

Chapter 2

The Sonin–Douglas Problem

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2.1 Introduction

Consider a system of second-order ordinary differential equations, solved with respect to the second derivatives of an unknown curve $x^i = x^i(t)$,

$$\ddot{x}^j - F^j = 0, \quad (2.1)$$

where $i, j = 1, 2, \dots, m$, and $F = F^j(x^i, \dot{x}^i)$ are given functions. Any collection of functions $g_{jk} = g_{jk}(x^i, \dot{x}^i)$, such that $\det g_{ij} \neq 0$, defines an equivalent system $g_{ij}(\ddot{x}^j - F^j) = 0$. Our goal is to study the problem of existence of a function $\mathcal{L} = \mathcal{L}(x^i, \dot{x}^i)$ such that

$$g_{ij}(\ddot{x}^j - F^j) = -\frac{\partial \mathcal{L}}{\partial x^i} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i}, \quad (2.2)$$

known as the *inverse problem of the calculus of variations* for the system (2.1). For historical reasons, we also refer to this problem as the *Sonin–Douglas problem*; Eq. (2.2) are called the *Sonin–Douglas equations*. Having in mind the correspondence with classical mechanics, we sometimes call the system $F = (F^1, \dots, F^m)$ the *force*. The functions g_{jk} are called *variational multipliers*. If the function $\mathcal{L}$ exists, it is a *Lagrangian* for the Eq. (2.2). Denoting

$$\varepsilon_i = g_{ij}(F^j - \ddot{x}^j), \quad (2.3)$$

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we can equivalently say that the functions ε_i are the *Euler–Lagrange expressions* for $\mathcal{L}$, or that the system of functions $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ is *variational*.

First ideas related to the variational origin of differential equations appeared in Sonin [15] in 1866, who studied the inverse problem for *one* second-order equation and proved that every second-order equation admits a Lagrangian; this Lagrangian may depend on time. The same idea and approach has later appeared in Darboux [4] in 1894. In 1941, Douglas [5] derived a complete classification of systems (2.2) for two equations and provided numerous examples of non-variational systems. The results of Douglas have been further developed from geometrical point of view by many authors (see e.g. Anderson and Thompson [1], Bucătaru [2], Crampin [3], Krupka et al. [11], Krupková and Prince [12], Sarlet et al. [13], Urban and Krupka [17, 18], and many others).

The Sonin–Douglas problem is a special case of the *Helmholtz variability* problem for general systems of m ordinary second-order equations in an implicit form

$$\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j) = 0, \quad (2.4)$$

where $1 \leq i, j \leq m$ (Helmholtz [19], 1887) for m functions of one real variable $t \rightarrow x^j(t)$. For historical remarks on the *Helmholtz variability conditions* and their generalizations we refer the reader to Havas [6], Krupka [7, 9] and Krupka et al. [11].

In this chapter we suppose, in accordance with Douglas [5], that our underlying spaces are open sets in Euclidean spaces; this completely covers the use of the inverse problem theory for differential equations and some applications in manifold theory. We do not consider the systems of differential equations (2.3) and (2.4) such that the functions ε_i depend explicitly on the parameter t of the curves $t \rightarrow x^j(t)$. Also, we do not discuss geometric aspects of the theory related to semisprays on tangent bundles.

For general variational theory on smooth manifolds and different aspects of the inverse problem we refer to Krupka [10] and Zenkov [20]. The chapter includes an elementary introduction to the Helmholtz and Sonin–Douglas problems as well as a derivation of a necessary and sufficient variability condition in terms of a system of partial differential equations whose solutions characterize the variational equations. It should be pointed out, however, that our approach and basic results *differ* from those of Douglas in two ways.

First, in his paper [5], Douglas requires *analyticity* of the initial data of the problem; in our derivation of the basic system of differential equations only finite *differentiability* is needed. Second, our basic theorem, characterizing variational equations in terms of a system of partial differential equations, does *not* coincide with an analogous system used by Douglas. Our approach is based on the variability criteria of the *Helmholtz type* whereas Douglas did not utilize the Helmholtz theory.

2.2 Energy Lagrangians

Suppose we have a system of functions $h = h_{jk}(x^i, \dot{x}^i)$, such that $h_{jk} = h_{kj}$, defined on an open set $U \times \mathbb{R}^m$, where U is an open set in $\mathbb{R}^m$. If in addition $\det h_{jk} \neq 0$, we sometimes call h a *metric* (on the set $U \times \mathbb{R}^m$). Consider a variational principle for curves in U defined by the Lagrangian

$$\mathcal{L}_h = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j \quad (2.5)$$

(the *energy Lagrangian* associated with the metric h).

Lemma 2.1 *The Euler–Lagrange expressions of the Lagrangian (2.5) are*

$$\begin{aligned} \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} &= \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l \\ &\quad - \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + h_{lk} \right) \ddot{x}^l. \end{aligned}$$

Proof Straightforward:

$$\begin{aligned} \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} &= \frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + 2h_{ik} \dot{x}^i \right) \\ &= \frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j - \frac{1}{2} \left(\frac{d}{dt} \frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^j + 2 \frac{\partial h_{ij}}{\partial \dot{x}^k} \ddot{x}^i \dot{x}^j + 2 \frac{dh_{ik}}{dt} \dot{x}^i + 2h_{ik} \ddot{x}^i \right) \\ &= \frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j - \frac{1}{2} \left(\frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \dot{x}^i \dot{x}^j + \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \dot{x}^i \dot{x}^j \right. \\ &\quad \left. + 2 \frac{\partial h_{ij}}{\partial \dot{x}^k} \dot{x}^i \ddot{x}^j + 2 \frac{\partial h_{ik}}{\partial x^l} \dot{x}^l \dot{x}^i + 2 \frac{\partial h_{ik}}{\partial \dot{x}^l} \ddot{x}^l \dot{x}^i + 2h_{ik} \ddot{x}^i \right) \\ &= \frac{1}{2} \frac{\partial h_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j - \frac{1}{2} \left(\frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l + \frac{\partial h_{ik}}{\partial x^j} + \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j \\ &\quad - \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + h_{lk} \right) \ddot{x}^l. \quad \square \end{aligned}$$

Denote

$$C_{ijk} = \frac{1}{3} \left(\frac{\partial h_{ij}}{\partial \dot{x}^k} + \frac{\partial h_{jk}}{\partial \dot{x}^i} + \frac{\partial h_{ki}}{\partial \dot{x}^j} \right). \quad (2.6)$$

We call the system of functions $C = \{C_{ijk}\}$ the *Cartan tensor* associated with h . Note that we introduce C_{ijk} by the *Young decomposition*

$$\frac{\partial h_{ij}}{\partial \dot{x}^k} = C_{ijk} - \frac{1}{3} \left(\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) - \frac{1}{3} \left(\frac{\partial h_{ki}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right),$$

which defines the coefficient 1/3 in (2.6). Using this decomposition we can write the Euler–Lagrange expressions for the Lagrangian $\mathcal{L}_h$ in terms of the Cartan tensor and the complementary skew-symmetrized first derivatives $\partial h_{ij}/\partial \dot{x}^k$. Since

$$\left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l}\right)\dot{x}^i = 2C_{ilk}\dot{x}^i - \frac{1}{3}\left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{kl}}{\partial \dot{x}^i} - \frac{\partial h_{ik}}{\partial \dot{x}^l}\right)\dot{x}^i,$$

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial x^j} \frac{\partial h_{il}}{\partial \dot{x}^k} \dot{x}^l \dot{x}^j \dot{x}^i &= \frac{1}{2} \frac{\partial}{\partial x^j} \left(C_{ilk} - \frac{1}{3} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} \right) - \frac{1}{3} \left(\frac{\partial h_{ki}}{\partial \dot{x}^l} - \frac{\partial h_{il}}{\partial \dot{x}^k} \right) \right) \dot{x}^l \dot{x}^j \dot{x}^i \\ &= \frac{1}{2} \frac{\partial}{\partial x^j} \left(C_{ilk} - \frac{2}{3} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} \right) \right) \dot{x}^l \dot{x}^j \dot{x}^i \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j &= \frac{1}{4} \left(\frac{\partial}{\partial \dot{x}^l} \frac{\partial h_{ij}}{\partial \dot{x}^k} + \frac{\partial}{\partial \dot{x}^k} \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) \dot{x}^i \dot{x}^j \\ &= \frac{1}{4} \left(\frac{\partial}{\partial \dot{x}^l} \left(C_{ijk} - \frac{1}{3} \left(\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) - \frac{1}{3} \left(\frac{\partial h_{ki}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) \right) \right. \\ &\quad \left. + \frac{\partial}{\partial \dot{x}^k} \left(C_{ijl} - \frac{1}{3} \left(\frac{\partial h_{jl}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) - \frac{1}{3} \left(\frac{\partial h_{li}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) \right) \right) \dot{x}^i \dot{x}^j \\ &= \frac{1}{4} \left(\frac{\partial C_{ijk}}{\partial \dot{x}^l} + \frac{\partial C_{ijl}}{\partial \dot{x}^k} - \frac{2}{3} \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial h_{ik}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) \right. \\ &\quad \left. - \frac{2}{3} \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial h_{jl}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) \right) \dot{x}^i \dot{x}^j, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} &= \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j \\ &\quad - \frac{1}{2} \frac{\partial}{\partial x^j} \left(C_{ilk} - \frac{2}{3} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} \right) \right) \dot{x}^l \dot{x}^j \dot{x}^i - \frac{1}{4} \left(\frac{\partial C_{ijk}}{\partial \dot{x}^l} + \frac{\partial C_{ijl}}{\partial \dot{x}^k} \right) \dot{x}^i \dot{x}^j \dot{x}^l \\ &\quad + \frac{1}{6} \left(\frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial h_{ik}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) + \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial h_{jl}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) \right) \dot{x}^i \dot{x}^j \dot{x}^l \\ &\quad - \left(2C_{ilk} - \frac{1}{3} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{kl}}{\partial \dot{x}^i} - \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \right) \dot{x}^i \dot{x}^l - h_{lk} \ddot{x}^l. \end{aligned}$$

Remark 2.1 (The Cartan tensor) Note the identity

$$\frac{\partial \mathcal{L}_h}{\partial \dot{x}^i} \dot{x}^i = \frac{1}{2} \frac{\partial h_{rj}}{\partial \dot{x}^i} \dot{x}^r \dot{x}^j \dot{x}^i + h_{ij} \dot{x}^j \dot{x}^i = \frac{1}{2} C_{rij} \dot{x}^r \dot{x}^j \dot{x}^i + 2\mathcal{L}_h.$$

Thus, if the Cartan tensor satisfies

$$C_{rji}\dot{x}^r\dot{x}^j\dot{x}^i = 0, \quad (2.7)$$

then the energy Lagrangian $\mathcal{L}_h$ is positive homogeneous of degree 2,

$$\frac{\partial \mathcal{L}_h}{\partial \dot{x}^i} \dot{x}^i = 2\mathcal{L}_h.$$

The corresponding Euler–Lagrange expressions are

$$\begin{aligned} \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} &= \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j \\ &\quad + \frac{1}{3} \frac{\partial}{\partial x^j} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} \right) \dot{x}^l \dot{x}^j \dot{x}^i \\ &\quad + \frac{1}{6} \left(\frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial h_{ik}}{\partial \dot{x}^j} - \frac{\partial h_{ij}}{\partial \dot{x}^k} \right) + \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial h_{jl}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^l} \right) \right) \dot{x}^i \dot{x}^j \dot{x}^l \\ &\quad + \frac{1}{3} \left(\frac{\partial h_{lk}}{\partial \dot{x}^i} - \frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{kl}}{\partial \dot{x}^i} - \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i \dot{x}^l - h_{lk} \ddot{x}^l. \end{aligned}$$

Remark 2.2 If the functions h_{ij} satisfy

$$\frac{\partial h_{jk}}{\partial \dot{x}^i} - \frac{\partial h_{ij}}{\partial \dot{x}^k} = 0, \quad (2.8)$$

then

$$\begin{aligned} \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} &= \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j \\ &\quad - \frac{1}{2} \frac{\partial C_{ilk}}{\partial x^j} \dot{x}^l \dot{x}^j \dot{x}^i - \left(\frac{1}{2} \frac{\partial C_{ijk}}{\partial \dot{x}^l} \dot{x}^i \dot{x}^j + 2C_{ilk} \dot{x}^i + h_{lk} \right) \dot{x}^l. \end{aligned}$$

Remark 2.3 (Finsler geometry) If both conditions (2.7) and (2.8) are satisfied, then $\mathcal{L}_h$ is a *Finsler Lagrangian* and the Euler–Lagrange expressions

$$\frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_h}{\partial \dot{x}^k} = \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j - h_{lk} \ddot{x}^l$$

represent the geodesic equations in Finsler geometry.

Remark 2.4 By setting

$$g_{kl} = \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + h_{lk} \quad (2.9)$$

we can immediately check that this expression satisfies the integrability condition

$$\frac{\partial g_{kl}}{\partial \dot{x}^p} = \frac{\partial g_{kp}}{\partial \dot{x}^l}.$$

Indeed,

$$\begin{aligned} \frac{\partial g_{kl}}{\partial \dot{x}^p} &= \frac{\partial}{\partial \dot{x}^p} \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + h_{lk} \right) \\ &= \frac{1}{2} \frac{\partial^3 h_{ij}}{\partial \dot{x}^p \partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \frac{\partial^2 h_{ip}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i + \frac{\partial^2 h_{il}}{\partial \dot{x}^p \partial \dot{x}^k} \dot{x}^i \\ &\quad + \frac{\partial^2 h_{ik}}{\partial \dot{x}^p \partial \dot{x}^l} \dot{x}^i + \frac{\partial h_{pl}}{\partial \dot{x}^k} + \frac{\partial h_{pk}}{\partial \dot{x}^l} + \frac{\partial h_{lk}}{\partial \dot{x}^p} \\ &= \frac{\partial g_{kp}}{\partial \dot{x}^l}, \end{aligned}$$

so the derivative (2.9) is symmetric. The Cartan tensor is

$$\frac{1}{3} \left(\frac{\partial g_{kl}}{\partial \dot{x}^p} + \frac{\partial g_{pk}}{\partial \dot{x}^l} + \frac{\partial g_{lp}}{\partial \dot{x}^k} \right) = \frac{\partial g_{kl}}{\partial \dot{x}^p}.$$

If h_{kl} satisfies the homogeneity condition

$$\frac{\partial h_{kl}}{\partial \dot{x}^p} \dot{x}^p = 0,$$

then

$$\frac{\partial g_{kl}}{\partial \dot{x}^p} \dot{x}^p = 0.$$

Indeed,

$$\begin{aligned} \frac{\partial g_{kl}}{\partial \dot{x}^p} \dot{x}^p &= \frac{1}{2} \frac{\partial^3 h_{ij}}{\partial \dot{x}^p \partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^p + \frac{\partial^2 h_{ip}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^p + \frac{\partial^2 h_{il}}{\partial \dot{x}^p \partial \dot{x}^k} \dot{x}^i \dot{x}^p \\ &\quad + \frac{\partial^2 h_{ik}}{\partial \dot{x}^p \partial \dot{x}^l} \dot{x}^i \dot{x}^p + \left(\frac{\partial h_{pl}}{\partial \dot{x}^k} + \frac{\partial h_{pk}}{\partial \dot{x}^l} + \frac{\partial h_{lk}}{\partial \dot{x}^p} \right) \dot{x}^p, \end{aligned}$$

where

$$\frac{\partial^2 h_{ip}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^p = \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial h_{ip}}{\partial \dot{x}^k} \dot{x}^i \dot{x}^p \right) - \frac{\partial h_{ip}}{\partial \dot{x}^k} \frac{\partial}{\partial \dot{x}^l} (\dot{x}^i \dot{x}^p) = 0,$$

and similarly for the remaining summands.

2.3 Integrability Conditions

In this section we recall elementary theorems on integration of differential systems appearing in this chapter; essentially, we need only simple systems of Frobenius type in Euclidean spaces $\mathbb{R}^n$. All the functions that we consider here are defined on a star-shaped neighbourhood U of the origin $0 \in \mathbb{R}^n$.

Suppose we have a system of functions $A = \{A_k\}$, $1 \leq k \leq n$, defined on U , and consider the differential equations

$$A_k = \frac{\partial P}{\partial x^k} \quad (2.10)$$

for an unknown function P .

Lemma 2.2 (a) Equation (2.10) has a solution P if and only if the functions A_k satisfy

$$\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k} = 0. \quad (2.11)$$

(b) If condition (2.11) is satisfied, then a solution P is given by

$$P = x^k \int_0^1 A_k(\tau x^l) d\tau.$$

Proof Necessity of condition (2.11) is obvious. To prove the sufficiency, we differentiate P with respect to x^l . We have

$$\begin{aligned} \frac{\partial P}{\partial x^l} &= \int_0^1 A_p(\tau x^l) d\tau + x^k \int_0^1 \left(\frac{\partial A_k}{\partial x^p} \right)_{\tau x^l} \tau d\tau \\ &= \int_0^1 A_p(\tau x^l) d\tau + x^k \int_0^1 \left(\frac{\partial A_p}{\partial x^k} \right)_{\tau x^l} \tau d\tau \\ &= \int_0^1 \frac{d}{d\tau} \left(A_p(\tau x^l) \tau \right) d\tau \\ &= A_p(x^l). \end{aligned} \quad \square$$

Remark 2.5 If we have a system of the form

$$A_{(\alpha)k} = \frac{\partial P_{(\alpha)}}{\partial x^k},$$

the criterion (2.11) applies to each equation separately: We have

$$\frac{\partial A_{(\alpha)k}}{\partial x^l} - \frac{\partial A_{(\alpha)l}}{\partial x^k} = 0.$$

Now, suppose that we have a system of functions $A = \{A_{kl}\}$ defined on U , such that

$$A_{kl} = -A_{lk}.$$

Consider the differential equations

$$A_{kl} = \frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} \quad (2.12)$$

for an unknown system of functions $Q = Q_l$.

Lemma 2.3 (a) Equation (2.12) has a solution Q if and only if the functions A_{kl} satisfy

$$\frac{\partial A_{ks}}{\partial x^l} + \frac{\partial A_{sl}}{\partial x^k} + \frac{\partial A_{lk}}{\partial x^s} = 0. \quad (2.13)$$

(b) If condition (2.13) is satisfied, then a solution Q is given by

$$Q_l = x^p \int_0^1 A_{pl}(\tau x^i) \tau d\tau.$$

Proof Necessity of condition (2.13) is immediate. To prove the sufficiency, we differentiate Q_l with respect to x^k . We have

$$\frac{\partial Q_l}{\partial x^k} = \int_0^1 A_{kl}(\tau x^i) \tau d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} \right)_{\tau x^i} \tau^2 d\tau$$

and

$$\begin{aligned} \frac{\partial Q_l}{\partial x^k} - \frac{\partial Q_k}{\partial x^l} &= \int_0^1 A_{kl}(\tau x^i) \tau d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} \right)_{\tau x^i} \tau^2 d\tau \\ &\quad - \int_0^1 A_{lk}(\tau x^i) \tau d\tau - x^p \int_0^1 \left(\frac{\partial A_{pk}}{\partial x^l} \right)_{\tau x^i} \tau^2 d\tau \\ &= 2 \int_0^1 A_{kl}(\tau x^i) \tau d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} - \frac{\partial A_{pk}}{\partial x^l} \right)_{\tau x^i} \tau^2 d\tau \\ &= 2 \int_0^1 A_{kl}(\tau x^i) \tau d\tau + x^p \int_0^1 \left(\frac{\partial A_{pl}}{\partial x^k} + \frac{\partial A_{kp}}{\partial x^l} + \frac{\partial A_{lk}}{\partial x^p} \right)_{\tau x^i} \tau^2 d\tau \\ &\quad - x^p \int_0^1 \left(\frac{\partial A_{lk}}{\partial x^p} \right)_{\tau x^i} \tau^2 d\tau \\ &= \int_0^1 \left(\left(\frac{\partial A_{kl}}{\partial x^p} \right)_{\tau x^i} x^p \tau^2 + 2A_{kl}(\tau x^i) \tau \right) d\tau \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{d}{d\tau} \left(A_{kl}(\tau x^i) \tau^2 \right) d\tau \\
&= A_{kl}(x^i)
\end{aligned}$$

as required. $\square$

2.4 Variational Systems and the Helmholtz Conditions

We shall say that a system of functions $\varepsilon = \{\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)\}$ is *variational* if there exists a function $\mathcal{L} = \mathcal{L}(x^j, \dot{x}^j)$ such that

$$\varepsilon_i = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i}. \quad (2.14)$$

We give a straightforward proof of the well-known necessary and sufficient variationality conditions, based on the existence of a *second-order* Lagrangian for the system $\varepsilon = \{\varepsilon_i\}$ (the *Vainberg-Tonti Lagrangian*, cf. Krupka [8]).

Theorem 2.1 (Helmholtz conditions) *Let $\varepsilon = \{\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)\}$ be a system of functions. The following two conditions are equivalent:*

- (a) *The system $\varepsilon = \{\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)\}$ is variational.*
- (b) *The functions ε_i satisfy*

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} - \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} = 0, \quad (2.15)$$

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} - \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} \right) = 0, \quad (2.16)$$

$$\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) = 0. \quad (2.17)$$

Proof 1. We prove that (a) implies (b). Suppose that the functions ε_i are expressible in the form (2.14). Expressing ε_i as

$$\varepsilon_i = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k$$

and differentiating, we get the formulas

$$\begin{aligned}
\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} &= - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \ddot{x}^i}, \\
\frac{\partial \varepsilon_i}{\partial \dot{x}^l} &= \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k,
\end{aligned}$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^l} = \frac{\partial^2 \mathcal{L}}{\partial x^l \partial x^i} - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k.$$

From these expressions,

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = 0$$

and

$$\begin{aligned} & \frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} - \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) \\ &= \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial x^l} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l} \dot{x}^k \\ & \quad - \frac{\partial^2 \mathcal{L}}{\partial x^i \partial \dot{x}^l} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k \partial \dot{x}^l} \ddot{x}^k + 2 \left(\frac{\partial^3 \mathcal{L}}{\partial x^k \partial \dot{x}^l \partial \dot{x}^i} \dot{x}^k + \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^k \partial \dot{x}^l \partial \dot{x}^i} \ddot{x}^k \right) = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) \\ &= \frac{\partial^2 \mathcal{L}}{\partial x^l \partial x^i} - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k \\ & \quad - \frac{\partial^2 \mathcal{L}}{\partial x^i \partial x^l} + \frac{\partial^3 \mathcal{L}}{\partial x^i \partial x^k \partial \dot{x}^l} \dot{x}^k + \frac{\partial^3 \mathcal{L}}{\partial x^i \partial \dot{x}^k \partial \dot{x}^l} \ddot{x}^k \\ & \quad - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} - \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k \right. \\ & \quad \left. - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial x^l} + \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l} \dot{x}^k + \frac{\partial^2 \mathcal{L}}{\partial x^i \partial \dot{x}^l} + \frac{\partial^3 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^k \partial \dot{x}^l} \ddot{x}^k \right) \\ &= \frac{\partial^2 \mathcal{L}}{\partial x^l \partial x^i} - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^3 \mathcal{L}}{\partial x^l \partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k \\ & \quad - \frac{\partial^2 \mathcal{L}}{\partial x^i \partial x^l} + \frac{\partial^3 \mathcal{L}}{\partial x^i \partial x^k \partial \dot{x}^l} \dot{x}^k + \frac{\partial^3 \mathcal{L}}{\partial x^i \partial \dot{x}^k \partial \dot{x}^l} \ddot{x}^k \\ & \quad - \frac{\partial}{\partial t} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \right) - \frac{\partial}{\partial x^k} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \right) \dot{x}^k \\ & \quad - \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \right) \ddot{x}^k = 0, \end{aligned}$$

proving formulas (2.15)–(2.17).

2. We prove that (b) implies (a). Conditions (2.15)–(2.17) ensure the existence of a second-order Lagrangian $\mathcal{K} = \mathcal{K}(x^j, \dot{x}^j, \ddot{x}^j)$ such that

$$\varepsilon_i = \frac{\partial \mathcal{K}}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{x}^i} + \frac{d^2}{dt^2} \frac{\partial \mathcal{K}}{\partial \ddot{x}^i}.$$

The right-hand side is a polynomial in the variables $\ddot{x}^j$, $\ddot{\dot{x}}^j$, but ε_i depends only on x^j , $\dot{x}^j$, $\ddot{x}^j$. Thus, the 4th order term should vanish identically, so we have

$$\frac{\partial^2 \mathcal{K}}{\partial \ddot{x}^i \partial \ddot{x}^j} = 0,$$

hence $\mathcal{K} = \mathcal{K}_0 + A_l \ddot{x}^l$. Then,

$$\varepsilon_i = \frac{\partial \mathcal{K}_0}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{K}_0}{\partial \dot{x}^i} + \frac{\partial A^l}{\partial x^i} \ddot{x}^l - \frac{d}{dt} \frac{\partial A_l}{\partial \dot{x}^i} \ddot{x}^l + \frac{d^2 A_i}{dt^2}$$

and the coefficient of $\ddot{x}^l$ should vanish. We get the condition

$$\frac{\partial A_l}{\partial \dot{x}^i} - \frac{\partial A_i}{\partial \dot{x}^l} = 0,$$

ensuring that

$$A_l = \frac{\partial f}{\partial \dot{x}^l}$$

for some function $f = f(x^j, \dot{x}^j)$. Thus,

$$\mathcal{K} = \mathcal{K}_0 + \frac{\partial f}{\partial \dot{x}^l} \ddot{x}^l = \mathcal{K}_0 - \frac{\partial f}{\partial x^l} \dot{x}^k + \frac{df}{dt}$$

and by setting

$$\mathcal{L} = \mathcal{K}_0 - \frac{\partial f}{\partial x^l} \dot{x}^l,$$

we get a first order Lagrangian for $\varepsilon = \varepsilon_i$. □

Theorem 2.1 has the meaning of a *variationality criterion* for the given system of functions $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$; Eqs. (2.15)–(2.17) are the *integrability conditions* for the system (2.14). These equations are sometimes referred to as the *Helmholtz (variationality) conditions*.

Partial integration of the Helmholtz conditions shows that variational systems are always linear in the second derivatives $\ddot{x}^j$; the following theorem provides the integrability conditions for the coefficients.

Theorem 2.2 *The following conditions are equivalent:*

- (a) *The system $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$ is variational.*
- (b) *The functions ε_i are of the form*

$$\varepsilon_i = A_i - B_{ij} \ddot{x}^j,$$

where $A_i = A_i(x^k, \dot{x}^k)$ and $B_{ij} = B_{ij}(x^k, \dot{x}^k)$ are functions such that

$$B_{il} - B_{li} = 0, \quad (2.18)$$

$$\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} = 0, \quad (2.19)$$

$$\frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^j} \dot{x}^j = 0, \quad (2.20)$$

$$\frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j = 0. \quad (2.21)$$

Proof 1. Let ε_i be a solution of the Eqs. (2.15)–(2.17). Then, condition (2.16) together with (2.15) implies that each ε_i is of the form

$$\varepsilon_i = A_i - B_{ij} \ddot{x}^j, \quad (2.22)$$

where the coefficients are symmetric,

$$B_{il} = B_{li}.$$

Expression (2.22) already solves Eq. (2.15) and determines the unknown functions A_i and B_{ij} uniquely. The functions A_i and B_{ij} satisfy conditions (2.16) and (2.17) which reduce to

$$\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} - \left(\frac{\partial B_{ij}}{\partial \dot{x}^l} + \frac{\partial B_{lj}}{\partial \dot{x}^i} \right) \ddot{x}^j + 2 \frac{dB_{il}}{dt} = 0$$

and

$$\frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \left(\frac{\partial B_{ij}}{\partial x^l} - \frac{\partial B_{lj}}{\partial x^i} \right) \ddot{x}^j - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} - \left(\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} \right) \ddot{x}^j \right) = 0.$$

These conditions split into the following subsystems:

$$\frac{\partial B_{ij}}{\partial \dot{x}^l} + \frac{\partial B_{lj}}{\partial \dot{x}^i} - 2 \frac{\partial B_{il}}{\partial \dot{x}^j} = 0, \quad (2.23)$$

$$\frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^j} \dot{x}^j = 0, \quad (2.24)$$

$$\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} = 0, \quad (2.25)$$

$$-\frac{\partial B_{ij}}{\partial x^l} + \frac{\partial B_{lj}}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial \dot{x}^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) = 0, \quad (2.26)$$

$$\frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j = 0.$$

Condition (2.23) follows from (2.25). Condition (2.26) can also be omitted. Indeed, by differentiating (2.24), we obtain

$$\frac{1}{2} \frac{\partial}{\partial \dot{x}^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial^2 B_{il}}{\partial \dot{x}^j \partial x^s} \dot{x}^s + \frac{\partial B_{il}}{\partial x^j} = 0.$$

We write these equations as

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \dot{x}^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial^2 B_{il}}{\partial \dot{x}^j \partial x^s} \dot{x}^s + \frac{\partial B_{il}}{\partial x^j} &= 0, \\ \frac{1}{2} \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial A_j}{\partial \dot{x}^i} + \frac{\partial A_i}{\partial \dot{x}^j} \right) + \frac{\partial^2 B_{ji}}{\partial \dot{x}^l \partial x^s} \dot{x}^s + \frac{\partial B_{ji}}{\partial x^l} &= 0, \\ \frac{1}{2} \frac{\partial}{\partial \dot{x}^i} \left(\frac{\partial A_l}{\partial \dot{x}^j} + \frac{\partial A_j}{\partial \dot{x}^l} \right) + \frac{\partial^2 B_{lj}}{\partial \dot{x}^i \partial x^s} \dot{x}^s + \frac{\partial B_{lj}}{\partial x^i} &= 0. \end{aligned}$$

Combining these formulas gives

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \dot{x}^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial^2 B_{il}}{\partial \dot{x}^j \partial x^s} \dot{x}^s + \frac{\partial B_{il}}{\partial x^j} + \frac{1}{2} \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial A_j}{\partial \dot{x}^i} + \frac{\partial A_i}{\partial \dot{x}^j} \right) \\ + \frac{\partial^2 B_{ji}}{\partial \dot{x}^l \partial x^s} \dot{x}^s + \frac{\partial B_{ji}}{\partial x^l} - \frac{1}{2} \frac{\partial}{\partial \dot{x}^i} \left(\frac{\partial A_l}{\partial \dot{x}^j} + \frac{\partial A_j}{\partial \dot{x}^l} \right) - \frac{\partial^2 B_{lj}}{\partial \dot{x}^i \partial x^s} \dot{x}^s - \frac{\partial B_{lj}}{\partial x^i} = 0 \end{aligned}$$

and, after reindexing,

$$\begin{aligned} \frac{\partial^2 A_i}{\partial \dot{x}^j \partial \dot{x}^l} + \frac{\partial^2 B_{il}}{\partial \dot{x}^j \partial x^s} \dot{x}^s + \frac{\partial B_{il}}{\partial x^j} + \frac{\partial B_{ji}}{\partial x^l} - \frac{\partial B_{lj}}{\partial x^i} &= 0, \\ \frac{\partial^2 A_j}{\partial \dot{x}^i \partial \dot{x}^l} + \frac{\partial^2 B_{jl}}{\partial \dot{x}^i \partial x^s} \dot{x}^s + \frac{\partial B_{jl}}{\partial x^i} + \frac{\partial B_{ji}}{\partial x^l} - \frac{\partial B_{li}}{\partial x^j} &= 0. \end{aligned}$$

Subtracting the second equation from the first one, we get

$$\begin{aligned} \frac{\partial^2 A_i}{\partial \dot{x}^j \partial \dot{x}^l} + \frac{\partial^2 B_{il}}{\partial \dot{x}^j \partial x^s} \dot{x}^s + \frac{\partial B_{il}}{\partial x^j} + \frac{\partial B_{ji}}{\partial x^l} - \frac{\partial B_{lj}}{\partial x^i} - \frac{\partial^2 A_j}{\partial \dot{x}^i \partial \dot{x}^l} - \frac{\partial^2 B_{jl}}{\partial \dot{x}^i \partial x^s} \dot{x}^s \\ - \frac{\partial B_{jl}}{\partial x^i} - \frac{\partial B_{ji}}{\partial x^l} + \frac{\partial B_{li}}{\partial x^j} = \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial A_i}{\partial \dot{x}^j} - \frac{\partial A_j}{\partial \dot{x}^i} \right) + 2 \left(\frac{\partial B_{il}}{\partial x^j} - \frac{\partial B_{lj}}{\partial x^i} \right) = 0, \end{aligned}$$

which is exactly condition (2.26). This formula completes the proof of (b).

Conversely, if (b) is satisfied, then conditions (2.15)–(2.17) can be verified by direct substitutions. Indeed, (2.15) follows from (2.18); (2.16) follows from (2.18), (2.19) and (2.20),

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} \right) &= \frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial B_{ij}}{\partial \dot{x}^l} \ddot{x}^j + \frac{\partial A_l}{\partial \dot{x}^i} - \frac{\partial B_{lj}}{\partial \dot{x}^i} \ddot{x}^j + 2 \frac{dB_{il}}{dt} \\ &= \frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} + 2 \frac{\partial B_{il}}{\partial x^j} \dot{x}^j = 0, \end{aligned}$$

and (2.17) follows from (2.19) and (2.20). Indeed, by differentiating (2.20), we get

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial^2 B_{il}}{\partial \dot{x}^k \partial x^j} \dot{x}^j + \frac{\partial B_{il}}{\partial x^k} &= 0, \\ \frac{1}{2} \frac{\partial}{\partial \dot{x}^i} \left(\frac{\partial A_k}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^k} \right) + \frac{\partial^2 B_{kl}}{\partial \dot{x}^i \partial x^j} \dot{x}^j + \frac{\partial B_{kl}}{\partial x^i} &= 0. \end{aligned}$$

Thus, subtracting the second formula from the first one,

$$\frac{1}{2} \frac{\partial}{\partial \dot{x}^l} \left(\frac{\partial A_i}{\partial \dot{x}^k} + \frac{\partial A_k}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^k} - \frac{\partial B_{kl}}{\partial x^i} = 0.$$

Now,

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) &= \frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial B_{ij}}{\partial \dot{x}^l} \ddot{x}^j - \frac{\partial A_l}{\partial \dot{x}^i} + \frac{\partial B_{lj}}{\partial \dot{x}^i} \ddot{x}^j - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \\ &= \frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} - \frac{1}{2} \frac{\partial}{\partial x^p} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^p \\ &\quad + \left(-\frac{\partial B_{ij}}{\partial \dot{x}^l} + \frac{\partial B_{lj}}{\partial \dot{x}^i} - \frac{1}{2} \frac{\partial}{\partial \dot{x}^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \right) \ddot{x}^j = 0. \end{aligned}$$

where we also used (2.26). □

Example 2.1 We can apply the Helmholtz variability conditions (2.15)–(2.17) to the system $\varepsilon_i = -B_{ij}\ddot{x}^j$, where $B_{ij} = B_{ij}(x^k, \dot{x}^k)$. By a direct calculation,

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} - \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} = -B_{il} + B_{li}, \quad (2.27)$$

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} \right) &= -\frac{\partial B_{ij}}{\partial \dot{x}^l} \ddot{x}^j - \frac{\partial B_{lj}}{\partial \dot{x}^i} \ddot{x}^j + \frac{\partial (B_{il} + B_{li})}{\partial x^p} \dot{x}^p + \frac{\partial (B_{il} + B_{li})}{\partial \dot{x}^p} \ddot{x}^p, \quad (2.28) \end{aligned}$$

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i} \right) &= \left(-\frac{\partial B_{ij}}{\partial \dot{x}^l} + \frac{\partial B_{lj}}{\partial \dot{x}^i} \right) \ddot{x}^j + \frac{1}{2} \frac{d}{dt} \left(\frac{\partial B_{ij}}{\partial \dot{x}^l} \ddot{x}^j - \frac{\partial B_{lj}}{\partial \dot{x}^i} \ddot{x}^j \right) \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{\partial B_{ij}}{\partial x^l} + \frac{\partial B_{lj}}{\partial x^i} \right) \ddot{x}^j \\
&\quad + \frac{1}{2} \frac{d}{dt} \left(\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} \right) \ddot{x}^j + \frac{1}{2} \left(\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} \right) \ddot{x}^j, \tag{2.29}
\end{aligned}$$

so that the Helmholtz conditions yield

$$\begin{aligned}
B_{il} &= B_{li}, \\
2 \frac{\partial B_{il}}{\partial \dot{x}^p} - \frac{\partial B_{ip}}{\partial \dot{x}^l} - \frac{\partial B_{lp}}{\partial \dot{x}^i} &= 0, \quad \frac{\partial B_{il}}{\partial x^p} \dot{x}^p = 0, \\
\frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} &= 0, \quad -\frac{\partial B_{ij}}{\partial x^l} + \frac{\partial B_{lj}}{\partial x^i} = 0.
\end{aligned}$$

Differentiating with respect to $\dot{x}^j$, we obtain the Helmholtz conditions in the form of *integrability conditions*

$$B_{il} = B_{li}, \quad \frac{\partial B_{il}}{\partial x^j} - \frac{\partial B_{ij}}{\partial x^l} = 0, \quad \frac{\partial B_{il}}{\partial x^p} \dot{x}^p = 0.$$

2.5 The Structure of Variational Systems

Our objective now will be to investigate the *structure* of variational systems. To this purpose, we first prove some lemmas.

Lemma 2.4 *Let $B = B_{ij}(x^k, \dot{x}^k)$ be a system of functions. The following three conditions are equivalent:*

(a) *There exists an energy Lagrangian $\mathcal{L} = \mathcal{L}(x^k, \dot{x}^k)$ such that*

$$B_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}. \tag{2.30}$$

(b) *The functions B_{ij} satisfy*

$$B_{ij} = B_{ji}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k} = \frac{\partial B_{ik}}{\partial \dot{x}^j}. \tag{2.31}$$

Proof Since condition (b) obviously follows from (a), only the converse needs proof. The second condition in (2.31) implies that

$$B_{ij} = \frac{\partial h_i}{\partial \dot{x}^k} \tag{2.32}$$

for some functions h_i . These functions can be taken as

$$h_i = \dot{x}^r \int_0^1 B_{ir}(x^p, \lambda \dot{x}^p) d\lambda. \quad (2.33)$$

Indeed, h_i obviously satisfies condition (2.32):

$$\begin{aligned} \left(\frac{\partial h_i}{\partial \dot{x}^j} \right)_{(x^p, \dot{x}^p)} &= \int_0^1 B_{ij}(x^p, \lambda \dot{x}^p) d\lambda + \dot{x}^r \int_0^1 \left(\frac{\partial B_{ir}}{\partial \dot{x}^j} \right)_{(x^p, \kappa \dot{x}^p)} \lambda d\lambda \\ &= \int_0^1 \left(B_{ij}(x^p, \lambda \dot{x}^p) + \left(\frac{\partial B_{ir}}{\partial \dot{x}^j} \right)_{(x^p, \kappa \dot{x}^p)} \lambda \dot{x}^r \right) d\lambda \\ &= \int_0^1 \left(B_{ij}(x^p, \lambda \dot{x}^p) + \left(\frac{\partial B_{ij}}{\partial \dot{x}^r} \right)_{(x^p, \kappa \dot{x}^p)} \lambda \dot{x}^r \right) d\kappa \\ &= \int_0^1 \frac{d}{d\lambda} (B_{ij}(x^p, \lambda \dot{x}^p) \lambda) d\lambda = B_{ij}(x^p, \dot{x}^p). \end{aligned}$$

We now apply the condition $B_{ij} = B_{ji}$ (see (2.31)). We get the integrability condition

$$\frac{\partial h_i}{\partial \dot{x}^j} = \frac{\partial h_j}{\partial \dot{x}^i},$$

ensuring the existence of a function f such that

$$h_i = \frac{\partial f}{\partial \dot{x}^i}.$$

A solution of this equation can be taken as

$$f = \dot{x}^i \int_0^1 h_i(x^p, \tau \dot{x}^p) d\tau.$$

Substituting from formula (2.33), now expressed as

$$h_i(x^p, \tau \dot{x}^p) = \tau \dot{x}^r \int_0^1 B_{ir}(x^p, \lambda \tau \dot{x}^p) d\lambda,$$

we get

$$\begin{aligned} \mathcal{L} &= \dot{x}^i \int_0^1 \left(\tau \dot{x}^r \int_0^1 B_{ir}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) d\tau \\ &= \dot{x}^i \dot{x}^r \int_0^1 \left(\int_0^1 B_{ir}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau \\ &= \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j, \end{aligned}$$

where

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 B_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau. \quad (2.34)$$

The function $\mathcal{L}$ obviously satisfies Eq. (2.30). $\square$

Lemma 2.5 *Every solution of the system (2.30) is of the form*

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_0, \quad (2.35)$$

where $\mathcal{L}_h$ is the energy Lagrangian of the system $h = \{h_{ij}\}$ defined by (2.34),

$$\mathcal{L}_h = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j, \quad (2.36)$$

and

$$\mathcal{L}_0 = P + Q_i \dot{x}^i, \quad (2.37)$$

where $P = P(x^k)$, $Q_i = Q_i(x^k)$. The functions h_{ij} satisfy

$$h_{ij} = h_{ji}, \quad \frac{\partial h_{ij}}{\partial \dot{x}^k} = \frac{\partial h_{ik}}{\partial \dot{x}^j}. \quad (2.38)$$

Proof Formulas (2.35), (2.36) and (2.37) are immediate consequences of Lemma 2.4. Formula (2.38) follows from (2.36). $\square$

We find the Euler–Lagrange expressions for the Lagrangian $\mathcal{L}$ by a direct computation.

Lemma 2.6 *If B_{ij} satisfy condition (2.31), then*

$$\begin{aligned} & \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^k} \dot{x}^l - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \\ &= \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} - \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i} - \frac{\partial^2 B_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \tau d\tau - B_{lk} \ddot{x}^l. \end{aligned} \quad (2.39)$$

Proof Using Lemma 2.1, we have

$$\begin{aligned} & \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^k} \dot{x}^l - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \\ &= \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l \\ &\quad - \left(\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + h_{lk} \right) \ddot{x}^l, \end{aligned}$$

where

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 B_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau.$$

The derivatives of h_{ij} can be expressed as

$$\begin{aligned} \frac{\partial h_{ij}}{\partial x^k} &= 2 \int_0^1 \left(\int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \right) \tau d\tau, \\ \frac{\partial h_{ij}}{\partial \dot{x}^k} &= 2 \int_0^1 \left(\int_0^1 \left(\frac{\partial B_{ij}}{\partial \dot{x}^k} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda d\lambda \right) \tau^2 d\tau, \\ \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} &= 2 \int_0^1 \left(\int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial x^l \partial \dot{x}^k} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda d\lambda \right) \tau^2 d\tau, \\ \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} &= 2 \int_0^1 \left(\int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda^2 d\lambda \right) \tau^3 d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j \\ = \dot{x}^i \dot{x}^j \int_0^1 \left(\int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} - \frac{\partial B_{ik}}{\partial x_j} - \frac{\partial B_{jk}}{\partial x^i} \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \right) \tau d\tau, \end{aligned} \quad (2.40)$$

$$\frac{1}{2} \frac{\partial^2 h_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \dot{x}^l = \dot{x}^i \dot{x}^j \int_0^1 \left(\int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \right) \tau d\tau, \quad (2.41)$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial h_{il}}{\partial \dot{x}^k} + \frac{\partial h_{ik}}{\partial \dot{x}^l} \right) \dot{x}^l + h_{lk} \\ = \dot{x}^i \dot{x}^j \int_0^1 \left(\int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda^2 d\lambda \right) \tau^3 d\tau \\ + 2 \dot{x}^i \int_0^1 \left(\int_0^1 \left(\frac{\partial B_{il}}{\partial \dot{x}^k} + \frac{\partial B_{ik}}{\partial \dot{x}^l} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda d\lambda \right) \tau^2 d\tau \\ + 2 \int_0^1 \left(\int_0^1 B_{lk}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau \\ = \int_0^1 \left(\int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \right) \tau d\tau \end{aligned}$$

$$\begin{aligned}
& + 2 \int_0^1 \left(\int_0^1 \left(\left(\frac{\partial B_{il}}{\partial \dot{x}^k} + \frac{\partial B_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \right) \tau d\tau \\
& + 2 \int_0^1 \left(\int_0^1 B_{lk}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau \\
& = \int_0^1 \int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + 2 \left(\frac{\partial B_{il}}{\partial \dot{x}^k} + \frac{\partial B_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + 2 B_{lk} \right)_{(x^p, \lambda \tau \dot{x}^p)} \tau d\lambda d\tau.
\end{aligned}$$

Summarizing,

$$\begin{aligned}
& \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^k} \dot{x}^l - \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \\
& = \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} - \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i} \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \tau d\tau \\
& \quad - \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \tau d\tau \\
& \quad - \ddot{x}^l \int_0^1 \int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial \dot{x}^l \partial \dot{x}^k} \dot{x}^i \dot{x}^j + 2 \left(\frac{\partial B_{il}}{\partial \dot{x}^k} + \frac{\partial B_{ik}}{\partial \dot{x}^l} \right) \dot{x}^i + 2 B_{lk} \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \tau d\tau.
\end{aligned}$$

On the other hand, we suppose that

$$B_{ij} = B_{ji}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^k} = \frac{\partial B_{ik}}{\partial \dot{x}^j}.$$

Then

$$\frac{d}{d\tau} (B_{lk}(x^p, \lambda \tau \dot{x}^p) \tau^2 \lambda) = \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda \dot{x}^m \tau^2 \lambda + 2 B_{lk}(x^p, \lambda \tau \dot{x}^p) \tau \lambda$$

and

$$\begin{aligned}
& \frac{d}{d\lambda} \frac{d}{d\tau} (B_{lk}(x^p, \lambda \tau \dot{x}^p) \tau^2 \lambda) = \left(\frac{\partial^2 B_{lk}}{\partial \dot{x}^q \partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \tau \dot{x}^q \lambda \dot{x}^m \tau^2 \lambda \\
& \quad + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda \dot{x}^m \tau^2 + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \tau \dot{x}^m \tau \lambda + 2 B_{lk}(x^p, \lambda \tau \dot{x}^p) \tau \\
& = \left(\frac{\partial^2 B_{lk}}{\partial \dot{x}^q \partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda \tau \dot{x}^q \lambda \tau \dot{x}^m \tau + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda \tau \dot{x}^m \tau \\
& \quad + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \right)_{(x^p, \lambda \tau \dot{x}^p)} \lambda \tau \dot{x}^m \tau + 2 B_{lk}(x^p, \lambda \tau \dot{x}^p) \tau \\
& = \left(\left(\frac{\partial^2 B_{lk}}{\partial \dot{x}^q \partial \dot{x}^m} \dot{x}^q \dot{x}^m \right)_{(x^p, \lambda \tau \dot{x}^p)} + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \dot{x}^m \right)_{(x^p, \lambda \tau \dot{x}^p)} \right. \\
& \quad \left. + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} \dot{x}^m \right)_{(x^p, \lambda \tau \dot{x}^p)} + 2 B_{lk}(x^p, \lambda \tau \dot{x}^p) \right) \tau
\end{aligned}$$

$$= \left(\frac{\partial^2 B_{lk}}{\partial \dot{x}^q \partial \dot{x}^m} \dot{x}^q \dot{x}^m + 2 \left(\frac{\partial B_{lk}}{\partial \dot{x}^m} + \frac{\partial B_{lk}}{\partial \dot{x}^m} \right) \dot{x}^m + 2 B_{lk} \right)_{(x^p, \lambda, \tau, \dot{x}^p)} \tau.$$

Using this expression, we get formula (2.39) in the form

$$\begin{aligned} & \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^k} \dot{x}^l - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \\ &= \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial B_{ij}}{\partial x^k} - \frac{\partial B_{ik}}{\partial x^j} - \frac{\partial B_{jk}}{\partial x^i} \right)_{(x^p, \lambda, \tau, \dot{x}^p)} d\lambda d\tau \\ & \quad - \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial^2 B_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \right)_{(x^p, \lambda, \tau, \dot{x}^p)} d\lambda \cdot \tau d\tau - B_{lk} \ddot{x}^l, \end{aligned}$$

as desired. $\square$

Let us now consider a *first order* system $\varepsilon^0 = \{\varepsilon_i^0\}$.

Lemma 2.7 *Let $\varepsilon_i^0 = \varepsilon_i^0(x^k, \dot{x}^k)$ be a system of functions. The following three conditions are equivalent:*

- (a) *The system ε_i^0 is variational.*
- (b) *The components ε_i^0 satisfy*

$$\frac{\partial \varepsilon_i^0}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l^0}{\partial \dot{x}^i} = 0, \quad (2.42)$$

$$\frac{\partial \varepsilon_i^0}{\partial x^l} - \frac{\partial \varepsilon_l^0}{\partial x^i} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial \varepsilon_i^0}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l^0}{\partial \dot{x}^i} \right) = 0. \quad (2.43)$$

- (c) *There exist some functions $P = P(x^k)$ and $Q_i = Q_i(x^k)$ such that*

$$\varepsilon_l^0 = \frac{\partial P}{\partial x^i} + \left(\frac{\partial Q_i}{\partial x^j} - \frac{\partial Q_j}{\partial x^i} \right) \dot{x}^j. \quad (2.44)$$

Proof 1. (a) implies (b) by Theorem 2.1.

2. Let the system ε_i^0 satisfy Eqs. (2.42) and (2.43). Then ε_i^0 must be linear in $\dot{x}^i$, with skew-symmetric coefficients, that is, $\varepsilon_i^0 = R_i + S_{ij} \dot{x}^j$, where $S_{ij} = -S_{ji}$. The coefficients then satisfy

$$\frac{\partial R_i}{\partial x^l} - \frac{\partial R_l}{\partial x^i} + \left(\frac{\partial S_{ij}}{\partial x^l} - \frac{\partial S_{lj}}{\partial x^i} - \frac{\partial S_{il}}{\partial x^j} \right) \dot{x}^j = 0.$$

Hence,

$$\frac{\partial R_i}{\partial x^l} - \frac{\partial R_l}{\partial x^i} = 0, \quad \frac{\partial S_{ij}}{\partial x^l} + \frac{\partial S_{li}}{\partial x^j} + \frac{\partial S_{jl}}{\partial x^i} = 0.$$

These equations are the integrability conditions ensuring that

$$R_i = \frac{\partial P}{\partial x^i}, \quad S_{ij} = \frac{\partial Q_i}{\partial x^j} - \frac{\partial Q_j}{\partial x^i}$$

for some functions $P = P(x^k)$ and $Q = Q(x^k)$ (see Lemmas 2.2 and 2.3). This gives formula (2.31).

3. Suppose that ε_i^0 is of the form (2.44) and set

$$\mathcal{L} = P - Q_i \dot{x}^i.$$

Then

$$\frac{\partial \mathcal{L}}{\partial x^l} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^l} = \frac{\partial P}{\partial x^l} - \frac{\partial Q_i}{\partial x^l} \dot{x}^i + \frac{\partial Q_l}{\partial x^i} \dot{x}^i = \varepsilon_l.$$

Therefore ε_i^0 is variational. □

The structure of variational systems and the first order Lagrangians for them can now be completely characterized by means of the Helmholtz conditions (2.15)–(2.17) of Theorem 2.1 from Sect. 2.4, or, which is the same, by integrating system (2.18)–(2.21) of Theorem 2.2 from Sect. 2.4.

Recall that the Lagrangian $\mathcal{L}_h$, as used in the following theorem, is completely determined by the coefficients B_{ij} .

Theorem 2.3 *The following two conditions are equivalent:*

- (a) *The system $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$ is variational.*
- (b) *The functions ε_i are of the form*

$$\varepsilon_i = A_i - B_{ij} \ddot{x}^j, \tag{2.45}$$

where $\mathcal{L}_h$ is the energy Lagrangian (2.36) of the system (2.34), $B_{ij} = B_{ij}(x^k, \dot{x}^k)$ are functions such that

$$B_{il} = B_{li}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} = 0 \tag{2.46}$$

and the functions $A_i = A_i(x^k, \dot{x}^k)$ belong to the family

$$A_i = \Phi_i + \Psi_{ij} \dot{x}^j + \frac{\partial \mathcal{L}_h}{\partial x^i} - \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j,$$

where $\Phi_i = \Phi_i(x^k)$ and $\Psi_{ij} = \Psi_{ij}(x^k)$ are arbitrary functions such that

$$\frac{\partial \Phi_i}{\partial x^l} - \frac{\partial \Phi_l}{\partial x^i} = 0$$

and

$$\Psi_{ij} + \Psi_{ji} = 0, \quad \frac{\partial \Psi_{ij}}{\partial x^l} + \frac{\partial \Psi_{jl}}{\partial x^i} + \frac{\partial \Psi_{li}}{\partial x^j} = 0.$$

Proof We prove that (a) implies (b). Suppose that we have a variational system ε_i . Then ε_i must be of the form (2.45) and the coefficients B_{ij} satisfy conditions (2.46) (Theorem 2.2). Recall that these equations can be solved separately. The solutions are

$$B_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}, \quad (2.47)$$

where

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_0,$$

where $\mathcal{L}_h$ is the energy Lagrangian of the system

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 B_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau,$$

and where

$$\mathcal{L}_0 = P + Q_i \dot{x}^i$$

for some functions $P = P(x^k)$, $Q_I = Q_i(x^k)$ (Lemmas 2.4 and 2.5).

The coefficients A_i and B_{ij} also satisfy the system (2.20)

$$\frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) = - \frac{\partial B_{il}}{\partial x^j} \dot{x}^j.$$

Using this formula, as well as (2.47), we get

$$\begin{aligned} & \frac{1}{2} \left(\frac{\partial}{\partial \dot{x}^l} \left(A_i - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j \right) + \frac{\partial}{\partial \dot{x}^i} \left(A_l - \frac{\partial \mathcal{L}_h}{\partial x^l} + \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^l} \dot{x}^j \right) \right) \\ &= \frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) - \frac{1}{2} \left(\frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^l \partial x^i} - \frac{\partial^3 \mathcal{L}_h}{\partial \dot{x}^l \partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^i} \right. \\ & \quad \left. + \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^i \partial x^l} - \frac{\partial^3 \mathcal{L}_h}{\partial \dot{x}^i \partial x^j \partial \dot{x}^l} \dot{x}^j - \frac{\partial^2 \mathcal{L}_h}{\partial x^i \partial \dot{x}^l} \right) \\ &= \frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^l \partial \dot{x}^i} + \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^i \partial \dot{x}^l} \right) \dot{x}^j \\ &= \frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^j} \dot{x}^j = 0, \end{aligned}$$

from which we easily derive that

$$\frac{\partial^2}{\partial \dot{x}^k \partial \dot{x}^l} \left(A_i - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j \right) = 0.$$

By integration,

$$A_i - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j = \Phi_i + \Psi_{ij} \dot{x}^j,$$

where $\Phi_i = \Phi_i(x^k)$ and $\Psi_{ij} = \Psi_{ij}(x^k)$ are arbitrary functions such that

$$\Psi_{ij} + \Psi_{ji} = 0.$$

Consequently,

$$A_i = \frac{\partial \mathcal{L}_h}{\partial x^i} - \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j + \Phi_i + \Psi_{ij} \dot{x}^j. \quad (2.48)$$

Finally, the functions A_i satisfy condition (2.21). Substituting from (2.48),

$$\begin{aligned} & \frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j \\ &= \frac{\partial}{\partial x^l} \left(\frac{\partial \mathcal{L}_h}{\partial x^i} - \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j + \Phi_i + \Psi_{ij} \dot{x}^j \right) \\ & \quad - \frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}_h}{\partial x^l} - \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^l} \dot{x}^j + \Phi_l + \Psi_{lj} \dot{x}^j \right) \\ & \quad - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^l \partial x^i} - \frac{\partial^3 \mathcal{L}_h}{\partial \dot{x}^l \partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^i} + \Psi_{il} \right. \\ & \quad \left. - \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^i \partial x^l} + \frac{\partial^3 \mathcal{L}_h}{\partial \dot{x}^i \partial x^k \partial \dot{x}^l} \dot{x}^k + \frac{\partial^2 \mathcal{L}_h}{\partial x^i \partial \dot{x}^l} - \Psi_{li} \right) \dot{x}^j \\ &= \frac{\partial \Phi_i}{\partial x^l} - \frac{\partial \Phi_l}{\partial x^i} + \left(\frac{\partial \Psi_{ij}}{\partial x^l} + \frac{\partial \Psi_{jl}}{\partial x^i} + \frac{\partial \Psi_{li}}{\partial x^j} \right) \dot{x}^j \\ &= 0. \end{aligned}$$

Consequently, (2.41) implies

$$\frac{\partial \Phi_i}{\partial x^l} - \frac{\partial \Phi_l}{\partial x^i} = 0, \quad \frac{\partial \Psi_{ij}}{\partial x^l} + \frac{\partial \Psi_{jl}}{\partial x^i} + \frac{\partial \Psi_{li}}{\partial x^j} = 0. \quad (2.49)$$

2. To prove that (b) implies (a), we consider formulas (2.49) as the integrability condition; the corresponding integrals can be added to the Lagrangian $\mathcal{L}_h$ without changing the metric B_{ij} . Indeed, according to Lemmas 2.2 and 2.3, there exist functions $P = P(x^k)$ and $Q = Q(x^k)$ such that

$$\Phi_i = \frac{\partial P}{\partial \dot{x}^i}, \quad \Psi_{ij} = \frac{\partial Q_j}{\partial \dot{x}^i} - \frac{\partial Q_i}{\partial \dot{x}^j}.$$

Then

$$A_i = \frac{\partial \mathcal{L}_h}{\partial \dot{x}^i} - \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j + \frac{\partial P}{\partial \dot{x}^i} + \left(\frac{\partial Q_j}{\partial \dot{x}^i} - \frac{\partial Q_i}{\partial \dot{x}^j} \right) \dot{x}^j = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j,$$

where $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_0$ and $\mathcal{L}_0 = P + Q_j \dot{x}^j$. Then $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_0$ is a Lagrangian for ε_i (Lemma 2.7). $\square$

Theorem 2.4 *Let $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$ be a variational system. Then there exists a system $h = h(x^k, \dot{x}^k)$ and some systems of functions $S_j = S_j(x^k)$ and $R = R(x^k)$ such that the function*

$$\mathcal{L} = \mathcal{L}_h + S_p \dot{x}^p - R$$

is a Lagrangian for the system ε_i .

Proof Let ε_i be a solution of the Eqs. (2.15)–(2.17), Sect. 2.4. Then, according to Theorem 2.2,

$$\varepsilon_i = A_i - B_{ij} \ddot{x}^j, \quad (2.50)$$

where $A_i = A_i(x^k, \dot{x}^k)$ and $B_{ij} = B_{ij}(x^k, \dot{x}^k)$ are functions satisfying

$$B_{il} = B_{li}, \quad \frac{\partial B_{ij}}{\partial \dot{x}^l} - \frac{\partial B_{lj}}{\partial \dot{x}^i} = 0, \quad (2.51)$$

$$\begin{aligned} & \frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^j} \dot{x}^j = 0, \\ & \frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j = 0. \end{aligned} \quad (2.52)$$

The system (2.51) has already been solved (Lemmas 2.4 and 2.5). Given a solution B_{il} , set

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 B_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau \quad (2.53)$$

and denote by $\mathcal{L}_h$ the energy Lagrangian of the system $h = h_{ij}$,

$$\mathcal{L}_h = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j.$$

Then B_{il} can be expressed as

$$B_{il} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^l},$$

where $\mathcal{L} = \mathcal{L}(x^i, \dot{x}^i)$ is any element of the family $\mathcal{L} = \mathcal{L}_h + P + Q_p \dot{x}^p$, where $P = P(x^k)$ and $Q = Q(x^k)$ are arbitrary functions. We wish to show that one can always choose these functions in such a way that $\mathcal{L}$ is a Lagrangian for the system of functions ε_i (2.50). Using the Lagrangian $\mathcal{L}$, the functions ε_i can be expressed as

$$\begin{aligned} \varepsilon_i &= A_i - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \dot{x}^j \\ &= A_i - \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j + \frac{\partial \mathcal{L}}{\partial x_i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \dot{x}^j = \varepsilon_i^0 + E_i(\mathcal{L}), \end{aligned} \quad (2.54)$$

where $E_i(\mathcal{L})$ are the Euler–Lagrange expressions of $\mathcal{L}$ and $\varepsilon_i^0 = \varepsilon_i^0(x^k, \dot{x}^k)$ are the functions defined by

$$\begin{aligned} \varepsilon_i^0 &= A_i - \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j \\ &= A_i - \frac{\partial P}{\partial x^i} + \left(\frac{\partial Q_i}{\partial x^p} - \frac{\partial Q_p}{\partial x^i} \right) \dot{x}^p - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j. \end{aligned} \quad (2.55)$$

But by hypothesis, the functions ε_i (2.54) satisfy Eqs. (2.15)–(2.17). On the other hand, by Theorem 2.1 Eqs. (2.15)–(2.17), Sect. 2.4, are identically satisfied by the Euler–Lagrange expressions $E_i(\mathcal{L})$; thus, the functions ε_i^0 solve the system (2.52), (2.53), which reduces to

$$\frac{\partial \varepsilon_i^0}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l^0}{\partial \dot{x}^i} = 0 \quad (2.56)$$

and

$$\frac{\partial \varepsilon_i^0}{\partial x^l} - \frac{\partial \varepsilon_l^0}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial \varepsilon_i^0}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l^0}{\partial \dot{x}^i} \right) \dot{x}^j = 0. \quad (2.57)$$

But the summand

$$-\frac{\partial P}{\partial x^i} + \left(\frac{\partial Q_i}{\partial x^p} - \frac{\partial Q_p}{\partial x^i} \right) \dot{x}^p$$

in expression (2.55) satisfies the system (2.56), (2.57) identically (Lemma 2.7). Thus, Eqs. (2.56), (2.57) yield

$$A_i - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^j \partial \dot{x}^i} \dot{x}^j = \frac{\partial R}{\partial x^l} + \left(\frac{\partial S_i}{\partial x^j} - \frac{\partial S_j}{\partial x^i} \right) \dot{x}^j$$

for some functions $R = R(x^k)$ and $S_i = S_i(x^k)$. Returning to (2.55),

$$\varepsilon_i^0 = \frac{\partial R}{\partial x^l} + \left(\frac{\partial S_i}{\partial x^j} - \frac{\partial S_j}{\partial x^i} \right) \dot{x}^j - \frac{\partial P}{\partial x^i} + \left(\frac{\partial Q_i}{\partial x^p} - \frac{\partial Q_p}{\partial x^i} \right) \dot{x}^p$$

$$= \frac{\partial(R - P)}{\partial x^l} + \left(\frac{\partial(S_i + Q_i)}{\partial x^p} - \frac{\partial(S_p + Q_p)}{\partial x^i} \right) \dot{x}^p.$$

In this expression, P and Q_p are arbitrary. Specifying these functions as

$$P = -R, \quad Q_p = S_p,$$

we have $\varepsilon_I^0 = 0$, therefore, from (2.54),

$$\varepsilon_i = E_i(\mathcal{L}),$$

with

$$\mathcal{L} = \mathcal{L}_h - R + S_p \dot{x}^p.$$

□

Rephrasing Theorem 2.4, we get

Theorem 2.5 *Let $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$ be a variational system expressed as*

$$\varepsilon_i = A_i - B_{ij} \ddot{x}^j$$

and set

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 B_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau.$$

Then there exist some functions $R = R(x^k)$ and $S_i = S_i(x^k)$ such that

$$A_i = \frac{1}{2} \frac{\partial^2 h_{rs}}{\partial x^j \partial \dot{x}^i} \dot{x}^r \dot{x}^s \dot{x}^j + \frac{1}{2} \left(\frac{\partial h_{ri}}{\partial x^s} + \frac{\partial h_{si}}{\partial x^r} - \frac{\partial h_{rs}}{\partial x^i} \right) \dot{x}^r \dot{x}^s + \frac{\partial R}{\partial x^i} - \left(\frac{\partial S_p}{\partial x^i} - \frac{\partial S_i}{\partial x^p} \right) \dot{x}^p.$$

Remark 2.6 Any function $f = f(x^j, \dot{x}^j)$ can be expressed as

$$f = \mathcal{L}_h + P + Q_p \dot{x}^p + \frac{dg}{dt}$$

for some functions $h_{ij} = h_{ij}(x^k, \dot{x}^k)$, $P = P(x^k)$ and $g = g(x^k)$.

2.6 The Sonin–Douglas Problem

In this section we start by recalling Sonin's original ideas on the inverse problem for one second-order differential equation (see [5]). Then we present the general theory for systems of second-order equations, based on the Helmholtz variability conditions.

Given a function $F = F(t, x, \dot{x})$, the *Sonin's problem* consists in finding a function $g = g(t, x, \dot{x}) \neq 0$ for which there exists a solution $\mathcal{L} = \mathcal{L}(t, x, \dot{x})$ of the equation

$$g(F - \ddot{x}) = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}. \quad (2.58)$$

If $\mathcal{L}$ exists, then the function F , and also the function $g(F - \ddot{x})$, are called *variational*. Every solution $\mathcal{L}$ can be considered as the *Lagrange function* of a variational functional $\int \mathcal{L}(t, x(t), dx/dt)dt$, depending on real functions $t \rightarrow x(t)$, and condition (2.58) states that the differential equation

$$F - \ddot{x} = 0 \quad (2.59)$$

is equivalent with the corresponding *Euler–Lagrange equation*

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0.$$

In particular, solutions of equation (2.59) coincide with the *extremals* of the variational functional $\int \mathcal{L}(t, x(t), dx/dt)dt$.

Considering (2.58) as an equation for the pair $(g, \mathcal{L})$, we have the following result.

Theorem 2.6 (Sonin) *For every function F there exists a solution $(g, \mathcal{L})$ of Eq. (2.58).*

Proof Since g is supposed to be nonvanishing on its domain, Eq. (2.58) is equivalent to the system

$$g = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2}, \quad gF = \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{x}} - \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} \dot{x}. \quad (2.60)$$

The first equation can be solved immediately on star-shaped domains with center 0 in the variable x . We first solve the equation

$$g = \frac{\partial h}{\partial \dot{x}}.$$

Since

$$\begin{aligned} \left(\frac{\partial h}{\partial \dot{x}} \right)_{(x^p, \dot{x}^p)} &= \int_0^1 g(x, \kappa \dot{x}) d\kappa + \dot{x} \int_0^1 \left(\frac{\partial g}{\partial \dot{x}} \right)_{(x, \kappa \dot{x})} \kappa d\kappa \\ &= \int_0^1 \left(g(x, \kappa \dot{x}) + \left(\frac{\partial g}{\partial \dot{x}} \right)_{(x, \kappa \dot{x})} \kappa \dot{x} \right) d\kappa \\ &= \int_0^1 \frac{d}{d\kappa} (g(x, \kappa \dot{x}) \kappa) d\kappa \\ &= g(x, \dot{x}), \end{aligned}$$

the solution is

$$h = \dot{x} \int_0^1 g(x, \kappa \dot{x}) d\kappa.$$

Then we solve the equation

$$h = \frac{\partial \mathcal{L}}{\partial \dot{x}}.$$

A solution is

$$\mathcal{L} = \dot{x} \int_0^1 h(x, \tau \dot{x}) d\tau.$$

Substituting

$$h(x, \tau \dot{x}) = \tau \dot{x} \int_0^1 g(x, \kappa \tau \dot{x}) d\kappa,$$

we get

$$\mathcal{L} = \dot{x} \int_0^1 \left(\tau \dot{x} \int_0^1 g(x, \kappa \tau \dot{x}) d\kappa \right) d\tau = \dot{x}^2 \int_0^1 \left(\int_0^1 g(x, \kappa \tau \dot{x}) d\kappa \right) \tau d\tau.$$

The general solution to the first equation (2.60) is

$$\mathcal{L} = \frac{1}{2} \dot{x}^2 \int_0^1 \left(\int_0^1 g(x, \kappa \tau \dot{x}) d\kappa \right) \tau d\tau + A\dot{x} + B,$$

where the functions A and B do not depend on $\dot{x}$.

It is now sufficient to prove that the second equation (2.60) has a solution g . Following Sonin, we differentiate this equation with respect $\dot{x}$. We get

$$\begin{aligned} -gF + \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} - \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} \dot{x} \\ = -\frac{\partial g}{\partial \dot{x}} F - g \frac{\partial F}{\partial \dot{x}} + \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} - \frac{\partial^3 \mathcal{L}}{\partial t \partial \dot{x}^2} - \frac{\partial^3 \mathcal{L}}{\partial x \partial \dot{x}^2} \dot{x} - \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} = 0, \end{aligned}$$

hence g must satisfy

$$\frac{\partial g}{\partial \dot{x}} F + g \frac{\partial F}{\partial \dot{x}} + \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \dot{x} = 0. \quad (2.61)$$

This partial differential equation for the unknown function g can be solved by standard methods. $\square$

Consider now the class of second order ordinary differential equations, solved with respect to second derivatives for the unknown curve $x^i = x^i(t)$,

$$F^j - \ddot{x}^j = 0, \quad (2.62)$$

where $F^j(x^i, \dot{x}^i)$ are some given functions. Sometimes the system $F = \{F^j\}$ is called a (contravariant) *force*. Our main objective now will be to find the conditions to ensure the existence of a system of functions $g_{jk} = g_{jk}(x^i, \dot{x}^i)$ such that

$$\det g_{ij} \neq 0, \quad (2.63)$$

and a function $\mathcal{L} = \mathcal{L}(x^i, \dot{x}^i)$ such that

$$g_{ij}(F^j - \ddot{x}^j) = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i}. \quad (2.64)$$

The problem to solve this system is called the *Sonin–Douglas inverse problem of the calculus of variations*. The functions g_{jk} are called *variational multipliers* and $\mathcal{L}$ is the *Lagrangian* for Eq. (2.62). If there exists a solution $\mathcal{L}$, then system (2.62) is said to be *variational*; the force F^j is also said to be *variational*.

Suppose we are given a force $F^j = F^j(x^k, \dot{x}^k)$. Set

$$\varepsilon_i = g_{ij}(F^j - \ddot{x}^j).$$

The following is our basic theorem on the Sonin–Douglas problem.

Theorem 2.7 *The following two conditions are equivalent:*

- (a) *The system ε_i (2.62) is variational.*
- (b) *The functions g_{ij} and F^j satisfy*

$$g_{ij} - g_{ji} = 0, \quad (2.65)$$

$$\frac{\partial g_{ij}}{\partial \dot{x}^l} - \frac{\partial g_{lj}}{\partial \dot{x}^i} = 0, \quad (2.66)$$

$$\frac{1}{2} \left(g_{ij} \frac{\partial F^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j + \frac{\partial g_{il}}{\partial \dot{x}^j} F^j = 0, \quad (2.67)$$

$$\begin{aligned} & \left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) F^j + g_{ij} \frac{\partial F^j}{\partial x^l} - g_{lj} \frac{\partial F^j}{\partial x^i} \\ & - \frac{1}{2} \frac{\partial}{\partial x^j} \left(g_{ik} \frac{\partial F^k}{\partial \dot{x}^l} - g_{lk} \frac{\partial F^k}{\partial \dot{x}^i} \right) \dot{x}^j = 0. \end{aligned} \quad (2.68)$$

Proof In order to derive these formulas we substitute $B_{ij} = g_{ij}$ and $A_i = g_{ij} F^j$ from Theorem 2.2, Sect. 2.4, into Eqs. (2.18)–(2.21). Indeed,

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} \right) + \frac{\partial B_{il}}{\partial x^j} \dot{x}^j &= \frac{1}{2} \left(\frac{\partial g_{ij} F^j}{\partial \dot{x}^l} + \frac{\partial g_{lj} F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j \\ &= \frac{\partial g_{ij}}{\partial \dot{x}^l} F^j + \frac{1}{2} \left(g_{ij} \frac{\partial F^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j = 0 \end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial A^i}{\partial x^l} - \frac{\partial A^l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A^i}{\partial \dot{x}^l} - \frac{\partial A^l}{\partial \dot{x}^i} \right) \dot{x}^j \\
&= \frac{\partial g_{im} F^m}{\partial x^l} - \frac{\partial g_{lm} F^m}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial g_{im} F^m}{\partial \dot{x}^l} - \frac{\partial g_{lm} F^m}{\partial \dot{x}^i} \right) \dot{x}^j \\
&= \left(\frac{\partial g_{im}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^i} \right) F^m + g_{im} \frac{\partial F^m}{\partial x^l} - g_{lm} \frac{\partial F^m}{\partial x^i} \\
&\quad - \frac{1}{2} \frac{\partial}{\partial x^j} \left(g_{im} \frac{\partial F^m}{\partial \dot{x}^l} - g_{lm} \frac{\partial F^m}{\partial \dot{x}^i} \right) \dot{x}^j \\
&= 0.
\end{aligned}$$

□

Remark 2.7 The system of differential equations (2.65)–(2.68) differs from an analogous system used by Douglas [5].

Example 2.2 (Variational forces compatible with metric fields) Suppose that the system (2.60) is variational and has integrating factors forming a metric field $g = g_{jk}(x^i)$. Then the Helmholtz conditions (2.65)–(2.68) reduce to the system

$$g_{ij} - g_{ji} = 0, \quad (2.69)$$

$$\frac{1}{2} \left(g_{ij} \frac{\partial F^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j = 0, \quad (2.70)$$

$$\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) F^j + g_{ij} \frac{\partial F^j}{\partial x^l} - g_{lj} \frac{\partial F^j}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(g_{ik} \frac{\partial F^k}{\partial \dot{x}^l} - g_{lk} \frac{\partial F^k}{\partial \dot{x}^i} \right) \dot{x}^j = 0. \quad (2.71)$$

We determine all the solutions of this system. First we show that a solution is given by the functions

$$F^j = -\frac{1}{2} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s.$$

Indeed, by a direct substitution in (2.70) and (2.71),

$$\begin{aligned}
\frac{1}{2} \left(g_{ij} \frac{\partial F^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j &= -\frac{1}{2} g_{ij} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^l} + \frac{\partial g_{lk}}{\partial x^r} - \frac{\partial g_{rl}}{\partial x^k} \right) \dot{x}^r \\
&\quad - \frac{1}{2} g_{lj} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^r} - \frac{\partial g_{ri}}{\partial x^k} \right) \dot{x}^r + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j \\
&= -\frac{1}{2} \left(\frac{\partial g_{li}}{\partial x^r} + \frac{\partial g_{il}}{\partial x^r} \right) \dot{x}^r + \frac{\partial g_{il}}{\partial x^r} \dot{x}^r = 0
\end{aligned}$$

and

$$\begin{aligned}
& \left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) F^j + g_{ij} \frac{\partial F^j}{\partial x^l} - g_{lj} \frac{\partial F^j}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(g_{ik} \frac{\partial F^k}{\partial x^l} - g_{lk} \frac{\partial F^k}{\partial x^i} \right) \dot{x}^j \\
&= -\frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) g^{kj} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s \\
&\quad - \frac{1}{2} g_{ij} \frac{\partial g^{kj}}{\partial x^l} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s - \frac{1}{2} g_{ij} g^{kj} \frac{\partial}{\partial x^l} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s \\
&\quad + \frac{1}{2} g_{lj} \frac{\partial g^{kj}}{\partial x^i} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s + \frac{1}{2} g_{lj} g^{kj} \frac{\partial}{\partial x^i} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s \\
&\quad + \frac{1}{2} \frac{\partial}{\partial x^k} \left(g_{ij} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^l} + \frac{\partial g_{lk}}{\partial x^r} - \frac{\partial g_{rl}}{\partial x^k} \right) \dot{x}^r - g_{lj} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^r} - \frac{\partial g_{ri}}{\partial x^k} \right) \dot{x}^r \right) \dot{x}^k \\
&= \frac{1}{2} \left(g_{ij} \frac{\partial g^{kj}}{\partial x^l} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) - g_{lj} \frac{\partial g^{kj}}{\partial x^i} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \right. \\
&\quad \left. - g_{ij} \frac{\partial g^{kj}}{\partial x^l} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) + g_{lj} \frac{\partial g^{kj}}{\partial x^i} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \right) \dot{x}^r \dot{x}^s = 0.
\end{aligned}$$

Now, if F_1^j and F_2^j are two solutions and $H^j = F_1^j - F_2^j$, then (2.70) yields

$$g_{ij} \frac{\partial H^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial H^j}{\partial \dot{x}^i} = 0.$$

Solving these equations we get $g_{ij} H^j = \frac{1}{2} (Q_i + P_{ij} \dot{x}^j)$. Summarizing, the general solution of the system (2.69)–(2.71) is given by

$$F^j = -\frac{1}{2} g^{kj} \left(\left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s + P_{jp} \dot{x}^p + Q_j \right). \quad (2.72)$$

Clearly, the functions

$$\Gamma_{rs}^j = \frac{1}{2} g^{kj} \left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \quad (2.73)$$

are the components of the *Levi-Civita connection* associated with the metric field g_{ij} .

Remark 2.8 (*The inverse problem*) Formula (2.72), with given left-hand side, can be considered as an equation for the variational integrating factors g_{ij} . Note that if Γ_{rs}^j are considered as components of a linear connection, then the problem of determining g_{ij} from (2.73) is the *metrizable problem* of the affine connection (see e.g. Tanaka and Krupka [16]). Formula (2.72) also implies that if F^j is not a polynomial of degree 2, then there is no solution of the inverse problem in the form of a metric field $g = g_{jk}(x^i)$.

Example 2.3 Let $F^k = \dot{x}^k$. In this case Eqs. (2.65)–(2.68) reduce to the system

$$\begin{aligned} g_{ij} - g_{ji} &= 0, \quad \frac{\partial g_{ij}}{\partial \dot{x}^l} - \frac{\partial g_{lj}}{\partial \dot{x}^i} = 0, \\ g_{il} + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j + \frac{\partial g_{il}}{\partial \dot{x}^j} \dot{x}^j &= 0, \end{aligned} \quad (2.74)$$

$$\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) \dot{x}^j = 0. \quad (2.75)$$

Formula (2.74) immediately implies that there is no solution g_{il} depending on x^k only. Also, there is no solution in the form of a Finsler metric. Indeed, if

$$\frac{\partial g_{il}}{\partial \dot{x}^j} \dot{x}^j = 0,$$

then

$$g_{il} + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j = 0. \quad (2.76)$$

Differentiating,

$$\frac{\partial g_{il}}{\partial \dot{x}^p} \dot{x}^p + \frac{\partial}{\partial x^j} \left(\frac{\partial g_{il}}{\partial \dot{x}^p} \dot{x}^j \dot{x}^p \right) + \frac{\partial g_{il}}{\partial x^p} \dot{x}^p = \frac{\partial g_{il}}{\partial x^p} \dot{x}^p = 0,$$

hence, by (2.76), $g_{il} = 0$. This is, however, a contradiction, because $\det g_{il} \neq 0$.

Remark 2.9 (Sonin's inverse problem) Theorem 2.7, applied to one differential equation $F - \ddot{x} = 0$, states that the function $\varepsilon = g(F - \ddot{x})$ is variational if and only if

$$g \frac{\partial F}{\partial \dot{x}} + \frac{\partial g}{\partial x} \dot{x} + \frac{\partial g}{\partial \dot{x}} F = 0.$$

This formula agrees with Eq. (2.61), solving the Sonin's problem.

Now return to the general Sonin–Douglas problem, expressed by formulas (2.63) and (2.64). Consider a system $g = g_{ij}$ such that

$$g_{ij} - g_{ji} = 0, \quad \frac{\partial g_{ij}}{\partial \dot{x}^l} - \frac{\partial g_{lj}}{\partial \dot{x}^i} = 0$$

and the associated Lagrangian

$$\mathcal{L} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j,$$

where

$$h_{ij} = 2 \int_0^1 \left(\int_0^1 g_{ij}(x^p, \lambda \tau \dot{x}^p) d\lambda \right) \tau d\tau$$

(Lemmas 2.4, 2.5, Sect. 2.5). This Lagrangian satisfies

$$g_{ij} = \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^i \partial \dot{x}^j}$$

and its Euler–Lagrange expressions are

$$\begin{aligned} & \frac{\partial \mathcal{L}_h}{\partial x^k} - \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^k} \dot{x}^l - \frac{\partial^2 \mathcal{L}_h}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^l \\ &= \dot{x}^i \dot{x}^j \int_0^1 \int_0^1 \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial^2 g_{ij}}{\partial x^l \partial \dot{x}^k} \dot{x}^l \right)_{(x^p, \lambda \tau \dot{x}^p)} d\lambda \tau d\tau - g_{lk} \ddot{x}^l \end{aligned}$$

(Lemma 2.6, Sect. 2.5).

Theorem 2.8 *The following two conditions are equivalent:*

(a) *The variational multipliers g_{ij} satisfy*

$$g_{ij} - g_{ji} = 0, \quad \frac{\partial g_{ij}}{\partial \dot{x}^l} - \frac{\partial g_{lj}}{\partial \dot{x}^i} = 0,$$

and

$$g_{is} F^s = \frac{\partial P}{\partial x^i} + \left(\frac{\partial Q_s}{\partial x^i} - \frac{\partial Q_i}{\partial x^s} \right) \dot{x}^s$$

for some functions $P = P(x^k)$ and $Q_i = Q_i(x^k)$.

(b) *The pair (g_{ij}, F^k) solves the Sonin–Douglas problem.*

Proof The left-hand side of Eq. (2.64) can be expressed as

$$\begin{aligned} g_{ij}(F^j - \ddot{x}^j) &= g_{ij} F^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^l \partial \dot{x}^k} \ddot{x}^j \\ &= \frac{\partial \mathcal{L}_h}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \dot{x}^l - \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \ddot{x}^j + g_{ij} F^j - \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial^2 \mathcal{L}}{\partial x^l \partial \dot{x}^i} \dot{x}^l. \end{aligned} \tag{2.77}$$

Therefore, the Sonin–Douglas problem reduces to the problem of existence of a function $\mathcal{L}_0 = \mathcal{L}_0(x^k, \dot{x}^k)$ such that

$$g_{ij} F^j - \frac{\partial \mathcal{L}_h}{\partial x^i} + \frac{\partial^2 \mathcal{L}_h}{\partial x^l \partial \dot{x}^i} = \frac{\partial \mathcal{L}_0}{\partial x^i} - \frac{\partial^2 \mathcal{L}_0}{\partial x^l \partial \dot{x}^i} \dot{x}^l.$$

This means in particular that the functions $g_{ij} F^j$ are variational, and satisfy the Helmholtz conditions. Since these functions do not depend on $\ddot{x}^j$, we get, by setting $A_i = g_{is} F^s$ in Theorem 2.2 from Sect. 2.4,

$$\begin{aligned} \frac{\partial A_i}{\partial \dot{x}^l} + \frac{\partial A_l}{\partial \dot{x}^i} &= 0, \\ \frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j &= 0. \end{aligned} \quad (2.78)$$

Equation (2.78) implies that

$$A_i = a_i + b_{is} \dot{x}^s, \quad (2.79)$$

where $a_i = a_i(x^k)$, $b_{is} = b_{is}(x^k)$ and $b_{is} + b_{si} = 0$. By substituting in (2.79),

$$\begin{aligned} \frac{\partial A_i}{\partial x^l} - \frac{\partial A_l}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial \dot{x}^l} - \frac{\partial A_l}{\partial \dot{x}^i} \right) \dot{x}^j \\ = \frac{\partial a_i}{\partial x^l} - \frac{\partial a_l}{\partial x^i} + \frac{\partial b_{is}}{\partial x^l} \dot{x}^s - \frac{\partial b_{ls}}{\partial x^i} \dot{x}^s - \frac{1}{2} \frac{\partial (b_{il} - b_{li})}{\partial x^j} \dot{x}^j \\ = \frac{\partial a_i}{\partial x^l} - \frac{\partial a_l}{\partial x^i} + \left(\frac{\partial b_{is}}{\partial x^l} - \frac{\partial b_{ls}}{\partial x^i} - \frac{\partial b_{li}}{\partial x^s} \right) \dot{x}^s = 0. \end{aligned} \quad (2.80)$$

Thus, the coefficients of A_i satisfy

$$\frac{\partial a_i}{\partial x^l} - \frac{\partial a_l}{\partial x^i} = 0, \quad b_{is} + b_{si} = 0, \quad \frac{\partial b_{is}}{\partial x^l} + \frac{\partial b_{sl}}{\partial x^i} + \frac{\partial b_{li}}{\partial x^s} = 0.$$

By integration (cf. Lemma 2.3, Sect. 2.3),

$$a_i = \frac{\partial P}{\partial x^i}, \quad b_{is} = \frac{\partial Q_s}{\partial x^i} - \frac{\partial Q_i}{\partial x^s}$$

and substituting in (2.80), we get (2.77). $\square$

Formula (2.77) is the *compatibility condition* for the variational integrating factors g_{ij} and the force F^j .

Example 2.4 If $m = 2$, then Theorem 2.8 reduces to the equations

$$g_{12} - g_{21} = 0, \quad \frac{\partial g_{11}}{\partial \dot{x}^2} - \frac{\partial g_{12}}{\partial \dot{x}^1} = 0, \quad \frac{\partial g_{22}}{\partial \dot{x}^1} - \frac{\partial g_{12}}{\partial \dot{x}^2} = 0$$

and

$$g_{11} F^1 + g_{12} F^2 = \frac{\partial P}{\partial x^1} + \left(\frac{\partial Q_2}{\partial x^1} - \frac{\partial Q_1}{\partial x^2} \right) \dot{x}^2, \quad (2.81)$$

$$g_{21} F^1 + g_{22} F^2 = \frac{\partial P}{\partial x^2} + \left(\frac{\partial Q_1}{\partial x^2} - \frac{\partial Q_2}{\partial x^1} \right) \dot{x}^1. \quad (2.82)$$

This system can be considered as a system of non-homogeneous equations for the pair (F^1, F^2) : given g_{11} , g_{12} , and g_{22} , a solution (F^1, F^2) always exists and is unique. In this sense, letting g_{ij} run through all solutions of the system (2.81), we get all solutions of Sonin–Douglas problem in an implicit form.

2.7 Finsler Metrics

In this section we introduce, in the simplified setting for open subsets of Euclidean spaces, the basic concepts of Finsler geometry as needed in our discussion of the Sonin–Douglas inverse problem. For the general theory see e.g. Shen [14].

By a *Finsler metric* on an open set $U \subset \mathbb{R}^n$ we mean a system $g = \{g_{jk}\}$ of functions $g_{ij} : U \times \mathbb{R}^n \rightarrow \mathbb{R}$, $g_{ij} = g_{ij}(x^k, \dot{x}^k)$, satisfying the following two conditions:

- (a) The matrix g_{ij} is symmetric and non-singular on $U \times \mathbb{R}^n$:

$$g_{ij} = g_{ji}, \quad \det g_{ij} \neq 0, \quad (2.83)$$

and positive definite.

- (b) The derivatives satisfy

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j} \quad (2.84)$$

and

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \quad (2.85)$$

Using the *Cartan tensor* $C = \{C_{ijk}\}$ of the Finsler metric g (see Sect. 2.2),

$$C_{ijk} = \frac{1}{3} \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} + \frac{\partial g_{jk}}{\partial \dot{x}^i} + \frac{\partial g_{ki}}{\partial \dot{x}^j} \right),$$

and condition (2.84), condition (2.85) can also be expressed as

$$C_{ijk} \dot{x}^k = 0.$$

Note that condition (a) implies that for every vector field ξ on U , expressed as

$$\xi = \xi^k \frac{\partial}{\partial x^k},$$

the system of functions $g^\xi = \{g_{ij}^\xi\}$, defined by the condition

$$g_{ij}^\xi(x^l) = g_{ij}(x^l, \xi^k(x^l)),$$

is a regular metric field on U . This metric field satisfies at every point in its domain of definition

$$\left(\frac{\partial g_{ij}^\xi}{\partial x^m} \right)_{(x^p)} = \left(\frac{\partial g_{ij}}{\partial x^m} \right)_{(x^l, \xi^k(x^l))} + \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \right)_{(x^l, \xi^k(x^l))} \left(\frac{\partial \xi^k}{\partial x^m} \right)_{(x^l)}.$$

Since

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = C_{ijk} - \frac{1}{3} \left(\frac{\partial g_{jk}}{\partial \dot{x}^i} - \frac{\partial g_{ij}}{\partial \dot{x}^k} \right) - \frac{1}{3} \left(\frac{\partial g_{ki}}{\partial \dot{x}^j} - \frac{\partial g_{ij}}{\partial \dot{x}^k} \right),$$

we can also write

$$\frac{\partial g_{ij}^\xi}{\partial x^m} = \frac{\partial g_{ij}}{\partial x^m} + C_{ijk} \frac{\partial \xi^k}{\partial x^m}.$$

The following lemma states that a Finsler metric can be defined by means of the second derivatives of a function.

Lemma 2.8 *Let $g = \{g_{jk}\}$ be a system of functions. The following two conditions are equivalent:*

- (a) g is a Finsler metric.
- (b) There exists a function $\mathcal{L} = \mathcal{L}(x^i, \dot{x}^i)$ such that

$$g_{ij} = \frac{\partial^2 \mathcal{L}^2}{\partial \dot{x}^i \partial \dot{x}^j}$$

and

$$\mathcal{L}(x^i, \lambda \dot{x}^i) = \lambda \mathcal{L}(x^i, \dot{x}^i)$$

for all $\lambda > 0$.

Proof 1. Suppose we have a Finsler metric g . Consider the energy Lagrangian $\mathcal{L}_g$, associated with g . By Lemma 2.1, Sect. 2.2, the corresponding Euler-Lagrange expressions are

$$\begin{aligned} \frac{\partial \mathcal{L}_g}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} &= -\frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^s \partial \dot{x}^k} \dot{x}^s \dot{x}^i \dot{x}^j \\ &\quad - \left(\frac{1}{2} \frac{\partial^2 g_{ij}}{\partial \dot{x}^s \partial \dot{x}^k} \dot{x}^i \dot{x}^j + \left(\frac{\partial g_{is}}{\partial \dot{x}^k} + \frac{\partial g_{ik}}{\partial \dot{x}^s} \right) \dot{x}^i + g_{sk} \right) \ddot{x}^s. \end{aligned}$$

Using conditions (2.83)–(2.85), we get

$$\begin{aligned} \frac{\partial^2 g_{ij}}{\partial x^s \partial \dot{x}^k} \dot{x}^s \dot{x}^i \dot{x}^j &= 0, \quad \left(\frac{\partial g_{is}}{\partial \dot{x}^k} + \frac{\partial g_{ik}}{\partial \dot{x}^s} \right) \dot{x}^i = 0, \\ \frac{\partial^2 g_{ij}}{\partial \dot{x}^s \partial \dot{x}^k} \dot{x}^i \dot{x}^j &= \frac{\partial}{\partial \dot{x}^k} \left(\frac{\partial g_{ij}}{\partial \dot{x}^s} \dot{x}^i \dot{x}^j \right) - 2 \frac{\partial g_{ik}}{\partial \dot{x}^s} \dot{x}^i = 0. \end{aligned}$$

Thus, the Euler–Lagrange expressions are

$$\frac{\partial \mathcal{L}_g}{\partial x^k} - \frac{d}{dt} \frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} = -\frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i \dot{x}^j - g_{sk} \ddot{x}^s,$$

and hence

$$\frac{\partial^2 \mathcal{L}_g}{\partial \dot{x}^k \partial \dot{x}^l} = g_{kl}.$$

The Lagrange function $\mathcal{L}_g$ satisfies, for any $\lambda > 0$,

$$\mathcal{L}_g(x^p, \lambda \dot{x}^p) = \frac{1}{2} g_{ij}(x^p, \lambda \dot{x}^p) \lambda^2 \dot{x}^i \dot{x}^j.$$

Differentiating with respect to λ ,

$$\left(\frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} \right)_{(x^p, \lambda \dot{x}^p)} \dot{x}^k = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \right)_{(x^p, \lambda \dot{x}^p)} \dot{x}^k \lambda^2 \dot{x}^i \dot{x}^j + g_{ij}(x^p, \lambda \dot{x}^p) \lambda \dot{x}^i \dot{x}^j,$$

hence

$$\begin{aligned} \left(\frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} \right)_{(x^p, \lambda \dot{x}^p)} \lambda \dot{x}^k &= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \right)_{(x^p, \lambda \dot{x}^p)} \lambda^3 \dot{x}^k \dot{x}^i \dot{x}^j + g_{ij}(x^p, \lambda \dot{x}^p) \lambda^2 \dot{x}^i \dot{x}^j \\ &= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial \dot{x}^k} \right)_{(x^p, \lambda \dot{x}^p)} \lambda^3 \dot{x}^k \dot{x}^i \dot{x}^j + 2\mathcal{L}_g(x^p, \lambda \dot{x}^p) \end{aligned}$$

or, for $\lambda = 1$,

$$\frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} \dot{x}^k = \frac{1}{2} C_{ijk} \dot{x}^k \dot{x}^i \dot{x}^j + 2\mathcal{L}_g.$$

But by hypothesis, $C_{ijk} \dot{x}^k = 0$, so we have

$$\frac{\partial \mathcal{L}_g}{\partial \dot{x}^k} \dot{x}^k = 2\mathcal{L}_g.$$

Now, since $\mathcal{L}_g$ is everywhere positive, the formula

$$\mathcal{L} = \sqrt{\mathcal{L}_g}$$

defines a function such that

$$\mathcal{L}(x^p, \lambda \dot{x}^p) = \sqrt{\mathcal{L}_g(x^p, \lambda \dot{x}^p)} = \sqrt{\lambda^2 \mathcal{L}_g(x^p, \dot{x}^p)} = \lambda \mathcal{L}(x^p, \dot{x}^p).$$

This proves Lemma 2.8. □

Remark 2.10 If a Finsler metric $g_{ij} = g_{ij}(x^k, \dot{x}^k)$ on $U \subset \mathbb{R}^n$ admits a *diagonal* form, then it is a regular metric field on U . Indeed, in this case formula (2.84) is identically satisfied for any distinct indices i, j, l ; if $i = j$, then

$$\frac{\partial g_{ii}}{\partial \dot{x}^l} - \frac{\partial g_{li}}{\partial \dot{x}^i} = 0,$$

and if $l \neq i$ we get

$$\frac{\partial g_{ii}}{\partial \dot{x}^i} = 0, \quad l \neq i$$

for all $i, 1 \leq i \leq n$, and l . Formula (2.85) now yields, for diagonal terms,

$$\frac{\partial g_{ii}}{\partial \dot{x}^k} \dot{x}^k = \frac{\partial g_{ii}}{\partial \dot{x}^i} \dot{x}^i = 0,$$

proving that

$$\frac{\partial g_{ii}}{\partial \dot{x}^i} = 0.$$

Obviously, the same is true for the metrics whose non-diagonal elements are constant.

Remark 2.11 We can find a relation between the Euler–Lagrange expressions of the Lagrangians $\mathcal{L}$ and $\mathcal{L}^2$ where

$$\mathcal{L} = \sqrt{g_{pq} \dot{x}^p \dot{x}^q}.$$

Differentiating, we have

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{1}{2\mathcal{L}} \frac{\partial g_{pq}}{\partial x^i} \dot{x}^p \dot{x}^q, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{g_{pi} \dot{x}^p}{\mathcal{L}}$$

and

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} &= \frac{\frac{\partial g_{pi}}{\partial x^j} \dot{x}^p \mathcal{L} - g_{si} \dot{x}^s \frac{\partial \mathcal{L}}{\partial x^j}}{\mathcal{L}^2} \\ &= \frac{\frac{\partial g_{pi}}{\partial x^j} \dot{x}^p \mathcal{L} - \frac{1}{2\mathcal{L}} g_{si} \dot{x}^s \frac{\partial g_{pq}}{\partial x^j} \dot{x}^p \dot{x}^q}{\mathcal{L}^2} \\ &= \frac{1}{\mathcal{L}} \frac{\partial g_{pi}}{\partial x^j} \dot{x}^p - \frac{1}{2\mathcal{L}^3} \frac{\partial g_{pq}}{\partial x^j} g_{si} \dot{x}^p \dot{x}^q \dot{x}^s \\ &= \frac{1}{\mathcal{L}} \frac{\partial g_{pi}}{\partial x^j} \dot{x}^p - \frac{1}{2\mathcal{L}^3} \frac{\partial (g_{pq} g_{si})}{\partial x^j} \dot{x}^p \dot{x}^q \dot{x}^s + \frac{1}{2\mathcal{L}} \frac{\partial g_{si}}{\partial x^j} \dot{x}^s, \\ \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j} &= \frac{g_{ji} \mathcal{L} - g_{pi} \dot{x}^p \frac{\partial \mathcal{L}}{\partial \dot{x}^j}}{\mathcal{L}^2} = \frac{g_{ji}}{\mathcal{L}^2} - \frac{g_{pi} \dot{x}^p}{\mathcal{L}^2} \frac{\partial \mathcal{L}}{\partial \dot{x}^j} = \frac{g_{ji}}{\mathcal{L}} - \frac{g_{pi} g_{qj} \dot{x}^p \dot{x}^q}{\mathcal{L}^3}. \end{aligned}$$

Calculating the Euler–Lagrange expressions for $\mathcal{L}^2 = g_{pq}\dot{x}^p\dot{x}^q$, we get

$$\begin{aligned}
 & \frac{1}{2} \left(\frac{\partial \mathcal{L}^2}{\partial x^i} - \frac{\partial^2 \mathcal{L}^2}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}^2}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) \\
 &= \mathcal{L} \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial}{\partial x^j} \left(\mathcal{L} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) \dot{x}^j - \frac{\partial}{\partial \dot{x}^j} \left(\mathcal{L} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) \ddot{x}^j \\
 &= \mathcal{L} \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) - \frac{\partial \mathcal{L}}{\partial x^j} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \dot{x}^j - \frac{\partial \mathcal{L}}{\partial \dot{x}^j} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \ddot{x}^j \\
 &= \mathcal{L} \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) - \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \left(\frac{\partial \mathcal{L}}{\partial x^j} \dot{x}^j - \frac{\partial \mathcal{L}}{\partial \dot{x}^j} \ddot{x}^j \right) \\
 &= \mathcal{L} \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) - \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \frac{d\mathcal{L}}{dt}.
 \end{aligned}$$

Calculating the Euler–Lagrange expressions for $\mathcal{L}$, we get

$$\begin{aligned}
 & \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \\
 &= \frac{1}{2\mathcal{L}} \frac{\partial g_{pq}}{\partial x^i} \dot{x}^p \dot{x}^q - \frac{1}{\mathcal{L}} \frac{\partial g_{pi}}{\partial x^j} \dot{x}^p \dot{x}^j - \frac{g_{ji}}{\mathcal{L}} \ddot{x}^j + \frac{1}{2\mathcal{L}^3} \frac{\partial g_{pq}}{\partial x^j} g_{si} \dot{x}^p \dot{x}^q \dot{x}^s \dot{x}^j \\
 &\quad + \frac{g_{pi} g_{qj} \dot{x}^p \dot{x}^q}{\mathcal{L}^3} \ddot{x}^j \\
 &= \frac{1}{\mathcal{L}} \left(-g_{ji} \ddot{x}^j + \frac{1}{2} \left(\frac{\partial g_{pq}}{\partial x^i} - \frac{\partial g_{pi}}{\partial x^q} - \frac{\partial g_{qi}}{\partial x^p} \right) \dot{x}^p \dot{x}^q \right) \\
 &\quad + g_{si} \dot{x}^s \frac{1}{\mathcal{L}^2} \left(\frac{1}{2\mathcal{L}} \frac{\partial g_{pq}}{\partial x^j} \dot{x}^p \dot{x}^q \dot{x}^j + \frac{g_{qj} \dot{x}^q}{\mathcal{L}} \ddot{x}^j \right) \\
 &= \frac{1}{\mathcal{L}} \left(- \left(g_{ji} \ddot{x}^j + \frac{1}{2} \left(\frac{\partial g_{pi}}{\partial x^q} + \frac{\partial g_{qi}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^i} \right) \dot{x}^p \dot{x}^q \right) + g_{si} \dot{x}^s \frac{1}{\mathcal{L}} \frac{d\mathcal{L}}{dt} \right).
 \end{aligned}$$

Combining these formulas we have the following relation with the Euler–Lagrange expressions for $\mathcal{L} = \sqrt{g_{pq}\dot{x}^p\dot{x}^q}$:

$$\begin{aligned}
 & \frac{1}{2} \left(\frac{\partial \mathcal{L}^2}{\partial x^i} - \frac{\partial^2 \mathcal{L}^2}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}^2}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) \\
 &= \mathcal{L} \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j \right) - \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \frac{d\mathcal{L}}{dt} \\
 &= - \left(g_{ji} \ddot{x}^j + \frac{1}{2} \left(\frac{\partial g_{pi}}{\partial x^q} + \frac{\partial g_{qi}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^i} \right) \dot{x}^p \dot{x}^q \right) + g_{si} \dot{x}^s \frac{1}{\mathcal{L}} \frac{d\mathcal{L}}{dt} - \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \frac{d\mathcal{L}}{dt} \\
 &= - \left(g_{ji} \ddot{x}^j + \frac{1}{2} \left(\frac{\partial g_{pi}}{\partial x^q} + \frac{\partial g_{qi}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^i} \right) \dot{x}^p \dot{x}^q \right).
 \end{aligned}$$

In particular, for any fixed immersed curve $t \rightarrow \zeta(t) = \{x^i(t)\}$ we may use parametrization by the *arc s* of ζ , defined by

$$s = \int_{t_0}^t \mathcal{L}(x^k(t), \dot{x}^k(t)) dt, \quad \frac{ds}{dt} = \mathcal{L}(x^k(t), \dot{x}^k(t)).$$

Then, setting $s = t$, we get, along ζ ,

$$2g_{iq}\dot{x}^q \frac{1}{\mathcal{L}} \frac{d\mathcal{L}}{dt} = 2g_{iq}\dot{x}^q \left(\frac{ds}{dt} \right)^{-1} \frac{d^2s}{dt^2} = 0.$$

Thus, a curve parametrized by arc length is a solution of the Euler–Lagrange equation for the Lagrangian $\mathcal{L}$ if and only if it is a solution of the Euler–Lagrange equation for $\mathcal{L}^2$. Given a Finsler metric g_{ij} , one can easily determine variational forces F^i , compatible with g_{ij} . To this purpose, we shall consider the corresponding Sonin–Douglas problem

$$g_{ij}(F^j - \ddot{x}^j) = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i}$$

together with an additional condition

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \quad (2.86)$$

We proceed in a manner similar to that of Example 2.2 of Sect. 2.6. Consider the system consisting of the equations

$$g_{ij} - g_{ji} = 0, \quad (2.87)$$

$$\frac{\partial g_{ij}}{\partial \dot{x}^l} - \frac{\partial g_{lj}}{\partial \dot{x}^i} = 0, \quad (2.88)$$

$$\frac{1}{2} \left(g_{ij} \frac{\partial F^j}{\partial \dot{x}^l} + g_{lj} \frac{\partial F^j}{\partial \dot{x}^i} \right) + \frac{\partial g_{il}}{\partial x^j} \dot{x}^j \frac{\partial g_{il}}{\partial \dot{x}^j} F^j = 0, \quad (2.89)$$

$$\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} \right) F^j + g_{ij} \frac{\partial F^j}{\partial x^l} - g_{lj} \frac{\partial F^j}{\partial x^i} - \frac{1}{2} \frac{\partial}{\partial x^j} \left(g_{ik} \frac{\partial F^k}{\partial \dot{x}^l} - g_{lk} \frac{\partial F^k}{\partial \dot{x}^i} \right) \dot{x}^j = 0, \quad (2.90)$$

for F^i and g_{ij} , and Eq. (2.86).

Theorem 2.9 *The general solution of the system (2.86)–(2.90) is given by*

$$F^j = -\frac{1}{2} g^{kj} \left(\left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k} \right) \dot{x}^r \dot{x}^s + P_{jp} \dot{x}^p + Q_j \right),$$

where $P = P(x^k)$ and $Q_i = Q_i(x^k)$ are some arbitrary functions.

Proof We claim that the functions

$$F^j = -\frac{1}{2}g^{kj}\left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k}\right)\dot{x}^r\dot{x}^s$$

solve Eqs. (2.89) and (2.90). Indeed,

$$\begin{aligned} \frac{1}{2}\left(g_{ij}\frac{\partial F^j}{\partial \dot{x}^l} + g_{lj}\frac{\partial F^j}{\partial \dot{x}^i}\right) &= \frac{1}{2}\left(\frac{\partial g_{ij}F^j}{\partial \dot{x}^l} + \frac{\partial g_{lj}F^j}{\partial \dot{x}^i}\right) \\ &= -\frac{1}{4}\left(\frac{\partial}{\partial \dot{x}^l}\left(\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right)\dot{x}^r\dot{x}^s\right)\right. \\ &\quad \left.+ \frac{\partial}{\partial \dot{x}^i}\left(\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^l}\right)\dot{x}^r\dot{x}^s\right)\right) \\ &= -\frac{\partial g_{li}}{\partial x^r}\dot{x}^r. \end{aligned}$$

Hence

$$\frac{1}{2}\left(g_{ij}\frac{\partial F^j}{\partial \dot{x}^l} + g_{lj}\frac{\partial F^j}{\partial \dot{x}^i}\right) + \frac{\partial g_{il}}{\partial x^j}\dot{x}^j = 0,$$

proving (2.89). Similarly, since

$$\begin{aligned} g_{ij}F^j &= -\frac{1}{2}g_{ij}g^{kj}\left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k}\right)\dot{x}^r\dot{x}^s \\ &= -\frac{1}{2}\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right)\dot{x}^r\dot{x}^s, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial g_{ij}F^j}{\partial x^l} - \frac{\partial g_{lj}F^j}{\partial x^i} &= -\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right)\dot{x}^r\dot{x}^s \\ &\quad + \frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^l}\right)\dot{x}^r\dot{x}^s \\ &= -\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r}\right)\dot{x}^r\dot{x}^s + \frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r}\right)\dot{x}^r\dot{x}^s \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_{ik}F^k}{\partial \dot{x}^l} - \frac{\partial g_{lk}F^k}{\partial \dot{x}^i} &= -\frac{1}{2}\frac{\partial}{\partial \dot{x}^l}\left(\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right)\dot{x}^r\dot{x}^s\right) \\ &\quad + \frac{1}{2}\frac{\partial}{\partial \dot{x}^i}\left(\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^l}\right)\dot{x}^r\dot{x}^s\right) \end{aligned}$$

$$\begin{aligned}
&= -\left(\frac{\partial g_{ri}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^r} - \frac{\partial g_{rl}}{\partial x^i}\right)\dot{x}^r + \left(\frac{\partial g_{rl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^r} - \frac{\partial g_{ri}}{\partial x^l}\right)\dot{x}^r \\
&= 2\left(-\frac{\partial g_{ri}}{\partial x^l} + \frac{\partial g_{rl}}{\partial x^i}\right)\dot{x}^r.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i}\right)F^j + g_{ij}\frac{\partial F^j}{\partial x^l} - g_{lj}\frac{\partial F^j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^j}\left(g_{ik}\frac{\partial F^k}{\partial \dot{x}^l} - g_{lk}\frac{\partial F^k}{\partial \dot{x}^i}\right)\dot{x}^j \\
&= \frac{\partial g_{ij}F^j}{\partial x^l} - \frac{\partial g_{lj}F^j}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{ik}F^k}{\partial \dot{x}^l} - \frac{\partial g_{lk}F^k}{\partial \dot{x}^i}\right)\dot{x}^j \\
&= -\frac{1}{2}\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ri}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^r}\right)\dot{x}^r\dot{x}^s + \frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial g_{rl}}{\partial x^s} + \frac{\partial g_{sl}}{\partial x^r}\right)\dot{x}^r\dot{x}^s \\
&\quad - \frac{\partial}{\partial x^s}\left(\frac{\partial g_{ri}}{\partial x^l} + \frac{\partial g_{rl}}{\partial x^i}\right)\dot{x}^r\dot{x}^s \\
&= -\frac{\partial^2 g_{ri}}{\partial x^l\partial x^s}\dot{x}^r\dot{x}^s + \frac{\partial^2 g_{rl}}{\partial x^i\partial x^s}\dot{x}^r\dot{x}^s + \frac{\partial^2 g_{ri}}{\partial x^s\partial x^l}\dot{x}^r\dot{x}^s - \frac{\partial^2 g_{rl}}{\partial x^s\partial x^i}\dot{x}^r\dot{x}^s \\
&= 0
\end{aligned}$$

as desired. $\square$

Now, if F_1^j and F_2^j are two solutions and $H^j = F_1^j - F_2^j$, then (2.89) implies

$$\frac{1}{2}\left(g_{ij}\frac{\partial H^j}{\partial \dot{x}^l} + g_{lj}\frac{\partial H^j}{\partial \dot{x}^i}\right) + \frac{\partial g_{il}}{\partial \dot{x}^j}H^j = \frac{1}{2}\left(\frac{\partial g_{ij}H^j}{\partial \dot{x}^l} + \frac{\partial g_{lj}H^j}{\partial \dot{x}^i}\right) = 0, \quad (2.91)$$

hence $g_{ij}H^j = \frac{1}{2}(Q_i + P_{ij}\dot{x}^j)$ for some functions Q_i and P_{ij} depending only on x^k . Thus, $H^k = \frac{1}{2}g^{ik}(Q_i + P_{ij}\dot{x}^j)$.

Formula (2.91) is the *compatibility condition* for the variational integrating factors and the force. The functions

$$\Delta_{rs}^j = \frac{1}{2}g^{kj}\left(\frac{\partial g_{rk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^k}\right)$$

are the *geodesic coefficients* of the Finsler metric g_{ij} .

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