Chapter 2
An Electrostatically Actuated Micro-electro-mechanical System

Micro-electro-mechanical systems (MEMS) are miniaturized devices made up of mechanical and electro-mechanical elements. The industrial interest in decreasing the scale of electronic components has made the study of MEMS a hot topic in Electronic Engineering, including of course modeling and simulation. The monograph [12] presents a comprehensive treatment of MEMS and an updated bibliography. Here, we consider the existence and stability of periodic oscillations of an idealized mass-spring model of MEMS that has become canonical in the related literature.

The system under study is illustrated in Fig. 2.1 and consists of two parallel electrodes separated by a gap d; one of them is fixed and the second one is movable and attached to a linear spring with stiffness coefficient $k > 0$. When an AC-DC voltage $V(t) = v_{dc} + v_{ac}cos(\omega t)$ is applied, the Coulomb force between the plates makes the system highly nonlinear. Oscillations are ruled by the second order differential equation

$$my'' + cy' + ky = \frac{\varepsilon_0 A}{2} \frac{V^2(t)}{(d-y)^2}, \tag{2.1}$$

where y is the vertical displacement of the moving plate (y is always assumed to be less than d), m is its mass, c is a viscous damping coefficient, ε_0 is the absolute dielectric constant of vacuum and A is the electrode area.

This basic model was introduced by Nathanson et al. [10] in 1967 and has been studied since then by many authors (see the references in [12]) in connection with the phenomenon of *pull-in instability*: experimentally, for a small DC force it is observed that the upper electrode is in equilibrium, keeping a distance with the lower electrode, but if v_{dc} is gradually increased, it reaches a bifurcation value beyond which the structure collapses suddenly, hitting the lower electrode. Alternatively, one may leave the voltage fixed and gradually decrease the gap width d, obtaining the same effect.

In the case of a pure DC voltage $V(t) \equiv v_{dc}$, pull-in instability is easily explained as a saddle-node bifurcation. Equilibria of (2.1) correspond to the roots of

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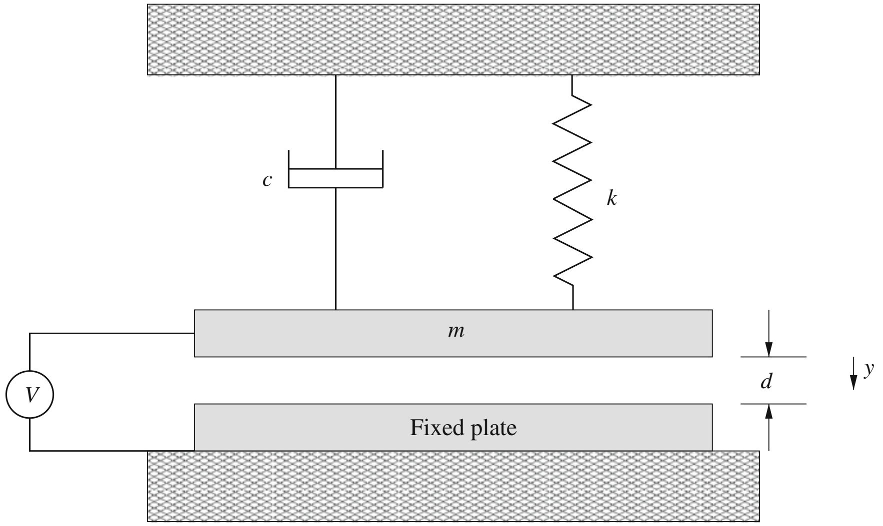

Fig. 2.1 Idealized mass-spring model of electrostatically actuated MEMS

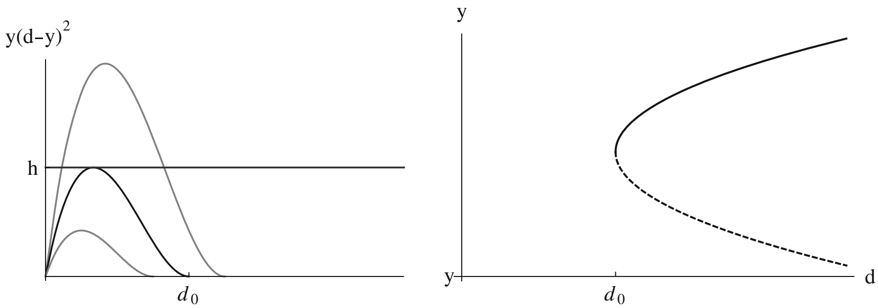

Fig. 2.2 Illustration of the saddle-node bifurcation in the case of DC voltage

$y(d-y)^2 = h$, where $h = \frac{\varepsilon_0 A v_{dc}^2}{2k} > 0$. This equation always has a root greater than d and hence without physical meaning. A direct analysis provides a threshold value

$$d_0 = \frac{3}{2}(2h)^{1/3}, \tag{2.2}$$

such that when $d > d_0$ there are two branches (saddle and node) collapsing at $d = d_0$, as shown in Fig. 2.2.

When the AC voltage is included, (2.1) becomes a non-autonomous differential equation and the objective is to prove rigorously the pull-in effect. This was done in [1] for the so-called *viscosity dominated regime*, when the damping coefficient is very high and damping effects dominate over inertial effects. This hypothesis leads to a reduced approximate first-order equation, revealing the saddle-node bifurcation.

Here we are going to outline the approach presented in [5], which has the advantage of being valid for the original (2.1), without any kind of approximation.

2.1 A Non-autonomous Saddle-Node Bifurcation

Hereafter, $V(t)$ is a continuous, positive, T-periodic function with $T = \frac{2\pi}{\omega}$. For convenience, we call $V_m = \min_{[0,T]} V(t)$, $V_M = \max_{[0,T]} V(t)$.

Theorem 2.1 *There exists $d_0 > 0$ such that*

1. *If $d < d_0$, (2.1) has no T-periodic solutions.*
2. *If $d = d_0$, (2.1) has at least one T-periodic solution.*
3. *If $d > d_0$, (2.1) has at least two T-periodic solutions.*

Besides, d_0 admits the following quantitative estimate

$$\frac{3}{2}\left(\frac{\varepsilon_0 A V_m^2}{k}\right)^{1/3} \le d_0 \le \frac{3}{2}\left(\frac{\varepsilon_0 A V_M^2}{k}\right)^{1/3}. \tag{2.3}$$

For the proof, first the singularity is moved to 0 by means of the change $u = d - y$. Then the proof follows five steps:

- Computation of explicit a priori bounds of the eventual T-periodic solutions
- Localization of the branch of unstable solutions by the method of upper and lower functions for a high d.
- It is shown that there are no solutions for a small d.
- By continuation and additivity of the degree, the second branch is proved to exist.
- The explicit bounds of the first step are used to estimate the bifurcation point by (2.3).

Remark 2.1 The inequality (2.3) is optimal because if $V(t)$ is constant (autonomous case), then the inequalities are in fact equalities, and d_0 is exactly the value $d_0 = \frac{3}{2}(2h)^{1/3}$ obtained before.

To assure the stability of the second branch, we need an additional hypothesis.

Theorem 2.2 *Given the conditions of Theorem 2.1, assume that*

$$4k < \frac{4\varepsilon_0 A c^3 \omega^3 V_m^8}{\left[\pi k c + c\omega V_m^2\right] d^3} + m\omega^2 + \frac{c^2}{m}. \tag{2.4}$$

Then,

1. *if $d = d_0$, (2.1) has a unique T-periodic solution which is not asymptotically stable,*

2. *if* $d > d_0$, (2.1) *has exactly two* T*-periodic solutions, one uniformly asymptotically stable and the other unstable.*

The proof is a direct application of Proposition A.3.

Remark 2.2 Concerning the physical meaning of condition (2.4), note that if $4k \leq \frac{c^2}{m}$ then (2.4) holds for any frequency w. This case can be related to the "viscosity dominated regime" studied in [1]. On the other hand, if $4k > \frac{c^2}{m}$, we can take $\omega^2 > 4k$. This resembles the paradigmatic phenomenon of "stabilization by high frequencies", which appears in a wide number of physical systems such as the inverted pendulum with vibrating support (see for instance [4]).

Example 2.1 For illustrative purposes, the results can be tested with the following realistic values of the physical parameters: $m = 3.5 \times 10^{-11}$ Kg, $k = 0.17$ N/m, $c = 1.78 \times 10^{-6}$ Kg/s, $A = 1.6 \times 10^{-9}\,\mathrm{m}^2$, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m (see [13]). If $V(t) = 10 + 2\cos(\omega t)$ V, then using Theorem 2.1 the bifurcation value is bounded by $2.62033 \times 10^{-6}\,\mathrm{m} < d_0 < 3.4336 \times 10^{-6}\,\mathrm{m}$. By Theorem 2.2, if $d > d_0$ and $\omega \geq 129777\,\mathrm{s}^{-1}$, then there are exactly two periodic solutions of (2.1), one asymptotically stable and the other unstable.

2.2 Further Remarks and Open Problems

The main result of this chapter (Theorem 2.1) only applies when the voltage $V(t)$ is positive for all t. For the model AC-DC voltage $V(t) = v_{dc} + v_{ac}\cos(\omega t)$, it means that $v_{dc} > v_{ac}$, that is, the DC voltage dominates over the DC voltage.

Open Problem 2.1 *To prove Theorem* 2.1 *for the model equation* (2.1) *with* $V(t) = v_{dc} + v_{ac}\cos(\omega t)$ *and* $v_{dc} \leq v_{ac}$.

The studied model is perhaps the simplest version of a MEMS and can be extended in many ways to include additional nonlinear effects that may influence the dynamics of the system. It is also possible to construct more complicated micro-structures with a higher number of moving components. We will briefly review some of these possible extensions below.

A first possibility is to consider a nonlinear damping (squeeze film damping) modeling the effect of air damping between the plates (see [2, 14], also [12, Sect. 4.3] and the references therein). This leads to the study of the equation

$$my'' + c(y)y' + ky = \frac{\varepsilon_0 A}{2}\frac{V^2(t)}{(d-y)^2}, \tag{2.5}$$

where (following [2, Sect. 4]), the damping coefficient has the form

$$c(y) = \frac{A}{(d-y)^3} + \frac{A}{d-y}.$$

Observe that the squeeze film damping is also a singular function.

In the design of a MEMS, it is natural to ask about the response of the device under the influence of external mechanical shocks. This effect has been considered in [7]. In the model equation, one has to include an additional external force $p(t)$ with zero mean value.

Yet, as we commented before, it is possible to consider more complicated microstructures. In the reference [3], the moving electrode is situated between two fixed capacitors, giving rise to a model with two singularities. Concretely, the relevant model is reduced to the equation

$$z'' + cz' + z = a_0 \left(\frac{1}{(1-z)^2} - \frac{1}{(1+z)^2} \right) + f \cos \omega t. \tag{2.6}$$

For this model, it is reasonable to conjecture a pitchfork bifurcation. A different model with two singularities representing a torsional actuator is considered in [8].

One important class of MEMS would be *microbeams*, where the moving electrode has one or two clamped ends, leading to the consideration of a beam equation of fourth order with singular nonlinearity (see [12, Chap. 6]). In turn, the analysis of the dynamical properties of arrays of nonlinearly coupled MEMS [6] is a challenging problem.

Finally, let us recall that at a scale of nanometers (nanoscale), besides the electromagnetic force it is necessary to consider intermolecular forces like the Van der Waals or Casimir force [8, 9, 11]. In the model equation, the consideration of such forces implies the inclusion of additional singular terms of higher order.

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