$$

where $\lambda_G^*(z_0) = g(z_0)/[G(z_0)]$ and C_{j_3} is given by (2.3.23).

So that

$$\begin{aligned}
\hat{\lambda}_{LNX}(z_0) &= -\frac{1}{\kappa}\ln\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_3}[\lambda_G^*(z_0)]^{j_2}\frac{(-\kappa)^{j_2}}{j_2!}A\sum_{j_1=0}^{n-r}C_{j_1} \\
&\qquad \int_0^\infty \alpha^{r+b_1+j_2-1}\mathrm{e}^{-[T_{0j_1}-(j_2+j_3)\ln G(z_0)]\alpha}d\alpha \\
\hat{\lambda}_{LNX}(x_0) &= -\frac{1}{\kappa}\ln A\sum C[\lambda_G^*(x_0)]^{j_2}\cdot\frac{\Gamma(r+b_1+j_2)}{[T_{0j_1}-(j_2+j_3)\ln G(x_0)]^{r+b_1+j_2}}.
\end{aligned}$$

Therefore

$$\hat{\lambda}_{LIN}(x_0) = -\frac{1}{\kappa}\ln\left(\frac{S_3^*}{S_0}\right),$$

where S_3^* is given by (2.3.21), $\sum$ and C by (2.3.22). □

2.4 Estimation of $(\alpha, \beta_1, \ldots, \beta_k)$, $R_H(x_0)$ and $\lambda_H(x_0)$ (All Parameters of H are Unknown)

This section is devoted to the estimation of the vector of parameters (α, β), $R_H(x_0)$ and $\lambda_H(x_0)$, where $\beta = (\beta_1, \ldots, \beta_k)$ using the ML and Bayes methods.

2.4.1 Maximum Likelihood Estimation of $(\alpha, \beta_1, \ldots, \beta_k), R_H(x_0), \lambda_H(x_0)$

In this section, G is assumed to depend on k-dimensional vector $\beta = (\beta_1, \ldots, \beta_k)$ of unknown parameters. So that H will depend on the $(k+1)$-dimensional vector of unknown parameters (α, β). All parameters are assumed to be positive. In this case, the LF is given by

$$\begin{aligned} L(\theta; \underline{x}) &\propto [\prod_{i=1}^{r} h(x_i|\theta)][R_H(x_r|\theta)]^{n-r} \\ &\propto [\prod_{i=1}^{r} \alpha\{G(x_i|\beta)\}^{\alpha-1} g(x_i|\beta)][1 - \{G(x_r|\beta)\}^{\alpha}]^{n-r}, \end{aligned} \tag{2.4.1}$$

where $\underline{x} = (x_1, \ldots, x_r)$ are the first r order statistics, $\theta = (\alpha, \beta)$, $\beta = (\beta_1, \ldots, \beta_k)$,
So that the LLF, denoted by $\ell(\theta; \underline{x})$ is given by

$$\begin{aligned} \ell(\theta; \underline{x}) \equiv \ln L(\theta; \underline{x}) &= r \ln \alpha + (\alpha - 1) \sum_{i=1}^{r} \ln G(x_i|\beta) + \sum_{i=1}^{r} \ln g(x_i|\beta) \\ &+ (n - r) \ln[1 - \{G(x_r|\beta)\}^{\alpha}] \end{aligned} \tag{2.4.2}$$

The likelihood equations (LEs) are then given by

$$\frac{\partial \ell}{\partial \alpha} : 0 = \frac{r}{\alpha} + \sum_{i=1}^{r} \ln G(x_i|\beta) - \frac{(n-r)\{G(x_r|\beta)\}^{\alpha} \ln G(x_r|\beta)}{1 - \{G(x_r|\beta)\}^{\alpha}} \tag{2.4.3}$$

and for $j = 1, \ldots, k$,

$$\begin{aligned} \frac{\partial \ell}{\partial \beta_j} : 0 = (\alpha - 1) \sum_{i=1}^{r} \frac{1}{G(x_i|\beta)} \frac{\partial G(x_i|\beta)}{\partial \beta_j} + \sum_{i=1}^{r} \frac{1}{g(x_i|\beta)} \frac{\partial g(x_i|\beta)}{\partial \beta_j} \\ - \frac{(n-r)\alpha\{G(x_r|\beta)\}^{\alpha-1}}{1 - \{G(x_r|\beta)\}^{\alpha}} \frac{\partial G(x_r|\beta)}{\partial \beta_j} \end{aligned} \tag{2.4.4}$$

By solving this system of equations, we obtain the MLEs of $\alpha, \beta_1, \ldots, \beta_k$. The invariance property of MLEs can then be applied to obtain the MLEs of $R_H(x_0)$ and $\lambda_H(x_0)$, for some x_0, by replacing the parameters by their MLEs.

2.4.2 Bayes Estimation of $(\alpha, \beta_1, \ldots, \beta_k), R_H(x_0), \lambda_H(x_0)$

AL-Hussaini (2010a) gave expressions for the Bayes estimators of the parameters, SF and HRF of H, under the SEL function. In what follows, such estimators are obtained under LINEX loss function.

Theorem 2.3 *Suppose that the CDF G depends on an unknown k-dimensional vector of parameters* $\beta = (\beta_1, \ldots, \beta_k)$, *so that H depends on the* $(k + 1)$ *unknown parameters* (α, β).

Given by

$$\pi(\theta) \equiv \pi(\alpha, \beta) = \pi_1(\alpha)\pi_2(\beta), \tag{2.4.5}$$

where

$$\pi_1(\alpha) \propto \alpha^{b_1 - 1} \exp(-b_2\alpha), \tag{2.4.6}$$

and $\pi_2(\beta)$ *is a k-variate PDF. Then*

1. *The LINEX estimators of the parameters are*

$$\hat{\alpha}_{LNX} = -\frac{1}{\kappa} \ln\left(S_1^{**}/S_0^{**}\right), \tag{2.4.7}$$

$$\hat{\beta}_{v,LNX} = -\frac{1}{\kappa} \ln\left(S_v^{**}/S_0^{**}\right), \quad v = 2, \ldots, k. \tag{2.4.8}$$

2. The LINEX estimators of the SF $R_H(x_0)$ and HRF $\lambda_H(x_0)$ are given, for some x_0, by

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa} \ln\left(S_{k+2}^{**}/S_0^{**}\right), \tag{2.4.9}$$

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa} \ln\left(S_{k+3}^{**}/S_0^{**}\right), \tag{2.4.10}$$

where

$$S_0^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_0, \quad I_0 = \int_\beta \frac{w(\beta, \underline{x})\pi_2(\beta)}{[T_{0j_1}(x_r, \beta)]^{r+b_1}} d\beta, \tag{2.4.11}$$

$$S_1^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_1, \quad I_1, = \int_\beta \frac{\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)}{[\kappa + T_{0j_1}(\beta)]^{r+b_1}} d\beta, \tag{2.4.12}$$

$$S_v^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_v, \quad I_v = \int_\beta \frac{\mathrm{e}^{-T_0(\beta)-\kappa\beta_v}\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}} d\beta, \quad v = 2, \ldots, k+1, \tag{2.4.13}$$

$$S_{k+2}^{**} = \mathrm{e}^{-\kappa} \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \frac{C_{j_1}}{j_2!} I_{k+2}, I_{k+2} = \int_\beta \frac{\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)}{[T_{j_1 j_2}(\beta)]^{r+b_1}} d\beta, \tag{2.4.14}$$

$$\begin{aligned} S_{k+3}^{**} &= \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \frac{C_{j_1}(-\kappa)^{j_2}\Gamma(r+b_1+j_2)}{j_2!\Gamma(r+b_1)} I_{k+3}, \\ I_{k+3} &= \int_\beta \frac{\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)[\lambda^*(x_0)]^{j_2}}{[T_{j_1 j_2 j_3}(\beta)]^{r+b_1+j_2}} d\beta. \end{aligned} \tag{2.4.15}$$

$$T_0(\beta) = \sum_{i=1}^{r} \ln G(x_i|\beta) + \sum_{i=1}^{r} \ln g(x_i|\beta), \tag{2.4.16}$$

$$T_{0j_1}(\beta) = b_2 + T_{j_1}(\beta), \tag{2.4.17}$$

$$T_{j_1}(\beta) = -\left[\sum_{i=1}^{r} \ln G(x_i|\beta) + j_1 \ln G(x_r|\beta)\right], \tag{2.4.18}$$

$$T_{j_1, j_2}(\beta) = T_{0j_1} - (j_2+1) \ln G(x_0|\beta), \tag{2.4.19}$$

$$T_{j_1, j_2, j_3}(\beta) = T_{0j_1}(\beta) - (j_2 + j_3) \ln G(x_0|\beta). \tag{2.4.20}$$

Proof The LF (2.4.1) can be written as

$$L(\theta; \underline{x}) \propto w(\beta) \sum_{j_1=0}^{n-r} C_{j_1} \alpha^r \exp[-\alpha T_{j_1}(\beta)], \tag{2.4.21}$$

where $\theta = (\alpha, \beta_1, \ldots, \beta_k), w(\beta) = \prod_{i=1}^{r} \lambda_G^*(x_i, \beta) = \prod_{i=1}^{r} \frac{g(x_i,\beta)}{G(x_i,\beta)}$ and $T_{j_1}(\beta)$ is given by (2.4.18).

The posterior PDF is given, from (2.4.5) and (2.4.21), by

$$\pi(\theta|\underline{x}) \propto L(\theta;\underline{x})\pi(\theta) = Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)], \qquad (2.4.22)$$

where A is a normalizing constant, given by

$$\begin{aligned} A &= \frac{1}{\Gamma(r+b_1)S_0^{**}}, \\ S_0^{**} &= \sum_{j_1=0}^{n-r} C_{j_1} I_0, \quad I_0 = \int_0^\infty \frac{w(\beta)\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}} d\beta, \end{aligned} \qquad (2.4.23)$$

$T_{0j_1}(\beta)$ is given by (2.4.17) and $T_{j_1}(\beta)$ by (2.4.18).

It then follows, from (1.3.2), that

$$\begin{aligned} \hat{\alpha}_{LNX} &= -\frac{1}{\kappa}\ln\int_\beta\int_0^\infty \mathrm{e}^{-\kappa\alpha}\pi(\alpha,\beta|\underline{x})d\alpha d\beta \\ &= -\frac{1}{\kappa}\ln\left[\int_\beta\int_0^\infty \mathrm{e}^{-\kappa\alpha}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)\right] d\alpha d\beta \\ &= -\frac{1}{\kappa}\ln\left[A\sum_{j_1=0}^{n-r} C_{j_1}\int_\beta w(\beta)\pi_2(\beta)\int_0^\infty \alpha^{r+b_1-1}\exp[-\alpha\{\kappa+T_{0j_1}(\beta)\}]\,d\alpha d\beta\right] \\ &= -\frac{1}{\kappa}\ln\left[A\Gamma(r+b_1)S_1^{**}\right] \\ &= -\frac{1}{\kappa}\ln\left[S_1^{**}/S_0^{**}\right] \end{aligned}$$

where S_0^{**} is given by (2.4.23),

$$S_1^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_1, \quad I_1 = \int_\beta \frac{w(\beta)\pi_2(\beta)}{[\kappa+T_{0j_1}(\beta)]^{r+b_1}} d\beta$$

For $v = 1, \ldots, k$, the LINEX Bayes estimator of β_v is given by

$$\begin{aligned}
\hat{\beta}_{v,LNX} &= -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-\kappa\beta_v}\pi(\alpha,\beta|\underline{x})d\alpha d\beta \\
&= -\frac{1}{\kappa}\ln\left[\int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-\kappa\beta_v}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)\right] d\alpha d\beta \\
&= -\frac{1}{\kappa}\ln\left[A\sum_{j_1=0}^{n-r} C_{j_1}\int_{\beta} \mathrm{e}^{-\kappa\beta_v}w(\beta)\pi_2(\beta)\int_{0}^{\infty}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]d\alpha d\beta\right] \\
&= -\frac{1}{\kappa}\ln\left[A\Gamma(r+b_1)S_i^{**}\right] \\
&= -\frac{1}{\kappa}\ln\left[S_i^{**}/S_0^{**}\right]
\end{aligned}$$

where, for $i = 2, \ldots, k+1$,

$$S_i^{**} = \sum_{j_1=0}^{n-r} C_{j_1} I_i, \quad I_i = \int_{\beta} \frac{\mathrm{e}^{-\kappa\beta_i}w(\beta)\pi_2(\beta)}{[T_{0j_1}(\beta)]^{r+b_1}}d\beta$$

The Bayes estimator of $R_H(x_0)$, at some x_0, is given by

$$\hat{R}_{LNX}(x_0) = -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty} \mathrm{e}^{-R(x_0)\kappa}\pi(\alpha,\beta|\underline{x})\, d\beta d\alpha$$

Since $R(x_0) = 1 - [G(x_0)]^{\alpha}$, then

$$\begin{aligned}
\mathrm{e}^{-\kappa R(x_0)} &= \mathrm{e}^{-\kappa\{1-[G(x_0)]^{\alpha}\}} = \mathrm{e}^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}[G(x_0)]^{\alpha j_2}}{j_2!} \\
&= \mathrm{e}^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}\mathrm{e}^{\alpha j_2\ln[G(x_0)]}}{j_2!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{R}_{LNX}(x_0) &= -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty}\left(Aw(\beta)\pi_2(\beta)\sum_{j_1=0}^{n-r} C_{j_1}\alpha^{r+b_1-1}\exp[-\alpha T_{0j_1}(\beta)]\right)\\ &\quad \mathrm{e}^{-\kappa}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}\mathrm{e}^{\alpha j_2 \ln[G(x_0)]}}{j_2!}d\alpha d\beta\\ &= -\frac{1}{\kappa}\ln A\mathrm{e}^{-\kappa}\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}C_{j_1}}{j_2!}\int_{\beta} w(\beta)\pi_2(\beta)\\ &\quad \int_{0}^{\infty}\alpha^{r+b_1-1}\mathrm{e}^{-\alpha[T_{0j_1}(\beta)-j_2\ln G(x_0)]}d\alpha d\beta\\ &= -\frac{1}{\kappa}\ln A\mathrm{e}^{-\kappa}\sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty}\frac{\kappa^{j_2}C_{j_1}}{j_2!}\int_{\beta} w(\beta)\pi_2(\beta)\frac{\Gamma(r+b_1)}{[T_{j_1,j_2}(\beta)]^{r+b_1}}d\alpha d\beta\\ &= -\frac{1}{\kappa}\ln\left(\frac{S_{k+2}^{**}}{S_0^{**}}\right)\end{aligned}$$

where

$T_{j_1,j_2}(\beta)$ is given by (2.4.19) and S_0^{**} by (2.4.23).

The Bayes estimator of $\lambda_H(x_0)$, at some x_0, is given by $\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln\int_{\beta}\int_{0}^{\infty}\mathrm{e}^{-\kappa\lambda(x_0)}\pi(\alpha,\beta|\underline{x})d\beta d\alpha$.

Since

$$\lambda_H(x_0) = \frac{h(x_0)}{R_H(x_0)} = \frac{\alpha[G(x_0)]^{\alpha-1}g(x_0)}{1-[G(x_0)]^{\alpha}} = \alpha\lambda^*(x_0)[G(x_0)]^{\alpha}\{1-[G(x_0)]^{\alpha}\}^{-1},$$

where $\lambda^*(x_0) = \frac{g(x_0)}{G(x_0)}$, then

$$\begin{aligned}\mathrm{e}^{-\kappa\lambda_H(x_0)} &= \sum_{j_2=0}^{\infty}\frac{[-\kappa\lambda_H(x_0)]^{j_2}}{j_2!}\\ &= \sum_{j_2=0}^{\infty}\frac{(-\kappa)^{j_2}\alpha^{j_2}[\lambda^*(x_0)]^{j_2}[G(x_0)]^{\alpha j_2}}{j_2!}\sum_{j_3=0}^{\infty}C_{j_3}[G(x_0)]^{\alpha j_3}\\ &= \sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}C_{j_2}C_{j_3}[\lambda^*(x_0)]^{j_2}\alpha^{j_2}\ \mathrm{e}^{(j_2+j_3)\alpha\ln G(x_0)}\end{aligned}$$

$$C_{j_2} = \frac{(-\kappa)^{j_2}}{j_2!},\ C_{j_3} = (-1)^{j_3}\binom{j_2+j_3-1}{j_3},\ \lambda^*(x_0) = \frac{g(x_0)}{G(x_0)}$$

Therefore,

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln \int_{\beta}\int_{0}^{\infty} \left(Aw(\beta)\pi_2(\beta) \sum_{j_1=0}^{n-r} C_{j_1} \alpha^{r+b_1-1} \exp[-\alpha T_{0j_1}(\beta)] \right)$$
$$\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_2} C_{j_3} [\lambda^*(x_0)]^{j_2}\, \alpha^{j_2} \mathrm{e}^{(j_2+j_3)\alpha \ln G(x_0)} d\alpha d\beta,$$
$$= -\frac{1}{\kappa} \ln A \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_1} C_{j_2} C_{j_3} \int_{\beta} w(\beta)\pi_2(\beta)[\lambda^*(x_0)]^{j_2} I(\beta) d\beta$$

$$I(\beta) = \int_0^{\infty} \alpha^{r+b_1+j_2-1}\, \mathrm{e}^{-\alpha[T_{0j_1}(\beta)-(j_2+j_3)\ln G(x_0)]} d\alpha$$
$$= \frac{\Gamma(r+b_1+j_2)}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}}.$$

So that

$$\hat{\lambda}_{LNX}(x_0) = -\frac{1}{\kappa}\ln A \sum C \int_{\beta} w(x_0)\pi_2(\beta)[\lambda^*(x_0)]^{j_2}$$
$$\frac{\Gamma(r+b_1+j_2)}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}} d\beta$$
$$= -\frac{1}{\kappa}\ln\left(\frac{S^{**}_{k+3}}{S^{**}_0}\right).$$

where, for $k = 0, 1, 2, \ldots,$

$$S^{**}_{k+3} = \sum C I_{k+3}, I_{k+3} = \frac{\Gamma(r+b_1+j_2)}{\Gamma(r+b_1)} \int_{\beta} \frac{w(x_0)\,\pi_2(\beta)[\lambda^*(x_0)]^{j_2}}{[T_{j_1,j_2,j_3}(\beta)]^{r+b_1+j_2}} d\beta,$$

and $T_{j_1,j_2,j_3}(\beta)$ is given by (2.4.20) and S^{**}_0 by (2.4.23). □

2.5 Bayes One-Sample Prediction of Future Observables (All Parameters of H are Unknown)

2.5.1 One-Sample Scheme

AL-Hussaini (2010a) obtained 100 $(1-\tau)$ % predictive intervals based on the two-sample scheme. In the one-sample scheme, the informative sample consists of the first r order statistics $X_1 < \cdots < X_r$ of a random sample of size n drawn from a population whose CDF is $H(x)$. The future sample consists of the remaining order statistics $X_{r+1} < \cdots < X_n$. Let $Y_s = X_{r+s}, s = 1, \ldots, n-r$. Write $f_r(y_s|\theta)$ to denote the PDF of the sth unit to fail, given that the rth unit had already failed. Then

$$f_r(y_s|\theta) \propto [H(y_s|\theta) - H(x_r|\theta)]^{s-1}[1 - H(y_s|\theta)]^{n-r-s} [R(x_r|\theta)]^{-(n-r)} h(y_s|\theta)$$

The binomial expansion of each of the first two terms on the right-hand side then yields

$$f_r(y_s|\theta) \propto \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} [H(y_s|\theta)]^{s-1-(j_1-j_2)} [H(x_r|\theta)]^{j_1} [R_H(x_r|\theta)]^{-(n-r)} h(y_s|\theta),$$

where
$\theta = (\alpha, \beta), \beta = (\beta_1, \ldots, \beta_k), D_{j_1}$ and D_{j_2} are given by

$$D_{j_1} = (-1)^{j_1} \binom{s-1}{j_1}, \quad D_{j_2} = (-1)^{j_2} \binom{n-r-s}{j_2}. \tag{2.5.1}$$

Substitution of $H(.|\theta) = [G(.|\beta)]^{\alpha})$ and $h(.|\theta) = \alpha G(.|\beta)]^{\alpha-1} g(.|\beta)$ then yields

$$\begin{aligned} f_r(y_s|\theta) &\propto \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} [G(y_s|\beta)]^{\alpha[s-1-j_1+j_2)]} \\ &\quad [G(x_r|\beta)]^{\alpha j_1} \{\alpha G(y_s|\beta)]^{\alpha-1} g(y_s|\beta)\} [R_H(x_r|\theta)]^{-(n-r)} \\ &\propto \lambda^*(y_s|\beta) \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} \alpha e^{\alpha[s-j_1+j_2)] \ln G(y_s|\beta)+j_1 \ln G(x_r|\beta)]} \\ &\quad [R_H(x_r|\theta)]^{-(n-r)}, \end{aligned} \tag{2.5.2}$$

where $\lambda^*(y_s|\beta) = \frac{g(y_s|\beta)}{G(y_s|\beta)]}$.

The posterior PDF is given, from (2.4.1) and (2.4.3), by

$$\begin{aligned}\pi(\theta|\underline{x}) &\propto L(\theta;\underline{x})\pi(\theta) \propto [\prod_{i=1}^{r} h(x_i|\theta)][R_H(x_r|\theta)]^{n-r}\alpha^{b_1-1}\exp(-b_2\alpha)\pi_2(\beta)\\ &= \{\prod_{i=1}^{r}\alpha[G(x_i|\beta)]^{\alpha-1}g(x_i|\beta)\}\alpha^{b_1-1}\exp(-b_2\alpha)\pi_2(\beta)[R_H(x_r|\theta)]^{n-r}\\ &= [\prod_{i=1}^{r}\lambda^*(x_i|\beta)][\prod_{i=1}^{r}[G(x_i|\beta)]^{\alpha}]\alpha^{r+b_1-1}\exp(-b_2\alpha)\pi_2(\beta)[R_H(x_r|\theta)]^{n-r}\\ &= w(\beta,\underline{x})\pi_2(\beta)\alpha^{r+b_1-1}\exp\{-[b_2-\sum_{i=1}^{r}\ln G(x_i|\beta)]\alpha\}[R_H(x_r|\theta)]^{n-r},\end{aligned} \tag{2.5.3}$$

where $w(\beta,\underline{x}) = \prod_{i=1}^{r}\lambda^*(x_i|\beta)]$.

It follows, from (2.5.2) and (2.5.3), that the predictive density function of Y_s is given by

$$\begin{aligned}f^*(y_s|\underline{x}) &= \int_{\Theta} f(y_s|\theta)\pi(\theta|\underline{x})d\theta\\ &= A\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}\int_{\beta} w(\beta,\underline{x})\lambda^*(y_s|\beta)\pi_2(\beta)\int_0^{\infty}\alpha^{r+b_1}\mathrm{e}^{-\alpha T^*_{j_1j_2}(\beta)}d\alpha d\beta\\ &= A\Gamma(r+b_1+1)\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}\int_{\beta}\frac{w(\beta,\underline{x})\lambda^*(y_s|\beta)\pi_2(\beta)}{T^*_{j_1j_2}(\beta)]^{r+b_1+1}}d\beta,\end{aligned} \tag{2.5.4}$$

where $\lambda^*(z|\beta) = \frac{g(z|\beta)}{G(z|\beta)}$ and $T_{0j_1}(\beta)$ is given by (2.4.12) and

$$T^*_{j_1j_2}(\beta) = T_{0j_1} - [s-(j_1-j_2)]\ln G(y_s|\beta), \tag{2.5.5}$$

The predictive SF is then given by

$$\begin{aligned}P[Y_s > v|\underline{x}] &= \int_v^{\infty} f^*(y_s|\underline{x})dy_s\\ &= A\Gamma(r+b_1+1)\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{n-r-s} D_{j_1}D_{j_2}I^*(v),\\ &= A\Gamma(r+b_1+1)S^{***}(v), \quad v > x_r,\end{aligned} \tag{2.5.6}$$

where

$$S^{***}(v) = \sum_{j_1=0}^{s-1} \sum_{j_2=0}^{n-r-s} D_{j_1} D_{j_2} I^*(v), \tag{2.5.7}$$

$$I^*(v) = \int_{\beta} w(\beta, \underline{x}) \pi_2(\beta) \int_{v}^{\infty} \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1 j_2}(\beta)]^{r+b_1+1}} dy_s d\beta.$$

The inner integral of $I^*(v)$ is given by

$$\int_{v}^{\infty} \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1 j_2}(\beta)]^{r+b_1+1}} dy_s = \int_{v}^{\infty} \frac{\lambda^*(y_s|\beta)}{[T_{0j_1}(\beta) - \{s-(j_1-j_2)\} \ln G(y_s|\beta)]^{r+b_1+1}} dy_s$$

By applying the transformation $z = T_{0j_1}(\beta) - \{s-(j_1-j_2)\} \ln G(y_s|\beta)$, $dz = -\{s-j_1+j_2\}\lambda^*(y_s|\beta)dy_s$ and $(v, \infty) \to (z_1, z_2)$, where $z_1 = T_{0j_1}(\beta) - (s-j_1+j_2) \ln G(v|\beta)$ and $z_2 = T_{0j_1}(\beta)$

Therefore

$$\begin{aligned}
\int_{v}^{\infty} \frac{\lambda^*(y_s|\beta)}{[T^*_{j_1 j_2}(\beta)]^{r+b_1+1}} dy_s &= \frac{-1}{s-j_1+j_2} \int_{z_1}^{z_2} z^{-r-b_1-1} dz \\
&= \frac{1}{(r+b_1)(s-j_1+j_2)} z^{-r-b_1} \Big|_{z_1}^{z_2} = \frac{1}{(r+b_1)(s-j_1+j_2)} [z_2^{-r-b_1} - z_1^{-r-b_1}] \\
&= \frac{[T_{0j_1}(\beta)]^{-r-b_1} - [T_{0j_1}(\beta) - \{s-j_1+j_2\} \ln G(v|\beta)]^{-r-b_1}}{(r+b_1)(s-j_1+j_2)}.
\end{aligned}$$

So that

$$I^*(v) = \int_{\beta} w(\beta, \underline{x}) \pi_2(\beta) \left(\frac{[T_{0j_1}(\beta)]^{-r-b_1} - [T_{0j_1}(\beta) - \{s-j_1+j_2\} \ln G(v|\beta)]^{-r-b_1}}{(r+b_1)(s-j_1+j_2)} \right) d\beta \tag{2.5.8}$$

Since, from (2.5.6),
$1 = P[Y_s > x_r|\underline{x}] = A\Gamma(r+b_1+1)S^{***}(x_r)$, then $A = \frac{1}{\Gamma(r+b_1+1)S^{***}(x_r)}$.
where $S^{***}(v)$ is given by (2.5.7).
It then follows, from (2.5.6), that

$$P[Y_s > v|\underline{x}] = \frac{S^{***}(v)}{S^{***}(x_r)}, \quad v > x_r. \tag{2.5.9}$$

A 100 $(1-\tau)$ % two-sided predictive interval for the sth future order statistic Y_s has lower and upper bounds L and U, given by

$$1-(\tau/2) = P[Y_s > L|\underline{x}] = \frac{S^{***}(L)}{S^{***}(x_r)} \quad \text{and} \quad \tau/2 = P[Y_s > U|\underline{x}] = \frac{S^{***}(U)}{S^{***}(x_r)}.$$

Equivalently, L and U are given by the solution of the following equations

$$\left.\begin{aligned} S^{***}(L) - [1-(\tau/2)]S^{***}(x_r) = 0, \\ S^{***}(U) - (\tau/2)S^{***}(x_r) = 0. \end{aligned}\right\} \tag{2.5.10}$$

where $S^{***}(v)$ is given by (2.5.7).

Remarks

1. The one-sided predictive interval of the form $Y_s < L$ is such that

$$0 = S^{***}(L) - (1-\tau)S^{***}(x_r)$$

and of the form $Y_s > U$ is such that

$$0 = S^{***}(U) - \tau S^{***}(x_r).$$

2. Two-sample scheme

In the case of two-sample scheme, we have two independent samples of sizes n and m. The informative sample consists of the first r order statistics $X_1 < \cdots < X_r$ of a random sample of size n. The future sample is assumed to consist of the order statistics $Y_\ell, \ell = 1, \ldots, m$. It is also assumed that all observations are drawn from the same population whose CDF is $H(x) = [G(x)]^\alpha$. Derivations of the estimators and predictive interval of the future observable $Y_\ell, \ell = 1, \ldots, m$ are the same as in the one-sample case, by replacing $f_r(y_s|\theta)$, given by (2.5.2) by

$$f(y_\ell|\theta) \propto [H(y_\ell|\theta)]^{\ell-1}[1-H(y_\ell|\theta)]^{m-\ell}h(y_\ell|\theta).$$

Proceeding as in the one-sample case, we finally obtain the estimators of $\alpha, \beta_i, (i = 1, \ldots, k)$, $R(x_0)$ and $\lambda(x_0)$ are given by

$$\begin{aligned}\hat{\alpha} &= \frac{(r+b_1)S_1^*}{S_0^*}, \beta_i = \frac{S_{i+1}^*}{S_0^*}, (i=1,\ldots,k),\\ R(x_0) &= 1 - \frac{S_{k+2}^*}{S_0^*}, \lambda(x_0) = \frac{(r+b_1)S_{k+3}^*}{S_0^*}\end{aligned} \tag{2.5.11}$$

The predictive SF of the future observable $Y_\ell, \ell = 1, \ldots, k$, is given by

$$P[Y_\ell > v|\underline{x}] = \frac{S_{k+5}^*(v)}{S_{02}}, \tag{2.5.12}$$

where, for $i = 0, 1, \ldots, k+2$,

$$S_i^* = \sum_{j_1=0}^{n-r} C_{j_1} I_{ij_1} \quad \text{and} \quad S_{k+3}^* = \sum_{j_1=0}^{n-r}\sum_{j_2=0}^{\infty} C_{j_1} I_{k+3,j_1,j_2}, \tag{2.5.13}$$

$$\begin{aligned}I_{0j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)}\pi_2(\beta)/T_{0j_1}^{r+b_1}(\beta)d\beta,\\ I_{1j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)}\pi_2(\beta)/T_{0j_1}^{r+b_1+1}(\beta)d\beta,\\ I_{i+1,j_1} &= \int_\beta \beta_i \mathrm{e}^{-T_0(\beta)}\pi_2(\beta)/T_{0j_1}^{r+b_1+1}(\beta)d\beta, \quad i = 1,\ldots,k,\\ I_{k+2,j_1} &= \int_\beta \mathrm{e}^{-T_0(\beta)}\pi_2(\beta)/T_{0j_1}^{r+b_1}(\beta)d\beta,\\ I_{k+3,j_1,j_2} &= \int_\beta \frac{g(x_0|\beta)\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)}{G(x_0|\beta)T_{j_1,j_2}^{r+b_1+1}(\beta)}d\beta,\\ I_{k+4,j_1,j_2} &= \int_\beta \frac{g(y_\ell|\beta)\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)}{G(y_\ell|\beta)T_{j_1,j_2}^{r+b_1+1}(\beta)}d\beta,\\ I_{k+5,j_1,j_3}(v) &= \int_\beta \left[\{T_{0j_1}(\beta)\}^{-(r+b_1)} - \{T_{0j_1}(\beta) - (\ell+j_3)\ln G(v|\beta)\}^{-(r+b_1)}\right]\end{aligned} \tag{2.5.14}$$

$$\mathrm{e}^{-T_0(\beta)}\pi_2(\beta)d\beta. \tag{2.5.15}$$

$T_0(\beta)$ and $T_{0j_1}(\beta)$ are given by (2.4.11) and (2.4.12) and $T_{j_1,j_2}(\beta)$ by (2.4.14).

So that the lower and upper bounds L and U of the $(1-\tau)$ % predictive interval of $Y_\ell, \ell = 1, \ldots, m$ are given by the solution of the equations

$$\left.\begin{aligned} S^*_{k+5}(L) - (1-(\tau/2))S_{02} = 0, \\ S^*_{k+5}(U) - (\tau/2)S_{02} = 0, \end{aligned}\right\} \tag{2.5.16}$$

where

$$S_{02} = \sum_{j_1=0}^{n-r}\sum_{j_3=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_3}}{\ell + j_3}\right] I_{0j_1} \quad \text{and} \quad S^*_{k+5}(v) = \sum_{j_1=0}^{n-r}\sum_{j_3=0}^{m-\ell}\left[\frac{C_{j_1}C_{j_3}}{\ell + j_3}\right] I_{k+5,j_1,j_3} \tag{2.5.17}$$

$$C_{j_1} = (-1)^{j_1}\binom{n-r}{j_1}, C_{j_3} = (-1)^{j_3}\binom{m-\ell}{j_3}, \tag{2.5.18}$$

I_{0j_1} and I_{k+5,j_1,j_3} are given by (2.5.14) and (2.5.15).

For details, See AL-Hussaini (2010a).

2.6 Numerical Computations Applied to Three Examples

Three examples are given: in one of which the base distribution G is in standard form (with no parameters involved), the second depends on one parameter β and, in the third, G depends on two parameters (β_1, β_2). The computations using Bayes method in the three examples are based on the square error loss function.

1. $G(x) = 1 - \mathrm{e}^{-x}$: the base distribution G does not depend on any unknown parameters.
2. $G(x) = 1 - \mathrm{e}^{-\beta x}$: the base distribution G depends on one parameter. In this case, the exponentiated distribution depends on the two parameters (α, β).
3. $G(x) = 1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}$: the base distribution depends on two parameters. In this case, the exponentiated distribution depends on the three parameters $(\alpha, \beta_1, \beta_2)$.

Example 2.1 $G(x) = 1 - \mathrm{e}^{-x}, x > 0$, so that $H(x) = (1 - \mathrm{e}^{-x})^{\alpha}$, where α is unknown.

- **Maximum likelihood estimation**

  To compute the MLEs of $\alpha, R(x_0), H(x_0)$ at some x_0,

  (i) Generate $n = 20$ uniform (0, 1) random numbers $u_1, \ldots, u_{20}$.
  (ii) Compute the corresponding $x_1, \ldots, x_{20}$, where $x_i = -\ln(1 - u_i^{1/\alpha})$, where U_i is uniform on the interval (0, 1). Choose $\alpha = 2.5$.
  (iii) Order the $x's$ and censor at r (r = 20, 18, 15).
  (iv) Use (2.3.4), with $G(x) = 1 - \mathrm{e}^{-x}$, to compute $\hat{\alpha}_{ML}$. The MLEs $\hat{\alpha}, \hat{R}(x_0), \hat{H}(x_0)$ and their MSEs are displayed in Table 2.3a–c.

Table 2.3 $n = 20, b_1 = 3, b_2 = 0.6, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) r = 20 (*complete sample case*)							
α	2.5890	0.0091	2.5780	0.0091	2.5920	0.0092	2.5
$R(x_0)$	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.986
$\lambda(x_0)$	0.165	0.006	0.165	0.006	0.166	0.006	0.16
(**b**) r = 18							
α	2.678	0.0943	2.678	0.0943	2.6820	0.0950	2.5
$R(x_0)$	0.988	0.0001	0.9850	0.0001	0.9850	0.0001	0.986
$\lambda(x_0)$	0.165	0.0008	0.165	0.0008	0.166	0.0008	0.16
(**c**) r = 15							
α	2.7095	0.165	2.7096	0.1652	2.7125	0.169	2.5
$R(x_0)$	0.9980	0.0001	0.9833	0.0001	0.9840	0.0001	0.986
$\lambda(x_0)$	0.1663	0.0042	0.1662	0.0042	0.1680	0.0050	0.16

The following Bayes methods are based on SEL.

- **Standard Bayes Method (SBM)**

Given b_1, b_2 and x_0, the Bayes estimates $\hat{\alpha}, \hat{R}(x_0), \hat{H}(x_0)$ are computed by using the expressions in (2.3.11). Based on 1,000 samples, each of size $n = 20$, censored at r = 20, 18, 15, when $b_1 = 3, b_2 = 0.6, x_0 = 0.2$, the average values of the estimates and their mean square errors (MSEs) over the 1,000 samples are given in Table 2.3a–c. We mean by the MSE, in the Bayes case, the overall risk function.

- **MCMC**

The data set is analyzed by applying Gibbs sampler and Metropolis-Hastings algorithm using WinBUGS 1.4 (http://www.nrc-bsu.cam.ac.uk/bugs/winbugs/contents.smtm1) which can be downloaded and used.

Step 0: Take some initial guess of $\alpha^{(0)}$.

Step 1: From $i = 1$ to N, generate $\alpha^{(i)}$ from the posterior PDF $\pi(\alpha|\underline{z})$, given by (2.3.8).

Step 2: Calculate the Bayes estimator of α by: $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}$, where M is the burn-in period.

Step 3: For a given time x_0, the Bayes estimators of the SF and HRF are given, respectively, by

$$\hat{R}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} [1 - (1 - e^{-x_0})^{\alpha^{(i)}}]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N}\left(\frac{a^{(i)}(1-\mathrm{e}^{-x_0})^{\alpha^{(i)}-1}\mathrm{e}^{-x_0}}{1-(1-\mathrm{e}^{-x_0})^{\alpha^{(i)}}}\right),$$

where M is the burn-in period.

The estimates and their MSEs, obtained by the MCMC method, are reported in Table 2.3a–c. The same parameter, and hyper-parameter values, used in SBM, are used here and computations are based on 1,000 samples.

- **Bayes prediction (two-sample scheme)**

The 95 % predictive intervals $(\tau = 0.05), n = 20, r = 20, 18, 15$ when $b_1 = 3, b_2 = 0.6$ for the first future observable Y_1 in a sample of size $m = 10$ future observables, are obtained by solving the two equations, given by (2.5.11). The intervals are found to be:

$$\begin{aligned} &0.0821 < Y_1 < 1.00845, \quad \text{length} = 1.0024, \quad (r = 20)\\ &0.0854 < Y_1 < 1.0921, \quad \text{length} = 1.0067, \quad (r = 18)\\ &0.0862 < Y_1 < 1.0949, \quad \text{length} = 1.0087, \quad (r = 15) \end{aligned}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively and similarly for the upper bounds.

Example 2.2 $G(x) = 1 - \mathrm{e}^{-\beta x}$, $x > 0$, $(\beta > 0)$, the base distribution G depends on a single parameter β.

- **Maximum likelihood estimation**

With $k = 1$, Eqs. (2.4.3) and (2.4.4) reduce to only two equations and we write β for β_1. The solution of the two equations, using some iteration scheme, such as Newton-Raphson, yields MLEs $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ of α and β. The MLEs of $\hat{R}_{ML}(x_0)$ and $\hat{\lambda}_{ML}(x_0)$ are obtained by applying the invariance property of MLEs. The average values of the estimates and their mean square errors (MSEs) over the 1,000 samples are given in Table 2.4a–c.

- **Standard Bayes method**

Suppose that α and β are independent and that α is distributed as gamma (b_1, b_2) whose PDF is given by (2.3.7) and β is distributed as gamma (b_3, b_4) whose PDF is given by

$$\pi_2(\beta) \propto \beta^{b_3-1}\exp(-b_4\beta), \quad \beta > 0, (b_3, b_4 > 0). \tag{2.6.1}$$

Table 2.4 $n = 20, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) *r = 20 (complete sample case)*							
α	2.5480	0.0254	2.5261	0.0071	2.5920	0.0092	2.5
β	1.523	0.0006	1.519	0.0004	1.574	0.0013	1.5
$R(x_0)$	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9855	8.17×10^{-5}	0.9658
$\lambda(x_0)$	0.3776	0.0006	0.3771	0.0006	0.3702	0.0022	0.3795
(**b**) $r = 18$							
α	2.6078	0.1325	2.5628	0.0442	2.6640	0.5821	2.5
β	1.4523	0.0154	1.4569	0.0152	1.4491	0.0283	1.5
$R(x_0)$	0.9618	1.2×10^{-4}	0.9618	1.2×10^{-4}	0.9608	1.5×10^{-4}	0.9658
$\lambda(x_0)$	0.3761	0.0092	0.3755	0.0092	0.3681	0.0191	0.3795
(**c**) $r = 15$							
α	2.6875	0.5960	2.6940	0.5573	2.7930	0.6956	2.5
β	1.4375	0.0687	1.4436	0.0638	1.4017	0.0854	1.5
$R(x_0)$	0.9580	0.0004	0.9583	0.0004	0.9574	5.88×10^{-4}	0.9658
$\lambda(x_0)$	0.3758	0.0130	0.3750	0.0130	0.3651	0.0282	0.3795

Accordingly, the Bayes estimators of $\alpha, \beta, R(x_0), \lambda(x_0)$ are given by

$$\hat{\alpha} = \frac{(r+b_1)S_1^*}{S_0^*},\ \hat{\beta} = \frac{S_2^*}{S_0^*},\ \hat{R}(x_0) = 1 - \frac{S_3^*}{S_0^*},\ \hat{\lambda}(x_0) = \frac{(r+b_1)S_4^*}{S_0^*}. \tag{2.6.2}$$

The predictive PDF and SF of the future ℓ th observable $Y_\ell, \ell = 1, \ldots, k$, are given by

$$f^*(y_\ell|\underline{x}) = \frac{(r+b_1)S_5^*}{S_{02}} \text{ and } P[Y_\ell > v|\underline{x}] = \frac{S_6^*(v)}{S_{02}},$$

where, for $v = 0, 1, 2, 3$, S_v^* and $S_4^* - S_6^*(.)$ are given by (2.5.17) with $k = 1$.

The integrals involved are given as follows, from (2.5.17), when $k = 1$ and $\pi_2(\beta)$ is given by (2.6.1). So that

$$I_{0j_1} = \int_0^\infty [\beta^{b_3-1} e^{-T^*(\beta)} / T_{0j_1}^{r+b_1}(\beta)] d\beta,$$

$T^*(\beta) = b_4\beta + T_0(\beta), T_0(\beta)$
is given by (2.4.11)

$$I_{1j_1} = \int_0^\infty [\beta^{b_3-1}\mathrm{e}^{-T^*(\beta)}/T_{0j_1}^{r+b_1+1}(\beta)]d\beta,$$

$$I_{2j_1} = \int_0^\infty [\beta^{b_3}\mathrm{e}^{-T^*(\beta)}/T_{0j_1}^{r+b_1}(\beta)]d\beta,$$

$$I_{3j_1} = \int_0^\infty [\beta^{b_3-1}\mathrm{e}^{-T^*(\beta)}/T_{1j_1}^{r+b_1}(\beta)]d\beta,$$

$$I_{4j_1j_2} = \int_0^\infty \frac{g(x_0|\beta)\beta^{b_3-1}\mathrm{e}^{-T^*(\beta)}}{G(x_0|\beta)T_{j_1j_2}^{r+b_1+1}(\beta)}d\beta,$$

$$I_{5j_1j_2} = \int_0^\infty \frac{g(y_\ell|\beta)\beta^{b_3-1}\mathrm{e}^{-T^*(\beta)}}{G(y_\ell|\beta)T_{j_1j_2}^{r+b_1+1}(\beta)}d\beta,$$

$$I_{6j_1j_2} = \int_0^\infty \left[\frac{1}{\{T_{0j_1}(\beta)\}^{r+b_1}} - \frac{1}{\{T_{0j_1}(\beta) - (\ell+j_3)\ln G(v|\beta)\}^{r+b_1}}\right]\beta^{b_3-1}\mathrm{e}^{-T^*(\beta)}d\beta$$

Computations are carried out in the same manner as in Example 2.1, with the obvious changes in which $H(x) = [G(x)]^\alpha, G(x) = 1 - \mathrm{e}^{-\beta x}$, with $n = 20$, censored at r = 20, 18, 15, when $b_1 = 3, b_2 = 0.6$. So that $x_i = -\frac{1}{\beta}\ln(1 - u_i^{1/\alpha})$, $i = 1, \ldots, 20$, where α and β are chosen to be $\alpha = 2.5$ and $\beta = 1.5$. Computations are also based on chosen values of $x_0 = 0.2, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3$.

- **MCMC**

In this case, samples are generated from the posterior distributions. Bayes estimates of α and β and their functions are computed according to the following steps:

Step 0: Take some initial guess of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$.
Step 1: Generate $\alpha^{(1)}$ and $\beta^{(1)}$ from the posterior PDFs, given by

$$\pi(\alpha|\beta, \underline{x}) = \frac{\pi(\alpha, \beta|\underline{x})}{\int_0^\infty \pi(\alpha, \beta|\underline{x})d\alpha} = \frac{\alpha^{r+b_1-1}\sum_{j_1=0}^{n-r} C_{j_1}\mathrm{e}^{-\alpha T_{0j_1}(\beta)}}{\Gamma(r+b_1)\sum_{j_1=0}^{n-r} C_{j_1}/[T_{0j_1}(\beta)]^{r+b_1}},$$

$$\pi(\beta|\alpha, \underline{x}) = \frac{\pi(\alpha, \beta|\underline{x})}{\int_0^\infty \pi(\alpha, \beta|\underline{x})d\beta} = \frac{\sum_{j_1=0}^{n-r} C_{j_1}\beta^{b_3-1}\mathrm{e}^{-\alpha T_{0j_1}(\beta)-T^*(\beta)}}{\sum_{j_1=0}^{n-r} C_{j_1}\int_0^\infty \beta^{b_3-1}\mathrm{e}^{-\alpha T_{0j_1}(\beta)-T^*(\beta)}d\beta},$$

where C_{j_1} and $T_{0j_1}(\beta)$ are given by (2.4.12), $T^*(\beta) = b_4\beta + T_0(\beta)$, $T_0(\beta)$ is given by (2.4.11).

Step 2: From $i = 1$ to $N - 1$, generate $\alpha^{(i+1)}$ and $\beta^{(i+1)}$ from $\pi(\alpha|\,\beta^{(i+1)}, \underline{x})$ and $\pi(\beta|\,\alpha^{(i+1)}, \underline{x})$, respectively.

Step 3: Calculate the Bayes estimates of α and β from $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N} \alpha^{(i)}$ and $\hat{\beta} = \frac{1}{N-M}\sum_{i=M+1}^{N} \beta^{(i)}$, respectively.

For a given x_0, calculate the Bayes estimates of the SF and HRF from:

$$\hat{R}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} [1 - (1 - \mathrm{e}^{-\beta^{(i)} x_0})^{\alpha^{(i)}}]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N} \left(\frac{a^{(i)}\beta^{(i)}(1 - \mathrm{e}^{-\beta^{(i)}x_0})^{\alpha^{(i)}-1}\mathrm{e}^{-\beta^{(i)}x_0}}{1 - (1 - \mathrm{e}^{-\beta^{(i)}x_0})^{\alpha^{(i)}}} \right).$$

It may be noted that in Kundu and Gupta (Kundu and Gupta 2008), the Bayes estimates of the parameters α and β (α and λ in their notation) were developed in the complete sample case, assuming that α and β are independent and each has a gamma prior.

The estimates using the ML method, SBM, MCMC are displayed in Table 2.4a–c.

- **Bayes prediction (two-sample scheme)**

The 95 % predictive interval $(\tau = 0.05), n = 20, r = 20, 18, 15$ when $b_1 = 3, b_2 = 0.6$, $b_3 = 2, b_4 = 3$, for the first future observable Y_1, in a sample of size $m = 10$ future observables are found to be

$$\begin{aligned} 0.0321 < Y_1 < 0.6542, &\quad \text{length} = 0.6221, \quad (r = 20) \\ 0.0412 < Y_1 < 0.7320, &\quad \text{length} = 0.6908, \quad (r = 18) \\ 0.0447 < Y_1 < 0.7397, &\quad \text{length} = 0.6950, \quad (r = 15) \end{aligned}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively. Similarly for the upper bounds.

Example 2.3 $G(x) = 1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}, x > 0$, $(\beta_1, \beta_2 > 0)$, the base (Weibull) distribution depends on two unknown parameters β_1, β_2.

- **Maximum likelihood estimation**

With $k = 2$, the system of LEs (2.4.3) and (2.4.4) reduce to three equations in the three unknowns α, β_1, β_2. By solving such equations, using some iteration method we obtain the MLEs of these parameters. The MLEs of $R(x_0)$ and $\lambda(x_0)$ are computed by applying the invariance property of MLEs.

- Standard Bayes method

Suppose that α and $\beta = (\beta_1, \beta_2)$ are independent and that α is distributed as gamma (b_1, b_2) whose PDF is given by (2.3.7) and β is such that

$$\begin{aligned}\pi_2(\beta) &= \pi_3(\beta_2|\beta_1)\pi(\beta_1)\\ &\propto [\beta_2^{b_4-1}\mathrm{e}^{-\beta_1\beta_2}][\beta_1^{b_3-1}\mathrm{e}^{-b_5\beta_1}],\\ &\quad \beta_1, \beta_2 > 0, (b_3, b_4, b_5 > 0)\\ &\propto \beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-\beta_1(b_5+\beta_2)}.\end{aligned} \tag{2.6.3}$$

So that the prior PDF of θ, is given, from (2.3.7) and (2.5.17), by

$$\pi(\theta) = \pi_1(\alpha)\pi_2(\beta) \propto \alpha^{b_1-1}\,\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-b_2\,\alpha-\beta_1(b_5+\beta_2)}. \tag{2.6.4}$$

According to Theorem 2.3, the Bayes estimators of $\alpha, \beta_1, \beta_2, R(x_0), \lambda(x_0)$ are given by

$$\hat{\alpha} = \frac{(r+b_1)\,S_1^*}{S_0^*},\ \hat{\beta}_1 = \frac{S_2^*}{S_0^*},\ \hat{\beta}_2 = \frac{S_3^*}{S_0^*},\ \hat{R}(x_0) = 1 - \frac{S_4^*}{S_0^*},\ \hat{\lambda}(x_0) = \frac{(r+b_1)S_5^*}{S_0^*} \tag{2.6.5}$$

where, for $\ell = 0, 1, \ldots, 5$, S_ℓ^* is given by (2.5.13), in which

$$I_{0j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,$$

$$T^*(\beta_1, \beta_2) = T_0(\beta_1, \beta_2) + \beta_1(b_5 + \beta_2),$$

$$I_{1j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1+1}}d\beta_1 d\beta_2,$$

$$I_{2j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,$$

$$I_{3j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{0j_1}^{r+b_1}}d\beta_1 d\beta_2,$$

$$I_{4j_1} = \int_0^\infty\int_0^\infty \frac{\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{T_{1j_1}^{r+b_1}}d\beta_1 d\beta_2,$$

$$I_{5j_1j_2} = \int_0^\infty\int_0^\infty \frac{g(x_0|\beta_1,\beta_2)\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{G(x_0|\,\beta_1,\beta_2)T_{j_1j_2}^{r+b_1+1}(\beta_1,\beta_2)}d\beta_1 d\beta_2,$$

In this example, $G(x) = 1 - \mathrm{e}^{-\beta_1 x^{\beta_2}}$, so that $X_i = [-\frac{1}{\beta_1}\ln(1 - U_i^{1/\alpha})]^{1/\beta_2}$.

- **MCMC**

Bayes estimates of α, β_1, β_2 and their functions are computed according to the following steps:

Step 0: Take some initial guess of α, β_1, β_2, say $\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}$.

Step 1: Generate $\alpha^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}$ from the posterior PDFs, given, respectively, by

$$\begin{aligned}
\pi(\alpha|\beta_1,\beta_2,\underline{x}) &= \frac{\pi(\alpha,\beta_1,\beta_2|\underline{x})}{\int_0^\infty \pi(\alpha,\beta_1,\beta_2|\underline{x})d\alpha} \\
&= \frac{\alpha^{r+b_1-1}\sum_{j_1=0}^{n-r} C_{j_1} e^{-\alpha T_{0j_1}(\beta_1,\beta_2)}}{\Gamma(r+b_1)\sum_{j_1=0}^{n-r} C_{j_1}/[T_{0j_1}(\beta_1,\beta_2)]^{r+b_1}} \\
\pi(\beta_1|\alpha,\beta_2,\underline{x}) &= \frac{\pi(\alpha,\beta_1,\beta_2|\underline{x})}{\int_0^\infty \pi(\alpha,\beta_1,\beta_2|\underline{x})d\beta_1} \\
&= \frac{\sum_{j_1=0}^{n-r} C_{j_1}\beta_1^{b_3-1} e^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}}{\sum_{j_1=0}^{n-r} C_{j_1}\int_0^\infty \beta_1^{b_3-1} e^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}d\beta_1}, \\
\pi(\beta_2|\alpha,\beta_1,\underline{x}) &= \frac{\pi(\alpha,\beta_1,\beta_2|\underline{x})}{\int_0^\infty \pi(\alpha,\beta_1,\beta_2|\underline{x})d\beta_2} \\
&= \frac{\sum_{j_1=0}^{n-r} C_{j_1}\beta_2^{b_4-1} e^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}}{\sum_{j_1=0}^{n-r} C_{j_1}\int_0^\infty \beta_2^{b_4-1} e^{-\alpha T_{0j_1}(\beta_1,\beta_2)-T^*(\beta_1,\beta_2)}d\beta_2}.
\end{aligned}$$

where C_{j_1} is given by (2.3.11), $T_{0j_1}(\beta)$ by (2.4.17) and $T^*(\beta_1,\beta_2) = b_4\beta + T_0(\beta)$, $T_0(\beta)$ is given by (2.4.16).

Step 2: From $i = 1$ to $N-1$, generate $\alpha^{(i+1)}, \beta_1^{(i+1)}$ and $\beta_2^{(i+1)}$ from $\pi(\alpha|\beta_1^{(i)}, \beta_2^{(i)}, \underline{x})$, $\pi(\beta_1|\alpha^{(i)}, \beta_2^{(i)}, \underline{x})$ and $\pi(\beta_2|\alpha^{(i)}, \beta_1^{(i)}, \underline{x})$, respectively.

Step 3: Calculate the Bayes estimates of α,β_1 and β_2 from $\hat{\alpha} = \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}$, $\hat{\beta}_1 = \frac{1}{N-M}\sum_{i=M+1}^{N}\beta_1^{(i)}$ and $\hat{\beta}_2 = \frac{1}{N-M}\sum_{i=M+1}^{N}\beta_2^{(i)}$, respectively.

For a given x_0, calculate the Bayes estimates of the SF and HRF from:

$$\hat{R}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N}\left[1-\left(1-e^{-\beta_1^{(i)}x_0^{\beta_2^{(i)}}}\right)^{\alpha^{(i)}}\right]$$

and

$$\hat{\lambda}(x_0) = \frac{1}{N-M}\sum_{i=M+1}^{N}\left(\frac{a^{(i)}\beta_1^{(i)}\beta_2^{(i)}x_0^{\beta_2^{(i)}-1}\left[1-e^{-\beta_1^{(i)}x_0^{\beta_2^{(i)}}}\right]^{\alpha^{(i)}-1}e^{-\beta_1^{(i)}x_0^{\beta_2^{(i)}}}}{1-\left[1-e^{-\beta_1^{(i)}x_0^{\beta_2^{(i)}}}\right]^{\alpha^{(i)}}}\right)$$

Table 2.5 $n = 20, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$ and 1,000 repetitions

Parameters	SBM		MCMC		MLE		Actual values
	Bayes	MSE	Bayes	MSE	ML	MSE	
(**a**) r = 20 (*complete sample case*)							
α	2.4012	0.0054	2.4865	0.0005	2.3122	0.0731	2.5
β_1	1.3942	0.0121	1.4572	0.0013	1.2647	0.0596	1.5
β_2	0.6012	0.0143	0.5231	0.0018	0.6412	0.0596	0.5
$R(x_0)$	0.8341	0.0002	0.8333	0.0001	0.8631	0.0012	0.833
$\lambda(x_0)$	0.8371	0.0082	0.8568	0.0023	0.8134	0.0121	0.8792
(**b**) r = 18							
α	2.2143	0.1251	2.4431	0.0851	2.0122	0.2845	2.5
β_1	1.2491	0.0649	1.4025	0.0056	1.1831	0.1314	1.5
β_2	0.6582	0.0423	0.6021	0.0214	0.7342	0.0921	0.5
$R(x_0)$	0.8352	0.0009	0.8.339	0.0002	0.8921	0.0116	0.833
$\lambda(x_0)$	0.8021	0.0142	0.8352	0.0092	0.7985	0.0163	0.8792
(**c**) r = 15							
α	2.0051	0.3325	2.4258	0.1369	1.9683	0.3452	2.5
β_1	1.1638	0.1272	1.3253	0.0509	1.1276	0.2134	1.5
β_2	0.7420	0.0883	0.6807	0.0655	0.8535	0.1049	0.5
$R(x_0)$	0.8374	0.0015	0.8344	0.0016	0.9840	0.0145	0.833
$\lambda(x_0)$	0.7786	0.0295	0.8081	0.0270	0.7621	0.0352	0.8792

where M is the burn-in period.

Computations are carried out as before for $n = 20, r = 20, 18, 15, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 0.5, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$. The estimates obtained by the above methods and their MSEs are displayed in Table 2.5.

- Bayes prediction (two-sample scheme)

The 95 % predctive intervals, for the first future observable Y_1 in a sample of size $m = 10$ future observables, when $n = 20, r = 20, 18, 15, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 0.5, b_1 = 3, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 0.8, x_0 = 0.2$. are obtained by solving the two equations, given by (2.5.16). The intervals are found to be:

$$\begin{aligned} &0.00193 < Y_1 < 0.54554, \quad \text{length} = 0.54361, \quad (r = 20) \\ &0.00215 < Y_1 < 0.57045, \quad \text{length} = 0.56830, \quad (r = 18) \\ &0.00309 < Y_1 < 0.58464, \quad \text{length} = 0.58155, \quad (r = 15) \end{aligned}$$

where the lower bound of each interval is the average of the lower bounds L computed to satisfy the first equation of (2.5.16) for each one of the 1,000 samples, respectively and similarly for the upper bounds, where

$$I_{6j_1j_2} = \int_0^\infty \int_0^\infty \frac{g(x_\ell|\beta_1,\beta_2)\beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)}}{G(x_\ell|\beta_1,\beta_2)T_{j_1j_2}^{r+b_1+1}(\beta_1,\beta_2)} d\beta_1 d\beta_2$$

$$I_{7j_1j_3'} = \int_0^\infty \int_0^\infty \left[\{T_{0j_1}(\beta_1,\beta_2)\}^{-(r+b_1)} - \{T_{0j_1}(\beta_1,\beta_2) - (\ell+j_3)\ln G(v|\beta_1,\beta_2)\}^{-(r+b_1)}\right] \times \beta_1^{b_3-1}\beta_2^{b_4-1}\mathrm{e}^{-T^*(\beta_1,\beta_2)} d\beta_1 d\beta_2.$$

Remarks

1. It may be noticed, in the three examples, that the Bayes estimates, using the MCMC, performs best in most cases, in the sense of having smallest MSEs then comes the estimates using SBM and finally those based on MLEs.
2. Even with censored samples (r = 15), the estimates are still reasonable.
3. Indeed, predictive intervals for $Y_\ell, \ell = 2, \ldots, k$, can be obtained as that computed for Y_1, by following the same steps.

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